Algebraic Aspects of Deformation Quantization

Lecture Notes [1]

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Here is the plan of my lecture:

1. What is Deformation Quantization?
2. The Solution by Kontsevich
3. Deforming Algebraic Varieties
4. About the Proofs
5. Final Remarks

1. What is Deformation Quantization?

Let $\mathbb{K}$ be a field of characteristic 0 and let C be a commutative $\mathbb{K}$-algebra.

A *Poisson bracket* on C is a $\mathbb{K}$-bilinear function

$$\alpha : C \times C \to C$$

which makes C into a Lie algebra, and is a biderivation (i.e. a derivation in each argument).

We usually write

$$\{f, g\}_\alpha := \alpha(f, g)$$

for $f, g \in C$.

The pair (C, α) is called a *Poisson algebra*.

Poisson brackets arise in several ways.

Example 1. Classical Hamiltonian mechanics. Here $\mathbb{K} = \mathbb{R}$, X is an even dimensional differentiable manifold (the phase space), and $C = \mathrm{C}^\infty(X)$, the ring of differentiable $\mathbb{R}$-valued functions on X.

There is a symplectic (i.e. nondegenerate) Poisson bracket $\{-, -\}$ on C.

Given a function $H \in C$ (the energy), the Hamiltonian vector field $\{H, -\}$ on X determines how points $x \in X$ move. See the book [GS] for a discussion.

Example 2. Here is an explicit formula. Take $C := \mathbb{K}[t_1, \ldots, t_{2m}]$, the polynomial algebra in $2m$ variables. We may view C as $\mathcal{O}(X)$, the ring of algebraic function on the affine variety $X := \mathbf{A}^{2m}$.

Define a biderivation α by the formula

$$\{f, g\}_\alpha = \tfrac{1}{2} \sum_{i=1}^{m} \big(\tfrac{\partial f}{\partial t_i} \cdot \tfrac{\partial g}{\partial t_{i+m}} - \tfrac{\partial g}{\partial t_i} \cdot \tfrac{\partial f}{\partial t_{i+m}} \big).$$

This is a symplectic Poisson bracket on C.

[1] written: 19 May 2007

Let $\mathbb{K}[[\hbar]]$ be the ring of formal power series in the variable $\hbar$. We call $\hbar$ the *deformation parameter*.

Definition 3. Let $C[[\hbar]]$ be the set of formal power series with coefficients in C, which we view only as a $\mathbb{K}[[\hbar]]$-module.

A *star product* on $C[[\hbar]]$ is an associative, unital, $\mathbb{K}[[\hbar]]$-bilinear multiplication

$$\star : C[[\hbar]] \times C[[\hbar]] \to C[[\hbar]],$$

such that

$$f \star g \equiv fg \mod \hbar$$

for any $g, f \in C$.

The pair $\big(C[[\hbar]], \star\big)$ is also called an *associative deformation* of C.

Example 4. Suppose $\big(C[[\hbar]], \star\big)$ is an associative deformation of C.

Given $f, g \in C$, we know that

$$f \star g - g \star f \equiv 0 \mod \hbar.$$

Hence there is a unique element

$$\{f, g\}_\star \in C$$

such that

$$\tfrac{1}{2\hbar}\big(f \star g - g \star f\big) \equiv \{f, g\}_\star \mod \hbar.$$

It is quite easy to show that $\{-,-\}_\star$ is a Poisson bracket on C.

Thus an associative deformation of C induces a Poisson bracket on C.

Deformation quantization seeks to reverse Example 4.

Definition 5. Given a Poisson bracket α on the algebra C, a *deformation quantization* of α is an associative deformation $\big(C[[\hbar]], \star\big)$ of C such that

$$\{f, g\}_\star = \{f, g\}_\alpha$$

for all $f, g \in C$.

In physics $\hbar$ is the *Planck constant*.

For a quantum phenomenon depending on $\hbar$, the limit as $\hbar \to 0$ is thought of the as the classical limit of this phenomenon.

The original idea by the physicists Flato et. al. ([BFFLS], 1978) was that in the setup of Example 1, deformation quantization should model the transition from classical to quantum mechanics.

For a symplectic manifold X and $C = \mathrm{C}^\infty(X)$ the problem was solved by De Wilde and Lacomte ([DL], 1983). A more geometric solution was discovered by Fedosov ([Fe], 1994).

The general case, i.e. $C = \mathrm{C}^\infty(X)$ for a Poisson manifold X, was solved by Kontsevich ([Ko1], 1997). See surveys in the book [CKTB] and the article [CI].

2. The Solution by Kontsevich

Kontsevich, in his paper [Ko1], actually did more than proving existence of deformation quantization – he in fact classified all of them.

To state Kontsevich's result we need the notion of *gauge equivalence*.

A gauge equivalence of the $\mathbb{K}[[\hbar]]$-module $C[[\hbar]]$ is a $\mathbb{K}[[\hbar]]$-linear automorphism

$$\gamma : C[[\hbar]] \xrightarrow{\simeq} C[[\hbar]]$$

that is the identity modulo $\hbar$; namely

$$\gamma(f) \equiv f \mod \hbar$$

for all $f \in C$.

Two associative deformations $\big(C[[\hbar]], \star\big)$ and $\big(C[[\hbar]], \star'\big)$ of C are called *gauge equivalent* if there is some gauge equivalence γ such that

$$\gamma(f \star g) = \gamma(f) \star' \gamma(g)$$

for all $f, g \in C$.

Definition 6. A *formal Poisson bracket* on $C[[\hbar]]$ is a $\mathbb{K}[[\hbar]]$-bilinear Poisson bracket α on the commutative algebra $C[[\hbar]]$ that vanishes modulo $\hbar$, i.e.

$$\{f, g\}_\alpha \equiv 0 \mod \hbar.$$

We also call the pair $\big(C[[\hbar]], \alpha\big)$ a *Poisson deformation* of C.

There is a similar notion of gauge equivalence between Poisson deformations.

Given a formal Poisson bracket α, it can be expanded into a power series

$$\{f, g\}_\alpha = \alpha(f, g) = \sum_{j=1}^{\infty} \alpha_j(f, g)\hbar^j$$

for $f, g \in C$.

The first order term α_1 turns out to be a Poisson bracket on C. Conversely:

Example 7. If α_1 is a Poisson bracket on C, then $\alpha := \alpha_1 \hbar$ is a formal Poisson bracket on $C[[\hbar]]$.

Thus a Poisson bracket on C is a special case of a Poisson deformation of C.

Theorem 8. (Kontsevich) *Let X be a differentiable $\mathbb{R}$-manifold and $C := \mathrm{C}^\infty(X)$. Then there is a canonical bijection*

$$\mathrm{quant} : \frac{\{\text{Poisson deformations of } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{associative deformations of } C\}}{\text{gauge equivalence}}$$

preserving first order terms.

The function quant is called the *quantization map*.

Here is what I mean by "preserving first order terms": given a formal Poisson bracket $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$ and corresponding star product $\star = \mathrm{quant}(\alpha)$, one has

$$\{-,-\}_\star = \{-,-\}_{\alpha_1}$$

as Poisson brackets on C.

3. Deforming Algebraic Varieties

Suppose $C = \mathcal{O}(X)$, the ring of functions on an affine smooth algebraic variety over the field $\mathbb{K}$. It is natural to ask whether a Poisson bracket on C can be quantized.

This sort of question is central in current ring theory – cf. the fundamental paper by Artin, Tate and Van den Bergh ([ATV], 1990), and the survey paper [SV].

Regarding complete varieties, there are classical (i.e. mid 20-th century) results on commutative deformations. But very little has been done on noncommutative deformations.

A more recent paper by Kontsevich ([Ko3], 2001) is about noncommutative deformations of algebraic varieties. It contains many new ideas, but still it does not settle the affine quantization problem mentioned above.

I would like to state our main result, which goes in a different direction than [Ko3].

In order to state this result I must use some fancy algebro-geometric language in the next two slides.

Let X be an n-dimensional smooth variety over the field $\mathbb{K}$, with structure sheaf $\mathcal{O}_X$.

A *Poisson deformation of* $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat, complete, commutative Poisson $\mathbb{K}[[\hbar]]$-algebras on X, such that

$$\mathcal{A}/(\hbar) \cong \mathcal{O}_X.$$

An *associative deformation of* $\mathcal{O}_X$ is a sheaf $\mathcal{A}$ of flat, complete, associative, unital $\mathbb{K}[[\hbar]]$-algebras on X, such that

$$\mathcal{A}/(\hbar) \cong \mathcal{O}_X.$$

Here is our main result from [Ye2], with enhancement in [LY]:

Theorem 9. (Y, Leitner-Y) *Assume $\mathbb{R} \subset \mathbb{K}$, and let X be a smooth algebraic variety satisfying the cohomological vanishing condition*

$$\mathrm{H}^1(X, \mathcal{O}_X) = \mathrm{H}^2(X, \mathcal{O}_X) = 0.$$

Then there is a canonical bijection

$$\mathrm{quant} : \frac{\{\text{Poisson deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{associative deformations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

which preserves first order terms.

As corollary we solve the affine quantization problem:

Corollary 10. *Assume $\mathbb{R} \subset \mathbb{K}$. Let X be a smooth affine algebraic variety and let $C := \mathcal{O}(X)$. Then there is a canonical bijection*

$$\mathrm{quant} : \frac{\{\text{Poisson deformations of } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{associative deformations of } C\}}{\text{gauge equivalence}}$$

preserving first order terms.

Theorem 9 applies also to a large class of complete varieties: the flag varieties $X = G/P$, where G is a connected reductive algebraic group and P is a parabolic subgroup. This class of varieties includes the projective spaces $\mathbf{P}^n_{\mathbb{K}}$.

4. About the Proofs

The proofs of Theorems 8 and 9 are extremely long and technical, so I will only mention a few highlights in order to give the flavor.

The key to it all is the *Kontsevich Formality Theorem* ([Ko1], 1997).

It gives an explicit quantization map

$$\mathrm{quant} : \{\text{Poisson deformations of } C\} \to \{\text{associative deformations of } C\}$$

for the commutative $\mathbb{K}$-algebra

$$C := \mathbb{K}[[t_1, \ldots, t_n]],$$

the ring of formal power series in n variables.

Here is a list of "buzz words" that occur in the proof of the Formality Theorem:

- DG Lie algebras and the Maurer-Cartan equation,
- L_∞ morphisms,
- poly vector fields and poly differential operators,
- trees and configuration spaces.

The passage from the local setup (the power series ring $\mathbb{K}[[t_1, \ldots, t_n]]$) to the global setup (an n-dimensional smooth algebraic variety X, or a C^∞ manifold X) is done using *formal geometry.*

There is an infinite dimensional bundle

$$\pi : \operatorname{Coor} X \to X,$$

called the *coordinate bundle* of X, which is a moduli space for coordinate systems on X.

Given a point $x \in X$, the points in the fiber $\pi^{-1}(x)$ correspond to coordinate systems at the point x.

See Figure 1 for an illustration.

On $\operatorname{Coor} X$ there is a universal coordinate system. So working on $\operatorname{Coor} X$ we "pretend that we are working locally on X", and invoke the Formality Theorem. (All this can be made precise!)

The next step is to try to find a section of the coordinate bundle. Having such a section will finish the proof.

We are allowed to replace the coordinate bundle $\operatorname{Coor} X$ with a slightly modified bundle, denoted by $\operatorname{LCC} X$. One can show that in the C^∞ case this new bundle has a global section. This is where Kontsevich's proof of Theorem 8 ends.

In the case of an algebraic variety we must work harder.

The crucial fact is that the bundle $\operatorname{LCC} X$ is (almost) a torsor under a pro-unipotent group.

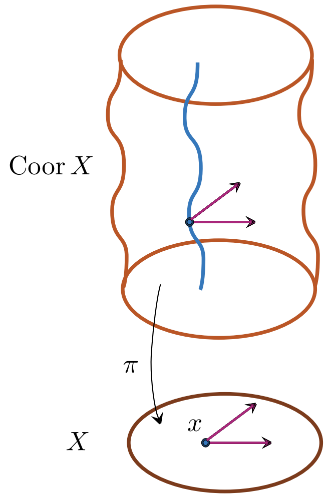

FIGURE 1. The coordinate bundle of X

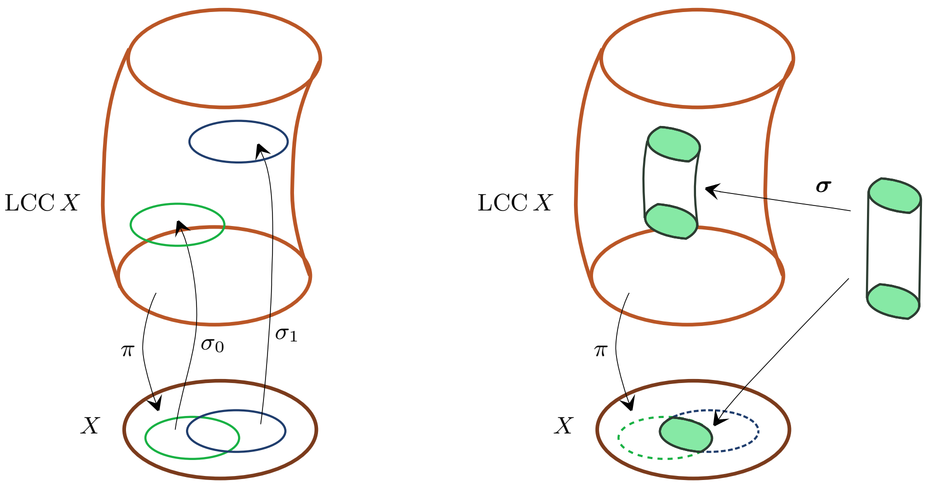

FIGURE 2. From two usual sections to a single simplicial section

Using an averaging process for pro-unipotent group actions [Ye3] we prove that there exists a *simplicial section* of the bundle LCC X.

See Figure 2 for an illustration.

By combining the simplicial section with a technique called *mixed resolutions* [Ye4] we can finish the proof of Theorem 9.

As usually happens when dealing with open coverings and resolutions, cohomological obstructions crop up. This is why we need the cohomological vanishing condition in Theorem 9.

5. Final Remarks

Feynman Diagrams. The explicit formula for the universal quantization map (in the Kontsevich Formality Theorem) contains countably many real constants (the weights). These arise as integrals on configuration spaces of graphs.

Kontsevich [Ko2] says that the idea of the proof came from considerations of string theory: Feynman diagrams of some quantum field theories.

Number Theory. The Formality Theorem has an intriguing arithmetic side to it. The weights mentioned above belong the ring of *motivic periods*, a subring of $\mathbb{C}$. They are the reason we required $\mathbb{R} \subset \mathbb{K}$ in the quantization results.

The structure of the ring of motivic periods is related to some very deep conjectures in number theory. For a discussion see Kontsevich's paper [Ko2].

It is claimed that because of certain results of Drinfeld (on the Grothendieck-Teichmuller group) it is possible to find an alternative universal quantization map that is defined over $\mathbb{Q}$. But I don't know of a published proof.

Algebroids. The last remark I want to make is about quantizing an algebraic variety X for which the cohomological vanishing condition in Theorem 9 does not hold; for instance an abelian surface.

As suggested in [Ko3], and from ongoing research of mine with Leitner [LY], it seems that in these more difficult cases there will be deformation quantization, but in a twisted way, involving *stacks of algebroids* and *nonabelian gerbes*.

Strangely, the latter are now studied mostly by mathematical physicists, in the context of *higher gauge theory*. See [BS] and [BM]. At present there does seem to be any direct link between that and deformation quantization.

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