THE AUSLANDER CONDITION AND PERVERSE SHEAVES

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Introduction

I will discuss a few notions from commutative algebraic geometry, and show how they can be successfully implemented in noncommutative ring theory.

The point of view that ring theory is affine noncommutative algebraic geometry is conceptually helpful here. It is expected that when noncommutative algebraic geometry shall be sufficiently developed, the notions discussed today ought to extend to noncommutative schemes that are not affine.

Dualizing complexes

Throughout k is a field, A is a noetherian associative k-algebra (sometimes noncommutative), and X is usually a finite type (commutative) k-scheme.

Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ denote the bounded derived category of $\mathcal{O}_X$-modules with coherent cohomologies.

For two complexes $\mathcal{M}, \mathcal{N}$ we let $\mathrm{R}\mathcal{H}om_X(\mathcal{M}, \mathcal{N})$ be the derived functor of sheaf $\mathcal{H}om$.

The following goes back to Grothendieck (1960's).

Definition 1. **(sketch)** A complex $\mathcal{R} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is *dualizing* if the functor $\mathcal{M} \mapsto \mathrm{R}\mathcal{H}om_X(\mathcal{M}, \mathcal{R})$ is a duality (contravariant equivalence) of $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$.

Now look at a noncommutative algebra A. We shall prefer to work with left A-modules. Let us write A° for the opposite algebra, and $A^{\mathrm{e}} := A \otimes_k A^{\circ}$. Thus an A°-module is a right A-module M_A, and an A^{e}-module is a bimodule ${}_A M_A$, central over k.

Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ the bounded derived category of A-modules with finitely generated cohomologies. The next definition is mine, from 1990.

Definition 2. **(sketch)** A complex of bimodules $R \in \mathsf{D}^{\mathrm{b}}(A^{\mathrm{e}})$ is *dualizing* if the functors

$$\mathrm{R}\,\mathrm{Hom}_A(-, R)$$

and

$$\mathrm{R}\,\mathrm{Hom}_{A^{\circ}}(-,R)$$

are a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A^{\circ})$.

Such dualizing complex do exist, and a little later I'll mention some things we know about them.

Residue Complexes - Commutative

One of the problems encountered by Grothendieck was gluing dualizing complexes. Existence of dualizing complexes on affine schemes (and smooth schemes) was quite easy, but not on an arbitrary scheme X. There was also the deep question of functoriality and relative duality.

The data for gluing also existed: one could choose on each affine open set $U \subset X$ a unique (up to isomorphism) dualizing complex $\mathcal{R}_U$, by requiring local duality to hold. This is duality relative to the base field k. Lets us call the dualizing complex $\mathcal{R}_U$ *balanced.*

But gluing of complexes in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ is in general impossible.

The solution of Grothendieck was the use of Cousin complexes, and I would like to say a few words about them.

Aside. Deligne found a direct construction of dualizing complexes, and recently Neeman discovered yet another approach. The latter, via Brown representability, works in the noncommutative realm too. Cf. work of Jørgensen, and Van den Bergh-Bondal.

The Cousin functor of Grothendieck $\mathcal{M} \mapsto \mathrm{E}\mathcal{M}$ sends a complex $\mathcal{M}$ to a sum of local cohomologies, and it is a functor

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X) \to \mathsf{C}(\mathsf{QCoh}\, X),$$

the latter being the abelian category of complexes.

If $\mathcal{R}$ is a dualizing complex then $\mathcal{R} \cong \mathrm{E}\mathcal{R}$ in the derived category.

Example 3. Take $A := k[t]$, $X := \operatorname{Spec} A$. Then $\mathcal{O}_X$ itself is dualizing, and

$$\mathrm{E}\mathcal{O}_X = \left(\mathrm{H}^0_w \mathcal{O}_X \to \bigoplus_x \mathrm{H}^1_x \mathcal{O}_X \right)$$

where w is the generic point and x runs over the closed points.

The affine version looks like this:

$$\mathrm{E}A = \left(k(t) \to \bigoplus_{\mathfrak{m}} \frac{k(t)}{k[t]_{\mathfrak{m}}} \right).$$

We see that $\mathrm{E}A$ is the minimal injective resolution of the module A.

The balanced dualizing complex $\mathcal{R}_X$ here is $\Omega^1_{X/k}[1]$.

For the balanced dualizing complex $\mathcal{R}_U$ it is customary to write

$$\mathcal{K}_U := \mathrm{E}\mathcal{R}_U,$$

and to call it the *residue complex* of U.

Now the gluing data can take effect: we can glue the residue complexes $\mathcal{K}_U$ in the nicely behaved category $\mathsf{C}(X)$ and get a global residue complex $\mathcal{K}_X$.

Residue Complexes - Noncommutative

Over a noncommutative algebra A there is also a canonical choice for a dualizing complex R_A. If A is graded or local, this is the *balanced dualizing complex*; it is the one satisfying noncommutative local duality. In the graded case it was introduced my myself in 1990, and the complete local case was studied by Chan and Wu-Zhang very recently.

Van den Bergh (1997) found a condition that applies to any finitely generated k-algebra A.

Definition 4. A dualizing complex R is called *rigid* if

$$R \cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R \otimes_k R)$$

in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(A^{\mathrm{e}})$.

Such R is unique, so we denote it by R_A. There is no conflict: when there are a balanced complex and a rigid one, they coincide.

The Cousin functor makes sense also for noncommutative A (given a dimension function dim – more on this later). It turns out that residue complexes $\mathcal{K}_A$ exist for many kinds of noncommutative algebras:

- Twisted homogeneous coordinate rings (Y., 92).
- 3 dimensional Sklyanin algebras (Y., 96).
- PI algebras, with an auxiliary hypothesis, like graded or filtered. This includes all algebras finite over the center (Y.-Zhang, 98).
- An algebra with Auslander rigid dualizing complex that has a pure injective resolution on either side (Y.-Zhang, 98).

(The first 2 hold in the graded sense).

The residue complex $\mathcal{K}_A$ is a complex of bimodules that are injective modules on either side. And often there are functorial properties, such as for a finite algebra homomorphism $A \to B$.

In the first 3 cases above we even have a very detailed knowledge of the structure of the bimodule complex $\mathcal{K}_A$. For instance, in the PI case $\mathcal{K}_A$ has a canonical decomposition into indecomposable bimodules, indexed by the cliques of prime ideals of A.

Let us look at an example of gluing residue complexes on a noncommutative scheme.

Example 5. Suppose $\mathcal{A}$ is a coherent $\mathcal{O}_X$-algebra. Then $\mathsf{QCoh}\,\mathcal{A}$ can be viewed as a noncommutative scheme finite over its center.

There is a complex $\mathcal{K}_{\mathcal{A}}$ of quasi-coherent $\mathcal{A} \otimes_{\mathcal{O}_X} \mathcal{A}^\circ$-modules on X, injective on either side, such that the functors $\mathrm{R}\mathcal{H}om(-, \mathcal{K}_{\mathcal{A}})$ are a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathcal{A}^\circ)$.

However we soon run into trouble.

Many familiar algebras do not have a residue complex – because of obvious obstructions. For example $U(\mathfrak{sl}_2)$.

Another annoying problem is we cannot prove existence of dualizing complexes in sufficient generality. For example, we do not know if every affine PI algebra has a dualizing complex. So we are looking for additional tools to tackle the problem.

Perverse Sheaves

These were introduced around 1980 by Goresky-MacPherson and Beilinson-Bernstein-Deligne-Gabber. Originally perverse sheaves were complexes of sheaves on a space X with constructible cohomology sheaves. For example sheaves of $\mathbb{Q}$-modules on a real analytic manifold.

Among the good properties of perverse sheaves are that they form an abelian category, and can be glued (hence the name "sheaf").

An exercise in the book by Kashiwara-Schapira discussed perverse sheaves in $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$ with X smooth affine (commutative), and this prompted the definitions below. I recently learned that Bondal is also studying perverse coherent sheaves.

Again X is a scheme over k. Define

$$\mathsf{M}_i := \{\mathcal{M} \in \mathsf{Coh}\,X \mid \dim \operatorname{supp} \mathcal{M} \leq i\}$$

and for any $\mathcal{M}$ let

$$\begin{aligned} \Gamma_{\mathsf{M}_i}\mathcal{M} &:= \{\text{sections with } \dim \operatorname{supp} \leq i\} \\ &= \bigcup_{\dim Z \leq i} \Gamma_Z \mathcal{M}. \end{aligned}$$

A perversity is a function $p : \mathbb{Z} \to \mathbb{Z}$ obeying

$$-1 \leq p(n+1) - p(n) \leq 0.$$

Definition 6. Given p we define

$$\begin{aligned} {}^p\mathsf{D}^{\leq 0}_{\mathrm{c}}(X) := \{\mathcal{M} \in \mathsf{Coh}\,X \mid \\ \dim \operatorname{supp} \mathrm{H}^j \mathcal{M} < i \text{ for all } j > p(i)\} \end{aligned}$$

and

$$\begin{aligned} {}^p\mathsf{D}^{\geq 0}_{\mathrm{c}}(X) := \{\mathcal{M} \in \mathsf{Coh}\,X \mid \\ \mathrm{H}^j \mathrm{R}\Gamma_{\mathsf{M}_i} \mathcal{M} = 0 \text{ for all } j < p(i)\}. \end{aligned}$$

Then (at least for the extreme cases $p(n) = 0$ and $p(n) = -n$), the pair

$$({}^p\mathsf{D}^{\leq 0}_{\mathrm{c}}(X), {}^p\mathsf{D}^{\geq 0}_{\mathrm{c}}(X))$$

is a *t-structure* on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. The *heart*

$${}^p\mathsf{D}^{0}_{\mathrm{c}}(X) := {}^p\mathsf{D}^{\leq 0}_{\mathrm{c}}(X) \cap {}^p\mathsf{D}^{\geq 0}_{\mathrm{c}}(X)$$

is then an abelian category.

Furthermore $U \mapsto {}^p\mathsf{D}^{0}_{\mathrm{c}}(U)$ is a stack (a "sheaf") of abelian categories on X (for the Zariski topology). So we call the objects of ${}^p\mathsf{D}^{0}_{\mathrm{c}}(X)$ *perverse sheaves.*

For the trivial perversity $p(n) = 0$ we recover ${}^p\mathsf{D}^{0}_{\mathrm{c}}(X) = \mathsf{Coh}\, X$.

On the other hand:

Proposition 7. For any affine open set $U = \operatorname{Spec} A$ let $\mathcal{R}_U$ be its rigid dualizing complex. Then

$$\mathcal{R}_U \in {}^p\mathsf{D}^{0}_{\mathrm{c}}(U)$$

for the perversity $p(n) = -n$. Consequently we can glue these complexes to obtain the perverse sheaf $\mathcal{R}_X$.

We see that we have an alternative gluing method, other than residue complexes.

Auslander Dualizing Complexes

The Auslander condition on Gorenstein rings (rings with finite injective dimension) was studied by Björk and Levasseur. In 1996 I found a very straightforward generalization to dualizing complexes.

Let R be a dualizing complex over the algebra A. For a module M we write

$$j(M) := \inf\{j \mid \operatorname{Ext}^j_A(M, R) \neq 0\}.$$

Likewise for a right module.

Definition 8. Let R be a dualizing complex over the algebra A. Suppose that for every finitely generated A-module M, every i and every A°-submodule

$$N \subset \operatorname{Ext}^i_A(M, R),$$

one has

$$j(N) \geq i.$$

And also with A and A° exchanged. Then R is called an *Auslander dualizing complex.*

Example 9. If A is an Auslander Gorenstein algebra then $A = R$ is an Auslander dualizing complex.

Let

$$\operatorname{Cdim} M := -j(M).$$

It is called the *canonical dimension* of M.

The work of Björk and Levasseur, with some mild modifications, implies:

Theorem 10. (Y., '96) If R is Auslander then Cdim is an exact dimension function on $\mathsf{Mod}\, A$.

With Zhang we showed that many of the results proved earlier for Auslander-Gorenstein algebras hold also under the much weaker assumption of existence of Auslander dualizing complex. Further results along this direction were obtained by Wu-Zhang (cf. Zhang's lecture).

The next theorem is quite useful.

Theorem 11. (Y.-Zhang, '98) A has a rigid Auslander dualizing complex R_A when:

1. A is finite over its center.
2. A is PI and has a filtration s.t. $\mathrm{gr}\, A$ is noetherian connected graded.
3. A has a filtration s.t. $\mathrm{gr}\, A$ is a graded quotient of the skew polynomial algebra $k_{(q_{ij})}[x_1, \ldots, x_n]$, $q_{ij} \in k$.

Example 12. Any quotient B of an enveloping algebra $A = U(\mathfrak{g})$ has an Auslander dualizing complex, since $\mathrm{gr}\, B$ is commutative ($q_{ij} = 0$).

Example 13. Many known examples of quantum algebras A fall into case 3. So any quotient $B = A/I$ does too.

Actually in all cases of the theorem we get more: there is a strong compatibility between the dualizing complexes R_A and R_B for any quotient $B = A/I$.

Here is the connection with perverse sheaves.

Given an Auslander dualizing complex R over the algebra A, there is a dimension function Cdim. If we think of $X = \mathsf{Mod}\, A$ as a noncommutative space, then Cdim M is the "dimension of the support" of the module M.

At any rate the functors Γ_{M_i} are defined, and hence we can define ${}^p\mathsf{D}^{\leq 0}_{\mathrm{c}}(A)$ and ${}^p\mathsf{D}^{\geq 0}_{\mathrm{c}}(A)$ just like before (Definition 6).

Theorem 14. (Y.-Zhang, '99) Suppose R is an Auslander dualizing complex. Then

$$({}^p\mathsf{D}^{\leq 0}_{\mathrm{c}}(A), {}^p\mathsf{D}^{\geq 0}_{\mathrm{c}}(A))$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$. Furthermore $R \in {}^p\mathsf{D}^{0}_{\mathrm{c}}(A)$, so R is a perverse left module over A. The same is true on the right side.

Finally two questions:

Question 15. How to define perverse bimodules? This is crucial if we want to glue dualizing complexes.

Question 16. For which dimension functions dim on $\mathsf{Mod}\, A$, or more generally on a quasi-scheme X, do we get a t-structure?

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