RESEARCH SUMMARY: 2001 - 2006

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This summary covers the research in mathematics I have done from 2001 to 2006. For earlier work please consult [71]. The summary is organized by topics. Section 4 is about current work and plans for future research.

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1. Deformation Quantization in Algebraic Geometry

Subsections 1.1 and 1.2 give some background. My own work is explained in Subsections 1.3 - 1.7.

1.1. **Overview of Deformation Quantization.** The origin of *deformation quantization* is in the paper [9] from 1978 by the physicists Flato et. al. They asked whether the ring $\mathrm{C}^{\infty}(X)$ of functions on a Poisson differentiable manifold X can be quantized. Namely, does there exist an associative $\mathbb{R}[[\hbar]]$-bilinear multiplication $\star$ on the $\mathbb{R}[[\hbar]]$-module $\mathrm{C}^{\infty}(X)[[\hbar]]$, satisfying

$$f \star g \equiv fg \mod \hbar \tag{1.1.1}$$

and

$$\tfrac{1}{2}(f \star g - g \star f) \equiv \{f, g\}\hbar \mod \hbar^2 \tag{1.1.2}$$

for any $f, g \in \mathrm{C}^{\infty}(X)$. Here $\hbar$ is a formal parameter (the "Planck constant") and $\{-,-\}$ is the Poisson bracket. The multiplication $\star$ is called a *star product.* Furthermore, there should be a sequence $\{\beta_j\}_{j\geq 1}$ of bidifferential operators on X such that

$$f \star g = fg + \sum_{j\geq 1} \beta_j(f, g)\hbar^j. \tag{1.1.3}$$

The physical reasoning goes like this: the noncommutative algebra $\big(\mathrm{C}^{\infty}(X)[[\hbar]], \star\big)$ is a model for the quantization of the classical system whose phase space is X.

The problem of existence of a deformation quantization turned out to be a difficult one. For a symplectic manifold X it was solved by De Wilde and Lacomte [21] in 1983. A more geometric proof, using the formal geometry of Gelfand-Kazhdan [28], was discovered by Fedosov [24] in 1994. The problem was finally solved by Kontsevich [45] in 1997.

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Deformation quantization is also closely related to index theorems; this was already observed by Fedosov. There are some results in this direction by Schapira et. al. [62] and Tsygan et. al. [59].

1.2. **The Work of Kontsevich.** Kontsevich did more than prove existence of a deformation quantization of the algebra $C := \mathrm{C}^{\infty}(X)$, given a Poisson bracket $\{-,-\}$. He showed how to obtain a canonical deformation quantization, up to gauge equivalence. To be more precise, Kontsevich considered formal Poisson brackets on C, which are by definition Poisson brackets on the commutative $\mathbb{R}[[\hbar]]$-algebra $C[[\hbar]]$ that are congruent to 0 modulo $\hbar$. A usual Poisson bracket $\{-,-\}$ on C extends to a formal Poisson bracket $\{-,-\}\hbar$ in an obvious way. There is a group acting on the set of all formal Poisson brackets, called the group of *gauge equivalences*. Likewise there is a group of gauge equivalences acting on the set of all deformation quantizations of C (i.e. star products on $C[[\hbar]]$). The result of Kontsevich [45] is that there is a canonical bijection

$$Q : \frac{\{\text{formal Poisson brackets on } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } C\}}{\text{gauge equivalence}}$$

called the *quantization map*. The map Q preserves first order terms, in the following sense. Given a formal Poisson bracket $\alpha = \{-,-\}$ we can expand it into a series of biderivations, namely $\{f, g\} = \sum_{j=1}^{\infty} \alpha_j(f, g)\hbar^j$ for $f, g \in C$. Then, if $\star = Q(\alpha)$, one has the relation

$$\tfrac{1}{2}(f \star g - g \star f) \equiv \alpha_1(f, g)\hbar \mod \hbar^2.$$

The crux of Kontsevich's solution of the deformation quantization problem was his celebrated *Formality Theorem* [45]. It asserted the existence of an L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}]) \tag{1.2.1}$$

between the differential graded (DG) Lie algebras $\mathcal{T}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{R}[\boldsymbol{t}])$. Here $\mathbb{R}[\boldsymbol{t}] := \mathbb{R}[t_1, \ldots, t_n]$, the algebra of polynomial functions on $\mathbb{R}^n$. The morphism $\mathcal{U}$ has some important invariance properties. The formulas for $\mathcal{U}$ were inspired by considerations of string theory (see [46]).

Let us recall the roles of the DG Lie algebras mentioned above. Let $\mathbb{K}$ be a field of characteristic 0, and let C be a commutative $\mathbb{K}$-algebra. The DG Lie algebra $\mathcal{T}_{\mathrm{poly}}(C)$ of poly derivations controls formal Poisson structures on C, in the sense that formal Poisson structures are precisely solutions $\alpha = \sum_{j \geq 1} \alpha_j \hbar^j \in \mathcal{T}^1_{\mathrm{poly}}(C)[[\hbar]]^+$ of the Maurer-Cartan equation $\mathrm{d}(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0$, and gauge equivalence are parameterized by the Lie algebra $\mathcal{T}^0_{\mathrm{poly}}(C)[[\hbar]]^+$. Similarly the DG Lie algebra $\mathcal{D}_{\mathrm{poly}}(C)$ of poly differential operators (which is a subalgebra of the Hochschild complex) controls deformation quantizations of C (this goes back to Gerstenhaber [27]). An L_∞ quasi-isomorphism $\Psi : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$ is a sequence of homomorphisms

$$\Psi_j : \textstyle\bigwedge^j_{\mathbb{K}} \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C),$$

where Ψ_1 is a DG Lie algebra quasi-isomorphism, up to the higher homotopies $\Psi_2, \Psi_3, \ldots$. By standard facts from deformation theory [45, 26] the L_∞ quasi-isomorphism Ψ induces a bijection between gauge equivalence classes of solutions of the Maurer-Cartan equation in $\mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^+$ and $\mathcal{D}_{\mathrm{poly}}(C)[[\hbar]]^+$.

The Formality Theorem solves the deformation problem locally (for small open sets in the C^∞ manifold X). In order to globalize Kontsevich used Fedosov's method, i.e. by going to formal geometry. (This is briefly explained, in the algebraic context, in Subsection 1.4 below.) Thus he obtained an L_∞ quasi-isomorphism

$$\Psi : \mathcal{T}_{\mathrm{poly}}\big(\mathrm{C}^\infty(X)\big) \to \mathcal{D}_{\mathrm{poly}}\big(\mathrm{C}^\infty(X)\big),$$

which gave rise to the quantization map Q.

The universal deformation formula (1.2.1) has other amazing consequences, of which I'd like to mention one. Consider a finite dimensional real Lie algebra $\mathfrak{g}$, and let $X := \mathfrak{g}^*$. The symmetric algebra $\mathrm{S}(\mathfrak{g})$ coincides with the ring of polynomial functions on X, and it has the Kostant-Kirillov Poisson bracket. The canonical deformation quantization of $\mathrm{S}(\mathfrak{g})$, when evaluated at $\hbar = 1$, recovers the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. This provides a new proof of the Duflo isomorphism. Recently Alexeev and Meinrenken [3], using results of Torossian [65] on an extended Kontsevich deformation formula, proved the longstanding Kashiwara-Vergne conjecture (which is a generalization of the Duflo result).

1.3. **Deforming Algebraic Varieties.** Let $\mathbb{K}$ be a field of characteristic 0, and let X be a smooth n-dimensional variety over $\mathbb{K}$. Assume X is endowed with a Poisson structure α; namely there is a biderivation $\alpha \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$ such that the bracket $\{f, g\} := \langle \alpha, \mathrm{d}f \wedge \mathrm{d}g \rangle$, for local sections $f, g \in \mathcal{O}_X$, satisfies the Jacobi identity. What is the correct notion of deformation of the Poisson variety (X, α)?

In case X is affine, say $X = \operatorname{Spec} C$, then the obvious notion of deformation quantization is a star product $\star$ on the $\mathbb{K}[[\hbar]]$-module $C[[\hbar]]$, as in Subsection 1.1. (There is a more delicate definition in the affine setting, called semi-formal deformation [47], which we will not consider here.) When X is not affine this has an immediate generalization: a star product $\star$ on the sheaf $\mathcal{O}_X[[\hbar]]$. This means that $\star$ makes $\mathcal{O}_X[[\hbar]]$ into a sheaf of associative $\mathbb{K}[[\hbar]]$-algebras, and there is a sequence $\{\beta_j\}_{j\geq 1}$ of global bidifferential operators on $\mathcal{O}_X$ such that equations (1.1.1), (1.1.2), (1.1.3) hold locally.

There is a more refined notion of deformation quantization of $\mathcal{O}_X$, which we introduced in [79]. Here we are looking for a sheaf $\mathcal{A}$ of $\mathbb{K}[[\hbar]]$-algebras on X, which admits local differential trivializations. This means that X can be covered by open sets $\{U_i\}$, and on each U_i there is an isomorphism of sheaves of $\mathbb{K}[[\hbar]]$-algebras $\mathcal{A}|_{U_i} \cong \mathcal{O}_{U_i}[[\hbar]]$, where on the right side the multiplication is a star product. On a double intersection the resulting automorphism of $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$ has to be a gauge equivalence, namely of the form $f \mapsto f + \sum_{j\geq 1} \gamma_j(f)\hbar^j$, where the γ_j are differential operators. The "naive" deformation of the previous paragraph now becomes a *globally trivialized deformation quantization.*

In general one must allow this more refined definition of deformation. For instance, if Y is any smooth variety and $X := \mathrm{T}^*Y$, the cotangent bundle, endowed with its canonical symplectic Poisson structure, then X has a deformation quantization (see [79, Example 1.7]); but it is unlikely that this deformation can be globally trivialized. (Our definition of deformation quantization was adopted by several authors; cf. [8].)

One of the things we show in [79] is that when X is affine, or more generally when $\mathrm{H}^1(X, \mathcal{D}_X) = 0$, then any deformation quantization can be globally trivialized (see [79, Theorem 1.13]). Here $\mathcal{D}_X$ is the sheaf of differential operators on the variety X. This result also explains why in the C^∞ setting it is enough to consider globally trivialized deformation quantizations (i.e. deformations of the algebra $\mathrm{C}^{\infty}(X)$): on a differentiable manifold X the sheaf $\mathcal{D}_X$ has vanishing higher cohomologies.

1.4. **Existence of Deformations.** The question of existence of a deformation quantization of a Poisson variety X was open until recently, even for an affine variety $X = \operatorname{Spec} C$. Only a few cases were known explicitly. Things changed after the Kontsevich Formality Theorem: it easily implies that when X is an affine open set in $\mathbf{A}^n_{\mathbb{K}}$, or more generally an affine scheme admitting an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$, then deformation quantizations exist, and moreover they can be classified up to gauge equivalence (see [79, Corollary 3.24]). Still for a general affine variety there was no satisfactory answer.

Other cases were treated too. For instance, Artin [5] considered deformations via an obstruction theory approach. And in [10] the authors found sufficient and necessary conditions for deforming symplectic varieties.

The problem was solved to a large extent in our paper [79]. Recall that X is said to be $\mathcal{D}$-*affine* if any quasi-coherent left $\mathcal{D}_X$-module has vanishing higher cohomologies. This class of varieties includes affine varieties (of course), but also the flag varieties (such as the projective spaces $\mathbf{P}^n$ and the Grassmannians). Note that for such varieties any deformation quantization can be globally trivialized (see Subsection 1.3). Our main result [79, Theorem 0.1] asserts that when X is a $\mathcal{D}$-affine, and $\mathbb{R} \subset \mathbb{K}$, there is a canonical function

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \to \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

which preserves first order terms. The quantization map Q commutes with étale morphisms $X' \to X$. When X is affine the map Q is actually bijective; so that we have a complete solution of the affine case.

Let me outline the proof. The strategy in [79] is to adopt Kontsevich's proof to the algebro-geometric setup as much as possible, with input from other sources such as [18]. Thus we study the bundle $\pi : \operatorname{Coor} X \to X$ of formal coordinate systems on X as an infinite dimensional scheme. Surprisingly, in doing so we had to quote results from our older papers [76, 78]. Let $\mathcal{P}_X$ be the sheaf of *principal parts* on X (cf. [23]); this sheaf is also called the sheaf of infinite order jets, or the sheaf of sections of the jet bundle of X. The complete pullback $\pi^{\widehat{*}}\, \mathcal{P}_X$ is canonically isomorphic to $\mathcal{O}_{\operatorname{Coor} X}[[\boldsymbol{t}]]$, and this isomorphism is the *universal Taylor expansion* of $\mathcal{O}_X$, in a way which we make precise in [79].

We prove that the pullbacks $\pi^{\widehat{*}}\, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X})$ and $\pi^{\widehat{*}}\, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$ also have universal Taylor expansions. Thus there is a canonical isomorphism of graded Lie algebras

$$\pi^{\widehat{*}}\, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \cong \mathcal{O}_{\operatorname{Coor} X} \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]), \tag{1.4.1}$$

and likewise for $\mathcal{D}_{\mathrm{poly},X}$. The universal Kontsevich deformation $\mathcal{U}$ of (1.2.1) can be applied to the DG Lie algebras on the right. However the isomorphisms (1.4.1) do not respect differentials, due to the presence of the *Grothendieck connection* (see Subsection 1.6) on the left side. This forces us to twist the universal L_{∞} morphism (in a sense explained in Subsection 1.5).

The new L_{∞} morphism Ψ descends to the quotient bundle $\operatorname{Coor} X / \mathrm{GL}_n(\mathbb{K})$. This bundle is "almost a torsor" under a pro-unipotent group, and hence by the method explained in Subsection 1.6 we can find a *simplicial section* of the bundle.

This simplicial section $\boldsymbol{\sigma}$ gives rise to an L_∞ quasi-isomorphism

$$\Psi_{\boldsymbol{\sigma}} : \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}) \tag{1.4.2}$$

between the sheaves of DG Lie algebras $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$ and $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ on X. Here $\operatorname{Mix}_{\boldsymbol{U}}(-)$ is the *mixed resolution*, which is explained in Subsection 1.6. Passing to global sections it follows that

$$\Gamma(X, \Psi_{\boldsymbol{\sigma}}) : \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) \to \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)$$

is an L_∞ quasi-isomorphism. There are also two global DG Lie algebra homomorphism

$$\Gamma(X, \mathcal{D}_{\mathrm{poly},X}) \to \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big) \tag{1.4.3}$$

and

$$\Gamma(X, \mathcal{T}_{\mathrm{poly},X}) \to \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big). \tag{1.4.4}$$

When X is $\mathcal{D}$-affine we know that (1.4.3) is a quasi-isomorphism; and hence we obtain the quantization map Q. And when X is affine the DG Lie algebra homomorphism (1.4.4) is also a quasi-isomorphism, implying that Q is bijective.

Here are several papers citing our paper [79]: [22], [16], [8], [42], [55], [10], [20], [25]. In the very recent preprint [69] by Van den Bergh there is an alternative proof of our main result.

1.5. **Continuous and Twisted L_∞ Morphisms.** This work [80] is one of the companions to our paper on deformation quantization. The purpose is to establish two technical aspects of L_∞ morphisms. First we consider the question of continuity of L_∞ morphisms. In the setup of deformation quantization we encounter a complicated situation: adic completions of quasi-coherent sheaves, and nontrivial maps between them.

The existing methods of commutative algebra and sheaf theory do not suffice here. In earlier treatments often such problems were either "shoved under the rug" or bypassed using ad hoc solutions.

In [80] we introduced the notion of *dir-inv modules*. As the name suggests, these are modules (or sheaves) equipped with filtrations, resembling the topology of a one-dimensional local field. We worked out some basic properties of these "topological modules", most importantly completions, direct sums and tensor products.

The second technical aspect treated in [80] was twisting of L_∞ morphisms. Consider the universal L_∞ quasi-isomorphism $\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ of Kontsevich, where $\mathbb{K}[\boldsymbol{t}] := \mathbb{K}[t_1, \ldots, t_n]$, the polynomial algebra. Then $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ have the $\boldsymbol{t}$-adic dir-inv structures, and every component $\mathcal{U}_j$ of $\mathcal{U}$ is continuous. It follows that for any super-commutative associative unital DG $\mathbb{K}$-algebra A there is an induced continuous A-multilinear L_∞ quasi-isomorphism

$$\mathcal{U}_A : A \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to A \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

By twisting a DG Lie algebra such as $A \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ we mean changing the differential from d to $\mathrm{d} + \mathrm{ad}(\omega)$, where $\omega \in A^1 \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ is a solution of the Maurer-Cartan equation. This is the case in the universal Taylor expansion

$$\Omega_{\mathrm{Coor}\,X} \mathbin{\widehat{\otimes}}_{\mathcal{O}_{\mathrm{Coor}\,X}} \widehat{\pi^*}\,(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \cong \Omega_{\mathrm{Coor}\,X} \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]),$$

where $\Omega_{\operatorname{Coor} X} = \bigoplus_{p \geq 0} \Omega^p_{\operatorname{Coor} X}$ is the de Rham complex of the scheme $\operatorname{Coor} X$ (cf. (1.4.1)). The Grothendieck connection on the left side corresponds to $\operatorname{ad}(\omega_{\mathrm{MC}})$ on the right side, for a canonical element ω_{MC} called the *Maurer-Cartan form.*

In [80, Theorem 0.1] we prove that the DG Lie algebra $A \mathbin{\widehat{\otimes}}_{\mathbb{K}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and the L_∞ quasi-isomorphism $\mathcal{U}_A$ can be twisted by ω, and we give an explicit formula for the twisted L_∞ quasi-isomorphism $\mathcal{U}_{A,\omega}$.

1.6. **Mixed Resolutions and Simplicial Sections.** In the C^∞ context the bundle $\operatorname{Coor} X/\mathrm{GL}_n(\mathbb{K})$ has contractible fibers, and therefore it admits global C^∞ sections. This does not work in the algebro-geometric context. As a way of getting around this problem we introduced the notion of *simplicial section* of a bundle, in [81]. This concept is inspired by work of Bott [13], and by our own work with Hübl [41].

Suppose $\pi : Z \to X$ is a morphism of schemes. A simplicial section of π, based on an open covering $\{U_i\}$ of X, consists of a family of morphisms

$$\sigma_{\boldsymbol{i}} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to Z,$$

for $\boldsymbol{i} = (i_0, \ldots, i_q)$ and $q \in \mathbb{N}$, that commute with π and satisfy the simplicial relations (see [81]). Here $\boldsymbol{\Delta}^q_{\mathbb{K}}$ is the geometric q-dimensional simplex

$$\boldsymbol{\Delta}^q_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1),$$

and $U_{\boldsymbol{i}} := U_{i_0} \cap \cdots \cap U_{i_q}$.

We show (in [79]) that the bundle $\pi : \operatorname{Coor} X \to X$ is a torsor under the group $\mathrm{GL}_n \ltimes G$, where G is a pro-unipotent group. Locally π has sections, so we can choose an open covering $\boldsymbol{U} = \{U_i\}$ of X with sections $U_i \to \operatorname{Coor} X$. Due to the averaging process described in Subsection 1.7, these sections can be extended to a simplicial section $\boldsymbol{\sigma}$ of $\operatorname{Coor} X/\mathrm{GL}_n(\mathbb{K})$ based on $\boldsymbol{U}$.

Let's move to the notion of *mixed resolutions.* Suppose the open covering $\boldsymbol{U}$ consists of affine open sets. Given any sheaf $\mathcal{M}$ of $\mathbb{K}$-modules on X one can form the *commutative Čech resolution* $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. This is a complex of sheaves on X, made up of the Čech resolution, to which one applies the Thom-Sullivan normalization (cf. [41] and [35]). There is a functorial quasi-isomorphism $\mathcal{M} \to \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$, and globally this gives an isomorphism $\mathrm{R}\Gamma(X, \mathcal{M}) \xrightarrow{\simeq} \Gamma(X, \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}))$ in $\mathsf{D}(\mathsf{Mod}\, \mathbb{K})$. Since the commutative Čech resolution involves the family of simplices $\boldsymbol{\Delta}^q_{\mathbb{K}}$, it follows that sometimes operations on $\pi^* \mathcal{M}$, combined with the simplicial section $\boldsymbol{\sigma}$, can induce similar operations on $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$.

Recall the sheaf of principal parts $\mathcal{P}_X$. There is a canonical integrable connection

$$\nabla_{\mathcal{P}} : \mathcal{P}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

called the *Grothendieck connection.* For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ there is a corresponding de Rham complex $\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$ with differential $\nabla_{\mathcal{P}}$, and the map

$$\mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

is a quasi-isomorphism. We can now define the mixed resolution of $\mathcal{M}$ to be the total complex

$$\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X).$$

Then $\mathcal{M} \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is a functorial quasi-isomorphism of sheaves, and $\mathrm{R}\Gamma(X, \mathcal{M}) \cong \Gamma(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}))$. Moreover, certain operations on $\pi^{\widehat{*}} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$

will induce similar operations on $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$. This is how the L_∞ morphism $\Psi_{\boldsymbol{\sigma}}$ of (1.4.2) is constructed.

1.7. **An Averaging Process for Unipotent Group Actions.** This is a new result about the structure of torsors under unipotent groups in characteristic 0.

Suppose $\mathbb{K}$ is a field of characteristic 0. By a *weight sequence* in $\mathbb{K}$ we mean a sequence $\boldsymbol{w} = (w_0, \ldots, w_q)$ of elements of $\mathbb{K}$ such that $\sum w_i = 1$. Thus $\boldsymbol{w}$ is a $\mathbb{K}$-rational point of $\boldsymbol{\Delta}^q_{\mathbb{K}}$. Let G be a unipotent group over $\mathbb{K}$, and suppose Z is a set with $G(\mathbb{K})$-action which is transitive and has trivial stabilizers. Let $\boldsymbol{z} = (z_0, \ldots, z_q)$ be a sequence of points in Z, and let $\boldsymbol{w}$ be a weight sequence in $\mathbb{K}$. We prove in [82] that there is a point $\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z$ called the *weighted average*. The operation wav_G is symmetric, functorial, simplicial, and is the identity for $q = 0$. In case G is the abelian group $\mathbf{A}^n_{\mathbb{K}}$, i.e. a vector space, then Z is an affine space, and the averaging process is the familiar one.

Actually we prove a more geometric result: with G as above, let X be a $\mathbb{K}$-scheme, let Z be a G-torsor over X, and let Y be any X-scheme. Suppose $\boldsymbol{f} = (f_0, \ldots, f_q)$ is a sequence of X-morphisms $f_i : Y \to Z$. Then there is an X-morphism

$$\mathrm{wav}_G(\boldsymbol{f}) : \boldsymbol{\Delta}^q_{\mathbb{K}} \times Y \to Z$$

called the *weighted average*. The operation wav_G is symmetric, simplicial, functorial in the data (G, X, Y, Z), and is the identity for $q = 0$. (In the previous paragraph we had the special case $X = Y = \operatorname{Spec} \mathbb{K}$).

Due to functoriality the averaging result extends to pro-unipotent groups, and also to semi-direct products, such as $\mathrm{GL}_n \ltimes G$, which occurs in our main application (i.e. $\pi : \operatorname{Coor} X \to X$).

It is interesting to note that our averaging process implies that a unipotent group in characteristic 0 is special (i.e. all torsors have rational sections). This was observed by Reichstein.

2. Rigid Dualizing Complexes on Schemes

In Subsections 2.1 and 2.3 I review some of the background. My own work is explained in Subsections 2.2 and 2.4.

2.1. **Duality for Commutative Rings – Overview.** Grothendieck duality theory for commutative rings [33] says that for a finite type homomorphism $f^* : A \to B$ between noetherian commutative rings there is a *twisted inverse image* functor $f^! : \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)$. Here $\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A)$ is the derived category of bounded below complexes of A-modules with finitely generated cohomologies. The assignment $f^* \mapsto f^!$ has to be a 2-*functor* (i.e. it should be compatible with compositions; this is also called a *pseudofunctor*). Moreover, when f^* is either smooth or finite, $f^!$ has to have particular formulas. If f^* is smooth of relative dimension n one defines

$$f^\sharp M := \Omega^n_{B/A}[n] \otimes_A M;$$

and when f^* is finite one defines

$$f^\flat M := \mathrm{RHom}_A(B, M).$$

Then there should be a 2-functorial isomorphism $f^! \cong f^\sharp$ for smooth homomorphisms, and a 2-functorial isomorphism $f^! \cong f^\flat$ for finite homomorphisms.

A word on notation. We denoted the ring homomorphism $A \to B$ by f^*, and this is to indicate that the corresponding morphism of affine schemes is $f : \operatorname{Spec} B \to \operatorname{Spec} A$. In the larger context (of algebraic geometry) this notation makes good sense, even though it might seem peculiar when applied to rings only.

Known constructions of this duality theory have to put some restriction on the category of rings, e.g. looking only at the category of finite type $\mathbb{K}$-algebras, where $\mathbb{K}$ is some noetherian base ring admitting a *dualizing complex* $R_{\mathbb{K}}$. We remind that a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called dualizing if it has finite injective dimension, and $\mathrm{RHom}_A(R, R) = A$. Suppose A is a finite type $\mathbb{K}$-algebra, with structural homomorphism $\pi^* : \mathbb{K} \to A$. It is known that $R_A := \pi^! R_{\mathbb{K}}$ is a dualizing complex over A, and hence it induces a duality (i.e. an auto-equivalence) $\mathrm{D}_A := \mathrm{RHom}_A(-, R)$ of $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$. For any homomorphism $f^* : A \to B$ one can recover the functor $f^!$ using the formula

$$f^! \cong \mathrm{D}_B \, \mathrm{L}f^* \, \mathrm{D}_A. \tag{2.1.1}$$

Thus to construct the 2-functor $f^!$ it suffices to find a collection of dualizing complexes R_A with suitable variance properties.

It might be a surprise to non-experts, but there is no easy way to construct the affine duality theory described above. The original treatment in [33] first builds the duality theory for all schemes (and in particular it needs to look at proper morphisms), and then restricts attention to affine schemes. There are alternative approaches (mainly in the work of Kunz and Lipman, and their respective students and collaborators [50, 51, 37, 52, 37, 34], where the concrete aspects are stressed), but they are neither easy nor complete.

2.2. **Rigid Dualizing Complexes over Commutative Rings.** In the preprint [92] Zhang and I apply ideas from noncommutative algebraic geometry to the problem. From our earlier work on rigid dualizing complexes over noncommutative rings [88] we knew that much of the commutative affine duality theory can be obtained quite easily, if we consider essentially finite type algebras over a base field $\mathbb{K}$. (Recall that a $\mathbb{K}$-algebra is called essentially finite type if it is a localization of a finitely generated algebra.) The challenge was to extend this method to include, at the very least, the case $\mathbb{K} = \mathbb{Z}$. We succeeded to do so in [93] for any finite dimensional regular noetherian ring $\mathbb{K}$. Our main result is that any essentially finite type $\mathbb{K}$-algebra A has a rigid dualizing complex R_A, which is unique up to a unique rigid isomorphism. If $f^* : A \to B$ is a finite (resp. essentially smooth homomorphism) then there is a canonical isomorphism $f^\flat R_A \cong R_B$ (resp. $f^\sharp R_A \cong R_B$). As explained in Subsection 2.1, this data is enough to construct the inverse image 2-functor $f^* \mapsto f^!$.

Here is a quick explanation of how rigidity is defined and used. Let A be any commutative ring and let B be a commutative A-algebra. In [92] we construct a functor

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B)$$

called the *squaring operation*. This is a quadratic functor, in the sense that given a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$ and an element $b \in B$ one has

$$\mathrm{Sq}_{B/A}(b\phi) = b^2 \, \mathrm{Sq}_{B/A}(\phi) \in \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(\mathrm{Sq}_{B/A} M, \mathrm{Sq}_{B/A} N).$$

In case B is flat over A then

$$\mathrm{Sq}_{B/A} M = \mathrm{RHom}_{B \otimes_A B}(B, M \otimes^{\mathrm{L}}_A M);$$

but in general we have to choose a K-flat DG algebra resolution $\tilde{B} \to B$ relative to A, and to use the DG algebra $\tilde{B} \otimes_A \tilde{B}$ to define $\mathrm{Sq}_{B/A}$.

Following Van den Bergh [67] we define a *rigid complex* over B relative to A to be a pair (M, ρ), where $M \in \mathsf{D}(\mathsf{Mod}\, B)$, and

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A} M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, B)$, called a *rigidifying isomorphism.* (I am suppressing some finiteness conditions.) A *rigid morphism* from (M, ρ) to (M', ρ') is a morphism $\phi : M \to M'$ in $\mathsf{D}(\mathsf{Mod}\, B)$ such that $\rho' \circ \phi = \mathrm{Sq}_{B/A}(\phi) \circ \rho$. We prove several properties of rigid complexes. In particular, we show that for an essentially smooth homomorphism $f^* : B \to C$ (i.e. f^* is essentially finite type and formally smooth) there is an induced rigidifying isomorphism $f^\sharp(\rho)$ on $f^\sharp M$. And for a finite homomorphism $f^* : B \to C$ there is an induced rigidifying isomorphism $f^\flat(\rho)$ on $f^\flat M$.

Now fix a regular noetherian ring $\mathbb{K}$ of finite Krull dimension. Let A be an essentially finite type $\mathbb{K}$-algebra. A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R, ρ) with R a dualizing complex. Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(R, R) = A$ it follows that the only rigid automorphism of R is the identity automorphism $\mathbf{1}_R$. From this it is not hard to prove that any two rigid dualizing complexes (R, ρ) and (R', ρ') over A relative to $\mathbb{K}$ are *uniquely isomorphic.* This fact, together with the induction operations $\rho \mapsto f^\sharp(\rho)$ and $\rho \mapsto f^\flat(\rho)$, make our method work.

2.3. **Duality for Schemes – Overview.** By Grothendieck duality for schemes we mean a twisted inverse image 2-functor from the category of schemes (say of finite type over a noetherian base ring $\mathbb{K}$) to the 2-category of all categories. Thus to a scheme X we assign the category $\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$, and to a morphism $f : X \to Y$ there is a functor $f^! : \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$, compatible with compositions. There should be 2-functorial isomorphisms $f^! \cong f^\sharp$ and $f^! \cong f^\flat$ for smooth homomorphisms and finite homomorphisms respectively (see notation in Subsection 2.1). Also there should be a nondegenerate trace morphism $\mathrm{Tr}_f : \mathrm{R}f_*\, f^! \to \mathbf{1}$ on $\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y)$ when f is proper.

There are two major obstacles to establishing such a Grothendieck duality theory for schemes (assuming that we already have at our disposal a satisfactory affine duality theory, as explained in Subsection 2.1). The first is the problem of defining the functors $f^!$ correctly, and the second is constructing the trace maps for proper morphisms. The first problem amounts to finding a suitable dualizing complex $\mathcal{R}_X$ on each scheme X (cf. formula (2.1.1)), whereas the second problem amounts to constructing $\mathrm{Tr}_f : \mathrm{R}f_*\, \mathcal{R}_X \to \mathcal{R}_Y$ when f is proper.

In [33] both these problems were tackled using Cousin complexes. Other treatments (like Deligne in [33, Appendix], or Neeman [58]) used representability arguments to obtain $f^!$ directly. More explicit approaches to Grothendieck duality on schemes can be found in the papers [4], [19], [36], [37], [38], [39], [44], [51], [52], [53], [73], [75].

2.4. **Rigid Dualizing Complexes on Schemes.** Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let X be a finite type $\mathbb{K}$ scheme. Recall that a dualizing complex on X is a complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ that has finite injective dimension, and $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R}) = \mathcal{O}_X$. In [85] we define a *rigid dualizing complex* over X relative to $\mathbb{K}$ to be a pair $(\mathcal{R}, \boldsymbol{\rho})$, where $\mathcal{R}$ is a dualizing complex, and $\boldsymbol{\rho} = \{\rho_U\}$ is a rigid structure on $\mathcal{R}$. This means that for any affine open set $U = \operatorname{Spec} A \subset X$ the dualizing complex $R_A := \mathrm{R}\Gamma(U, \mathcal{R})$ is equipped with a rigidifying isomorphism $\rho_U : R_A \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}} R_A$, and for any inclusion $g : U' = \operatorname{Spec} A' \subset U$ the isomorphism $g^* R_A \cong R_{A'}$ is rigid (i.e. $g^\sharp(\rho_U) = \rho_{U'}$).

We prove in [85] that X has a rigid dualizing complex $(\mathcal{R}, \boldsymbol{\rho})$, and it is unique up to a unique rigid isomorphism. Given the affine theory of rigid dualizing complexes (see Subsection 2.2) this becomes a problem of gluing (complexes and morphisms between them). Our strategy is as follows. The rigid dualizing complexes on affine pieces give rise to a dimension function dim on the points of X. Let us recall (from [33]) that a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ is said to be *Cohen-Macaulay* if the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ vanish unless $\dim(x) = -i$. Define $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_X)_{\mathrm{CM}}$ to be the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_X)$ consisting of Cohen-Macaulay complexes. We prove that the assignment $U \mapsto \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_U)$, for $U \subset X$ open, is a stack of categories on X. Now for an affine open set $U = \operatorname{Spec} A$ let $\mathcal{R}_U$ be the sheafification of the rigid dualizing complex R_A. Almost by definition $\mathcal{R}_U$ is a Cohen-Macaulay complex. For an inclusion $g : U' \to U$ of affine open sets there is an isomorphism $g^* \mathcal{R}_U \cong \mathcal{R}_{U'}$, coming from the rigidity, and these isomorphisms satisfy the cocycle condition on triple inclusions. So by the stack property we obtain a global complex $\mathcal{R}_X$, together with a rigid structure, and it is unique up to a unique isomorphism.

The fact that we are dealing with stacks implies very easily that for a finite morphism $f : X \to Y$ there is a canonical isomorphism $f^\flat \mathcal{R}_Y \cong \mathcal{R}_X$; and for a smooth morphism there is a canonical isomorphism $f^\sharp \mathcal{R}_Y \cong \mathcal{R}_X$. It remains to find a trace map when f is proper. This we do using Cousin complexes (in the sense of [33]). We define $\mathcal{K}_X$ to be the Cousin complex of the rigid dualizing complex $\mathcal{R}_X$, with respect to the dimension function dim mentioned above. Since $\mathcal{K}_X \cong \mathcal{R}_X$ there is a rigid structure $\boldsymbol{\rho}_X$ on $\mathcal{K}_X$, and we call the pair $(\mathcal{K}_X, \boldsymbol{\rho}_X)$ the *rigid residue complex* of X. For any scheme morphism $f : X \to Y$ we get a map of graded sheaves $\mathrm{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$, whose formulas are local. Indeed, the trace Tr_f depends only on the homomorphisms $f^* : \mathcal{O}_{Y,y}/\mathfrak{m}_y^i \to \mathcal{O}_{X,x}/\mathfrak{m}_x^i$ between truncated local rings, for x closed in $f^{-1}(y)$. These are finite homomorphisms in the category of essentially finite type $\mathbb{K}$-algebras, for which we have functorial traces on rigid dualizing complexes. In order to show that Tr_f is a map of complexes when f is proper we reduce to the case of $\mathbf{P}^1_A$, where A is an essentially finite type artinian $\mathbb{K}$-algebra (very similarly to the way it was done in [33]). We then use rigidity we prove a residue theorem for $\mathbf{P}^1_A$.

The last part of the paper [85] is about base change for Cohen-Macaulay morphisms. We show how rigidity allows a significant simplification of Conrad's results [19].

3. Rigid Dualizing Complexes over Noncommutative Rings

In this section I present two new applications of dualizing complexes to ring theory.

3.1. Multiplicities of Indecomposable Injectives.

Let A be a (left) noetherian ring. Any injective left A-module I is a direct sum of indecomposable injective left modules. Some of these indecomposables are related to prime ideals in A. Indeed, for a prime $\mathfrak{p}$ there is an indecomposable injective left A-module $J(\mathfrak{p})$, which is characterized (up to isomorphism) by the property that it has a nonzero submodule that's isomorphic to a submodule of $A/\mathfrak{p}$. Furthermore, the injective hull of the left A-module $A/\mathfrak{p}$ is isomorphic to $J(\mathfrak{p})^{\operatorname{Grank} A/\mathfrak{p}}$, where $\operatorname{Grank} A/\mathfrak{p}$ is the Goldie rank of $A/\mathfrak{p}$. Thus given an injective module I we can write $I \cong J(\mathfrak{p})^{\mu} \oplus I'$, where μ is a cardinal number, $J(\mathfrak{p})^{\mu}$ denotes the direct sum of μ copies of $J(\mathfrak{p})$, and I' does not have any nonzero submodule isomorphic to a submodule of $A/\mathfrak{p}$. The number μ is called the *multiplicity* of $J(\mathfrak{p})$ in I.

Now suppose that A is a Gorenstein noetherian ring, and let $A \to I^0 \to I^1 \to \cdots$ be a minimal injective resolution of A as left module. (This resolution is unique up to non-unique isomorphism.) For any $i \geq 0$ and prime ideal $\mathfrak{p}$ let $\mu_i(\mathfrak{p})$ denote the multiplicity of $J(\mathfrak{p})$ in I^i. It is a classical fact that when A is commutative these multiplicities are either 0 or 1; and they are called *Bass numbers* in that context. The determination of the multiplicities for noncommutative rings was the subject of quite a few papers: Barou-Malliavin [7], Malliavin [56], and Brown-Levasseur [11] studies universal enveloping algebras of finite dimensional solvable Lie algebras; and Brown [14], Brown-Hajarnavis [15] and Stafford-Zhang [64] looked at Gorenstein noetherian PI algebras. The last term of the minimal injective resolution of A was also studied, by the above authors and by Ajitabh-Smith-Zhang [1].

The theme of our paper with Zhang [93] is to generalize and unify the existing results mentioned above. The tool we use is *rigid Auslander dualizing complexes.* Here is a reminder of the definitions. Suppose $\mathbb{K}$ is a base field, and A is some noetherian $\mathbb{K}$-algebra, possibly noncommutative. Let $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A^{\mathrm{op}}$, so that A^{e}-modules are A-A-bimodules. A *dualizing complex* over A [85] is a complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ which has finite injective dimension and finitely generated cohomologies on both sides, and such that $\mathrm{RHom}_A(R, R) = A$ and $\mathrm{RHom}_{A^{\mathrm{op}}}(R, R) = A$. As in Subsection 2.2, R is called *rigid* [67] if $R \cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes_{\mathbb{K}} R)$. The complex R is said to be *Auslander* [74] if $\mathrm{Ext}^q_{A^{\mathrm{op}}}(N, R) = 0$ for every A^{op}-submodule $N \subset \mathrm{Ext}^p_A(M, R)$ and every $q > p$; and if the same holds after exchanging A and A^{op}.

Suppose A is a $\mathbb{K}$-algebra admitting a filtration such that the associated graded algebra is connected, noetherian and commutative (or just PI). It is known from [87] that A has an Auslander rigid dualizing complex R_A. Suppose $R_A \to J$ is a minimal injective resolution of R_A as complex of left A-modules. We prove a theorem that calculates the multiplicities of indecomposable injectives in the terms of J. As one of the corollaries we determine the multiplicities $\mu_i(\mathfrak{p})$ in the minimal injective resolution of the algebra $A := \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$. The multiplicities turn out to be either $\mu_i(\mathfrak{p}) = 0$ or $\mu_i(\mathfrak{p}) = \operatorname{Grank} A/\mathfrak{p}$, depending on the Gelfand-Kirillov dimension of $A/\mathfrak{p}$. Note that the earlier results [56, 11] only treated the case of solvable Lie algebras.

3.2. **Homological Transcendence Degree.** Let $\mathbb{K}$ be a base field, and consider a division ring D which is an essentially finitely generated $\mathbb{K}$-algebra (i.e. finitely generated as division ring), but possibly infinite over its center. One would like to define the transcendence degree of D over $\mathbb{K}$. In case D is finite over its center $\mathrm{Z}(D)$, the obvious definition would be to take the transcendence degree of the field $\mathrm{Z}(D)$. But otherwise a good definition was missing. There are a few properties that would be expected of such a definition. First one would expect it to coincide with the usual transcendence degree when D is commutative. Second, if $D_1 \subset D_2$ is a finite extension of division algebras (on either side) then these two algebras should have the same transcendence degree. A third condition is that in case D is the ring of fractions of some finitely generated subalgebra $A \subset D$, then the transcendence degree of D should coincide with the Gelfand-Kirillov dimension $\operatorname{GKdim} A$, or with the global dimension $\operatorname{gl.dim} A$, if the latter if finite.

The earliest definition of such a transcendence degree goes back to Gelfand-Kirillov [29]. They used it to distinguish between the various Weyl division rings. The problem is that this invariant is hard to compute. Since then several authors have looked into this problem: [54], [94], [95], [61], [63], [66]. The problem of finding a good definition of transcendence degree for division rings became more urgent in recent years, with the developments in noncommutative algebraic geometry.

In [90] Zhang and I propose the following definition. Given a division ring D over $\mathbb{K}$ we write $D^{\mathrm{e}} := D \otimes_{\mathbb{K}} D^{\mathrm{op}}$, and we define the *homological transcendence degree* of D over $\mathbb{K}$ to be

$$\operatorname{Htr} D := \operatorname{inj.dim}_{D^{\mathrm{e}}} D,$$

the injective dimension of D as D^{e}-module. (It is easy to see that this definition is left-right symmetric, i.e. $\operatorname{Htr} D = \operatorname{Htr} D^{\mathrm{op}}$.) We prove that this invariant has the first two expected properties listed above. As for the third property, assume D is the ring of fractions of a domain A which admits a filtration, such that associated graded algebra $\operatorname{gr} A$ is connected, noetherian and commutative (or just PI). In this case we prove that $\operatorname{Htr} D = \operatorname{GKdim} A$. This case includes, of course, the Weyl algebras and the universal enveloping algebras of finite dimensional Lie algebras. Finally we prove that for an Artin-Schelter regular graded Ore domain A, the ring of fractions D satisfies $\operatorname{Htr} D = \operatorname{gl.dim} A$. We do not need to assume that A is noetherian in this last case. The proofs rely on various properties of rigid Auslander dualizing complexes.

4. Research in Progress and Future Plans

4.1. **Rigid Dualizing Complexes.** The rigid dualizing complex R_A of an algebra A (over a base field $\mathbb{K}$) is a very special and useful object (cf. Section 3). There are several questions about rigid dualizing complexes that we consider to be interesting and important. The first has to do with existence. To date, the best existence criterion for a rigid dualizing complex is the one due to Van den Bergh [67], which goes via local duality for connected graded algebras. This criterion is very useful, yet does not cover all cases (e.g. there are noetherian affine PI algebras for which this criterion does not apply). We wish to find new, alternative methods to prove existence of rigid dualizing complexes.

The second question is about the behavior of rigid dualizing complexes in families. Namely, suppose A is a flat algebra over a commutative ring C. Is there a good notion of *relative rigid dualizing complex* $R_{A/C}$? In case A is commutative the answer is of course yes: take $R_{A/C} := f^{!}C$, where $f^{*} : C \to A$ is the structural homomorphism, and $f^{!}$ is Grothendieck's twisted inverse image functor (see Subsection 2.1).

The third question concerns the Auslander property of a dualizing complex (see Subsection 3.1). In all known examples, the rigid dualizing complex R_A has the Auslander property. Is this true in general? If not, what is the condition?

4.2. **Duality for Abstract Noncommutative Spaces.** Duality for affine noncommutative schemes (i.e. rings) is quite well-developed by now. So is duality for projective noncommutative schemes (in the sense of Artin-Zhang; see [86], [43] and [60]), and for noncommutative quasi-coherent ringed schemes (see [91]). However, so far nothing was done for the abstract quasi-schemes of Van den Bergh [68], or the noncommutative spaces of Kontsevich and Rosenberg [48]. We propose to perform research in this direction. Some problems: (a) find a good formulation of Grothendieck duality on an abstract noncommutative space; (b) define a notion of dimension, perhaps using a suitable Auslander property of dualizing complexes (cf. Subsection 3.1); (c) study regularity properties of the space as reflected by its dualizing complexes; (d) try to construct (analogues) of canonical projective embeddings, building on the properties of the rigid dualizing complex. Note the potential similarity of this item to Subsection 4.3, since the Kontsevich-Rosenberg construction resembles an algebraic stack.

4.3. **Dualizing Complexes on Algebraic Stacks.** Despite the fact that algebraic stacks (i.e. Deligne-Mumford stacks and Artin stacks) are used a lot nowadays, and much is known about their structure, still there is no theory of Grothendieck duality for stacks. This omission might be because the conventional approach to Grothendieck duality does not generalize well to algebraic stacks. On the face of it, the approach of [93] (see Subsection 2.4) might be suitable for algebraic stacks. Indeed, the core ingredient of [93], namely the theory of rigid dualizing complexes for commutative algebras, has a very well-understood variance behavior with respect to smooth homomorphisms. Presumably the second main ingredient of [85], which is the stack property of Cohen-Macaulay complexes for the Zariski topology, could be extended to bigger sites (such as the étale or smooth topologies), thus permitting gluing Cohen-Macaulay complexes defined locally on a suitable atlas of an algebraic stack.

Assuming success in formulating a Grothendieck duality theory for stacks, we shall want to apply it in particular to stacks of stable maps, which are the underlying geometric objects for the quantum cohomology of a scheme.

4.4. **The Derived Picard Group.** Let $\mathbb{K}$ be a field, and let A be a $\mathbb{K}$-algebra. The *derived Picard group* $\operatorname{DPic}(A)$ is the group of isomorphism classes of two-sided tilting complexes, and the operation is derived tensor product. It is known [77] that $\operatorname{DPic}(A)$ parameterizes the isomorphism classes of dualizing complexes over A.

An intriguing question about $\operatorname{DPic}(A)$ is its relation to the link graph on $\operatorname{Spec} A$, where A is a noetherian $\mathbb{K}$-algebra. If A is finite dimensional hereditary this was solved in [57]. In general, do any properties of the group $\operatorname{DPic}(A)$ constitute obstructions to noncommutative localization? What do conditions on dualizing complexes, like the Auslander condition, mean in terms of the group $\operatorname{DPic}(A)$?

4.5. **Homological Transcendence Degree and Noncommutative Birational Geometry.** Let $\mathbb{K}$ be a field. Division rings of transcendence degree 1 over $\mathbb{K}$ (i.e. "function fields of noncommutative curves") were classified by Artin and Stafford [6]. The case of transcendence degree 2 (i.e. "function fields of noncommutative surfaces") is work in progress. We propose to apply homological transcendence degree (see Subsection 3.2), and related ideas, to study division rings, and to contribute to the classification project.

There are also some questions pertaining to the invariant Htr itself. Let D be a division ring, essentially finitely generated over $\mathbb{K}$, and suppose that $D \otimes_{\mathbb{K}} D^{\mathrm{op}}$ is noetherian. Is $\operatorname{Htr} D < \infty$? If $\operatorname{Htr} D = n$, is $D^{\sigma}[n]$ a rigid dualizing complex over D for some automorphism σ? Both questions have positive answers in all examples we looked at.

4.6. **Geometry of Hopf Algebras.** The structure of the prime spectrum of a Hopf algebra H is the subject of a few recent papers [30, 31, 32, 12]. We propose to contribute to this subject. Our idea is to try to use the rigid dualizing complex R_H as follows. Presumably H is a Gorenstein ring (this is known to be true is some cases, e.g. when H is PI [70]), and $R_H \cong \boldsymbol{\omega}_H[n]$. In analogy to the Haar measure on a Lie group, we think that the dualizing bimodule $\boldsymbol{\omega}_H$ should be "invariant," namely it should be a "comodule" over H in some generalized way. Also $R_H = \boldsymbol{\omega}_H[n]$ should be Auslander. These strong properties of $\boldsymbol{\omega}_H$ should allow us to gain new insights into the geometry associated to H.

4.7. **Algebraic Aspects of Deformation Quantization.** In connection with this topic we wish to look into four problems. First, we would like to understand the relation between deformation quantization and other deformation processes that occur in noncommutative algebraic geometry, most notably the Sklyanin process of deforming $\mathbf{P}^2$. The methods of [91] are expected to help in this investigation.

The second problem is taken from Kontsevich's paper [47]: to work out precise formulas for the star product given by his quantization map (see Subsection 1.2). For instance, consider the polynomial algebra $\mathbb{K}[s,t]$ with Poisson structure $st\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t}$. Kontsevich speculated that the quantization is the $\mathbb{K}[[\hbar]]$-algebra generated by s,t with single relation $t \star s = \exp(\hbar) s \star t$; see [47, Section 2.3].

The third problem also emerges from [47]. It is to study semi-formal deformations a commutative algebra C. Kontsevich proved existence of such deformations in case the Poisson scheme $\operatorname{Spec} C$ has a suitable compactification. We want to get a better understanding of the significance of semi-formal deformations; to try to find other existence criteria; and maybe even to obtain a classification.

Finally we wish to extend the results of [79] by removing the hypothesis that the scheme X is $\mathcal{D}$-affine. This would force us to allow weak deformation quantizations, i.e. instead of a sheaf of $\mathbb{K}[[\hbar]]$-algebras $\mathcal{A}$ we only have a $\mathbb{K}[[\hbar]]$-linear prestack of algebroids (cf. [47], [16] and [55]). This is work in progress, jointly with F. Leitner.

4.8. **The Two Multiplications on Hochschild Cohomology.** Suppose $\mathbb{K}$ is a field of characteristic 0 and X is a smooth separated n-dimensional scheme over $\mathbb{K}$. The ith *Hochschild cohomology* of X is

$$\mathrm{HH}^i(X) := \mathrm{Ext}^i_{X^2}(\mathcal{O}_X, \mathcal{O}_X),$$

where $X^2 := X \times_{\mathbb{K}} X$, and $\mathcal{O}_X$ is considered a coherent $\mathcal{O}_{X^2}$-module via the diagonal embedding. The ith *tangent cohomology* of X is

$$\mathrm{HT}^i(X) := \bigoplus\nolimits_{p+q=i} \mathrm{H}^q\big(X, \bigwedge\nolimits^p_{\mathcal{O}_X} \mathcal{T}_X\big),$$

where $\mathcal{T}_X$ is the tangent sheaf of X. There is a canonical isomorphism

$$\mathrm{HH}^i(X) \cong \mathrm{HT}^i(X), \tag{4.8.1}$$

which was first established by Swan [Sw], and a more general proof appeared in our paper [78]. Our proof goes like this: we introduced the continuous Hochschild cochain complex $\mathcal{C}_{\mathrm{cd},X} = \bigoplus_{p \geq 0} \mathcal{C}^p_{\mathrm{cd},X}$, and proved that it is canonically isomorphic to $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$ in the derived category of $\mathcal{O}_{X^2}$-modules. Next we proved that the canonical map

$$\bigoplus\nolimits_p \big(\bigwedge\nolimits^p_{\mathcal{O}_X} \mathcal{T}_X\big)[-p] \to \mathcal{C}_{\mathrm{cd},X}$$

is a quasi-isomorphism of complexes of $\mathcal{O}_X$-modules. (After shifting this is exactly the quasi-isomorphism $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly},X} \to \mathcal{D}_{\mathrm{poly},X}$ occurring in deformation quantization; see Subsection 1.2.) Applying global cohomology one obtains the isomorphisms $\mathrm{HH}^i(X) \cong \mathrm{HT}^i(X)$. Our proof also shows that the isomorphism (4.8.1) is compatible with étale morphisms $X' \to X$.

Note that when X is affine the isomorphism (4.8.1) becomes the Hochschild-Kostant-Rosenberg Theorem.

The graded $\mathbb{K}$-module $\mathrm{HH}(X) := \bigoplus_i \mathrm{HH}^i(X)$ is a graded algebra with the Yoneda product, whereas $\mathrm{HT}(X) := \bigoplus_i \mathrm{HT}^i(X)$ becomes a graded $\mathbb{K}$-algebra by combining the wedge product with the cup product on cohomology. In the last page of his paper [45], Kontsevich makes the following unproved claim: there is an isomorphism of graded algebras $\mathrm{HH}(X) \cong \mathrm{HT}(X)$, which is compatible with étale morphisms. Kontsevich states that this isomorphism is important for mirror symmetry. Some authors (e.g. [17]) say that the formula for the algebra isomorphism is as follows. There is an element c in the Hodge cohomology $\bigoplus_{p,q} \mathrm{H}^q(X, \Omega^p_X)$, which represents the square root of the Todd class. The algebra isomorphism $\mathrm{HT}(X) \to \mathrm{HH}(X)$ should be multiplication by c on $\mathrm{HT}(X)$, composed with the isomorphism (4.8.1).

I propose to prove this claim of Kontsevich. The idea is to use the methods of [79] to obtain an A_∞ quasi-isomorphism

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}[-1]) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}[-1]).$$

The twisting corresponding to the Grothendieck connection (cf. Subsection 1.4) is expected to give rise to the twisting of $\mathrm{HT}(X)$ by the element c. There are some preliminary results, and a collaboration with C. Torossian.

4.9. **Homological Mirror Symmetry and Noncommutative Algebraic Geometry.** There are several points of contact between homological mirror symmetry (and its surrounding mathematical envelope) and noncommutative algebraic geometry. We plan to see if some of the techniques that we have developed over the years may be applied to this area of research. In particular we are thinking about the following techniques: (a) duality for noncommutative projective schemes; (b) dualizing complexes over noncommutative ringed schemes, which by definition act on the derived category by a dual Fourier-Mukai transform; and (c) Auslander-Reiten quivers of derived categories [57]. Some evidence in this direction can be found in the recent paper [2], in which the homological mirror symmetry conjecture is proved for weighted projective spaces and their noncommutative deformations, and which uses our duality for noncommutative projective schemes [86].

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