On Deformation Quantization in Algebraic Geometry

Lecture Notes [1]

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Here is the plan of my lecture:

1. What is Deformation Quantization?
2. The Local Picture
3. Deforming Algebraic Varieties

1. What is Deformation Quantization?

Let $\mathbb{K}$ be a field of characteristic 0 and let C be a commutative $\mathbb{K}$-algebra. A *Poisson bracket* on C is a $\mathbb{K}$-bilinear function

$$\{-,-\} : C \times C \to C$$

which makes C into a Lie algebra, and is a bi-derivation (i.e. a derivation in each argument).

Let $\mathcal{T}_C = \operatorname{Der}_{\mathbb{K}}(C)$ be the module of derivations of C. Given $\alpha \in \bigwedge^2_C \mathcal{T}_C$ we can define a bi-derivation $\{-,-\}_\alpha$ by

$$\{f, g\}_\alpha := \langle \mathrm{d}f \wedge \mathrm{d}g, \alpha \rangle$$

for $f, g \in C$. If $\{-,-\}_\alpha$ is a Poisson bracket on C (i.e. the Jacobi identity holds) then we call α a *Poisson structure*.

Poisson brackets arise in several ways.

Example 1. Classical Hamiltonian mechanics. Here $\mathbb{K} = \mathbb{R}$, X is an even dimensional differentiable manifold (the phase space; often $X = \mathrm{T}^* Y$, the cotangent bundle of a manifold Y) and $C := \mathrm{C}^\infty(X)$, the ring of differentiable $\mathbb{R}$-valued functions. A Poisson structure is an element of $\bigwedge^2_C \mathcal{T}_C \cong \mathrm{C}^\infty(X, \bigwedge^2 \mathrm{T}X)$. See [GS].

Example 2. Lie theory: let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $C := \operatorname{Sym} \mathfrak{g}$ can be identified with $\mathcal{O}(\mathfrak{g}^*)$, the ring of algebraic functions on the dual $\mathfrak{g}^*$. There is an intrinsically defined Poisson bracket called the Kostant-Kirillov bracket.

Example 3. Here is an explicit formula. Take $n = 2m$ and $C := \mathbb{K}[t_1, \ldots, t_n]$, the polynomial algebra. Then

$$\alpha := \sum_{i=1}^{m} \tfrac{\partial}{\partial t_i} \wedge \tfrac{\partial}{\partial t_{i+m}}$$

is a Poisson structure on C. In fact this is a *symplectic structure*, namely for each maximal ideal $\mathfrak{m} \subset C$ the pairing

$$\mathfrak{m}/\mathfrak{m}^2 \times \mathfrak{m}/\mathfrak{m}^2 \to C/\mathfrak{m}$$

induced by α is nondegenerate.

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Example 4. Let $\hbar$ denote a central variable (the "Planck constant"). Suppose A is a flat $\mathbb{K}[[\hbar]]$-algebra with multiplication $\star$, and $\psi : A/(\hbar) \xrightarrow{\simeq} C$ is a $\mathbb{K}$-algebra isomorphism. Given $f, g \in C$ choose arbitrary lifts $\tilde{f}, \tilde{g} \in A$, and define

$$\{f, g\}_\star := \psi\big(\tfrac{1}{2\hbar}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f})\big).$$

This makes sense because $\hbar$ divides $\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f}$ in A. The element $\{f, g\}_\star \in C$ turns out to be independent of the choice of lifts, and this is a Poisson bracket.

Deformation quantization seeks to reverse Example 4.

Definition 5. A *deformation quantization* of C is a $\mathbb{K}[[\hbar]]$-bilinear associative multiplication $\star$ on the $\mathbb{K}[[\hbar]]$-module $C[[\hbar]]$, of the form

$$f \star g = fg + \sum_{j=1}^{\infty} \beta_j(f, g)\hbar^j$$

for $g, f \in C$, where

$$\beta_j : C \times C \to C$$

are bi-differential operators. The multiplication $\star$ is called a *star product.*

The reason that the β_j have to be bi-differential operators is to make the star product $\star$ local; otherwise the deformation has no geometric significance.

Note that for a deformation quantization $\star$ of C the induced Poisson structure is

$$\{f, g\}_\star = \tfrac{1}{2}\big(\beta_1(f, g) - \beta_1(g, f)\big).$$

Definition 6. Given a Poisson structure α on the algebra C, a deformation quantization of the Poisson algebra (C, α) is a deformation quantization $\star$ of C such that

$$\{f, g\}_\star = \{f, g\}_\alpha$$

for all $f, g \in C$.

The original idea by the physicists Flato et. al. [BFFLS] in 1978 was that in the setup of Example 1 the quantization process should model the transition from classical to quantum mechanics.

For a symplectic manifold X and $C = \mathrm{C}^\infty(X)$ the problem was solved by De Wilde and Lacomte in 1983 [DL]. A more geometric solution was discovered by Fedosov in 1994 [Fe]. The general case, i.e. $C = \mathrm{C}^\infty(X)$ for a Poisson manifold X, was solved by Kontsevich in 1997 [Ko1]. See the survey articles [Ke] and [CI].

To state Kontsevich's result we need the notion of *gauge equivalence.* A gauge equivalence is a $\mathbb{K}[[\hbar]]$-linear automorphism $\gamma : C[[\hbar]] \xrightarrow{\simeq} C[[\hbar]]$ of the form $\gamma = \mathbf{1} + \sum_{j=1}^{\infty} \gamma_j \hbar^j$, where $\mathbf{1} : C \xrightarrow{\simeq} C$ is the identity, and each $\gamma_j : C \to C$ is a differential operator. Two star products $\star$ and $\star'$ on $C[[\hbar]]$ are called gauge equivalent if

$$\gamma(f \star' g) = \gamma(f) \star \gamma(g)$$

for some gauge equivalence γ.

A *formal Poisson structure* on C is a series $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$, with $\alpha_j \in \bigwedge^2_C \mathcal{T}_C$, which is a Poisson structure on the commutative algebra $C[[\hbar]]$. There is a similar notion of gauge equivalence of formal Poisson structures.

Example 7. If α_1 is a Poisson structure on C then $\alpha := \alpha_1 \hbar$ is a formal Poisson structure.

Theorem 8. (Kontsevich) *Let X be a differentiable $\mathbb{R}$-manifold and $C := \mathrm{C}^\infty(X)$. Then there is a canonical bijection*

$$Q : \frac{\{\text{formal Poisson structures on } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } C\}}{\text{gauge equivalence}}$$

preserving first order terms.

The function Q is called the *quantization map*. By "preserving first order terms" I mean that given a formal Poisson structure $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$, and corresponding deformation quantization $\star = Q(\alpha)$, one has

$$\{-,-\}_\star = \{-,-\}_{\alpha_1}$$

as Poisson brackets on C.

The proof has a local aspect, which I will explain next. The global part of the proof uses formal geometry, and I will talk about it later, in the context of algebraic geometry.

2. The Local Picture

Suppose C is one of the following kinds of $\mathbb{K}$-algebras:

(i) $C = \mathcal{O}(U)$, the coordinate ring of some affine Zariski open set $U \subset \mathbf{A}^n_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[t_1, \ldots, t_n]$.
(ii) $C = \mathbb{K}[[t_1, \ldots, t_n]]$, the power series ring.
(iii) $\mathbb{K} = \mathbb{R}$, and $C = \mathrm{C}^\infty(U)$, the ring of differentiable functions on some open set $U \subset \mathbb{R}^n$ (in the classical topology), with coordinates $t_1, \ldots, t_n$.

It is crucial that C is equipped with *coordinates*.

Let α be a fixed Poisson structure on C.

If α is symplectic (so $n = 2m$) then locally, i.e. after passing from (i) to (ii), or shrinking U in case (iii) above, and a change of coordinates, the Poisson structure α looks like in Example 3. Then (C, α) has the so-called *Weyl deformation*, which is unique up to gauge equivalence.

In general, when the Poisson structure is degenerate, the local picture is much more complicated. It is best handled using *differential graded Lie algebras.*

A DG (differential graded) Lie algebra is a graded $\mathbb{K}$-module $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$, with a bracket $[-,-]$ satisfying the graded version of the Lie algebra identities, together with a differential $\mathrm{d} : \mathfrak{g}^i \to \mathfrak{g}^{i+1}$ compatible with $[-,-]$.

Given a DG Lie algebra $\mathfrak{g}$, let us define a new DG Lie algebra

$$\mathfrak{g}[[\hbar]]^+ := \bigoplus\nolimits_i \hbar \mathfrak{g}^i[[\hbar]] \subset \bigoplus\nolimits_i \mathfrak{g}^i[[\hbar]],$$

in which $\hbar$ is central. The *Maurer-Cartan equation* in $\mathfrak{g}[[\hbar]]^+$ is

$$\mathrm{d}(\alpha) + \tfrac{1}{2}[\alpha, \alpha] = 0\,, \quad \alpha = \textstyle\sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathfrak{g}^1[[\hbar]]^+.$$

Every element γ in the pro-nilpotent Lie algebra $\mathfrak{g}^0[[\hbar]]^+$ acts on the vector space $\mathfrak{g}^1[[\hbar]]^+$ by the affine transformation

$$\mathrm{af}(\gamma)(\alpha) := \mathrm{d}(\gamma) - [\gamma, \alpha]\,,\ \alpha \in \mathfrak{g}^1[[\hbar]]^+.$$

Let $\exp(\mathfrak{g}^0[[\hbar]]^+)$ be the pro-unipotent group associated to $\mathfrak{g}^0[[\hbar]]^+$. The Lie algebra action above integrates to an action of the group $\exp(\mathfrak{g}^0[[\hbar]]^+)$ on $\mathfrak{g}^1[[\hbar]]^+$ by affine transformations, called gauge equivalences. The gauge equivalences preserve the set of solutions of the Maurer-Cartan equation; see [GM]. One defines

$$\mathrm{MC}(\mathfrak{g}[[\hbar]]^+) := \frac{\{\text{solutions of MC equation in } \mathfrak{g}[[\hbar]]^+\}}{\text{gauge equivalences}}.$$

Let us return to our deformation problem, where C is one of the commutative $\mathbb{K}$-algebras (i)-(iii) above. Recall the module of derivations $\mathcal{T}_C$. For $i \geq -1$ define

$$\mathcal{T}^i_{\mathrm{poly}}(C) := \bigwedge\nolimits_C^{i+1} \mathcal{T}_C.$$

So $\mathcal{T}^{-1}_{\mathrm{poly}}(C) = C$ and $\mathcal{T}^0_{\mathrm{poly}}(C) = \mathcal{T}_C$. The direct sum

$$\mathcal{T}_{\mathrm{poly}}(C) := \bigoplus\nolimits_i \mathcal{T}^i_{\mathrm{poly}}(C)$$

is a DG Lie algebra, called the algebra of *poly derivations* of C. The Lie bracket is the Schouten-Nijenhuis bracket, and the differential is 0. The solutions of the Maurer-Cartan equation in $\mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^+$ are precisely the formal Poisson structures on C.

The second DG Lie algebra in this picture is that of the *poly differential operators*. For $i \geq -1$ one defines

$$\mathcal{D}^i_{\mathrm{poly}}(C) := \{\phi : C^{i+1} \to C \mid \phi \text{ is a differential operator in each argument}\}.$$

So $\mathcal{D}^{-1}_{\mathrm{poly}}(C) = C$ and $\mathcal{D}^0_{\mathrm{poly}}(C) = \mathcal{D}(C)$, the ring of differential operators. $\mathcal{D}_{\mathrm{poly}}(C)$ is a sub DG Lie algebra of the Hochschild cochain complex of C, with the Gerstenhaber bracket. A solution $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j$ of the Maurer-Cartan equation in $\mathcal{D}_{\mathrm{poly}}(C)[[\hbar]]^+$ is a deformation quantization of C, i.e.

$$f \star g := fg + \sum_{j=1}^{\infty} \beta_j(f,g)\hbar^j$$

is an associative deformation of the multiplication of C.

There is a canonical map of complexes

$$\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$$

given by

$$\mathcal{U}_1(\partial_1 \wedge \cdots \wedge \partial_k)(f_1, \ldots, f_k) := \tfrac{1}{k!} \sum\nolimits_{\sigma \in S_k} \mathrm{sgn}(\sigma) \partial_{\sigma(1)}(f_1) \cdots \partial_{\sigma(k)}(f_k)$$

for $f_i \in C$ and $\partial_i \in \mathcal{T}_C$. It is known that $\mathcal{U}_1$ is a quasi-isomorphism – see [Ko1] for the case $C = \mathrm{C}^\infty(U)$, and [Ye1] for the case $C = \mathcal{O}(U)$ – and it induces an isomorphism of graded Lie algebras in cohomology. But $\mathcal{U}_1$ is not a DG Lie algebra homomorphism!

Theorem 9. (Kontsevich Formality Theorem) *Let C be one of the algebras* (i)-(iii) *above. Assume $\mathbb{R} \subset \mathbb{K}$. Then $\mathcal{U}_1$ extends to an L_∞ quasi-isomorphism*

$$\mathcal{U} = \{\mathcal{U}_j\}_{j=1}^{\infty} : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C).$$

In other words, $\mathcal{U}_1$ is a DG Lie algebra quasi-isomorphism, up to specified higher homotopies $\mathcal{U}_2, \mathcal{U}_3, \ldots$. Each of the maps $\mathcal{U}_j$ is invariant under linear change of coordinates.

There is an induced L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^+ \to \mathcal{D}_{\mathrm{poly}}(C)[[\hbar]]^+,$$

and a calculation shows that we get a bijection

$$\mathrm{MC}(\mathcal{U}) : \mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^{+}\big) \xrightarrow{\simeq} \mathrm{MC}\big(\mathcal{D}_{\mathrm{poly}}(C)[[\hbar]]^{+}\big)$$

with an explicit formula. Therefore:

Corollary 10. *Assume* $\mathbb{R} \subset \mathbb{K}$. *In each of the cases* (i) - (iii) *above there is a canonical bijection of sets*

$$Q : \frac{\{\text{formal Poisson structures on } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } C\}}{\text{gauge equivalence}}$$

preserving first order terms.

3. Deforming Algebraic Varieties

Let X be an n-dimensional smooth variety over the field $\mathbb{K}$. A Poisson structure on X is an element $\alpha \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$ such that the bi-derivation $\{-,-\}_\alpha$ is a Poisson bracket on the sheaf of functions $\mathcal{O}_X$.

The aim is to deform the sheaf $\mathcal{O}_X$. So we are looking for a sequence of bi-differential operators

$$\beta_j : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

such that the formula

$$f \star g := fg + \sum_{j=1}^{\infty} \beta_j(f,g)\hbar^j,$$

for local sections $f, g \in \mathcal{O}_X$, defines an associative $\mathbb{K}[[\hbar]]$-algebra structure on the sheaf $\mathcal{O}_X[[\hbar]]$.

Remark 11. Actually there is a more refined notion of deformation quantization of $\mathcal{O}_X$; but when $\mathrm{H}^1(X, \mathcal{D}_X) = 0$, which holds in many cases, the two notions of deformation quantization coincide. Here $\mathcal{D}_X$ is the sheaf of differential operators on X.

Recall that X is called *$\mathcal{D}$-affine* if $\mathrm{H}^i(X, \mathcal{M}) = 0$ for any quasi-coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and any $i > 0$.

Here is our main result from [Ye2]:

Theorem 12. (Y) *Assume X is $\mathcal{D}$-affine variety and* $\mathbb{R} \subset \mathbb{K}$. *Then there is a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

which preserves first order terms, and respects étale morphisms $X' \to X$. If X is affine then Q is bijective. There is an explicit formula for Q.

Theorem 12 applies to two large classes of varieties.

1. Affine varieties. Note that if $X = \operatorname{Spec} C$ is affine, but does not admit an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$, then this result is not trivial, since there has to be some gluing. Cf. case (i) earlier.
2. The flag varieties $X = G/P$, where G is a connected reductive algebraic group and P is a parabolic subgroup. By the Beilinson-Bernstein Theorem the variety X is $\mathcal{D}$-affine. This class of varieties includes the projective spaces $\mathbf{P}^n_{\mathbb{K}}$.

Let me conclude with an explanation of the proof of Theorem 12. A formal coordinate system at a closed point $x \in X$ is an isomorphism of $\mathbb{K}$-algebras

$$\boldsymbol{k}(x)[[\boldsymbol{t}]] = \boldsymbol{k}(x)[[t_1, \ldots, t_n]] \xrightarrow{\simeq} \widehat{\mathcal{O}}_{X,x},$$

where $\boldsymbol{k}(x)$ is the residue field. There is an infinite dimensional scheme $\operatorname{Coor} X$, with a projection $\pi : \operatorname{Coor} X \to X$, which is a moduli space for formal coordinate systems. (In [Ko1] the notation for $\operatorname{Coor} X$ is X^{coor}.) In particular for every closed point $x \in X$ the $\boldsymbol{k}(x)$-rational points in the fiber $\pi^{-1}(x)$ stand in bijection to the set of formal coordinate systems at x.

To get an idea of how the bundle $\operatorname{Coor} X$ looks, let us note that $\operatorname{Coor} X = \lim_{\leftarrow} \operatorname{Coor}^i X$, where each $\operatorname{Coor}^i X$ is the finite dimensional bundle parameterizing formal coordinate systems up to order i.

Any function f on X has a universal Taylor expansion, when we pull it up to $\operatorname{Coor} X$ via π. Thus the pullback of the structure sheaf $\mathcal{O}_X$ embeds inside the power series algebra $\mathcal{O}_{\operatorname{Coor} X}[[\boldsymbol{t}]]$.

Likewise the pullbacks to $\operatorname{Coor} X$ of the sheaves $\mathcal{T}_{\mathrm{poly},X}$ and $\mathcal{D}_{\mathrm{poly},X}$ are embedded inside $\mathcal{O}_{\operatorname{Coor} X} \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{O}_{\operatorname{Coor} X} \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ respectively.

Due to the Formality Theorem we obtain an L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{O}_{\operatorname{Coor} X} \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to \mathcal{O}_{\operatorname{Coor} X} \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

If we had a section $\sigma : X \to \operatorname{Coor} X$ then we could pull $\mathcal{U}$ down to an L_∞ quasi-isomorphism on X. However usually there are no global sections of $\operatorname{Coor} X$.

The group GL_n acts on $\operatorname{Coor} X$ by linear change of coordinates. Let us define $\operatorname{LCC} X$ to be the quotient scheme $\operatorname{Coor} X / \mathrm{GL}_n$. ("LCC" stands for "linear coordinate classes".) Recall that the universal deformation of Kontsevich is invariant under linear change of coordinates. This implies that the L_∞ morphism $\mathcal{U}$ descends to $\operatorname{LCC} X$; and hence it suffices to work with sections $\sigma : X \to \operatorname{LCC} X$.

In the C^∞ context such global sections $\sigma : X \to \operatorname{LCC} X$ do exists (because the fibers of the bundle $\operatorname{LCC} X$ are contractible). But this is not the case in algebraic geometry. So we must use a trick.

Let G be the group of $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[[\boldsymbol{t}]]$. So $G \cong \mathrm{GL}_n \ltimes N$, where N is the subgroup of elements that act trivially modulo $(\boldsymbol{t})^2$. The group N is pro-unipotent. It turns out that $\operatorname{Coor} X$ is a G-torsor over X.

Suppose we are given a finite number of sections

$$\sigma_0, \ldots, \sigma_q : U \to \operatorname{LCC} X$$

over some open set $U \subset X$. Using an averaging process for unipotent group actions [Ye3], we show that there exists a canonical morphism

$$\sigma : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U \to \operatorname{LCC} X$$

which restricts to σ_j on the j-th vertex of $\boldsymbol{\Delta}^q_{\mathbb{K}}$. Here $\boldsymbol{\Delta}^q_{\mathbb{K}}$ is the q-dimensional geometric simplex.

Since sections exist locally, we can choose an open covering $X = \bigcup U_i$ with sections $\sigma_i : U_i \to \operatorname{LCC} X$. For any $i_0, \ldots, i_q$ we then obtain a morphism

$$\boldsymbol{\sigma}_{(i_0,\ldots,i_q)} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times (U_{i_0} \cap \cdots \cap U_{i_q}) \to \operatorname{LCC} X.$$

(See Figure 1 for an illustration of the case $q = 1$.) As q varies we have a *simplicial section* $\boldsymbol{\sigma}$ of $\pi : \operatorname{LCC} X \to X$.

Another device we use is *mixed resolutions* [Ye4]. The mixed resolution $\operatorname{Mix}(\mathcal{T}_{\mathrm{poly},X})$ is a sheaf of DG Lie algebras on X which is quasi-isomorphic to

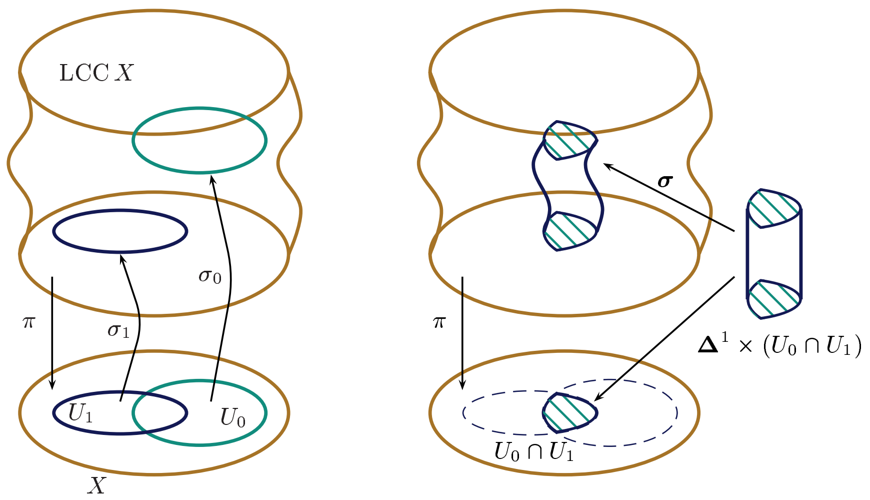

FIGURE 1. Simplicial sections, $q = 1$. We start with sections over two open sets U_0 and U_1 in the left diagram; and we pass to a simplicial section $\boldsymbol{\sigma}$ on the right.

$\mathcal{T}_{\mathrm{poly},X}$. Likewise for $\mathcal{D}_{\mathrm{poly},X}$. The simplicial section $\boldsymbol{\sigma}$ allows us to pull down the universal L_∞ quasi-isomorphism $\mathcal{U}$, and after twisting we obtain an L_∞ quasi-isomorphism

$$\Psi_{\boldsymbol{\sigma}} : \mathrm{Mix}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}(\mathcal{D}_{\mathrm{poly},X})$$

between sheaves of DG Lie algebras on X.

Finally we note that each sheaf $\mathcal{D}^p_{\mathrm{poly},X}$ is a quasi-coherent left $\mathcal{D}_X$-module. If X is $\mathcal{D}$-affine it is now pretty easy to deduce Theorem 12.

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