RESEARCH SUMMARY

AMNON YEKUTIELI

This summary covers the research in mathematics I have done since 1990. At the end I also mention several topics for future research.

Contents

1. Noncommutative Grothendieck Duality

1.1. **Duality for Noncommutative Rings.** Grothendieck Duality was originally formulated in (commutative) algebraic geometry, in the seminal text [22]. In my thesis, and later in [67], I introduced the noncommutative version of Grothendieck Duality. The main idea in [67] was that a *dualizing complex* R over a noncommutative $\mathbb{k}$-algebra A ($\mathbb{k}$ is the base field) should be a complex of *bimodules*, and the derived functors $\mathrm{RHom}_A(-, R)$ and $\mathrm{RHom}_{A^{\mathrm{op}}}(-, R)$ should induce a duality (anti-equivalence) between the derived categories of *left* and *right* A-modules. Here A^{op} stands for the opposite algebra, so right A-modules are naturally considered as left A^{op}-modules. Of course we must impose some finiteness conditions. Usually we assume A is noetherian, and the duality is between the bounded derived categories with finite cohomologies $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$.

The impetus to generalize Grothendieck Duality from commutative algebraic geometry to noncommutative rings came from my thesis advisor Michael Artin, in connection with his work on local duality for regular graded algebras [2], [4] and [5]. More on this in §1.2 below.

Noncommutative dualizing complexes have developed into a powerful method, with a variety of applications, reaching from ring theory to representations of Lie algebras, and even to theoretical physics. The paper [67] was cited so far in 22 papers by various authors.

1.2. **Balanced Dualizing Complexes.** Actually, in [67] I considered noetherian connected graded $\mathbb{k}$-algebras. It turns out that local duality is not automatic for a noncommutative graded algebra, even if a dualizing complex does exist. Something more is needed, which is why I defined the *balanced dualizing complex*. The existence

Date: 4 December 2001.

of a balanced dualizing complex was verified in some important cases, including the twisted homogeneous coordinate rings of [6].

For a few years the true meaning of balanced dualizing complexes was obscure. But in [80] J.J. Zhang and I proved that the existence of a balanced dualizing complex over the algebra A implies the condition χ of Artin-Zhang [7], and the finite dimensionality of the projective spectrum $\operatorname{Proj} A$. Then in 1996 M. Van den Bergh [62] proved the converse statement. This directly implied existence of balanced dualizing complexes for a wide class of graded $\Bbbk$-algebras, for instance PI (polynomial identity) algebras.

1.3. **Duality on Noncommutative Projective Schemes.** As mentioned above, in [80] J.J. Zhang and I proved that for a large class of graded algebras A, including regular Artin-Schelter algebras, the projective spectrum $\operatorname{Proj} A$ of [7] has a global Serre-Grothendieck duality. Among other applications, recently this duality featured in a paper in theoretical physics by A. Kapustin, A. Kuznetsov and D. Orlov [35].

1.4. **Canonical Deformations.** The paper [69] examined noncommutative deformations of the commutative polynomial algebra $A = \Bbbk[x_1, \ldots, x_n]$ and its de Rham complex $\Omega^{\cdot}_{A/\Bbbk}$. The deformed algebra A_q is called a quantum affine space. These deformations are related to work of Wess-Zumino, Maltsiniotis, Manin and Artin-Schelter-Tate [3]. I proved that there exists a deformation of $\Omega^{\cdot}_{A/\Bbbk}$, called the canonical deformation and denoted by $\Omega^{\cdot}_q$, such that $\Omega^0_q = A_q$, and $\Omega^n_q[n]$ is the balanced dualizing complex of A_q.

2. Rigid Auslander Dualizing Complexes

2.1. **Rigid Dualizing Complexes.** Another development due to Van den Bergh [62] is the introduction of *rigid* dualizing complexes. A dualizing complex R over a noncommutative $\Bbbk$-algebra A is called rigid if

$$R \cong \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \tag{1}$$

in the derived category $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, where $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$.

Observe that for a commutative ring A, letting $X := \operatorname{Spec} A$, $\pi : X \to \operatorname{Spec} \Bbbk$ the structural morphism and $R := \pi^! \Bbbk$ the twisted inverse image of the module $\Bbbk$, one has a canonical isomorphism (1) coming from flat base change, cf. [22].

Van den Bergh proved that a rigid dualizing complex is unique up to isomorphism. He also proved that if A is a graded algebra with balanced dualizing complex R, then R is also rigid.

In [81] Zhang and I proved that the rigid dualizing complex R_A is unique up to a unique isomorphism. Moreover for a finite homomorphism of rings $A \to B$ there is at most one rigid trace morphism $\operatorname{Tr}_{B/A} : R_B \to R_A$, i.e. a morphism in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ that is nondegenerate and compatible with the rigidifying isomorphisms (1).

2.2. **Auslander Dualizing Complexes.** The Auslander-Gorenstein property of a noncommutative algebra A is extremely useful for applications (cf. work of Björk [11] and Levasseur [44]). However it only makes sense when A is Gorenstein, which is a rather strong regularity condition. In [72] I suggested that instead one should look for a dualizing complex R which has the Auslander property. This property asserts that for every finite A-module M, every integer p, every A^{op}-submodule

$N \subset \mathrm{Ext}^p_A(M,R)$ and every integer $q < p$, the module $\mathrm{Ext}^q_{A^{\mathrm{op}}}(N,R)$ vanishes; and the same holds after exchanging A with A^{op}.

My basic observation was that the double Ext spectral sequence arguments used by Björk and Levasseur apply (with minor modifications) to the situation of an Auslander dualizing complex. And as shown in my paper with J.J. Zhang [81], a large class of algebras admit an Auslander dualizing complex, including many so-called quantum algebras.

Existence of an Auslander dualizing complex has many applications. For instance, we obtained the Gabber Maximality Principal for A-modules, the catenarity of the prime spectrum $\operatorname{Spec} A$ (extending work of Goodearl-Lenagan [21]), and the finiteness of the Krull dimension $\operatorname{Kdim} A$, for many algebras A.

Q.S. Wu and Zhang used Auslander dualizing complexes in their paper [66] to prove, among other things, that a PI Hopf algebra A admitting a noetherian connected filtration (e.g. any A finite over its center) is necessarily Gorenstein. This had been conjectured by K.A. Brown.

The paper [81] was already cited in 9 articles by various authors.

2.3. **The Residue Complex of a Noncommutative Ring.** The study of the *residue complex* $\mathcal{K}^{\cdot}_A$ of a noncommutative algebra began in [72], and better results are contained in my paper with Zhang [82]. The Cousin functor E, introduced by Grothendieck, is a rather subtle operation in commutative algebraic geometry. We showed that it exists also in the noncommutative setting, once interpreted correctly. The key is to consider the category $\mathsf{Mod}\, A$ of A-modules as a "space" filtered according to a suitable dimension function. Indeed when A has an Auslander rigid dualizing complex R we take the canonical dimension function Cdim determined by R, namely

$$\operatorname{Cdim} M := -\inf\{i \mid \mathrm{Ext}^i_A(M,R) \neq 0\}.$$

In nice cases the residue complex is $\mathcal{K}^{\cdot}_A = \mathrm{E}R$. Perhaps the most notable result of [82] is that an affine noetherian PI algebra A has a residue complex (we have to assume A has some noetherian connected filtration, but we believe this restriction can be removed). It should be noted however that if the ring A is "too noncommutative" then it does not have a residue complex.

2.4. **Duality for Universal Enveloping Algebras.** In [76] I proved that for a finite dimensional Lie algebra $\mathfrak{g}$, the rigid dualizing complex R of the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ is $R = \mathrm{U}(\mathfrak{g}) \otimes \big(\bigwedge^n \mathfrak{g}\big)[n]$, where $n = \operatorname{rank}_{\Bbbk} \mathfrak{g}$. This had been conjectured by Van den Bergh in 1996. As a corollary I deduced a Poincaré duality between the Hochschild homology and cohomology of $\mathrm{U}(\mathfrak{g})$-bimodules. There are also applications to representation theory of Lie algebras, such as the structure of duals of Verma modules, extending results of K.A. Brown and T. Levasseur [15]. In these applications the crucial ingredient is the rigid trace morphism

$$\mathrm{Tr} : B^* \to \mathrm{U}(\mathfrak{g}) \otimes \Big(\bigwedge^n \mathfrak{g}\Big)[n]$$

in $\mathsf{D}(\mathsf{Mod}\, \mathrm{U}(\mathfrak{g})^{\mathrm{e}})$, where $B = \mathrm{U}(\mathfrak{g})/I$ is any quotient ring of $\mathrm{U}(\mathfrak{g})$ that's finite over $\Bbbk$, and $B^* := \mathrm{Hom}_{\Bbbk}(B, \Bbbk)$, which is the rigid dualizing complex of B.

In [76] I also proved that if C is a smooth $\Bbbk$-algebra of dimension n, $\operatorname{char} \Bbbk = 0$ and $\mathcal{D}(C)$ is the ring of differential operators, then the rigid dualizing complex of $\mathcal{D}(C)$ is $\mathcal{D}(C)[2n]$. The proof used standard facts about $\mathcal{D}$-modules. This approach was extended by S. Chemla [18] to the case of the ring of differential operators

$\mathcal{D}(\mathcal{L})$ of a Lie algebroid $\mathcal{L}$ over a smooth commutative ring C, using her results in [17]. The rigid dualizing complex here is

$$\mathcal{D}(\mathcal{L}) \otimes_C \big(\textstyle\bigwedge_C^n \mathcal{L}\big) \otimes_C \Omega^m_{C/\Bbbk}[n+m]$$

where $n = \operatorname{rank}_C \mathcal{L}$ and $m = \dim C$. Note that Chemla's result recovers my original result for $\mathrm{U}(\mathfrak{g})$, since for $\mathcal{L} := \mathfrak{g}$ and $C := \Bbbk$ we get $\mathcal{D}(\mathcal{L}) = \mathrm{U}(\mathfrak{g})$.

2.5. **Perverse Modules.** Perverse sheaves first appeared in the context of geometry of singular spaces [8].

In [85] we prove that if A is a noncommutative $\Bbbk$-algebra with an Auslander dualizing complex R and p is the minimal perversity $p(n) = -n$, then there is a corresponding t-structure on the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. The formulas are similar to the topological setup (as in [8] and [34]), but the dimension of support of a module M is measured by the canonical dimension $\operatorname{Cdim} M$ determined by the complex R. The heart ${}^{p}\mathsf{D}^{0}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called the category of perverse modules.

Now suppose A is a differential $\Bbbk$-algebra of finite type (i.e. there is a filtration of A s.t. $\operatorname{gr} A$ is finite over its center, and the center is finitely generated over $\Bbbk$). Then we show that the rings A and A^{e} both have rigid Auslander dualizing complexes, and the rigid dualizing complex R_A of A, as complex of bimodules, is perverse, namely $R_A \in {}^{p}\mathsf{D}^{0}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{e}})$.

2.6. **Perverse Sheaves and Dualizing Complexes.** Let $(X, \mathcal{A})$ be a pair consisting of a scheme X and a sheaf of noncommutative rings $\mathcal{A}$ on X. Suppose X is separated finite type over $\Bbbk$, and $\mathcal{A}$ has a filtration F s.t. $F_{-1}\mathcal{A} = 0$, $\mathcal{A} = \bigcup F_i \mathcal{A}$, the center $\mathrm{Z}(\operatorname{gr}^F \mathcal{A})$ is a quasi-coherent, locally finitely generated $\mathcal{O}_X$-algebra, and $\operatorname{gr}^F \mathcal{A}$ is a coherent $\mathrm{Z}(\operatorname{gr}^F \mathcal{A})$-module. We call $(X, \mathcal{A})$ a differential quasi-coherent ringed scheme of finite type. Familiar examples are $\mathcal{A} := \mathcal{D}_X$, the ring of differential operators for X smooth in characteristic 0; or any coherent $\mathcal{O}_X$-algebra $\mathcal{A}$.

Restricting to an affine open set $U = \operatorname{Spec} C \subset X$ we obtain a differential $\Bbbk$-algebra of finite type $A := \Gamma(U, \mathcal{A})$. As explained above A has a rigid dualizing complex R_A. Furthermore, as C-bimodules the cohomologies $\mathrm{H}^i R_A$ are supported on the diagonal $U \subset U \times U$. Since the open sets $U \times U$ cover the diagonal $X \subset X^2 = X \times X$, one would like to glue the dualizing complexes R_A to a global complex $\mathcal{R}_{\mathcal{A}}$. This kind of gluing is tricky, since we are dealing with objects in triangulated categories. We remind the reader that in [22] the solution was to pass to residue complexes, which are well defined as sheaves and so can be glued. Yet in the noncommutative situation residue complexes seldom exist!

Our solution in [85] is to use perverse sheaves. First we show there is a product $(X^2, \mathcal{A}^{\mathrm{e}})$ that's also a differential quasi-coherent ringed scheme of finite type. Relying on our results for rings we show that there is a perverse t-structure on the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$ for the minimal perversity $p(n) = -n$. We then prove that the heart ${}^{p}\mathsf{D}^{0}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$ is a stack of abelian categories on X^2. For each affine open set U as above the rigid dualizing complex R_A sheafifies to a perverse sheaf on $U \times U$, supported on the diagonal. Now we can perform the gluing.

3. Derived Picard Groups

3.1. **The Derived Picard Group.** Let A be a $\Bbbk$-algebra. A two-sided tilting complex is a complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ s.t. there exists some other complex S satisfying $T \otimes^{\mathrm{L}}_A S \cong S \otimes^{\mathrm{L}}_A T \cong A$. The *derived Picard group* $\operatorname{DPic}_{\Bbbk}(A)$ consists of isomorphism classes of two-sided tilting complexes, and the operation is the derived tensor product. In [75] I proved that when A is commutative with connected spectrum, or when A is local, then $\mathrm{DPic}_{\Bbbk}(A) = \mathbb{Z} \times \mathrm{Pic}_{\Bbbk}(A)$. Here $\mathrm{Pic}_{\Bbbk}(A)$ is the usual (noncommutative) Picard group of Morita equivalences. This result was obtained independently by Rouquier-Zimmermann [57]. On the other hand already in the smallest nonlocal noncommutative $\Bbbk$-algebra – the ring of upper triangular 2×2 matrices – one has a proper inclusion $\mathbb{Z} \times \mathrm{Pic}_{\Bbbk}(A) \subsetneq \mathrm{DPic}_{\Bbbk}(A)$. Thus $\mathrm{DPic}_{\Bbbk}(A)$ is a genuine noncommutative geometric invariant.

In [75] I also proved that $\mathrm{DPic}_{\Bbbk}(A)$ classifies the isomorphism classes of dualizing complexes the algebra A.

B. Keller (lecture at CIRM, October 2001) proved that the Lie algebra of $\mathrm{DPic}_{\Bbbk}(A)$ is isomorphic as graded Lie algebra to the Hochschild cohomology $\mathrm{HH}^{\cdot}(A)$ of A. This implies that $\mathrm{HH}^{\cdot}(A)$ with its Gerstenhaber Lie algebra structure is invariant under derived Morita equivalence.

The paper [75] has an appendix by my summer student E. Kreines. This paper was already cited 7 times, including the review paper by P. May [48].

3.2. **Derived Picard Groups of Hereditary Algebras.** In [49] with J. Miyachi we look at the structure of $\mathrm{DPic}_{\Bbbk}(A)$ when A is a finite dimensional hereditary $\Bbbk$-algebra (the path algebra of a finite quiver). We work out the structure of the group $\mathrm{DPic}_{\Bbbk}(A)$ for many types of quivers, including all Dynkin and affine quivers. For that we analyze the natural action of $\mathrm{DPic}_{\Bbbk}(A)$ on the Auslander-Reiten quiver of the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$. We also get an interesting connection with the reflection functors of Bernstein-Gelfand-Ponomarev [10] and Auslander-Platzeck-Reiten, and the Weyl group of the quiver.

One should mention that for some commutative schemes X (e.g. $X = \mathbf{P}^1_{\Bbbk}$) the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$, where A is the path algebra of a suitable quiver. In a recent paper of Kontsevich-Rosenberg [41] it is shown that this phenomenon holds also for $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$, where $X = \mathbf{NP}^n_{\Bbbk}$ is their noncommutative projective space. Thus our calculations in [49] describe the group of auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$ in these cases.

3.3. **Derived Picard Groups of Finite Dimensional Algebras.** In the paper [77] I use recent deep work of Huisgen-Zimmermann and Saorín [30] and Rouqier [55] to prove that for any finite dimensional algebra A over an algebraically closed field $\Bbbk$ the group $\mathrm{DPic}_{\Bbbk}(A)$ is a locally algebraic group (i.e. it has a geometric structure). This is reminiscent of the Picard scheme of a (commutative) variety. The locally algebraic groups obtained are often of the form of a nontrivial semidirect product $D \ltimes G$, where D is a discrete group and G is a connected algebraic group. Interestingly, similar groups occur in D.O. Orlov's paper [50].

4. Residues on Schemes

4.1. **Topological Local Fields and Residues.** In [68] I worked on *topological local fields*, expanding earlier work of A.N. Parshin [51] and V.G. Lomadze [47]. A topological local field of dimension n is a field K endowed with a topology and a rank n valuation, that's isomorphic to an iterated Laurent series field $F((t_1)) \cdots ((t_n))$, where F is a finitely generated extension of the base field $\Bbbk$. To a topological local field K one assigns its separated differential forms $\Omega^{\cdot,\mathrm{sep}}_{K/\Bbbk}$. There is a notion of morphism of topological local fields $K \to L$. I proved that there is a functorial residue map $\mathrm{Res}_{L/K} : \Omega^{\cdot,\mathrm{sep}}_{L/\Bbbk} \to \Omega^{\cdot,\mathrm{sep}}_{K/\Bbbk}$, and studied its properties.

The results on topological local fields required a lot of foundational work on topological rings and differential operators, that was also done in [68]. One of my theorems on differential operators in characteristic $p > 0$ was used by K.E. Smith in several papers, including [61].

4.2. **Explicit Construction of the Residue Complex.** Let X be a scheme of finite type over a field $\Bbbk$. According to Grothendieck Duality theory [22], there is a special complex $\mathcal{K}^{\cdot}_X$ of quasi-coherent injective $\mathcal{O}_X$-modules called the residue complex. It is defined as follows. Denote the structural morphism by $\pi : X \to \operatorname{Spec} \Bbbk$. Then in the derived category $\mathsf{D}(\mathsf{Mod}\, X)$ there is a dualizing complex $\pi^!\Bbbk$, called the twisted inverse image of $\Bbbk$. Applying the Cousin functor E (which takes a sheaf to the direct sum of its local cohomologies) we obtain $\mathcal{K}^{\cdot}_X := \mathrm{E}\pi^!\Bbbk$.

The residue complex $\mathcal{K}^{\cdot}_X$ enjoys remarkable functorial properties, which are deduced from the corresponding properties of the functor $\pi^!$. To mention two, there is a homomorphism of complexes $\mathrm{Tr}_f : f_*\mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ when $f : X \to Y$ is a proper morphism; and there is a quasi-isomorphism $\Omega^n_{X/\Bbbk}[n] \to \mathcal{K}^{\cdot}_X$ when X is smooth of dimension n. Thus in the smooth case $\mathcal{K}^{\cdot}_X$ is the minimal injective resolution of the sheaf $\Omega^n_{X/\Bbbk}$ of top degree differential forms.

The importance of the residue complex $\mathcal{K}^{\cdot}_X$ (and the difficulty in constructing it explicitly) is its double nature: on the one hand it consists of concrete modules (which can be expressed as injective hulls, or as modules of differential forms with denominators); yet on the other hand it has the functorial variance properties, which are invisible from the naive concrete point of view. The problem of explicit construction of the residue complex can be stated as finding formulas for the coboundary operator $\delta : \mathcal{K}^q_X \to \mathcal{K}^{q+1}_X$ and the variance (Tr_f etc.), which involve the concrete description of the modules (necessarily with auxiliary data).

Attempts at an explicit description of Duality Theory were for a long time restricted to the dualizing sheaf $\boldsymbol{\omega}_X$, which is the lowest nonvanishing cohomology sheaf of $\mathcal{K}^{\cdot}_X$. Using *local cohomology residues*, J. Lipman, E. Kunz and their collaborators were able to give a pretty complete picture. See [26], [42], [43], [45] and [46] and their references.

In [68] I gave an explicit construction of the residue complex $\mathcal{K}^{\cdot}_X$, for X a reduced scheme of finite type over a perfect field $\Bbbk$. Here is the main idea. Let $\xi = (x_0, \ldots, x_n)$ be a saturated chain of points in X, namely each x_i is an immediate specialization of x_{i-1}. Beilinson [9] showed how the residue field $\boldsymbol{k}(x_0)$ can be completed to a ring $\boldsymbol{k}(x_0)_\xi$, and I proved that $\boldsymbol{k}(x_0)_\xi$ is a finite product of topological local fields of dimension n. Furthermore any lifting $\sigma : \boldsymbol{k}(x_n) \to \widehat{\mathcal{O}}_{X,x_n}$ gives rise to a morphism of topological local fields $\sigma : \boldsymbol{k}(x_n) \to \boldsymbol{k}(x_0)_\xi$, and hence to the Parshin residue map

$$\mathrm{Res}_{\sigma,\xi} : \Omega^{m+n}_{\boldsymbol{k}(x_0)/\Bbbk} \to \Omega^{m+n,\mathrm{sep}}_{\boldsymbol{k}(x_0)_\xi/\Bbbk} \to \Omega^{m}_{\boldsymbol{k}(x_n)/\Bbbk}.$$

To illustrate how the residue complex $\mathcal{K}^{\cdot}_X$ is constructed from the residue maps $\mathrm{Res}_{\sigma,\xi}$ let's examine the really easy case of an integral curve X (cf. Serre's [60]). Write x for the generic point of X and y for some closed point. Note that there is only one lifting $\sigma : \boldsymbol{k}(y) \to \widehat{\mathcal{O}}_{X,y}$. Define $\mathcal{K}_X(x) := \Omega^1_{\boldsymbol{k}(x)/\Bbbk}$ and $\mathcal{K}_X(y) :=$

$\mathrm{Hom}^{\mathrm{cont}}_{\boldsymbol{k}(y)}(\widehat{\mathcal{O}}_{X,y}, \boldsymbol{k}(y))$. Let $\delta : \mathcal{K}_X(x) \to \mathcal{K}_X(y)$ be the homomorphism

$$\delta(\alpha)(a) := \mathrm{Res}_{(x,y)}(a\alpha) \in \boldsymbol{k}(y)$$

for $\alpha \in \Omega^1_{\boldsymbol{k}(x)/\Bbbk}$ and $a \in \widehat{\mathcal{O}}_{X,y}$. Finally define $\mathcal{K}^{-1}_X := \mathcal{K}_X(x)$ and $\mathcal{K}^0_X := \bigoplus_y \mathcal{K}_X(y)$.

The explicit construction of the residue complex in [68] made it possible for the first time to compare Parshin residues to local cohomology residues (see my joint paper with P. Sastry [59]). Furthermore the papers by Hübl [24] and Sastry [58] are largely based on my work in [68]. In all there are at least 14 citations of [68] by other authors.

4.3. **Beilinson Completion Algebras.** Let X be a finite type scheme over a perfect field $\Bbbk$. Following Beilinson [9] I showed that a saturated chain $\xi = (x_0, \ldots, x_n)$ in X defines a completion $\mathcal{O}_{X,\xi}$ of the local ring $\mathcal{O}_{X,x_0}$. The completion $\mathcal{O}_{X,\xi}$ is a complete semilocal ring, and its residue fields are topological local fields. For $n = 0$ we simply get the adic completion $\mathcal{O}_{X,(x)} = \widehat{\mathcal{O}}_{X,x}$. A *Beilinson completion algebra* is any algebra A with topology and with valuations on its residue fields, that's isomorphic to some $\mathcal{O}_{X,\xi}$.

The main result in [70] is the existence of the dual module $\mathcal{K}(A)$ of a Beilinson completion algebra A, which is a canonical injective hull with a topology. Suppose A is local with maximal ideal $\mathfrak{m}$ and residue field K. Define $\boldsymbol{\omega}(K) := \Omega^{m,\mathrm{sep}}_{K/\Bbbk}$ where $m := \mathrm{rank}_K \Omega^{1,\mathrm{sep}}_{K/\Bbbk}$. Suppose $\sigma, \sigma' : K \to A$ are two liftings. I discovered a canonical isomorphism of A-modules

$$\Psi_{\sigma,\sigma'} : \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma}(A, \boldsymbol{\omega}(K)) \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma'}(A, \boldsymbol{\omega}(K)).$$

This isomorphism has an explicit formula, that surprisingly uses differential operators. Choose a sequence $(a_0, a_1, \ldots)$ of elements in A whose symbols are a K-basis of $\mathrm{gr}_{\mathfrak{m}} A$. Since for any element $a \in A$ the Taylor expansions $a = \sum \sigma'(\lambda_i) a_i = \sum \sigma(\mu_j) a_j$ are unique, we get an infinite matrix $D = [d_{i,j}]$ of differential operators $d_{i,j} \in \mathcal{D}(K)$ satisfying

$$\sigma'(\lambda) a_i = \sum_j \sigma(d_{i,j}(\lambda)) a_j \in A.$$

Now an element $\phi \in \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma}(A, \boldsymbol{\omega}(K))$ can be viewed as a row $[\phi(a_0), \phi(a_1), \ldots]$ of elements in $\boldsymbol{\omega}(K)$. It's known that $\boldsymbol{\omega}(K)$ is a right $\mathcal{D}(K)$-module, so we can multiply $\phi \cdot D^{\mathrm{t}}$; and this new row of elements of $\boldsymbol{\omega}(K)$ is the one representing $\Psi_{\sigma,\sigma'}(\phi)$. It's not hard to verify that for a third lifting $\sigma'' : K \to A$ one gets $\Psi_{\sigma,\sigma''} = \Psi_{\sigma',\sigma''} \circ \Psi_{\sigma,\sigma'}$, and so we can safely define $\mathcal{K}(A) := \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma}(A, \boldsymbol{\omega}(K))$.

I then showed that the dual modules are covariant w.r.t. intensifications homomorphisms $A \to A'$ (these include completions and étale extensions), and contravariant w.r.t. morphisms $A \to B$ (such as finite maps).

4.4. **Residues and Differential Operators.** A much improved construction of the residue complex $\mathcal{K}^{\cdot}_X$ is contained in my paper [73]. Here X is any finite type scheme over a perfect field $\Bbbk$. The construction is based on the method of Beilinson completion algebras discussed above. The idea is that for any point $x \in X$ we let $\mathcal{K}_X(x)$ be the constant sheaf with support $\overline{\{x\}}$ and stalk the dual module $\mathcal{K}(\widehat{\mathcal{O}}_{X,x})$. For an immediate specialization (x, y) there is an intensification homomorphism

$\widehat{\mathcal{O}}_{X,x} = \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,(x,y)}$ and a morphism $\widehat{\mathcal{O}}_{X,y} = \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,(x,y)}$, and thus we obtain an $\mathcal{O}_X$-module map

$$\delta : \mathcal{K}(x) \to \mathcal{K}(\mathcal{O}_{X,(x,y)}) \to \mathcal{K}(y).$$

Summing over all points in X we get the residue complex $\mathcal{K}^{\cdot}_X$.

This construction of $\mathcal{K}^{\cdot}_X$ implies some new results in algebraic geometry and ring theory, such as the contravariance of de Rham homology w.r.t. étale morphisms, and an explicit description of the intersection cohomology $\mathcal{D}$-module of a curve (in characteristic 0). Extending the latter to higher dimensions is one of my research goals (see below).

5. Beilinson Adeles on Schemes

5.1. **Adeles and Differential Forms.** Another topic of research is Beilinson adeles. Recall the classical ring of adeles $\mathbb{A}(X)$ of A. Weil and C. Chevalley, associated to a regular curve X. This is a "restricted product": $\mathbb{A}(X) \subset \prod_{x \in X} \boldsymbol{k}(X)_{(x)}$, where $\boldsymbol{k}(X)_{(x)}$ denotes the completion of the function field $\boldsymbol{k}(X)$ at a point x. A.N. Parshin [51] extended the definition to a surface X, and A.A. Beilinson [9] found a general definition for the complex of sheaves of adeles $\underline{\mathbb{A}}^{\cdot}(\mathcal{M})$, where $\mathcal{M}$ is any quasi-coherent sheaf on a noetherian scheme X.

In [28] R. Hübl and I found a formula for the homomorphism of complexes of sheaves $\underline{\mathbb{A}}^{\cdot}(\boldsymbol{\omega}_X) \to \mathcal{K}^{\cdot}_X[-n]$ on an integral n-dimensional scheme X. Here $\mathcal{K}^{\cdot}_X$ is the residue complex of X and $\boldsymbol{\omega}_X := \mathrm{H}^{-n}\mathcal{K}^{\cdot}_X$ is the dualizing sheaf. Such a formula was predicted by J. Lipman. We also showed that the double complex $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\Omega^{\cdot}_{X/\Bbbk})$ calculates de Rham cohomology for X smooth. For a smooth morphism $f : X \to S$ we used the complex $f_* \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\Omega^{\cdot}_{X/S})$ to describe explicitly the Gauss-Manin connection.

5.2. **Adelic Chern-Weil Theory.** In the paper [29] Hübl and I developed a Chern-Weil theory based on adeles. We showed that any vector bundle E on a variety X admits adelic connections ∇, and such connections calculate the usual Chern classes of E. We then went on to construct adelic secondary Chern-Simons classes, thereby extending the work of Bloch-Esnault [13]. An algebraic proof of the Bott Residue Formula for a vector field with finitely many zeroes was given. Global integration $\int_X : \mathrm{H}^n(X, \Omega^n_{X/\Bbbk}) \to \Bbbk$ was performed using the sum of residues over all maximal chains of points in X (see §4.2).

Using the action of adeles on the residue complex from [79] we proved a version of the Gauss-Bonnet Formula, taking place in the residue complex (i.e. before passing to cohomology).

The paper [29] was one of the subjects of a seminar at the Independent University of Moscow in 1998 devoted to current trends and problems of modern algebraic geometry.

6. Formal Schemes

6.1. **Formal Schemes and Duality.** In [74] I studied quasi-coherent sheaves on formal schemes, and introduced the notion of formally finite type morphism between formal schemes. I defined dualizing complexes on a formal scheme $\mathfrak{X}$, proved some of their properties (like uniqueness up to twists) and existence under some natural assumptions. A peculiar feature of my definition was that a dualizing complex $\mathcal{R}$

on $\mathfrak{X}$ has discrete (or torsion) quasi-coherent cohomology sheaves, that are often not coherent. This peculiarity was later explained in a paper by Alonso-Jeremías-Lipman [1], with whom I had fruitful exchanges (a section of their paper is based on my [74]). In [1] it was shown that there are actually two types of dualizing complexes on $\mathfrak{X}$: t-dualizing complexes (my definition) and c-dualizing complexes. These two types are related via Greenlees-May duality.

In [74] I gave a third construction of the residue complex. Here X is a finite type scheme over a regular base scheme S (e.g. $S = \operatorname{Spec} \mathbb{Z}$), and one is looking at the relative residue complex $\mathcal{K}^{\cdot}_{X/S}$. I used I-C. Huang's work on traces for local rings [23], in conjunction with my own results on smooth formal embeddings and duality on formal schemes. One of the consequences I got is the equality between $\mathrm{H}^{-n}\mathcal{K}^{\cdot}_{X/S}$ and the sheaf of degree n regular differentials of Kunz and Waldi [43], when X is generically smooth over S of relative dimension n.

6.2. **Hochschild Complex of a Scheme.** In [78] I considered a scheme X that's separated and of finite type over a noetherian base ring $\Bbbk$. The formal completion of the product scheme X^{q+2} along the diagonal embedding of X is a topological sheaf of $\mathcal{O}_{X^2}$-modules, supported on X, that we denote by $\widehat{\mathcal{B}}^{-q}(X)$. Taking all $q \geq 0$ we obtain the complete bar resolution $\widehat{\mathcal{B}}^{\cdot}(X)$ of $\mathcal{O}_X$. On any affine open set in X this resolution is the adic completion of the usual (unnormalized) bar resolution.

Define $\widehat{\mathcal{C}}^{\cdot}(X) := \mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \widehat{\mathcal{B}}^{\cdot}(X)$, the complete Hochschild chain complex. The continuous Hochschild cochain complex of X with values in an $\mathcal{O}_X$-module $\mathcal{M}$ is $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M})$. When X is smooth and $\mathcal{M} = \mathcal{O}_X$ the continuous Hochschild cochain complex turns out to be exactly the complex of poly differential operators of Kontsevich [39].

I proved that whenever X is smooth over $\Bbbk$ (regardless of characteristic) there is a functorial isomorphism

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M})$$

in the derived category $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$.

I also proved that if X is smooth over $\Bbbk$ of relative dimension n and $n!$ is invertible in $\Bbbk$, then there is a quasi-isomorphism

$$\bigoplus\nolimits_q \mathcal{H}om_{\mathcal{O}_X}(\Omega^q_{X/\Bbbk}, \mathcal{M})[-q] \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}).$$

This is a generalized Hochschild-Kostant-Rosenberg Theorem. When $\mathcal{M} = \mathcal{O}_X$ this is the quasi-isomorphism underlying the Kontsevich Formality Theorem. Indeed for a real C^∞ manifold X the quasi-isomorphism above is [39] Theorem 4.6.1.1.

7. Research in Progress and Future Plans

7.1. **Rigid Dualizing Complexes and DGAs.** The importance of rigid dualizing complexes has already been established. So far the definition involved an algebra A over a base field $\Bbbk$. I would like to extend the definition to the case where $\Bbbk$ is any (noetherian) commutative ring, and A is not necessarily flat over $\Bbbk$. There are some technical problems, that I think can be solved with DGAs. Then I would like to prove uniqueness and functoriality of these rigid dualizing complexes. Using this general notion of rigidity, together with gluing of perverse sheaves, one should be able to obtain a totally new approach to the classical Grothendieck Duality of schemes, including the delicate base change theorems.

7.2. **Rigid Dualizing Complexes and Hopf Algebras.** In [76] the rigid dualizing complex was used to establish a Poincaré duality for bimodules over an enveloping algebra $\mathrm{U}(\mathfrak{g})$. I would like to extend this result to other Hopf algebras, and in particular the quantum enveloping algebras $\mathrm{U}_q(\mathfrak{g})$.

Furthermore I wish to see if the rigid dualizing complex of a Hopf algebra is compatible in some way with comultiplication. The heuristic is that the rigid dualizing complex is an analog of the invariant Haar measure on a Lie group. If this approach succeeds we may be able to extend the powerful methods of bialgebra cohomology, so far used very effectively by S. Gelaki and P. Etingof [20] for finite dimensional Hopf algebras, also to some infinite dimensional Hopf algebras.

7.3. **Noncommutative Schemes.** Van den Bergh [63] proposed a theory of noncommutative algebraic geometry, which includes the projective geometry of Artin-Zhang. This new geometry talks about "sheaves" of algebras and bimodules on noncommutative quasi-schemes, and morphisms between quasi-schemes. Very recently another noncommutative algebraic geometry was proposed by Kontsevich-Rosenberg [41]. I would like to study these new geometries. Among other questions, is it possible to incorporate duality theory into these geometrical setups, and if so, what are the applications? In particular I want to look at canonical projective embeddings, the behavior of duality in blow-ups, perverse sheaves and group actions.

7.4. **The Derived Picard Group.** An intriguing question about $\mathrm{DPic}_{\Bbbk}(A)$ is its relation to the link graph on $\operatorname{Spec} A$, where A is a noetherian $\Bbbk$-algebra. If A is finite dimensional hereditary this was solved in [49]. In general, do any properties of the group $\mathrm{DPic}_{\Bbbk}(A)$ constitute obstructions to noncommutative localization? What do conditions on dualizing complexes, like the Auslander condition, mean in terms of the group $\mathrm{DPic}_{\Bbbk}(A)$?

On a broader perspective the derived Picard group is related to the geometry of commutative varieties and to mirror symmetry (conjectures of Konstevich [38] and work of Orlov [50] and Bondal-Polischchuk [14]). These relations should be explored.

7.5. **Intersection Cohomology.** As proved in [73], algebraic residues and Beilinson completion algebras give rise to a description of the intersection cohomology $\mathcal{D}$-module $\mathcal{L}(X, Y)$ of a curve X in characteristic 0. What is the explicit description of $\mathcal{L}(X, Y)$ when $\dim X \geq 2$? I would expect that residues along chains in desingularizations $X' \to X$ shall play a role in this description. What would that tell us about the singularities of X? What is the intersection cohomology version of the residue complex, and what information does it carry about the singularities of X?

Recently M. Blickle [12] worked on a characteristic p version of the intersection cohomology $\mathcal{D}$-module. What is the connection between his work and our approach of algebraic residues?

7.6. **Adeles, Arithmetic Geometry and Hodge Theory.** As demonstrated in [29], the complex of adeles $\mathcal{A}^{\cdot}_X = \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\Omega^{\cdot}_{X/\Bbbk})$ is a very good model for calculating de Rham cohomology, as far as intersection theory and characteristic classes are concerned. Indeed the complex $\mathcal{A}^{\cdot}_X$ behaves formally like the Dolbault complex of C^{∞} differential forms on a differentiable manifold. There exist adelic connections on any vector bundle. However, by [71], the filtration $\mathcal{A}^{\cdot;\geq q}_X$ doesn't induce the Hodge decomposition on $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$, when X is smooth projective over $\mathbb{C}$. The conjecture is that instead what we get is the coniveau filtration. Similarly on an arithmetic scheme, the adeles $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ do not account for the primes at infinity. Could it be possible to enrich Beilinson's adeles, and also the algebraic residues of [68], by some $\mathbb{R}$-metric data, adding the missing "archimedean part"? If this plan works out one could have very interesting applications both to arithmetic intersection theory and to Hodge theory.

7.7. **Adeles, Residues and Algebraic Stacks.** Recently there has been a flourish of research involving calculations of characteristic classes on algebraic spaces, and more generally on algebraic stacks (cf. Edidin-Graham [19]). Some of these calculations arise from physics (cf. Kontsevich-Manin [40]). As shown in [29], the method of adeles and algebraic residues is effective for dealing with intersection theory on schemes, and I propose to study generalizations of this method to stacks. In particular one would expect to obtain secondary characteristic classes for vector bundles on stacks.

References

[1] L. Alonso, A. Jeremías and J. Lipman, Duality and flat base change on formal schemes, in "Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes," Contemp. Math. **244**, Amer. Math. Soc., 1999, 3-90.

[2] M. Artin and W.F. Schelter, Graded algebras of global dimension 3, Adv. in Math. **66** (1987), 171-216.

[3] M. Artin, W. Schelter and J. Tate, Quantum deformations of GL_n, Comm. Pure Appl. Math. **44** (1991), 879-895.

[4] M. Artin, J. Tate and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, in: The Grothendieck Festschrift, vol. 1, Birkhäuser, Boston, 1990.

[5] M. Artin, J. Tate and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. **106** (1991), 335-388.

[6] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, J. Algebra **133** (1990), 249-271.

[7] M. Artin and J.J. Zhang, Noncommutative projective schemes, Adv. in Math. **109** (1994), 228-287.

[8] A.A. Beilinson, J. Bernstein and P. Deligne, Faisceaux perverse, in "Analyse et Topologie sur les Espaces Singulieres," Astérisque **100**, Soc. Math. France, 1982, 5-171.

[9] A.A. Beilinson, Residues and adeles, Funkt. Anal. Pril. **14** (1980) no. 1, 44-45; English trans. in Func. Anal. Appl. **14** (1980) no. 1, 34-35.

[10] I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev, Coxeter functors and Gabriel's theorem, Usp. Mat. Nauk **28** (1973) 19-23, Transl. Russ. Math. Surv. **28** (1973), 17-32.

[11] J.E. Björk, The Auslander condition on noetherian rings, "Séminaire Dubreil-Malliavin 1987-88," Lecture Notes in Math. **1404**, Springer, Berlin, 1989, 137-173.

[12] M. Blickle, The intersection homology $\mathcal{D}$-module in finite characteristic, eprint math.AG/0110244.

[13] S. Bloch and H. Esnault, Algebraic Chern-Simons Theory, Amer. J. Math. **119** (1997), 903-952.

[14] A.I. Bondal and A. E. Polishchuk, Homological properties of associative algebras: the method of helices. (Russian) Izv. Ross. Akad. Nauk Ser. Mat. **57** (1993), 3-50; translation in Russian Acad. Sci. Izv. Math. **42** (1994), 219-260.

[15] K.A. Brown and T. Levasseur, Cohomology of bimodules over enveloping algebras, Math. Z. **189** (1985), no. 3, 393–413.

[16] S. Chemla, Operations for modules on Lie-Reinhart superalgebras, Bull. Soc. Math. France **122** (1994).

[17] S. Chemla, A duality property for complex Lie algebroids, Math. Z. **232** (1999), 367-388.

[18] S. Chemla, Rigid dualizing complexes for Lie algebroids, in preparation.

[19] D. Edidin and W. Graham, Localization in equivariant intersection theory and the Bott residue formula, Amer. J. Math. **120** (1998), 619-636.

[20] P. Etingof and S. Gelaki, On finite dimensional semisimple and cosemisimple Hopf algebras in positive characteristic, Inter. Math. Res. Not. **16**, (1998), 851-864.

[21] K.R. Goodearl and T.H. Lenagan, Catenarity in quantum algebras, J. Pure Appl. Algebra, **111** (1996), 123-142.

[22] R. Hartshorne, "Residues and Duality", Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[23] I-C. Huang, "Pseudofunctors on Modules with Zero Dimensional Support," Memoirs AMS **548** (1995), AMS, Providence.

[24] R. Hübl, Residues of regular and meromorphic differential forms, Math. Annalen **300** (1994), 605-628.

[25] R. Hübl and E. Kunz, Integration of Differential Forms on Schemes, J. reine angew. Math. **410** (1990), 53-83.

[26] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. reine angew. Math. **410** (1990), 84-108.

[27] R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** no. 4 (1993), 749-787.

[28] R. Hübl and A. Yekutieli, Adeles and differential forms, J. reine angew. Math. **471** (1996), 1-22.

[29] R. Hübl and A. Yekutieli, Adelic Chern forms and applications, Amer. J. Math. **121** (1999), 797-839.

[30] B. Huisgen-Zimmermann and M. Saorín, Geometry of chain complexes and outer automorphism groups under derived equivalence, Trans. Amer. Math. Soc. **353** (2001), 4757-4777.

[31] P. Jørgensen, Serre duality for $Tails(A)$, Proc. Amer. Math. Soc. **125** (1997), 709-716.

[32] P. Jørgensen, Non-commutative graded homological identities, J. London Math. Soc. **57** (1998), 336-350.

[33] P. Jørgensen, Gorenstein homomorphisms of noncommutative rings, J. Algebra **211** (1999), 240-267.

[34] M. Kashiwara and P. Schapira, "Sheaves on Manifolds," Springer, 1990.

[35] A. Kapustin, A. Kuznetsov and D. Orlov, Noncommutative Instantons and Twistor Transform, Comm. Math. Phys. **221** (2001), 385-432.

[36] B. Keller, A remark on tilting theory and DG algebras, Manuscripta Math. **79** (1993), 247-252.

[37] B. Keller, Deriving DG categories, Ann. Scient. Éc. Norm. Sup. **27** (1994), 63-102.

[38] M. Kontsevich, Homological algebra of mirror symmetry, Proc. ICM Zurich 1994, Bikhauser, Basel, 1995, 120-139.

[39] M. Kontsevich, Deformation quantization of Poisson manifolds, eprint q-alg/9709040.

[40] M. Kontsevich and Yu. Manin, Quantum cohomology of a product, Invent. Math. **124** (1996), 313-339.

[41] M. Kontsevich and A. Rosenberg, Noncommutative smooth spaces, in "The Gelfand Mathematical Seminars, 1996-1999," Birkhaüser, Boston, MA, 2000, 85-108.

[42] E. Kunz, "Kähler Differentials", Vieweg, Braunschweig, Wiesbaden 1986.

[43] E. Kunz and R. Waldi, "Regular Differential Forms," Contemp. Math. **79** (1988), AMS, Providence.

[44] T. Levasseur, Some properties of noncommutative regular rings, Glasgow Math. J. **34** (1992), 277-300.

[45] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Astérisque **117** (1984).

[46] J. Lipman and P. Sastry, Regular differentials and equidimensional scheme-maps, J. Alg. Geom. **1** (1992), 101-130.

[47] V.G. Lomadze, On residues in algebraic geometry, Izv. Acad. Nauk SSSR Ser. Mat. **45** (1981), 1258-1287; English trans. in Math. USSR Izv. **19** (1982), 495-520.

[48] J.P. May, Picard groups, Grothendieck rings, and Burnside rings of categories, preprint.

[49] J. Miyachi and A. Yekutieli, Derived Picard groups of finite dimensional hereditary algebras, Compositio Math. **129** (2001), 341-368.

[50] D.O. Orlov, Equivalences of derived categories and $K3$ surfaces, J. Math. Sci. (New York) **84** (1997), no. 5, 1361-1381.

[51] A.N. Parshin, On the arithmetic of two dimensional schemes. I. Distributions and residues, Izv. Akad. Nauk SSSR Ser. Mat. **40** no. 4 (1976), 736-773. English trans. in Math. USSR Izv. **10** no. 4 (1976), 695-729.
[52] A.N. Parshin, Chern class, adeles and L-functions, J. reine angew Math. **341** (1983), 174-192.
[53] J. Rickard, Morita theory for derived categories, J. London Math. Soc. **39** (1989), 436-456.
[54] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **43** (1991), 37-48.
[55] A.L. Rosenberg. "Noncommutative Algebraic Geometry and Representations of Quantized Algebras," Kluwer, Dordrecht, 1995.
[56] R. Rouquier, Groupes d'automorphismes et équivalences stables ou dérivées, preprint.
[57] R. Rouquier and A. Zimmermann, Picard groups for derived module categories, preprint.
[58] P. Sastry, Residues and duality on algebraic schemes, Compositio Math. **101** (1996), 133-178.
[59] P. Sastry and A. Yekutieli, On residue complexes, dualizing sheaves and local cohomology modules, Israel J. Math. **90** (1995), 325-348.
[60] J.P. Serre, "Algebraic Groups and Class Fields," Springer, New York, 1988.
[61] K.E. Smith and M. Van den Bergh, Simplicity of rings of differential operators in prime characteristic, Proc. London Math. Soc. **75** (1997), 32-62.
[62] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered rings, J. Algebra **195** (1997), no. 2, 662-679.
[63] M. Van den Bergh, "Blowing up of Non-Commutative Smooth Surfaces," Mem. Amer. Math. Soc. **154** (2001).
[64] Q.S. Wu and J.J. Zhang, Some homological invariants of local PI algebras, J. Algebra **225** (2000), 904-935.
[65] Q.S. Wu and J.J. Zhang, Dualizing complexes over noncommutative local rings, J. Algebra **239** (2001), 513-548.
[66] Q.S. Wu and J.J. Zhang, Homological identities for noncommutative rings, J. Algebra **242** (2001), 516-535.
[67] A. Yekutieli, Dualizing complexes over noncommutative graded algebras, J. Algebra **153** (1992), 41-84.
[68] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).
[69] A. Yekutieli, Canonical deformations of De Rham complexes, Advances in Math. **115** (1995), 250-268.
[70] A. Yekutieli, Traces and differential operators over Beilinson completion algebras, Compositio Math. **99** (1995), 59-97.
[71] A. Yekutieli, Some remarks on Beilinson adeles, Proc. Amer. Math. Soc. **124** (1996), 3613-3618.
[72] A. Yekutieli, The residue complex of a noncommutative graded algebra, J. Algebra **186** (1996), 522-543.
[73] A. Yekutieli, Residues and differential operators on schemes, Duke Math. J. **95** (1998), 305-341.
[74] A. Yekutieli, Smooth formal embeddings and the residue complex, Canadian J. Math. **50** (1998), 863-896.
[75] A. Yekutieli, Dualizing complexes, Morita equivalence and the derived Picard group of a ring, J. London Math. Soc. **60** (1999) 723-746.
[76] A. Yekutieli, The rigid dualizing complex of a universal enveloping algebra, J. Pure Appl. Algebra **150** (2000), 85-93.
[77] A. Yekutieli, The derived Picard group is a locally algebraic group, to appear in Algebr. Represent. Theory; eprint: math.RA/0005005.
[78] A. Yekutieli, The continuous Hochschild cochain complex of a scheme, preprint; eprint math.AG/0111094.
[79] A. Yekutieli, The action of adeles on the residue complex, preprint.
[80] A. Yekutieli and J.J. Zhang, Serre duality for noncommutative projective schemes, Proc. Amer. Math. Soc. **125** (1997), 697-707.
[81] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.
[82] A. Yekutieli and J.J. Zhang, Residue complexes over noncommutative rings, preprint; eprint math.RA/0103075.

[83] A. Yekutieli and J.J. Zhang, Dualizing complexes and tilting complexes over simple rings, preprint; eprint math.RA/0110008.
[84] A. Yekutieli and J.J. Zhang, Multiplicities of indecomposable injectives, in preparation.
[85] A. Yekutieli and J.J. Zhang, Dualizing complexes and perverse sheaves on noncommutative ringed schemes, in preparation.

Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, ISRAEL
E-mail address: amyekut@math.bgu.ac.il
http://www.math.bgu.ac.il/∼amyekut