RESEARCH SUMMARY: 2007-2011

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This summary covers the research in mathematics I have done from 2007 to 2011. For earlier work please consult [32]. The summary is organized by topics. Section 3 is about current work and plans for future research.

Contents

1. Twisted Deformation Quantization in Algebraic Geometry

The study of deformation quantization of algebraic varieties was completed in my paper [33], published in 2005. Here is a quick overview. Let $(X, \mathcal{O}_X)$ be a smooth algebraic variety over a field $\mathbb{K}$ of characteristic 0. A Poisson (resp. associative) deformation of $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of Poisson (resp. associative) $\mathbb{K}[[\hbar]]$-algebras on X, such that $\mathcal{A}$ is flat and $\hbar$-adically complete, together with an isomorphism $\mathcal{A}/(\hbar) \cong \mathcal{O}_X$. (Actually the algebra $\mathbb{K}[[\hbar]]$ can be replaced by any complete noetherian local commutative $\mathbb{K}$-algebra R with residue field $\mathbb{K}$; I talk only about $\mathbb{K}[[\hbar]]$ to simplify the discussion.) I proved that for algebraic varieties X satisfying certain cohomological conditions (including affine varieties and the projective spaces $\mathbf{P}^n$), there is a bijection called the *quantization map* between Poisson deformations and associative deformations. This theorem is an analogue of the famous quantization result of Kontsevich [20] for differentiable manifolds.

After that I attempted to study *twisted deformation quantization of algebraic varieties*. A twisted deformation $\boldsymbol{\mathcal{A}}$ of $\mathcal{O}_X$ is a stacky version of a sheaf deformation. In the terminology of Kontsevich [21], a *twisted associative deformation* is very similar to a *stack of* $\mathbb{K}[[\hbar]]$*-algebroids*; and in the terminology of Kashiwara-Schapira [23] this is similar to a *DQ algebroid*. The idea that any Poisson deformation of $\mathcal{O}_X$ should admit a quantization to a twisted associative deformation is due to Kontsevich [21]; but this was quite sketchy, with no precise statements and no proofs.

In my paper [37] I introduced the notion of *twisted Poisson deformation* (this was never defined previously). The main idea behind my definition of twisted deformations is to break up a stacky deformation $\boldsymbol{\mathcal{A}}$ into two components: the *gauge gerbe* $\boldsymbol{\mathcal{G}}$, which controls the local objects and local isomorphisms between them; and the *representation*, which describes the actual algebra structure of the

Date: 30 Dec 2011.

local objects. Thus to understand how complicated a twisted deformation is (how far it is from being a sheaf) it is enough to study the gauge gerbe $\boldsymbol{\mathcal{G}}$. On the other hand, the gauge gerbe inherits a pronilpotent structure from the representation (the $\hbar$-adic filtration).

In the auxilliary paper [38] I introduced the concept of *pronilpotent gerbe*. I proved that there is an obstruction theory for such gerbes, similar to the well-known obstruction theory for abelian gerbes. As mentioned above, the gauge gerbe $\boldsymbol{\mathcal{G}}$ of a twisted deformation is pronilpotent; and the main result of [38] is that on any affine open set $U \subset X$ the groupoid $\boldsymbol{\mathcal{G}}(U)$ is nonempty and connected. This fact is crucial for making the descent data effectivce. See the lecture notes
http://www.math.bgu.ac.il/∼amyekut/lectures/ext-gerbes/notes.pdf
for more details.

In the auxilliary paper [40] I introduced the *reduced Deligne groupoid*, and proved that it is a reasonable object even for complete parameter algebras (such as $\mathbb{K}[[\hbar]]$). Until then all discussion of formal deformations (including in my own work!) was imprecise: the detailed proofs (cf. [9] and the references therein) only handled the nilpotent case (artinian parameter algebras such as $\mathbb{K}[\hbar]/(\hbar^m)$, $m \geq 1$). It was tacitly assumed that one could naively "go to the limit"; but in hindsight that is not true: the complete case required some nontrivial constructions and assumptions. I also extended the definition of the *Deligne* 2*-groupoid* to complete DG Lie algebras of quasi-quantum type; previously [15] the definition only applied to nilpotent algebras of quantum type, which was not good enough for our purposes. See the lecture notes
http://www.math.bgu.ac.il/∼amyekut/lectures/MC-complete/notes.pdf.

The twisted quantization map in [37] required *higher descent*, for gluing twisted deformations out of local information (descent data). For this purpose I wrote the short paper [41] on *cosimplicial crossed groupoids*, whose contents is all combinatorial. See the lecture notes
http://www.math.bgu.ac.il/∼amyekut/lectures/higher-descent/notes.pdf.

The final result in [37] is this: let $\mathbb{K}$ be a field containing the real numbers, and let X be a smooth algebraic variety over $\mathbb{K}$. Then there is a bijection of sets

$$\begin{aligned} \text{tw.quant} : \frac{\{\text{twisted Poisson deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}} \\ \xrightarrow{\simeq} \frac{\{\text{twisted associative deformations of } \mathcal{O}_X\}}{\text{twisted gauge equivalence}} \end{aligned}$$

called the *twisted quantization map*. This quantization map respects étale morphisms $X' \to X$, and also respects first order brackets (i.e. the Poisson brackets on $\mathcal{O}_X$ induced by twisted deformations). For a pretty detailed overview of this work please consult the survery article [34] (16 pages), or the lecture notes
http://www.math.bgu.ac.il/∼amyekut/lectures/twisted-defs/notes.pdf.

There remain several interesting open questions after [37]. Perhaps the most interesting question concerns the twisted quantization of a symplectic Poisson bracket on a Calaby-Yau surfaces: is this a really twisted associative deformation (i.e. not a sheaf)?

Let me also mention that my foundational work on the continuous Hochschild complex of a scheme in [31], and on the coordinate bundle in algebraic geometry in [33], have played central roles in many papers related to deformation quantization by other authors – notably the papers by Van den Bergh and Calaque [10] and by Caldararu [11].

Folllowing ideas of Kontsevich and Calaque, I constructed a theory of *nonabelian multiplicative surface integration* [39]. The purpose at the time was to enable gluing of gerbes; but this was eventually superseded by [41]. Still I believe this construction is valuable, in that it generalizes the classical 1-dimensional multiplicative integration (aka "path ordered exponential integration"), and is related to current work on nonabelian gauge theory (cf. [5, 7, 27]). See the lecture notes

http://www.math.bgu.ac.il/∼amyekut/lectures/multi-integ/notes.pdf

for more explanations.

Folllowing a question in a talk in a conference of deformation quantization in Scalea (Italy) in 2010, I wrote to short paper [36]. This paper proved that the derived Picard group of an associative deformation of a commutative ring is as small as possible (a direct product of the Picard group and $\mathbb{Z}$).

In total there are 7 papers in my project on twisted deformation quantization.

2. The Homology of Completion

In the process of work on deformation quantization [37] I realized that I needed some rather basic facts about completions. Specifically, let A be a commutative noetherian ring and $\mathfrak{a}$ an ideal in it. I needed to know about the structure of the $\mathfrak{a}$-adic completion of a free A-module of infinite rank. I also needed to know about sheaves of complete A-modules on a topological space. My results are in the paper [35]. There is some overlap with the book [28] on commutative algebra, and with the recent paper [23] of Kashiwara and Schapira, that also deals with deformations.

In a conference in Scalea (Italy) in 2010 I heard a very interesting talk by Kashiwara about [23], where he mentioned *cohomologically complete complexes*. This notion was very intriguing to me, and I wondered how it could be related the the work of Alonso, Jeremias and Lipman on the *Greenlees-May Duality*; see [16, 1, 2].

In the paper [26] with Porta and Shaul we gave an alternative proof of the GM duality, and some extensions of it, notably the *MGM Equivalence* between the category of cohomologically complete complexes and the category of cohomologically torsion complexes. We also showed that the definition of cohomologically complete complexes in [23] coincides with the one coming from the approach of [1]. We obtained a result on completion by derived double centralizer, which extends recent work of Efimov [13] (inspired by Kontsevich), and of Dwyer-Greenlees [12]. See the lecture notes

http://www.math.bgu.ac.il/∼amyekut/lectures/cohom-complete/notes.pdf

for more explanations.

The thesis of Liran Shaul (my Ph.D. student), about rigid dualizing complexes over complete rings, uses the paper [26] as its starting point.

3. Research in Progress and Future Plans

3.1. **Rigid Dualizing Complexes on Schemes and DM Stacks.** In my joint project with Zhang on rigid dualizing complexes over commutative rings [47, 48], we found an extremely effective way to handle Grothendieck duality on *affine schemes*. This utilizes Van den Bergh's *rigid dualizing complex*.

A followup paper [43] on *rigid dualizing complexes on schemes* has been in the works for a few years (see the survey paper [42]). The paper was completely written, but I was unhappy with the proof of the *residue theorem for proper maps.* I believe I now have an elegant proof (using rigidity in a subtle way), so the paper can finally be submitted for publication (this time it will be my own paper, without Zhang).

Despite the fact that algebraic stacks (i.e. Deligne-Mumford stacks and Artin stacks) are used a lot nowadays, and much is known about their structure, still there is no full theory of Grothendieck duality for stacks. The existence of dualizing complexes on DM stacks is known (see e.g. the recent paper by Arinkin and Bezrukavnikov [3]), as well as functoriality for smooth and finite maps of stacks. However *traces for proper maps of stacks* are not available so far.

This omission might be because the conventional approach to Grothendieck duality (representability of the adjoint to proper pushforward) does not generalize well to algebraic stacks. On the face of it, the approach of [47] might be suitable for algebraic stacks. Indeed, the core ingredient of [47], namely the theory of rigid dualizing complexes for commutative algebras, has a very well-understood variance behavior with respect to étale homomorphisms. Presumably the second main ingredient of [47], which is the stack property of Cohen-Macaulay complexes for the Zariski topology, could be extended to bigger sites (such as the étale topology), thus permitting gluing Cohen-Macaulay complexes defined locally on an étale atlas of a DM stack.

Assuming success in formulating a Grothendieck duality theory for stacks, including traces for proper maps, we hope to use it for studying invariants of stacks (such as *stacks of stable maps*, that give rise to *Gromov-Witten invariants*), which presently can only be done using analytic integration (over the complex numbers).

3.2. **Rigid Dualizing Complexes for DG Algebras and DG Schemes.** The notion of dualizing complex over a DG algebra has been studied in a few papers (mainly by students of Foxby). However nothing was done so far regarding rigid dualizing complexes over DG algebras; and of course not for DG schemes (or more complicated derived schemes). On the other hand the role of the diagonal embedding is central for DG algebras: this is used in Kontsevich's definition of smoothness of a DG algebra.

Here are a few of my aims: to establish rigidity for dualizing complexes over DG algebras; and to extend existing results on schemes by working with DG schemes (e.g. the relative dualizing complex: remove the flatness assumption).

In the case of smooth finite dimensional noncommutative algebras, and of smooth proper schemes (even noncommutative), we know that the rigid dualizing complex represents the Serre functor (cf. [46]). Perhaps for smooth DG algebras this should also be true. In this case, and if the rigid dualizing complex (or its inverse) is *ample*, then a projective embedding of sorts can be constructed (like in [6]).

3.3. **Geometry of Hopf Algebras.** The structure of the prime spectrum of a Hopf algebra H is the subject of a few recent papers (see [8] and its references). We propose to contribute to this subject. Our idea is to try to use the rigid dualizing complex R_H as follows. Presumably H is a Gorenstein ring (this is known to be true is many cases, e.g. when H is PI [30]), and $R_H \cong \boldsymbol{\omega}_H[n]$. In analogy to the Haar measure on a Lie group, we think that the dualizing bimodule $\boldsymbol{\omega}_H$ should be "invariant," namely it should be a "comodule" over H in some generalized way. Also $R_H = \boldsymbol{\omega}_H[n]$ should be Auslander. These strong properties of $\boldsymbol{\omega}_H$ should allow us to gain new insights into the geometry associated to H. One possible outcome would be the construction of a dual Hopf algebra (inside a suitable derived category perhaps), generalizing what happens in the finite dimensional case.

3.4. **Algebraic Aspects of Deformation Quantization.** In connection with this topic we wish to look into three problems. First, we would like to understand the relation between deformation quantization and other deformation processes that occur in noncommutative algebraic geometry, most notably the Sklyanin process of deforming $\mathbf{P}^2$. The methods of [46] are expected to help in this investigation.

The second problem is taken from Kontsevich's paper [21]: to work out precise formulas for the star product given by his quantization map. For instance, consider the polynomial algebra $\mathbb{K}[s,t]$ with Poisson structure $st\frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t}$. Kontsevich speculated that the quantization is the $\mathbb{K}[[\hbar]]$-algebra generated by s,t with single relation $t \star s = \exp(\hbar) s \star t$; see [21, Section 2.3].

The third problem also emerges from [21]. It is to study semi-formal deformations a commutative algebra C. Kontsevich proved existence of such deformations in case the Poisson scheme $\operatorname{Spec} C$ has a suitable compactification. We want to get a better understanding of the significance of semi-formal deformations; to try to find other existence criteria; and maybe even to obtain a classification.

3.5. **Homological Mirror Symmetry and Noncommutative Algebraic Geometry.** There are several points of contact between *homological mirror symmetry* (and its surrounding mathematical envelope) and noncommutative algebraic geometry. We plan to see if some of the techniques that we have developed over the years may be applied to this exciting new area of research. In particular we are thinking about the following techniques: (a) duality for noncommutative projective schemes [44]; (b) dualizing complexes over noncommutative ringed schemes, which by definition act on the derived category by a dual Fourier-Mukai transform [46]; and (c) Auslander-Reiten quivers of derived categories [17]. Some evidence in this direction can be found in the paper [4], in which the homological mirror symmetry conjecture is proved for weighted projective spaces and their noncommutative deformations, and which uses our duality for noncommutative projective schemes [44].

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