

Rigid Dualizing Complexes and Perverse Coherent Sheaves

Lecture Notes
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Here is the plan of my lecture:

1. Background on Dualizing Complexes
2. Rigid Complexes over Rings
3. Rigid Dualizing Complexes over Rings
4. Rigid Dualizing Complexes over Schemes
5. Perverse Coherent Sheaves
6. Cohen-Macaulay Complexes

This talk is about joint work with James Zhang (Seattle).

1. BACKGROUND ON DUALIZING COMPLEXES

Dualizing complexes over schemes were introduced by Grothendieck in the 1960's (see [RD]), as a vast generalization of Serre duality.

Suppose X is a noetherian scheme.

We denote by $\text{Mod } \mathcal{O}_X$ the category of sheaves of \mathcal{O}_X -modules, and by $D(\text{Mod } \mathcal{O}_X)$ its derived category.

The full subcategory of bounded complexes with coherent cohomologies is $D_c^b(\text{Mod } \mathcal{O}_X)$. It is equivalent to $D^b(\text{Coh } \mathcal{O}_X)$.

Definition 1.1. (Grothendieck [RD]) A *dualizing complex* on X is a complex $\mathcal{R} \in D_c^b(\text{Mod } \mathcal{O}_X)$ satisfying the two conditions:

- (i) \mathcal{R} has finite injective dimension.
- (ii) The canonical morphism $\mathcal{O}_X \rightarrow R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{R}, \mathcal{R})$ is an isomorphism.

It follows that the functor

$$\mathcal{M} \mapsto R\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R})$$

is an auto-duality of $D_c^b(\text{Mod } \mathcal{O}_X)$.

When $X = \text{Spec } A$ is affine, the complex $R := \text{R}\Gamma(X, \mathcal{R}) \in \text{D}_f^b(\text{Mod } A)$ is called a dualizing complex over A , and

$$M \mapsto \text{RHom}_A(M, R)$$

is an auto-duality of $\text{D}_f^b(\text{Mod } A)$.

Suppose \mathbb{K} is a regular noetherian ring of finite Krull dimension, and X is a finite type \mathbb{K} -scheme, with structural morphism $\pi : X \rightarrow \text{Spec } \mathbb{K}$.

Then, according to [RD], there is a special dualizing complex on X , namely the *Grothendieck dualizing complex* $\mathcal{R}_X := \pi^! \mathbb{K}$.

The proof of existence of this complex, and its properties, is very difficult.

In this lecture I will explain an alternative approach to Grothendieck duality.

For other approaches see the papers in the references, mainly by Joseph Lipman and his coauthors.

2. RIGID COMPLEXES OVER RINGS

Suppose A is a commutative ring, and B is a commutative A -algebra.

In [YZ4] we constructed a functor

$$\text{Sq}_{B/A} : \text{D}(\text{Mod } B) \rightarrow \text{D}(\text{Mod } B),$$

called the *squaring operation*.

When B is flat over A one has

$$\text{Sq}_{B/A} M = \text{RHom}_{B \otimes_A B}(B, M \otimes_A^L M)$$

for $M \in \text{D}(\text{Mod } B)$.

But in general one has to use DG algebras to define $\text{Sq}_{B/A} M$.

The functor $\text{Sq}_{B/A}$ is quadratic, in the following sense. Given a morphism $\phi : M \rightarrow N$ in $\text{D}(\text{Mod } B)$, and an element $b \in B$, one has

$$\text{Sq}_{B/A}(b\phi) = b^2 \text{Sq}_{B/A}(\phi)$$

in

$$\text{Hom}_{\text{D}(\text{Mod } B)}(\text{Sq}_{B/A} M, \text{Sq}_{B/A} N).$$

Definition 2.1. Let B be a noetherian A -algebra, and let M be a complex in $\text{D}_f^b(\text{Mod } B)$ that has finite flat dimension over A . Assume

$$\rho : M \xrightarrow{\cong} \text{Sq}_{B/A} M$$

is an isomorphism in $\text{D}(\text{Mod } B)$. Then the pair (M, ρ) is called a *rigid complex over B relative to A* .

Definition 2.2. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A . A morphism $\phi : M \rightarrow N$ in $\text{D}(\text{Mod } B)$ is called a *rigid morphism relative to A* if

the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathrm{Sq}_{B/A} M \\ \phi \downarrow & & \downarrow \mathrm{Sq}_{B/A}(\phi) \\ N & \xrightarrow{\sigma} & \mathrm{Sq}_{B/A} N \end{array}$$

is commutative.

3. RIGID DUALIZING COMPLEXES OVER RINGS

From now on \mathbb{K} denotes a fixed noetherian regular ring of finite Krull dimension (e.g. a field or the ring of integers).

Let A be a noetherian \mathbb{K} -algebra. The next definition is due to Michel Van den Bergh [VdB].

Definition 3.1. A *rigid dualizing complex* over A relative to \mathbb{K} is a rigid complex (R, ρ) , such that R is a dualizing complex.

Note that the only rigid automorphism of a rigid dualizing complex (R, ρ) is the identity $\mathbf{1}_R$. Indeed, any automorphism ϕ of R has to be of the form $\phi = a\mathbf{1}_R$ for some invertible element $a \in A$. If ϕ is rigid then $a^2 = a$, and hence $a = 1$.

Recall that an A -algebra B is called *essentially finite type* if it is a localization of some finitely generated A -algebra.

The next theorems are taken from [YZ5]

Theorem 3.2. *Let A be an essentially finite type \mathbb{K} -algebra. Then A has a rigid dualizing complex (R_A, ρ_A) , which is unique up to a unique rigid isomorphism.*

Recall that a ring homomorphism $f^* : A \rightarrow B$ is called *finite* if B is a finitely generated A -module.

Theorem 3.3. *Let A and B be essentially finite type \mathbb{K} -algebras, and let $f^* : A \rightarrow B$ be a finite homomorphism. Then the complex $\mathrm{RHom}_A(B, R_A)$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism*

$$\mathrm{RHom}_A(B, R_A) \cong R_B.$$

We say that B is *essentially smooth* of relative dimension n over A if it is essentially finite type, formally smooth, and the rank of the projective B -module $\Omega_{B/A}^1$ is n . When $n = 0$ we say B is *essentially étale*.

Example 3.4. If A' is a localization of A then $A \rightarrow A'$ is essentially étale. If $B = A[t_1, \dots, t_n]$ is a polynomial algebra then $A \rightarrow B$ is essentially smooth of relative dimension n .

Theorem 3.5. *Let A and B be essentially finite type \mathbb{K} -algebras, and let $f^* : A \rightarrow B$ be an essentially smooth homomorphism of relative dimension n . Then*

the complex $\Omega_{B/A}^n[n] \otimes_A R_A$ has an induced rigidifying isomorphism, and there is a unique rigid isomorphism

$$\Omega_{B/A}^n[n] \otimes_A R_A \cong R_B.$$

Taking $n = 0$ we get an important corollary:

Corollary 3.6. *Given an essentially étale homomorphism $f^* : A \rightarrow B$, there is a unique rigid isomorphism*

$$B \otimes_A R_A \cong R_B.$$

4. RIGID DUALIZING COMPLEXES OVER SCHEMES

Definition 4.1. Let X be a finite type separated \mathbb{K} -scheme. A *rigid dualizing complex over X* (relative to \mathbb{K}) is the data (\mathcal{R}, ρ) , where:

- (1) $\mathcal{R} \in \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)$ is a dualizing complex on X .
- (2) $\rho = \{\rho_U\}$ is a collection of rigidifying isomorphisms, indexed by the affine open sets of X . Namely, for any affine open set U , ρ_U is a rigidifying isomorphism for the dualizing complex $\mathbf{R}\Gamma(U, \mathcal{R})$ over the \mathbb{K} -algebra $A := \Gamma(U, \mathcal{O}_X)$.

The condition is:

- (†) For any inclusion $V \subset U$ of affine open sets, with $A := \Gamma(U, \mathcal{O}_X)$ and $B := \Gamma(V, \mathcal{O}_X)$, the canonical isomorphism

$$B \otimes_A \mathbf{R}\Gamma(U, \mathcal{R}) \cong \mathbf{R}\Gamma(V, \mathcal{R})$$

is rigid, with respect to the rigidifying isomorphisms ρ_U and ρ_V .

We would like to prove *existence and uniqueness of a rigid dualizing complex on X* .

Consider an affine open set $U \subset X$, and let $A := \Gamma(U, \mathcal{O}_X)$. According to Theorem 3.2 there exists a rigid dualizing complex (R_A, ρ_A) over A . Let us denote by \mathcal{R}_U the corresponding complex of sheaves on U , which is of course a dualizing complex over U . And let's write $\rho_U := \rho_A$.

Now suppose $V \subset U$ is another affine open set, with $B := \Gamma(V, \mathcal{O}_X)$ and rigid dualizing complex (R_B, ρ_B) . According to Corollary 3.6 there is a unique isomorphism

$$(4.2) \quad \phi_{V/U} : \mathcal{R}_U|_V \xrightarrow{\cong} \mathcal{R}_V$$

in $\mathbf{D}(\text{Mod } \mathcal{O}_V)$ which respects rigidity.

Therefore given an affine open set $W \subset V$, these isomorphisms satisfy

$$\phi_{W/V} \circ \phi_{V/U} = \phi_{W/U}.$$

The next step would be to try to glue the affine dualizing complexes \mathcal{R}_U to a global complex $\mathcal{R}_X \in \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)$.

But here we encounter a genuine problem: *usually objects in derived categories cannot be glued!*

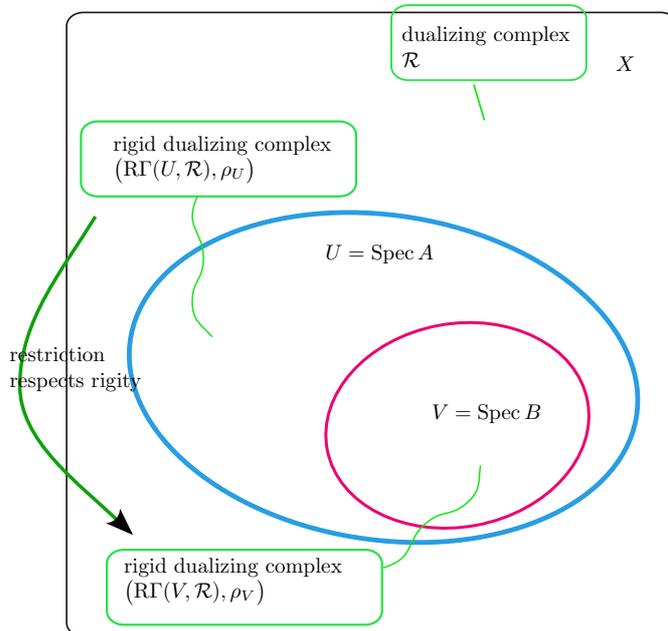


FIGURE 1

Grothendieck's solution in the commutative case, in [RD], was to use Cousin complexes. This solution can be used in our setup too, but it has a disadvantage: we are forced to leave the derived category, and then return to it.

We propose an alternative solution: *perverse coherent sheaves*.

Remark 4.3. For noncommutative ringed schemes one is forced to use perverse coherent sheaves, since Cousin complexes are ill-behaved. See [YZ3].

5. PERVERSE COHERENT SHEAVES

The notion of t-structures and perverse sheaves were introduced by Beilinson, Bernstein and Deligne [BBD] around 1980. This was in the context of intersection cohomology on singular spaces. For such a space X they were interested in t-structures on subcategories of $D(\text{Mod } \mathbb{K}_X)$, where \mathbb{K}_X is a constant sheaf of rings on X .

Perverse coherent sheaves came into the scene only very recently, independently in the work of Bezrukavnikov (after Deligne) [Bz], Bridgeland [Br], Kashiwara [Ka] and our paper [YZ3].

Let me recall what is a t-structure on a triangulated category D . It consists of the datum of two full subcategories $D^{\leq 0}$ and $D^{\geq 0}$ satisfying the axioms below, where $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$.

- (i) $D^{\leq -1} \subset D^{\leq 0}$ and $D^{\geq 1} \subset D^{\geq 0}$.
- (ii) $\text{Hom}_D(M, N) = 0$ for $M \in D^{\leq 0}$ and $N \in D^{\geq 1}$.

(iii) For any $M \in \mathbf{D}$ there is a distinguished triangle

$$M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$$

in \mathbf{D} with $M' \in \mathbf{D}^{\leq 0}$ and $M'' \in \mathbf{D}^{\geq 1}$.

When these conditions are satisfied one defines the *heart* of \mathbf{D} to be the full subcategory $\mathbf{D}^0 := \mathbf{D}^{\leq 0} \cap \mathbf{D}^{\geq 0}$. This is an abelian category.

Given a scheme X , the derived category $\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)$ has the *standard t-structure*, in which

$$\begin{aligned} \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^{\leq 0} &:= \\ &\{\mathcal{M} \in \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X) \mid H^i \mathcal{M} = 0 \text{ for all } i > 0\}, \\ \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^{\geq 0} &:= \\ &\{\mathcal{M} \in \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X) \mid H^i \mathcal{M} = 0 \text{ for all } i < 0\}. \end{aligned}$$

The heart $\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^0$ is equivalent to the category $\text{Coh } \mathcal{O}_X$ of coherent sheaves.

Other t-structures will be referred to as *perverse t-structures*.

Here is an observation. Suppose A is an essentially finite type \mathbb{K} -algebra, where as before \mathbb{K} is a finite dimensional regular noetherian ring. Let R_A be the rigid dualizing complex of A .

Then the duality $\mathbf{D} := \text{RHom}_A(-, R_A)$ gives rise to a perverse t-structure

$$\begin{aligned} {}^p\mathbf{D}_f^b(\text{Mod } A)^{\leq 0} &:= \\ &\{M \mid H^i \mathbf{D}M = 0 \text{ for all } i < 0\}, \\ {}^p\mathbf{D}_f^b(\text{Mod } A)^{\geq 0} &:= \\ &\{M \mid H^i \mathbf{D}M = 0 \text{ for all } i > 0\}. \end{aligned}$$

on $\mathbf{D}_f^b(\text{Mod } A)$.

We call it the *rigid perverse t-structure*. The heart is denoted by ${}^p\mathbf{D}_f^b(\text{Mod } A)^0$.

The next theorem was proved in [YZ3].

Theorem 5.1. *Let X be a finite type \mathbb{K} -scheme. Let \star denote either ≤ 0 , ≥ 0 or 0 .*

Define ${}^p\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^\star$ to be the class of complexes $\mathcal{M} \in \mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)$ such that

$$\text{R}\Gamma(U, \mathcal{M}) \in {}^p\mathbf{D}_f^b(\text{Mod } A)^\star$$

for any affine open set U , with $A := \Gamma(U, \mathcal{O}_X)$.

Then:

(1) *The pair*

$$({}^p\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^{\leq 0}, {}^p\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^{\geq 0})$$

is a t-structure on $\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)$.

(2) *The assignment $V \mapsto {}^p\mathbf{D}_c^b(\text{Mod } \mathcal{O}_V)^0$, for $V \subset X$ open, is a stack of abelian categories on X .*

Part (2) says that the objects of ${}^p\mathbf{D}_c^b(\text{Mod } \mathcal{O}_X)^0$, which we call *perverse coherent sheaves*, can be glued. They behave like sheaves, and hence the name.

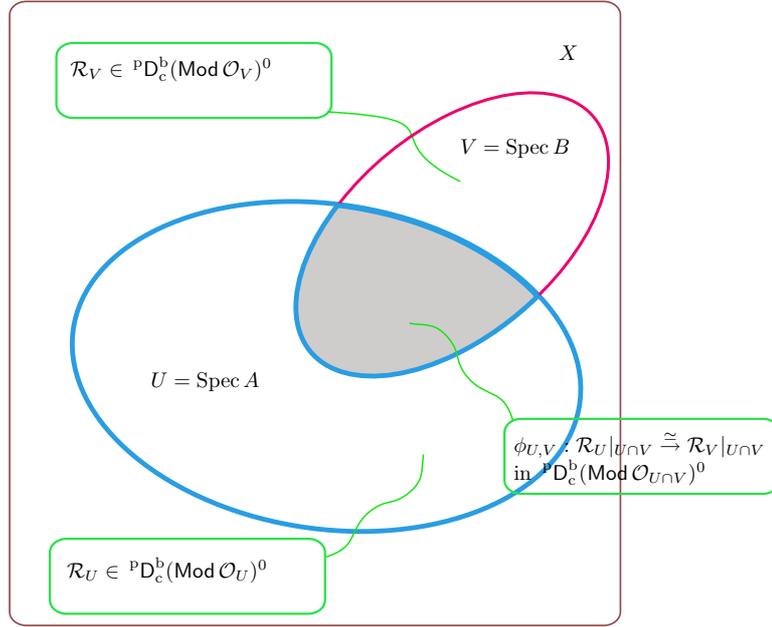


FIGURE 2

For any affine open set $U = \text{Spec } A$, the dualizing complex \mathcal{R}_U (the sheafification of the rigid dualizing complex R_A) is clearly a perverse coherent sheaf on U .

Thus we can use the isomorphisms

$$\phi_{U,V} : \mathcal{R}_U|_{U \cap V} \xrightarrow{\cong} \mathcal{R}_V|_{U \cap V},$$

deduced from equation (4.2), to glue the affine dualizing complexes.

In this way we obtain:

Theorem 5.2. *Let X be a finite type \mathbb{K} -scheme. There exists a rigid dualizing complex (\mathcal{R}_X, ρ_X) over X relative to \mathbb{K} , and it is unique up to a unique rigid isomorphism.*

Along the same lines, using Theorems 3.3 and 3.5, we can also prove:

Theorem 5.3. *Let X and Y be finite type \mathbb{K} -schemes, and let $f : X \rightarrow Y$ be a morphism.*

- (1) *If f is finite, then there is a unique isomorphism*

$$Rf_* \mathcal{R}_X \cong R\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{R}_Y)$$

which respects rigidity.

- (2) *If f is smooth of relative dimension n , then there is a unique isomorphism*

$$\Omega_{X/Y}^n[n] \otimes_{\mathcal{O}_X} f^* \mathcal{R}_Y \cong \mathcal{R}_X$$

which respects rigidity.

Proper morphisms and residues are treated in [Ye6].

Remark 5.4. Recently I discovered a totally new proof of the duality theorem for proper morphisms, which uses perverse sheaves only, avoiding residue calculations. If correct, the new proof will make the paper [Ye6] much shorter.

Remark 5.5. I think all the results here work also for essentially finite type \mathbb{K} -schemes.

6. COHEN-MACAULAY COMPLEXES

As before X is a finite type \mathbb{K} -scheme. Let \mathcal{R}_X be the rigid dualizing complex of X .

Given a point $x \in X$ let $\mathbf{k}(x)$ be its residue field. We denote by $\dim_{\mathbb{K}}(x)$ the unique integer i such that

$$\mathrm{Ext}_{\mathcal{O}_{X,x}}^{-i}(\mathbf{k}(x), \mathcal{R}_{X,x}) \neq 0.$$

Then the function

$$\dim_{\mathbb{K}} : X \rightarrow \mathbb{Z}$$

is a dimension function, i.e.

$$\dim_{\mathbb{K}}(y) = \dim_{\mathbb{K}}(x) - 1$$

when y is an immediate specialization of x .

Example 6.1. Take $\mathbb{K} := \mathbb{Z}$, the ring of integers, and $X := \mathbf{A}_{\mathbb{K}}^1 = \mathrm{Spec} \mathbb{K}[t]$, the affine line. Consider the following points in X : x_0 is the generic point; x_1 is the prime ideal (t) ; and x_2 is the maximal ideal $(t, 2)$. Then

$$\dim_{\mathbb{K}}(x_i) = 1 - i.$$

Recall from [RD] that a complex $\mathcal{M} \in \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Mod} \mathcal{O}_X)$ is called *Cohen-Macaulay* if for every point x the local cohomologies $\mathrm{H}_x^i \mathcal{M}$ all vanish except for $i = -\dim_{\mathbb{K}}(x)$.

Here is another result from [YZ3].

Theorem 6.2. *Let X be a finite type scheme over \mathbb{K} , let \mathcal{R}_X be the rigid dualizing complex of X , and let D be the duality functor $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$.*

Then the following conditions are equivalent for $\mathcal{M} \in \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Mod} \mathcal{O}_X)$.

- (i) \mathcal{M} is a perverse coherent sheaf (for the rigid perverse t -structure).
- (ii) $\mathrm{D}\mathcal{M}$ is a coherent sheaf, i.e. $\mathrm{H}^i \mathrm{D}\mathcal{M} = 0$ for all $i \neq 0$.
- (iii) \mathcal{M} is a Cohen-Macaulay complex.

In particular this implies the Cohen-Macaulay complexes form an abelian subcategory of $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(\mathrm{Mod} \mathcal{O}_X)$, a fact that seems to have eluded Grothendieck.

- END -

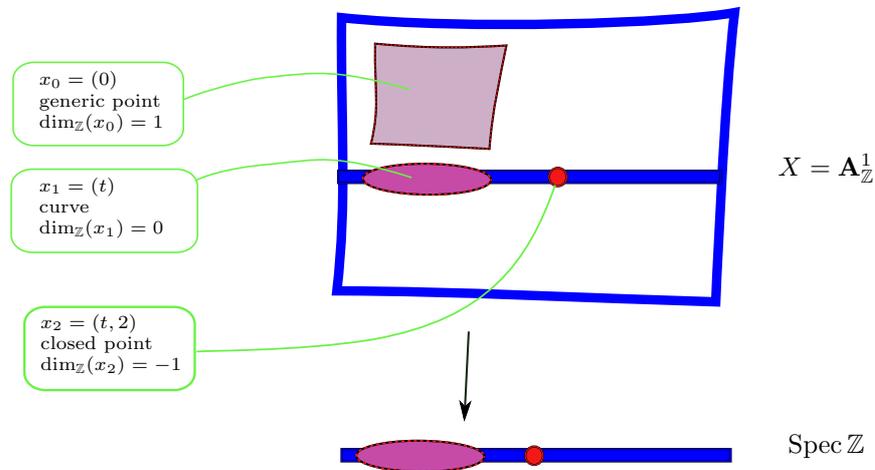


FIGURE 3

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