Deformation Quantization in Algebraic Geometry

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Lecture notes (with bibliography) at:
`http://www.math.bgu.ac.il/~amyekut/lectures/def-quant.html`

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3. Deforming Algebraic Varieties

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Let $\mathbb{K}$ be a field of characteristic 0 and let C be a commutative $\mathbb{K}$-algebra.

A *Poisson bracket* on C is a $\mathbb{K}$-bilinear function

$$\{-,-\} : C \times C \to C$$

which makes C into a Lie algebra, and is a bi-derivation (i.e. a derivation in each argument).

Let $\mathcal{T}_C = \mathrm{Der}_{\mathbb{K}}(C)$ be the module of derivations of C. Given $\alpha \in \bigwedge_C^2 \mathcal{T}_C$ we can define a bi-derivation $\{-,-\}_\alpha$ by

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If $\{-,-\}_\alpha$ is a Poisson bracket on C (i.e. the Jacobi identity holds) then we call α a *Poisson structure*.

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Example 1. Classical Hamiltonian mechanics. Here $\mathbb{K} = \mathbb{R}$, X is an even dimensional differentiable manifold (the phase space; often $X = \mathrm{T}^* Y$, the cotangent bundle of a manifold Y) and $C := \mathrm{C}^\infty(X)$, the ring of differentiable $\mathbb{R}$-valued functions. See [GS].

Example 2. Lie theory: let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $C := \operatorname{Sym} \mathfrak{g}$ can be identified with $\mathcal{O}(\mathfrak{g}^*)$, the ring of algebraic functions on the dual $\mathfrak{g}^*$. There is an intrinsically defined Poisson bracket called the Kostant - Kirillov bracket.

Example 3. Here is an explicit formula.
Take $n = 2m$ and $C := \mathbb{K}[t_1, \ldots, t_n]$, the polynomial algebra. Then

$$\alpha := \sum_{i=1}^{m} \frac{\partial}{\partial t_i} \wedge \frac{\partial}{\partial t_{i+m}}$$

is a Poisson structure on C.

Example 4.
Let $\hbar$ denote a central variable (the “Planck constant”). Suppose A is a flat $\mathbb{K}[[\hbar]]$-algebra with multiplication $\star$, and $\psi : A/(\hbar) \xrightarrow{\simeq} C$ is a $\mathbb{K}$-algebra isomorphism.

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Given $f, g \in C$ choose arbitrary lifts $\tilde{f}, \tilde{g} \in A$, and define

$$\{f, g\}_\star := \psi\big(\tfrac{1}{2\hbar}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f})\big).$$

This makes sense because $\hbar$ divides $\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f}$ in A.

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The element $\{f, g\}_\star \in C$ turns out to be independent of the choice of lifts, and this is a Poisson bracket.

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Definition 5. A deformation quantization of C is a $\mathbb{K}[[\hbar]]$-bilinear associative multiplication $\star$ on the $\mathbb{K}[[\hbar]]$-module $C[[\hbar]]$, of the form

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Definition 6. Given a Poisson structure α on the algebra C, a deformation quantization of the Poisson algebra (C, α) is a deformation quantization $\star$ of C such that

$$\{f, g\}_\star = \{f, g\}_\alpha$$

for all $f, g \in C$.

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The general case, i.e. $C = \mathrm{C}^{\infty}(X)$ for a Poisson manifold X, was solved by Kontsevich in 1997 [Ko1]. See the survey articles [Ke] and [CI].

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A gauge equivalence is a $\mathbb{K}[[\hbar]]$-linear automorphism $\gamma : C[[\hbar]] \xrightarrow{\simeq} C[[\hbar]]$ of the form $\gamma = \mathbf{1} + \sum_{j=1}^{\infty} \gamma_j \hbar^j$, where $\mathbf{1} : C \xrightarrow{\simeq} C$ is the identity, and each $\gamma_j : C \to C$ is a differential operator.

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Two star products $\star$ and $\star'$ on $C[[\hbar]]$ are called gauge equivalent if

$$\gamma(f \star' g) = \gamma(f) \star \gamma(g)$$

for some gauge equivalence γ.

A formal Poisson structure on C is a series $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$, with $\alpha_j \in \bigwedge_C^2 \mathcal{T}_C$, which is a Poisson structure on the commutative algebra $C[[\hbar]]$.

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Example 7. If α_1 is a Poisson structure on C then $\alpha := \alpha_1 \hbar$ is a formal Poisson structure.

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The function Q is called the *quantization map*.

By "preserving first order terms" I mean that given a formal Poisson structure $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$, and corresponding deformation quantization $\star = Q(\alpha)$, one has

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The proof has a local aspect, which I will explain next. The global part of the proof uses formal geometry, and I will talk about it later, in the context of algebraic geometry.

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(iii) $\mathbb{K} = \mathbb{R}$, and $C = \mathrm{C}^\infty(U)$, the ring of differentiable functions on some open set $U \subset \mathbb{R}^n$ (in the classical topology), with coordinates $t_1, \ldots, t_n$.

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It is crucial that C is equipped with coordinates.

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A DG (differential graded) Lie algebra is a graded $\mathbb{K}$-module $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}^i$, with a bracket $[-,-]$ satisfying the graded version of the Lie algebra identities, together with a differential $\mathrm{d} : \mathfrak{g}^i \to \mathfrak{g}^{i+1}$ compatible with $[-,-]$.

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Given a DG Lie algebra $\mathfrak{g}$, let us define a new DG Lie algebra

$$\mathfrak{g}[[\hbar]]^+ := \bigoplus_i \hbar \mathfrak{g}^i[[\hbar]] \subset \bigoplus_i \mathfrak{g}^i[[\hbar]],$$

in which $\hbar$ is central.

The Maurer-Cartan equation in $\mathfrak{g}[[\hbar]]^{+}$ is

$$\mathrm{d}(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0$$

for

$$\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathfrak{g}^1[[\hbar]]^{+}.$$

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One defines

$$\mathrm{MC}(\mathfrak{g}[[\hbar]]^+) := \frac{\{\text{solutions of MC equation in } \mathfrak{g}[[\hbar]]^+\}}{\text{gauge equivalences}}.$$

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Recall the module of derivations $\mathcal{T}_C$. For $i \geq -1$ define

$$\mathcal{T}^i_{\mathrm{poly}}(C) := \bigwedge\nolimits_C^{i+1} \mathcal{T}_C .$$

So $\mathcal{T}^{-1}_{\mathrm{poly}}(C) = C$ and $\mathcal{T}^0_{\mathrm{poly}}(C) = \mathcal{T}_C$.

The direct sum

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The solutions of the Maurer-Cartan equation in $\mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^+$ are precisely the formal Poisson structures on C.

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$$f \star g := fg + \sum_{j=1}^{\infty} \beta_j(f, g)\hbar^j$$

is an associative deformation of the multiplication of C.

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given by

$$\mathcal{U}_1(\partial_1 \wedge \cdots \wedge \partial_k)(f_1, \ldots, f_k) := \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \partial_{\sigma(1)}(f_1) \cdots \partial_{\sigma(k)}(f_k)$$

for $f_i \in C$ and $\partial_i \in \mathcal{T}_C$.

It is known that $\mathcal{U}_1$ is a quasi-isomorphism – see [Ko1] for the case $C = \mathrm{C}^{\infty}(U)$, and [Ye1] for the case $C = \mathcal{O}(U)$ – and it induces an isomorphism of graded Lie algebras in cohomology.

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But $\mathcal{U}_1$ is not a DG Lie algebra homomorphism!

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In other words, $\mathcal{U}_1$ is a DG Lie algebra quasi-isomorphism, up to specified higher homotopies $\mathcal{U}_2, \mathcal{U}_3, \ldots$.

Moreover, each of the maps $\mathcal{U}_j$ is invariant under linear change of coordinates.

There is an induced L_∞ quasi-isomorphism

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A calculation shows that we get a bijection

$$\begin{aligned}\mathrm{MC}(\mathcal{U}) : \mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(C)[[\hbar]]^+\big) \\ \xrightarrow{\simeq} \mathrm{MC}\big(\mathcal{D}_{\mathrm{poly}}(C)[[\hbar]]^+\big)\end{aligned}$$

with an explicit formula.

Therefore:

Corollary 10. *Assume* $\mathbb{R} \subset \mathbb{K}$*. In each of the cases* (i) - (iii) *above there is a canonical bijection of sets*

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3 Deforming Algebraic Varieties

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The aim is to deform the sheaf $\mathcal{O}_X$.

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such that the formula

$$f \star g := fg + \sum_{j=1}^{\infty} \beta_j(f,g)\hbar^j,$$

for local sections $f, g \in \mathcal{O}_X$, defines an associative $\mathbb{K}[[\hbar]]$-algebra structure on the sheaf $\mathcal{O}_X[[\hbar]]$.

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But when $\mathrm{H}^1(X, \mathcal{D}_X) = 0$, which holds in many cases, the two notions of deformation quantization coincide. Here $\mathcal{D}_X$ is the sheaf of differential operators on X.

Recall that X is called $\mathcal{D}$-*affine* if $\mathrm{H}^i(X, \mathcal{M}) = 0$ for any quasi-coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and any $i > 0$.

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Theorem 12. (Y) *Assume X is $\mathcal{D}$-affine and $\mathbb{R} \subset \mathbb{K}$. Then there is a canonical function*

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The function Q preserves first order terms, and respects étale morphisms $X' \to X$. If X is affine then Q is bijective. There is an explicit formula for Q.

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1. Affine varieties. Note that if $X = \operatorname{Spec} C$ is affine, but does not admit an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$, then this result is not trivial, since there has to be some gluing. Cf. case (i) earlier.

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2. The flag varieties $X = G/P$, where G is a connected reductive algebraic group and P is a parabolic subgroup. By the Beilinson-Bernstein Theorem the variety X is $\mathcal{D}$-affine. This class of varieties includes the projective spaces $\mathbf{P}^n_{\mathbb{K}}$.

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The universal Taylor expansion is a canonical embedding of algebras

$$\pi^{-1}\mathcal{O}_X \subset \mathcal{O}_{\operatorname{Coor} X}[[\boldsymbol{t}]].$$

Due to the Formality Theorem we obtain an L_∞ quasi-isomorphism

$$\mathcal{U} : \mathcal{O}_{\mathrm{Coor}\,X} \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to \mathcal{O}_{\mathrm{Coor}\,X} \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$$

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If we had a section $\sigma : X \to \mathrm{Coor}\,X$ then we could pull $\mathcal{U}$ down to an L_∞ quasi-isomorphism

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of sheaves of DG Lie algebras on X.

However usually there are no global sections of $\mathrm{Coor}\, X$.

Now the group GL_n acts on $\mathrm{Coor}\, X$ by linear change of coordinates. Let us define $\mathrm{LCC}\, X$ to be the quotient bundle $\mathrm{Coor}\, X / \mathrm{GL}_n$.

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In the C^∞ context such global sections $\sigma : X \to \mathrm{LCC}\, X$ do exists (because the fibers of this bundle are contractible).

But this is not the case in algebraic geometry. (Here is where our work diverges from that of Kontsevich.) So we must use a trick.

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We can choose an open covering $X = \bigcup U_i$ with sections $\sigma_i : U_i \to \mathrm{LCC}\, X$.

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Using an averaging process for unipotent group actions [Ye3], for any $i_0, \ldots, i_q$ we then obtain a morphism

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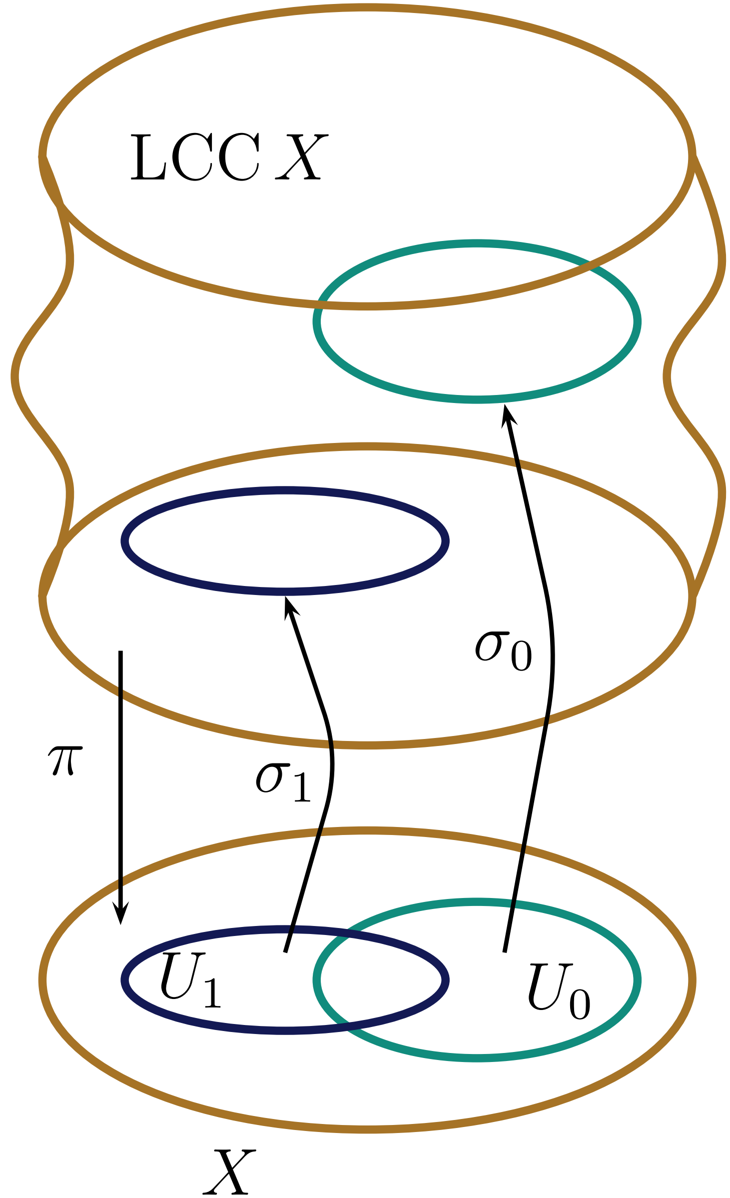
As q varies we have a simplicial section $\boldsymbol{\sigma}$, i.e. the simplicial relations are satisfied.

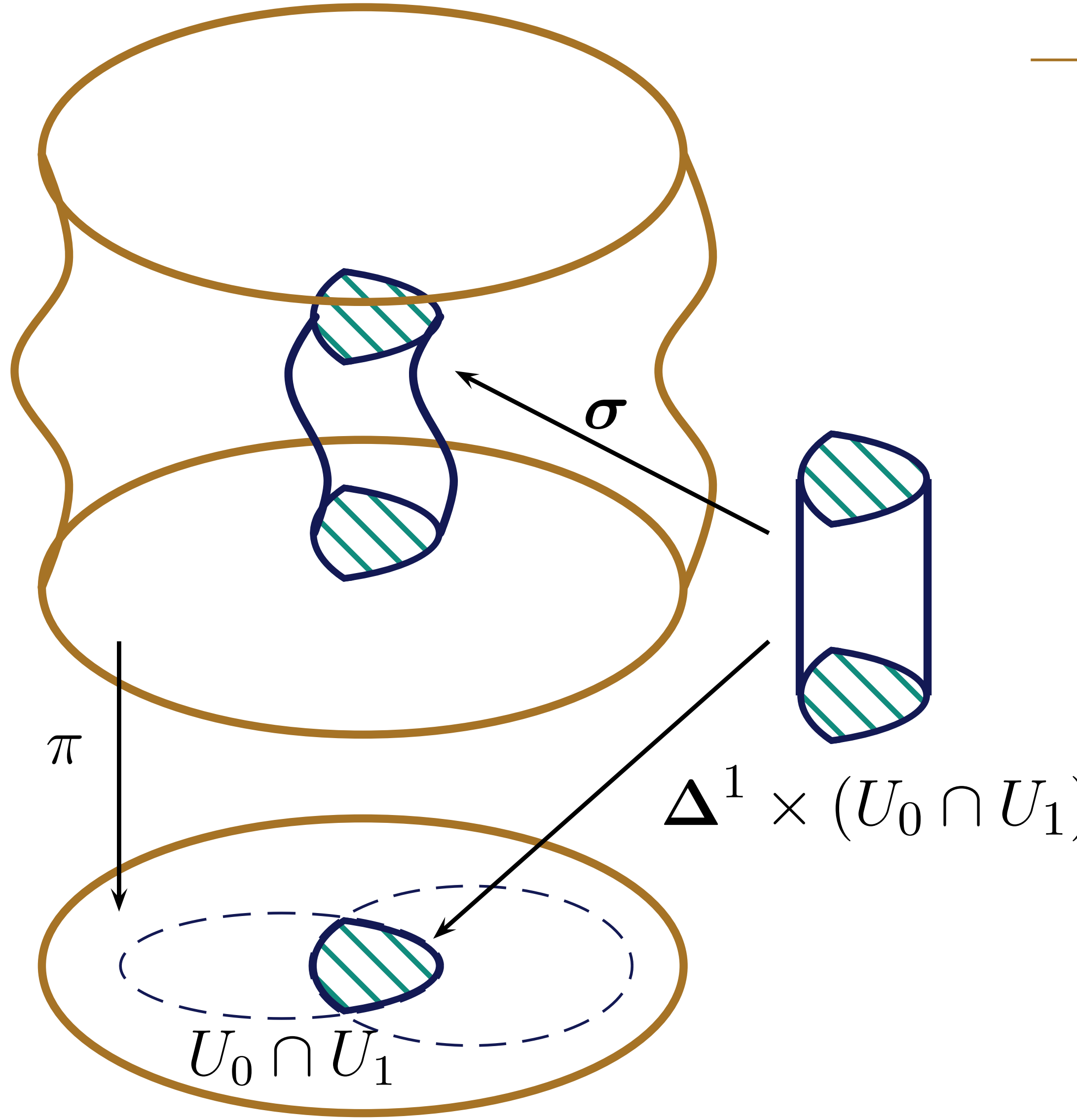
Deforming Algebraic Varieties

Here is an illustration
of the case $q = 1$.
We start with sections over
two open sets

$$\sigma_i : U_i \to \mathrm{LCC}\, X$$



and we pass to a simplicial section $\boldsymbol{\sigma}_{(0,1)}$ interpolating between them



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The simplicial section $\boldsymbol{\sigma}$ gives rise to an L_∞ quasi-isomorphism

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It is now pretty easy to deduce Theorem 12, including the cohomological conditions.

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– end –

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