Duality in Noncommutative Algebraic Geometry

Lecture Notes [1]

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This is a continuation of my talk "Dualizing Complexes over Noncommutative Rings". Here is the plan of the lecture:

1. Noncommutative Projective Schemes: Review
2. Grothendieck-Serre Duality for $\operatorname{Proj} A$
3. Quasi-Coherent Ringed Schemes
4. Dualizing Complexes on Noncommutative Ringed Schemes
5. Differential Quasi-Coherent Ringed Schemes
6. Perverse Coherent Sheaves
7. The Auslander Condition Revisited

Most of the work is joint with James Zhang (UW, Seattle).

1. Noncommutative Projective Schemes: Review

Let $\mathbb{K}$ be a field and let A be a noetherian connected graded $\mathbb{K}$-algebra. Recall that "connected" means that $A = \bigoplus_{i \geq 0} A_i$, with $A_0 = \mathbb{K}$ and each A_i a finitely generated $\mathbb{K}$-module. Artin and Zhang [AZ] define the *noncommutative projective spectrum* $\operatorname{Proj} A$ as follows.

Let $\mathsf{GrMod}\, A$ be the category of graded left A-modules. In it there is the subcategory $\mathsf{Tors}\, A$ consisting of $\mathfrak{m}$-torsion modules with respect to the augmentation ideal $\mathfrak{m}$ of A. Consider the quotient abelian category

$$\mathsf{Tails}\, A := \frac{\mathsf{GrMod}\, A}{\mathsf{Tors}\, A},$$

also denoted by $\mathsf{QGr}\, A$, with projection functor π. The "noncommutative space $\operatorname{Proj} A$" is the space such that "the category of quasi-coherent sheaves on $\operatorname{Proj} A$ is $\mathsf{Tails}\, A$". Thus if we write $X := \operatorname{Proj} A$ then

$$\mathsf{QCoh}\, X = \mathsf{Tails}\, A$$

tautologically. The subcategory $\mathsf{Coh}\, X$ of "coherent sheaves" is the image under π of the finitely generated modules.

Remark 1.1. $\operatorname{Proj} A$ is not a genuine space. It is only a metaphor, invented for heuristic purposes.

The category $\mathsf{QCoh}\, X$ is equipped with a "global sections functor"

$$\Gamma(X, -) := \operatorname{Hom}_{\mathsf{QCoh}\, X}(\mathcal{O}_X, -) : \mathsf{QCoh}\, X \to \mathsf{Mod}\, \mathbb{K},$$

where $\mathcal{O}_X$ is the class of the graded module A. There is also a Serre twist functor: if $\mathcal{M} \in \mathsf{QCoh}\, X$ is the class of a graded module M, then $\mathcal{M}(1)$ is the class of the shifted module $M(1)$.

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Recall Serre's Theorem on projective schemes: if X is a projective (commutative) scheme over $\mathbb{K}$ and $\mathcal{L}$ is an ample invertible sheaf then the commutative graded algebra

$$A := \bigoplus\nolimits_{i \geq 0} \Gamma(X, \mathcal{L}^i)$$

is noetherian, and $X \cong \operatorname{Proj} A$. The noncommutative definition of [AZ] in effect takes Serre's Theorem and turns it around.

2. Grothendieck-Serre Duality for $\operatorname{Proj} A$

There is a version of Grothendieck-Serre Duality for noncommutative projective schemes. Here we must assume that the algebra A admits a balanced dualizing complex R_A (see first lecture). By [YZ1, VdB] this is equivalent to A satisfying the χ condition on both sides, and the functors $\Gamma_{\mathfrak{m}}$, $\Gamma_{\mathfrak{m}^{\mathrm{op}}}$ having finite cohomological dimension. Let $\mathcal{R}_X$ be the class in $\mathsf{D}(\mathsf{QCoh}\, X)$ of $R_A[-1]$.

Theorem 2.1. ([YZ1]) *There is a functorial isomorphism*

$$\mathrm{RHom}_{\mathsf{QCoh}\, X}(\mathcal{M}, \mathcal{R}_X) \cong \mathrm{R}\Gamma(X, \mathcal{M})^*$$

for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)$.

In the theorem $(-)^*$ denotes $\mathbb{K}$-linear dual.

This duality has a few generalizations. Let $X^{\mathrm{op}} := \operatorname{Proj} A^{\mathrm{op}}$ be the opposite noncommutative projective scheme, with projection functor

$$\pi^{\mathrm{op}} : \mathsf{GrMod}\, A^{\mathrm{op}} \to \mathsf{QCoh}\, X^{\mathrm{op}}.$$

Let

$$\underline{\Gamma} : \mathsf{QCoh}\, X \to \mathsf{GrMod}\, A$$

be the functor

$$\underline{\Gamma}\, \mathcal{M} := \bigoplus\nolimits_{i \geq 0} \Gamma(X, \mathcal{M}(i)).$$

The next duality result appeared in the paper of Kapustin, Kuznetsov and Orlov (in a slightly weaker form). They used it to define vector bundles on $\operatorname{Proj} A$, in the study of noncommutative instantons.

Theorem 2.2. ([KKO]) *The functor*

$$\mathcal{M} \mapsto \pi^{\mathrm{op}}\, \mathrm{RHom}_A^{\mathrm{gr}}(\mathrm{R}\underline{\Gamma}\, \mathcal{M}, R_A[-1])$$

is a duality

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X^{\mathrm{op}}).$$

Van den Bergh and de Naeghel [NV] have recently proved yet another version of Serre Duality, this time in the bivariant formulation of Bondal-Kapranov [BK]. Define a functor

$$\mathcal{R}_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} - : \mathsf{D}^{-}(\mathsf{QCoh}\, X) \to \mathsf{D}^{-}(\mathsf{QCoh}\, X)$$

by

$$\mathcal{R}_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{M} := \pi(R[-1] \otimes^{\mathrm{L}}_A \mathrm{R}\underline{\Gamma}\, \mathcal{M}).$$

Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)_{\mathrm{fpd}}$ be the category of complexes with finite projective dimension (the compact objects in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)$).

Theorem 2.3. ([NV]) *There is a functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}(\mathcal{M}, \mathcal{R}_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} \mathcal{P}) \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{QCoh}\, X)}(\mathcal{P}, \mathcal{M})^*$$

for $\mathcal{P} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)_{\mathrm{fpd}}$ *and* $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)$*.*

The reduces to Theorem 2.1 when $\mathcal{P} = \mathcal{O}_X$.

Definition 2.4. The noncommutative projective scheme X is *smooth* if the category $\mathsf{Coh}\, X$ has finite global dimension; i.e. there is a number d such that

$$\mathrm{Ext}^i_{\mathsf{Coh}\, X}(\mathcal{M}, \mathcal{N}) = 0$$

for all objects $\mathcal{M}, \mathcal{N} \in \mathsf{Coh}\, X$ and all $i > d$.

If X is smooth then every object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)$ has finite projective dimension; and the functor $\mathcal{R}_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} -$ is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)$. Hence:

Corollary 2.5. ([NV]) *If X is smooth then the functor $\mathcal{R}_X \otimes^{\mathrm{L}}_{\mathcal{O}_X} -$ is a Serre functor of* $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{QCoh}\, X)$*.*

3. Quasi-Coherent Ringed Schemes

In this section we look at another kind of noncommutative space. Throughout $\mathbb{K}$ is a base field.

Definition 3.1. ([YZ4]) Suppose X is a $\mathbb{K}$-scheme, $\mathcal{A}$ is a sheaf of $\mathbb{K}$-algebras on X, and there is ring homomorphism $\mathcal{O}_X \to \mathcal{A}$ making $\mathcal{A}$ into a quasi-coherent $\mathcal{O}_X$-module on both sides. We call $(X, \mathcal{A})$ a *quasi-coherent ringed scheme over* $\mathbb{K}$.

Such ringed schemes are abundant; some prototypical examples are:

(1) X is any $\mathbb{K}$-scheme and $\mathcal{A} = \mathcal{O}_X$.
(2) $X = \operatorname{Spec} \mathbb{K}$ and $\mathcal{A}$ is any $\mathbb{K}$-algebra.
(3) X is smooth, $\operatorname{char} \mathbb{K} = 0$ and $\mathcal{A} := \mathcal{D}_X$, the sheaf of differential operators.
(4) X is any $\mathbb{K}$-scheme and $\mathcal{A}$ is any quasi-coherent $\mathcal{O}_X$-algebra.

Given a ringed scheme $(X, \mathcal{A})$ its opposite ringed scheme is $(X, \mathcal{A}^{\mathrm{op}})$, and the product is a quasi-coherent ringed scheme which we denote by $(X^2, \mathcal{A}^{\mathrm{e}})$. The definition of the product is pretty obvious. See Figure 1.

It is not hard to show uniqueness of $(X^2, \mathcal{A}^{\mathrm{e}})$. Surprisingly existence is not automatic! In [YZ4] we proved that the product exists iff certain Ore conditions are satisfied. (There are counterexamples.)

4. Dualizing Complexes on Noncommutative Ringed Schemes

Let $(X, \mathcal{A})$ be a noetherian quasi-coherent ringed scheme. By noetherian I mean that X is noetherian, and for any affine open set U the ring $A := \Gamma(U, \mathcal{A})$ is noetherian. Assume the product $(X^2, \mathcal{A}^{\mathrm{e}})$ exists and is noetherian too.

Let $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$ be some complex. We may define a functor

$$\mathrm{D} : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{op}})$$

as follows:

$$\mathrm{D}\mathcal{M} := \mathrm{R}\mathrm{p}_{2*}\, \mathrm{R}\mathcal{H}om_{\mathrm{p}_1^{-1}\mathcal{A}}(\mathrm{p}_1^{-1}\mathcal{M}, \mathcal{R}).$$

This is a contravariant Fourier-Mukai transform. See Figure 2.

There is an opposite version of this functor:

$$\mathrm{D}^{\mathrm{op}} : \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{A}).$$

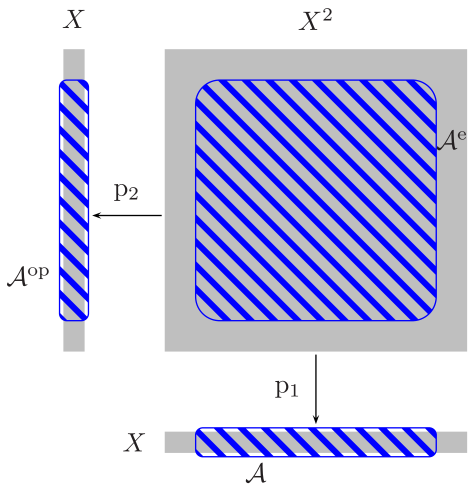

FIGURE 1. The product of $(X, \mathcal{A})$ and $(X, \mathcal{A}^{\mathrm{op}})$.

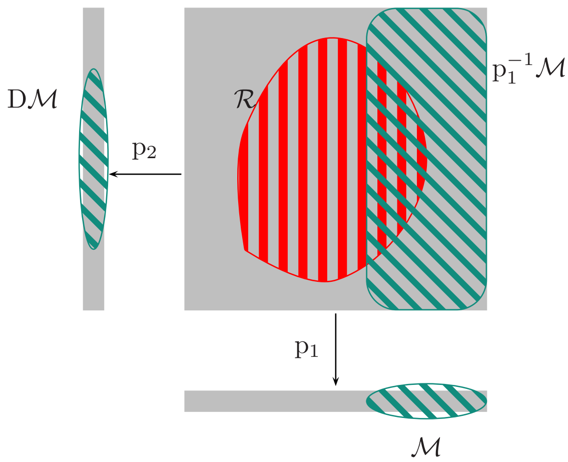

FIGURE 2. The duality functor D.

Definition 4.1. A complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})$ is called a *dualizing complex* if the adjunction morphisms $\mathbf{1} \to \mathrm{D}^{\mathrm{op}}\mathrm{D}$ and $\mathbf{1} \to \mathrm{D}\mathrm{D}^{\mathrm{op}}$ are both isomorphisms. (I am suppressing some details.)

The definition I just gave allows for all kinds of exotic dualizing complexes.

Example 4.2. Say X is an elliptic curve and take $\mathcal{A} := \mathcal{O}_X$. Then the product is $(X^2, \mathcal{A}^{\mathrm{e}}) = (X^2, \mathcal{O}_{X^2})$. It turns out that the Poincaré bundle $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}})$ is a dualizing complex over $(X, \mathcal{O}_X)$ in the noncommutative sense.

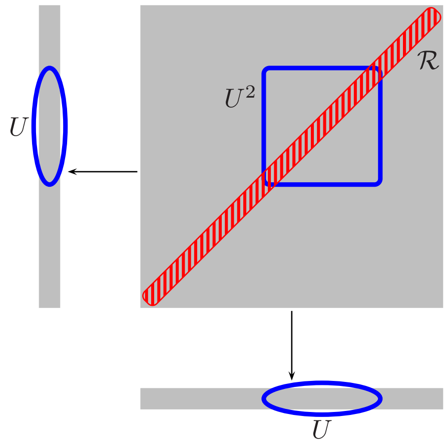

FIGURE 3. A dualizing complex $\mathcal{R}$ supported on the diagonal.

We are interested in dualizing complexes $\mathcal{R}$ that behave similarly to Grothendieck's dualizing complex $\pi^{!}\mathbb{K}$. Hence the definition below.

Definition 4.3. A *rigid dualizing complex over* $(X, \mathcal{A})$ is a pair $(\mathcal{R}, \boldsymbol{\rho})$, where:

(1) $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$ is a dualizing complex supported on the diagonal in X^2.
(2) $\boldsymbol{\rho} = \{\rho_U\}$ is a collection of rigidifying isomorphisms. Namely for any affine open set $U \subset X$, letting $A := \Gamma(U, \mathcal{A})$ and
$$R := \mathrm{R}\Gamma(U^2, \mathcal{R}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, A^{\mathrm{e}}),$$
the pair (R, ρ_U) is a rigid dualizing complex over A.
(3) The collection $\boldsymbol{\rho}$ satisfies a compatibility condition for inclusions of affine open sets.

See Figure 3 for an illustration.

Example 4.4. Consider a separated finite type $\mathbb{K}$-scheme X with structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$. If one looks carefully at the variance properties of the dualizing complex $\mathcal{R} := \pi^{!}\mathbb{K}$ that are worked out in [RD], one sees that this is in fact a rigid dualizing over $(X, \mathcal{O}_X)$.

5. Differential Quasi-Coherent Ringed Schemes

All quasi-coherent ringed schemes $(X, \mathcal{A})$ that "occur naturally" are of the following kind.

Definition 5.1. A *differential quasi-coherent ringed scheme of finite type over* $\mathbb{K}$ is a quasi-coherent ringed scheme $(X, \mathcal{A})$ such that:

(1) X is a finite type $\mathbb{K}$-scheme.
(2) There is an exhaustive nonnegative filtration $G = \{G_i\mathcal{A}\}_{i \in \mathbb{Z}}$ on the sheaf of rings $\mathcal{A}$ such that $\mathrm{gr}^G \mathcal{A}$ is a coherent module over its center $\mathrm{Z}(\mathrm{gr}^G \mathcal{A})$, and $\mathrm{Z}(\mathrm{gr}^G \mathcal{A})$ is a quasi-coherent $\mathcal{O}_X$-algebra of finite type.

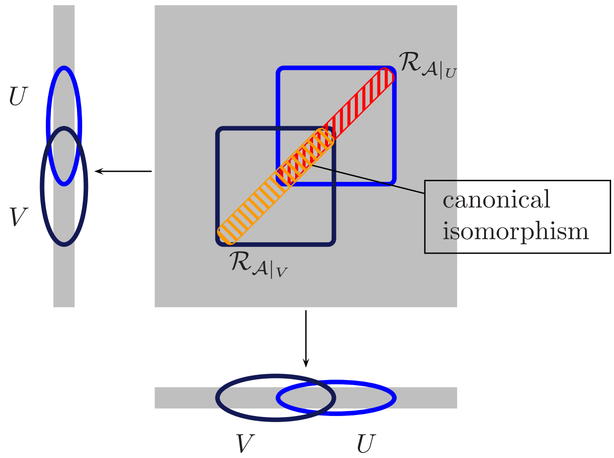

FIGURE 4. Gluing data for a rigid complex

Example 5.2. If X is smooth in characteristic 0 and $\mathcal{A} = \mathcal{D}_X$ is the ring of differential operators, then we can take G to be the order filtration on $\mathcal{A}$.

Example 5.3. If $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra (e.g. an Azumaya algebra) then we can take G to be the trivial filtration.

Let $(X, \mathcal{A})$ be a separated differential quasi-coherent ringed scheme of finite type over $\mathbb{K}$. The fact that $(X, \mathcal{A})$ is differential implies that the product $(X^2, \mathcal{A}^{\mathrm{e}})$ exists. It is not hard to show that the product itself is a differential quasi-coherent ringed scheme of finite type.

For any affine open set $U \subset X$ the ring $A := \Gamma(U, \mathcal{A})$ is a differential $\mathbb{K}$-algebra of finite type. As explained in the first lecture, Van den Bergh's existence criterion implies that A has a rigid dualizing complex R_A. Furthermore, we proved in [YZ3] that R_A is supported on the diagonal in U^2.

The fact that R_A is supported on the diagonal implies that it sheafifies to a complex $\mathcal{R}_{\mathcal{A}|_U} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}}|_{U^2})$, which is a dualizing complex over the affine ringed scheme $(U, \mathcal{A}|_U)$.

Because of the uniqueness of rigid dualizing complexes we obtain canonical isomorphisms

$$\mathcal{R}_{\mathcal{A}|_U}|_{U^2 \cap V^2} \cong \mathcal{R}_{\mathcal{A}|_V}|_{U^2 \cap V^2}$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}}|_{U^2 \cap V^2})$ for any two affine open sets U and V. See Figure 4

We would like to glue the affine dualizing complexes $\mathcal{R}_{\mathcal{A}|_U}$ into a global complex $\mathcal{R}_{\mathcal{A}} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$. But here we encounter a genuine problem: *usually objects in derived categories cannot be glued!*

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. However, as explained in [YZ2], this solution seldom applies in the noncommutative context.

The main discovery in [YZ4] is that *perverse coherent sheaves can be used instead of Cousin complexes to glue dualizing complexes.*

6. Perverse Coherent Sheaves

T-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $\mathsf{D}(\mathsf{Mod}\,\mathbb{K}_X)$, where $\mathbb{K}_X$ is a constant sheaf of rings on X.

Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ4].

Let me recall what is a t-structure on a triangulated category D. It consists of the datum of two full subcategories $\mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 0}$ satisfying the axioms below, where $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$.

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.
(ii) $\operatorname{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.
(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle
$$M' \to M \to M'' \to M'[1]$$
in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

When these conditions are satisfied one defines the *heart of* D to be the full subcategory $\mathsf{D}^0 := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$. This is an abelian category.

Given a noetherian ringed scheme $(X, \mathcal{A})$ the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})$ has the *standard t-structure*, in which
$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\leq 0} := \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \mathrm{H}^i\mathcal{M} = 0 \text{ for all } i > 0\},$$
$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\geq 0} := \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \mathrm{H}^i\mathcal{M} = 0 \text{ for all } i < 0\}.$$
The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^0$ is equivalent to the category $\mathsf{Coh}\,\mathcal{A}$ of coherent sheaves.

Other t-structures will be referred to as *perverse t-structures*.

Here is an observation. Suppose the ring A has a rigid dualizing complex R_A. Then the duality $\mathrm{D} := \mathrm{RHom}_A(-, R_A)$ gives rise to a perverse t-structure
$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\leq 0} := \{M \mid \mathrm{H}^i\mathrm{D}M = 0 \text{ for all } i < 0\},$$
$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\geq 0} := \{M \mid \mathrm{H}^i\mathrm{D}M = 0 \text{ for all } i > 0\}.$$
We call it the *rigid perverse t-structure*. The heart is denoted by ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^0$.

Theorem 6.1. ([YZ4]) *Let $(X, \mathcal{A})$ be a separated differential noetherian quasi-coherent ringed $\mathbb{K}$-scheme of finite type. Define*
$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\star} := \{\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}) \mid \mathrm{R}\Gamma(U, \mathcal{M}) \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}\big(\mathsf{Mod}\,\Gamma(U, \mathcal{A})\big)^{\star} \text{ for all affine open sets } U\}.$$
Then:

(1) *The pair*
$$\big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})^{\geq 0}\big)$$
is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A})$.
(2) *The assignment $V \mapsto {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{A}|_V)^0$, for $V \subset X$ open, is a stack of abelian categories on X.*

The last piece in the puzzle is the fact that the affine dualizing complexes $\mathcal{R}_{\mathcal{A}|_U}$ are perverse bimodules, namely they lie in ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathcal{A}^{\mathrm{e}}|_{U^2})^0$. Therefore the gluing data (arising from uniqueness of rigid dualizing complexes) becomes effective. We thus obtain:

Theorem 6.2. ([YZ4]) *Let $(X, \mathcal{A})$ be a separated differential quasi-coherent ringed $\mathbb{K}$-scheme of finite type. Then there exists a rigid dualizing complex $(\mathcal{R}_{\mathcal{A}}, \boldsymbol{\rho})$ over $\mathcal{A}$. It is unique up to a unique isomorphism in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{A}^{\mathrm{e}})$.*

7. The Auslander Condition Revisited

Let A be a differential algebra of finite type over $\mathbb{K}$. In the first lecture I stated that the rigid dualizing complex R_A of A satisfies the Auslander condition.

Definition 7.1. The canonical dimension $\operatorname{Cdim} M$ of an A-module M is defined by

$$\operatorname{Cdim} M := -\inf\{q \mid \operatorname{Ext}^q_A(M, R_A) \neq 0\} \in \mathbb{Z} \cup \{-\infty\}$$

for a finitely generated A-module M, and by

$$\operatorname{Cdim} M := \sup\{\operatorname{Cdim} M' \mid M' \subset M \text{ is finitely generated}\}$$

in general.

The Auslander condition implies that Cdim is a "nice" dimension function.

Here is an alternative characterization of the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, which resembles the original definition in [BBD]. For a module M and any integer i define $\Gamma_{\mathsf{M}_i} M$ to be the biggest submodule of M with $\operatorname{Cdim} \leq i$. This is a functor $\Gamma_{\mathsf{M}_i} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$, and we denote by $\mathrm{H}^j_{\mathsf{M}_i}$ its jth right derived functor. Thus $\mathrm{H}^j_{\mathsf{M}_i} M$ is the "i-th cohomology of M with supports in M_i".

Theorem 7.2. ([YZ3]) *Let A be a differential $\mathbb{K}$-algebra of finite type and $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.*

(1) *$M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}$ iff $\operatorname{Cdim} \mathrm{H}^j M < i$ for all integers i, j such that $j > -i$.*
(2) *$M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}$ iff $\mathrm{H}^j_{\mathsf{M}_i} M = 0$ for all integers i, j such that $j < -i$.*

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