The Action of Adeles on the Residue Complex

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Dedicated to Professor Steven Kleiman on the Occasion of his Sixtieth Birthday

Abstract

Let X be a scheme of finite type over a perfect field $\Bbbk$. In this paper we study the relation between two objects associated to X: the *Grothendieck residue complex* $\mathcal{K}^{\cdot}_X$ and the *Beilinson adeles complex* $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$. The latter is a differential graded algebra (DGA). Our first main result (Theorem 0.1) is that $\mathcal{K}^{\cdot}_X$ is a right differential graded (DG) module over $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$.

We give an application to de Rham theory. Define graded sheaves $\mathcal{F}^{\cdot}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^{\cdot}_{X/\Bbbk}, \mathcal{K}^{\cdot}_X)$ and $\mathcal{A}^{\cdot}_X := \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/\Bbbk}$. It is known that $\mathcal{A}^{\cdot}_X$ is a DGA. Our second main result (Theorem 0.2) is that $\mathcal{F}^{\cdot}_X$ is a right DG $\mathcal{A}^{\cdot}_X$-module. When X is smooth then $\mathcal{F}^{\cdot}_X$ calculates de Rham homology, $\mathcal{A}^{\cdot}_X$ calculates cohomology, and the action induces the cap product. We extend these constructions to singular schemes in characteristic 0 using smooth formal embeddings.

0 Introduction

Let X be a scheme of finite type over a field $\Bbbk$. In this paper we study the relation between two objects associated to X: the *Grothendieck residue complex* $\mathcal{K}^{\cdot}_X$ and the *Beilinson adeles complex* $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$.

Grothendieck duality theory, developed around 1960 (cf. [RD]), is a vast generalization of Serre duality. It is a deep and complicated theory, fully expressible only in the language of derived categories. Attempts to simplify it or find some explicit presentation of it attracted a considerable amount

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of research (a partial list of references is [AK], [Kl], [Li], [HK], [KW], [LS], [Ye1], [Ne], [AJL] and [Co]).

In our situation denote by $\pi : X \to \operatorname{Spec} \Bbbk$ the structural morphism. Then there is a functor $\pi^! : \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \Bbbk) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ between derived categories, called the twisted inverse image. The object $\pi^! \Bbbk$ is a dualizing complex on X. It has a canonical representative, namely its Cousin complex $\mathcal{K}^{\cdot}_X$, which is called the *residue complex* of X. This is a bounded complex of quasi-coherent injective $\mathcal{O}_X$-modules, and as a sheaf $\mathcal{K}^{\cdot}_X = \bigoplus_{x \in X} \mathcal{K}_X(x)$, where $\mathcal{K}_X(x)$ is a constant sheaf with support $\overline{\{x\}}$. $\mathcal{K}^{\cdot}_X$ enjoys some remarkable properties, that are deduced from corresponding properties of $\pi^!$.

Almost twenty years later Beilinson introduced his scheme theoretic adeles (see [Be]). This high dimensional generalization of the classical adeles of a curve is actually pretty easy to define (see Section 1). Given any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, the complex adeles with values in $\mathcal{M}$ is a bounded complex $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ of flasque $\mathcal{O}_X$-modules, and there is a canonical quasi-isomorphism $\mathcal{M} \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$. The sheaf $\underline{\mathbb{A}}^{q}_{\mathrm{red}}(\mathcal{M})$ is a "restricted product" of local factors, each such local factor corresponding to the geometric data of a chain $(x_0, \ldots, x_q)$ of points in X. Taking $\mathcal{M} = \mathcal{O}_X$ we obtain a DGA (differential graded algebra) $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$.

Here is the first main result of this paper.

Theorem 0.1. *Suppose $\Bbbk$ is a perfect field and X is a finite type $\Bbbk$-scheme. Then $\mathcal{K}^{\cdot}_X$ is a right DG $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$-module.*

The theorem is restated in more detail in Section 1 (Theorem 1.14) and proved there, using the explicit construction of $\mathcal{K}^{\cdot}_X$ described in [Ye3]. This construction is based on the theory of *Beilinson completion algebras* (BCAs) developed in [Ye2]. The action of the adeles is by "taking residues": multiplication by an adele supported on a chain $(x_0, \ldots, x_q)$ is a map from $\mathcal{K}_X(x_0)$ to $\mathcal{K}_X(x_q)$. In Question 1.20 we speculate on a generalization of Theorem 0.1.

In Section 2 we move on to de Rham theory. Let $(\Omega^{\cdot}_{X/\Bbbk}, \mathrm{d})$ be the the algebraic de Rham complex of X, which is a sheaf of commutative DGAs over $\Bbbk$. The graded sheaf

$$\mathcal{F}^{\cdot}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^{\cdot}_{X/\Bbbk}, \mathcal{K}^{\cdot}_X)$$

is a graded $\Omega^{\cdot}_{X/\Bbbk}$-module. According to [Ye3] $\mathcal{F}^{\cdot}_X$ has a coboundary operator D that's a differential operator of order ≤ 1 over $\mathcal{O}_X$, and $(\mathcal{F}^{\cdot}_X, \mathrm{D})$ is a DG $\Omega^{\cdot}_{X/\Bbbk}$-module.

On the other hand we have the graded algebra

$$\mathcal{A}^{\cdot}_X := \bigoplus\nolimits_{p,q} \underline{\mathbb{A}}^{q}_{\mathrm{red}}(\Omega^{p}_{X/\Bbbk}) \cong \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^{\cdot}_{X/\Bbbk}.$$

By [HY1] there is a differential D that makes $(\mathcal{A}^{\cdot}_X, \mathrm{D})$ into a DGA, and moreover $\Omega^{\cdot}_{X/\Bbbk} \to \mathcal{A}^{\cdot}_X$ is a DGA quasi-isomorphism. Theorem 0.1 implies that $\mathcal{F}^{\cdot}_X$ is a graded right $\mathcal{A}^{\cdot}_X$-module. Our second main result is:

Theorem 0.2. *Suppose $\Bbbk$ is a perfect field and X is a finite type $\Bbbk$-scheme. Then $\mathcal{F}^{\cdot}_X$ is a DG right $\mathcal{A}^{\cdot}_X$-module.*

Theorem 0.2 is used in [HY2] to state the adelic Gauss-Bonnet formula, which is proved there. See also Remark 2.6 below.

Consider a finite type $\Bbbk$-scheme Y and a closed subscheme $X \subset Y$. Define $\mathfrak{X} := Y_{/X}$, the formal completion of Y along X (cf. [EGA] I). Let X_i be the ith infinitesimal neighborhood of $X = X_0$ in Y. Define $\mathcal{A}^{\cdot}_{\mathfrak{X}} := \lim_{\leftarrow i} \mathcal{A}^{\cdot}_{X_i}$ and $\mathcal{F}^{\cdot}_{\mathfrak{X}} := \lim_{i \to} \mathcal{F}^{\cdot}_{X_i}$.

Corollary 0.3. *$\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is a right DG $\mathcal{A}^{\cdot}_{\mathfrak{X}}$-module. If $\operatorname{char} \Bbbk = 0$ and Y is smooth over $\Bbbk$ then de Rham cohomology is $\mathrm{H}^p_{\mathrm{DR}}(X) = \mathrm{H}^p \Gamma(X, \mathcal{A}^{\cdot}_{\mathfrak{X}})$, de Rham homology is $\mathrm{H}^{\mathrm{DR}}_p(X) = \mathrm{H}^{-p} \Gamma(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$, and the right action of $\mathcal{A}^{\cdot}_{\mathfrak{X}}$ on $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ induces the cap product.*

The embedding $X \subset \mathfrak{X}$ is an instance of a smooth formal embedding in the sense of [Ye4].

Theorem 0.2 and Corollary 0.3 are proved at the end of Section 2.

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1 The Action

Let us begin with a review of *Beilinson adeles* on a noetherian scheme X. A chain of length q of points in X is a sequence $\xi = (x_0, \ldots, x_q)$ of points with $x_{i+1} \in \overline{\{x_i\}}$. Denote by $S(X)_q$ the set of length q chains, so $\{S(X)_q\}_{q \geq 0}$ is a simplicial set. For a subset $T \subset S(X)_q$ and a point $x \in X$ let

$$\hat{x}T := \{(x_1, \ldots, x_q) \mid (x, x_1, \ldots, x_q) \in T\}.$$

Denote by $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ the maximal ideal. If $\mathcal{M}$ is a coherent $\mathcal{O}_X$-module then for any $n \geq 1$ the $\mathcal{O}_{X,x}$-module $\mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x$ can be thought of as a quasi-coherent sheaf, constant on the closed set $\overline{\{x\}}$. According to [Be] there is a unique collection of functors $\mathbb{A}(T, -) : \mathsf{QCoh}\, \mathcal{O}_X \to \mathsf{Ab}$, indexed by subsets $T \subset S(X)_q$, each of which commuting with direct limits, and satisfying

$$\mathbb{A}(T, \mathcal{M}) = \begin{cases} \prod_{(x) \in T} \lim_{\leftarrow n} \mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x & \text{if } q = 0 \\ \prod_{x \in X} \lim_{\leftarrow n} \mathbb{A}(\hat{x}T, \mathcal{M}_x / \mathfrak{m}_x^n \mathcal{M}_x) & \text{if } q > 0 \end{cases} \tag{1.1}$$

for $\mathcal{M}$ coherent. Furthermore each $\mathbb{A}(T, -)$ is exact.

For a single chain ξ one also writes $\mathcal{M}_\xi := \mathbb{A}(\{\xi\}, \mathcal{M})$, and this is the *Beilinson completion* of $\mathcal{M}$ along ξ. Then

$$\mathbb{A}(T, \mathcal{M}) \subset \prod_{\xi \in T} \mathcal{M}_\xi. \tag{1.2}$$

In view of this we shall say that $\mathbb{A}(T, \mathcal{M})$ is the group of adeles combinatorially supported on T and with values in $\mathcal{M}$.

Observe that for $q = 0$ and $\mathcal{M}$ coherent we have $\mathcal{M}_{(x)} = \widehat{\mathcal{M}}_x$, the $\mathfrak{m}_x$-adic completion, and (1.2) is an equality.

Define a presheaf $\underline{\mathbb{A}}(T, \mathcal{M})$ by

$$\Gamma(U, \underline{\mathbb{A}}(T, \mathcal{M})) := \mathbb{A}(T \cap S(U)_q, \mathcal{M}) \tag{1.3}$$

for $U \subset X$ open. Then $\underline{\mathbb{A}}(T, \mathcal{M})$ is in fact a flasque sheaf. Also $\underline{\mathbb{A}}(T, \mathcal{O}_X)$ is a flat $\mathcal{O}_X$-algebra, and $\underline{\mathbb{A}}(T, \mathcal{M}) \cong \underline{\mathbb{A}}(T, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M}$. Given a local section $m \in \underline{\mathbb{A}}(T, \mathcal{M})$ we shall often use the inclusion (1.2) to write $m = (m_\xi)$ where ξ runs over T and $m_\xi \in \mathcal{M}_\xi$.

Let $S(X)_q^{\mathrm{red}}$ be the set of reduced chains (i.e. without repeated points), and define

$$\underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M}) := \underline{\mathbb{A}}(S(X)_q^{\mathrm{red}}, \mathcal{M}).$$

For $0 \leq i \leq q$ the ith face map ∂_i, which omits the point x_i from a chain $(x_0, \ldots, x_q)$, induces a homomorphism

$$\partial^i : \underline{\mathbb{A}}^{q-1}_{\mathrm{red}}(\mathcal{M}) \to \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M}).$$

Then $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is a complex with coboundary operator $\partial := \sum (-1)^i \partial^i$, and $\mathcal{M} \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is a quasi-isomorphism. Since $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})$ is a complex of flasque sheaves we get

$$\mathrm{H}^q \Gamma(X, \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{M})) = \mathrm{H}^q(X, \mathcal{M}).$$

The complex $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ is a DGA, with the Alexander-Whitney product. For local sections $a \in \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$ and $b \in \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$ the product is

$$a \cdot b := \partial^-(a) \cdot \partial^+(b) \in \underline{\mathbb{A}}^{q+q'}_{\mathrm{red}}(\mathcal{O}_X),$$

where ∂^- and ∂^+ correspond respectively to the initial and final segments of $(0, \ldots, q, \ldots, q + q')$. This algebra is not (graded) commutative. For proofs and more details see [Hr], [Ye1] Chapter 3 and [HY1] Section 1.

Example 1.4. Suppose X is a nonsingular curve. The relation to the classical ring of adeles $\mathbb{A}(X)$ of Chevalley and Weil associated to X is

$$\mathbb{A}(X) = \Gamma(X, \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{O}_X)).$$

From now we assume X is a finite type scheme over a perfect field $\mathbb{k}$.

Next let us recall the construction of the residue complex $\mathcal{K}^{\cdot}_X$ in [Ye3]. It starts with the theory of Beilinson completion algebras (BCAs) developed in [Ye2]. A BCA A is a semilocal $\mathbb{k}$-algebra with a topology and with valuations on its residue fields. Each local factor of A is a quotient of $K((s_1, \ldots, s_n))[[t_1, \ldots, t_m]]$ where K is some finitely generated extension field of $\mathbb{k}$, and $K((s_1, \ldots, s_n)) = K((s_n)) \cdots ((s_1))$ is the field of iterated Laurent series. One considers two kinds of homomorphisms between BCAs: morphisms $f : A \to B$ and intensifications $u : A \to \widehat{A}$.

Each BCA A has a dual module $\mathcal{K}(A)$, which is functorial w.r.t. these homomorphisms; namely there are maps $\mathrm{Tr}_f : \mathcal{K}(B) \to \mathcal{K}(A)$ and $\mathrm{q}_u : \mathcal{K}(A) \to \mathcal{K}(\widehat{A})$. Algebraically $\mathcal{K}(A)$ is an injective hull of $A/\mathfrak{r}$, where $\mathfrak{r}$ is the Jacobson radical.

Example 1.5. Let $A := \mathbb{k}[[s_1, s_2]]$, $B := \mathbb{k}(s_1, s_2)$ and $\widehat{B} := \mathbb{k}((s_1, s_2))$. The inclusions $f : A \to \widehat{B}$ and $u : B \to \widehat{B}$ are respectively a morphism and an intensification. The dual modules are

$$\mathcal{K}(A) = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{k}}(A, \mathbb{k}),$$

$$\mathcal{K}(B) = \Omega^2_{B/\mathbb{k}} = B \cdot \mathrm{d}s_1 \wedge \mathrm{d}s_2$$

and

$$\mathcal{K}(\widehat{B}) = \Omega^{2,\mathrm{sep}}_{\widehat{B}/\mathbb{k}} = \widehat{B} \cdot \mathrm{d}s_1 \wedge \mathrm{d}s_2.$$

The homomorphism $\mathrm{q}_u : \mathcal{K}(B) \to \mathcal{K}(\widehat{B})$ is the inclusion. For a form $\beta \in \Omega^{2,\mathrm{sep}}_{\widehat{B}/\mathbb{k}}$ the functional $\mathrm{Tr}_f(\beta) \in \mathcal{K}(A)$ is described as follows. Given an element $a \in A$ write

$$a\beta = \sum_{i,j} \lambda_{i,j} s_1^i s_2^j \cdot \mathrm{d}s_1 \wedge \mathrm{d}s_2 \in \Omega^{2,\mathrm{sep}}_{\widehat{B}/\mathbb{k}}$$

with $\lambda_{i,j} \in \mathbb{k}$. Then

$$\mathrm{Tr}_f(\beta)(a) = \lambda_{-1,-1},$$

namely the residue of $a\beta$.

Suppose $\xi = (x, \ldots, y)$ is a saturated chain of points in X (i.e. each point is an immediate specialization of the previous one). Then the Beilinson completion $\mathcal{O}_{X,\xi}$ is a BCA. The natural algebra homomorphisms $\partial^- : \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,\xi}$ and $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ are an intensification and a morphism, respectively. So there are homomorphisms on dual modules

$$\mathrm{q}_{\partial^-} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{X,\xi})$$

and

$$\mathrm{Tr}_{\partial^+} : \mathcal{K}(\mathcal{O}_{X,\xi}) \to \mathcal{K}(\mathcal{O}_{X,(y)}).$$

The composition

$$\mathrm{Tr}_{\partial^+} \circ \mathrm{q}_{\partial^-} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{X,(y)})$$

is denoted by δ_ξ. We regard $\mathcal{K}_X(x) := \mathcal{K}(\mathcal{O}_{X,(x)})$ as a quasi-coherent $\mathcal{O}_X$-module, constant on the closed set $\overline{\{x\}}$. Define

$$\mathcal{K}_X^{-q} := \bigoplus_{\dim \overline{\{x\}}=q} \mathcal{K}_X(x) \tag{1.6}$$

and

$$\delta := (-1)^{q+1} \sum_{(x,y)} \delta_{(x,y)} : \mathcal{K}_X^{-q} \to \mathcal{K}_X^{-q+1}. \tag{1.7}$$

Then the pair $(\mathcal{K}_X^{\cdot}, \delta)$ is the residue complex of X. That is to say, in the notation of the introduction, there is a canonical isomorphism $\mathcal{K}_X^{\cdot} \cong \pi^! \Bbbk$ in the derived category $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ (see [Ye3] Corollary 2.5).

Let x be a point of dimension q in X, and consider a local section $\phi_x \in \mathcal{K}_X(x) \subset \mathcal{K}_X^{-q}$. Let $\xi = (x_0, \ldots, x_{q'})$ be any chain of length q' in X, and let $a_\xi \in \mathcal{O}_{X,\xi}$. Define an element $\phi_x \cdot a_\xi \in \mathcal{K}_X^{-q+q'}$ as follows. If $x = x_0$ and ξ is saturated then there are homomorphisms $\mathrm{q}_{\partial^-} : \mathcal{K}_X(x) \to \mathcal{K}(\mathcal{O}_{X,\xi})$ and $\mathrm{Tr}_{\partial^+} : \mathcal{K}(\mathcal{O}_{X,\xi}) \to \mathcal{K}_X(x_{q'})$. Since $\mathcal{K}(\mathcal{O}_{X,\xi})$ is an $\mathcal{O}_{X,\xi}$-module the product $a_\xi \cdot \mathrm{q}_{\partial^-}(\phi_x) \in \mathcal{K}(\mathcal{O}_{X,\xi})$ exists, and we set

$$\phi_x \cdot a_\xi := \mathrm{Tr}_{\partial^+}(a_\xi \cdot \mathrm{q}_{\partial^-}(\phi_x)) \in \mathcal{K}_X(x_{q'}). \tag{1.8}$$

Otherwise we set $\phi_x \cdot a_\xi := 0$.

Remark 1.9. In order to apply the Koszul sign rule we consider the "local" objects $\mathcal{K}_X(x)$ and $\mathcal{O}_{X,\xi}$ as ungraded; whereas the "global" objects $\mathcal{K}_X^{\cdot}$ and $\underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{O}_X)$ are considered as graded. Ungraded local elements shall be decorated with suitable subscripts, such as $a_\xi \in \mathcal{O}_{X,\xi}$ or $\phi_x \in \mathcal{K}_X(x)$.

Suppose $\xi = (x_0, \ldots, x_p)$ and $\eta = (y_0, \ldots, y_q)$ are chains such that $y_0 \in \overline{\{x_p\}}$. Then we denote by $\xi \vee \eta$ the concatenated chain $(x_0, \ldots, x_p, y_0, \ldots, y_q)$.

Lemma 1.10. *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains in X and let $\phi_x \in \mathcal{K}_X(x)$, $a_\xi \in \mathcal{O}_{X,\xi}$ and $b_\eta \in \mathcal{O}_{X,\eta}$ be some elements. Writing $\xi \vee \partial_0(\eta) := (x, \ldots, y, \ldots, z)$, one has $a_\xi \cdot b_\eta \in \mathcal{O}_{X,\xi\vee\partial_0(\eta)}$. Then*

$$(\phi_x \cdot a_\xi) \cdot b_\eta = \phi_x \cdot (a_\xi \cdot b_\eta) \in \mathcal{K}_X(z).$$

Proof. Consider the commutative diagram with intensifications u, v, w, u'

and morphisms f, f', g, h:

$$\begin{tikzcd}
\mathcal{O}_{X,(x)} \arrow[r, "w"] \arrow[dr, "v"'] & \mathcal{O}_{X,\xi} \arrow[d, "u'"] & \mathcal{O}_{X,(y)} \arrow[l, "f"'] \arrow[d, "u"] \\
& \mathcal{O}_{X,\xi\vee\partial_0(\eta)} & \mathcal{O}_{X,\eta} \arrow[l, "f'"'] \\
& & \mathcal{O}_{X,(z)} \arrow[u, "g"'] \arrow[ul, "h"]
\end{tikzcd}$$

According to [Ye3] Lemma 1.12 there is an isomorphism

$$\mathcal{O}_{X,\xi\vee\partial_0(\eta)} \cong \mathcal{O}_{X,\xi} \otimes^{(\wedge)}_{\mathcal{O}_{X,(y)}} \mathcal{O}_{X,\eta}$$

(intensification base change), so by [Ye2] Thm. 7.4 (ii) one has

$$\mathrm{Tr}_{f'} \circ \mathrm{q}_{u'} = \mathrm{q}_u \circ \; \mathrm{Tr}_f \, .$$

Now the BCA $\mathcal{O}_{X,\xi\vee\partial_0(\eta)}$ is commutative, $\mathrm{Tr}_{f'}$ is $\mathcal{O}_{X,\eta}$-linear and $\mathrm{q}_{u'}$ is $\mathcal{O}_{X,\xi}$-linear. Hence

$$\begin{aligned}
\phi_x \cdot (a_\xi \cdot b_\eta) &= \mathrm{Tr}_h\big((a_\xi \cdot b_\eta) \cdot \mathrm{q}_v(\phi_x)\big) \\
&= \mathrm{Tr}_g\big(\mathrm{Tr}_{f'}\big(f'(b_\eta) \cdot u'(a_\xi) \cdot \mathrm{q}_{u'}(\mathrm{q}_w(\phi_x))\big)\big) \\
&= \mathrm{Tr}_g\big(b_\eta \cdot \mathrm{Tr}_{f'}\big(\mathrm{q}_{u'}(a_\xi \cdot \mathrm{q}_w(\phi_x))\big)\big) \\
&= \mathrm{Tr}_g\big(b_\eta \cdot \mathrm{q}_u\big(\mathrm{Tr}_f(a_\xi \cdot \mathrm{q}_w(\phi_x))\big)\big) \\
&= (\phi_x \cdot a_\xi) \cdot b_\eta.
\end{aligned}$$

□

Suppose $x \in X$ is a point and $b \in \mathcal{O}_{X,(x)}$ is some element. Using the inclusion $\mathcal{O}_{X,(x)} \subset \underline{\mathbb{A}}^0_{\mathrm{red}}(\mathcal{O}_X)$ we consider b as an adele.

Lemma 1.11 (Approximation). *Let $U \subset X$ be an affine open set, $x \in U$ a point and $q \geq 1$. Define*

$$T := \{\xi \in S(U)^{\mathrm{red}}_q \mid \xi = (x, \ldots)\}.$$

Let $a \in \mathbb{A}(T, \mathcal{O}_X)$ be some adele.

(1) *Given an integer $n \geq 1$ there exists adeles $b \in \mathcal{O}_{X,(x)}$ and $c = (c_\eta) \in \mathbb{A}(\partial_0(T), \mathcal{O}_X)$ such that*

$$a - \partial^1(b) \cdot c \in \mathfrak{m}_x^n \cdot \mathbb{A}(T, \mathcal{O}_X).$$

(2) *Given an element* $\phi_x \in \mathcal{K}_X(x)$ *there is an integer* $n \geq 1$ *such that* $\mathfrak{m}_x^n \cdot \phi_x = 0$. *For such* n *the adeles* b *and* c *from part* (1) *satisfy*

$$\phi_x \cdot a_\xi = \phi_x \cdot (b_{(x,y)} \cdot c_\eta) \in \mathcal{K}_X(z)$$

for all $\xi \in T$, *where we write* $\eta = (y, \ldots, z) := \partial_0(\xi)$, *and* $b_{(x,y)}$ *is the* (x, y) *component of* $\partial^1(b)$.

Proof. (1) There is a ring homomorphism $\mathcal{O}_{X,(x)} \to \mathbb{A}(T, \mathcal{O}_X)$ and via this homomorphism we obtain an ideal $\mathfrak{m}_x^n \cdot \mathbb{A}(T, \mathcal{O}_X)$. Because the functor $\mathbb{A}(T, -)$ is exact we have

$$\frac{\mathbb{A}(T, \mathcal{O}_X)}{\mathfrak{m}_x^n \cdot \mathbb{A}(T, \mathcal{O}_X)} \cong \mathbb{A}(T, \mathcal{O}_{X,x}/\mathfrak{m}_x^n).$$

By the definition of adeles with values in a quasi-coherent sheaf there is an isomorphism

$$\mathbb{A}(T, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \cong \lim_{\rightarrow} \mathbb{A}(T, \mathcal{L})$$

where the limit runs over the coherent $\mathcal{O}_U$-submodules $\mathcal{L} \subset \mathcal{O}_{X,x}/\mathfrak{m}_x^n$. We note that $\partial_0(T) = \hat{x}T$, so for $\mathcal{L}$ large enough (i.e. $\mathcal{L}_x = \mathcal{O}_{X,x}/\mathfrak{m}_x^n$)

$$\begin{aligned} \mathbb{A}(T, \mathcal{L}) &= \lim\nolimits_{\leftarrow i} \mathbb{A}(\partial_0(T), \mathcal{L}_x/\mathfrak{m}_x^i \mathcal{L}_x) \\ &\cong \mathbb{A}(\partial_0(T), \mathcal{O}_{X,x}/\mathfrak{m}_x^n). \end{aligned}$$

The conclusion is that

$$\mathbb{A}(T, \mathcal{O}_{X,x}/\mathfrak{m}_x^n) \cong \mathbb{A}(\partial_0(T), \mathcal{O}_{X,x}/\mathfrak{m}_x^n).$$

Again we go to coherent subsheaves. Write $C := \Gamma(U, \mathcal{O}_X)$ and $\mathcal{O}_Z := \operatorname{Im}(\mathcal{O}_X \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$. Then

$$\begin{aligned} \mathbb{A}(\partial_0(T), \mathcal{O}_{X,x}/\mathfrak{m}_x^n) &= \lim\nolimits_{\rightarrow} \mathbb{A}(\partial_0(T), \mathcal{L}) \\ &\cong (\mathcal{O}_{X,x}/\mathfrak{m}_x^n) \otimes_C \mathbb{A}(\partial_0(T), \mathcal{O}_Z) \\ &\cong (\mathcal{O}_{X,x}/\mathfrak{m}_x^n) \otimes_C \mathbb{A}(\partial_0(T), \mathcal{O}_X). \end{aligned}$$

So there is a ring surjection

$$\mathcal{O}_{X,x} \otimes_C \mathbb{A}(\partial_0(T), \mathcal{O}_X) \twoheadrightarrow \mathbb{A}(T, \mathcal{O}_{X,x}/\mathfrak{m}_x^n). \tag{1.12}$$

Consider the image $\bar{a}$ of the adele a in $\mathbb{A}(T, \mathcal{O}_{X,x}/\mathfrak{m}_x^n)$. Using the surjection (1.12) we can write $\bar{a} = \sum_{i=1}^r b_i \otimes c_i$ with $b_i \in \mathcal{O}_{X,x}$ and $c_i \in \mathbb{A}(\partial_0(T), \mathcal{O}_X)$. By bringing the b_i to a common denominator we can assume $r = 1$.

(2) Because $\mathcal{K}(\mathcal{O}_{X,(x)})$ is an $\mathfrak{m}_x$-torsion module the element ϕ_x is annihilated by some power $\mathfrak{m}_x^n$. Pick adeles b and c as in part (1). Then for any chain $\xi = (x) \vee \eta = (x, y, \ldots) \in T$ we obtain

$$a_\xi - b_{(x,y)} \cdot c_\eta \in \mathfrak{m}_x^n \cdot \mathcal{O}_{X,\xi}.$$

Therefore

$$\phi_x \cdot a_\xi = \phi_x \cdot (b_{(x,y)} \cdot c_\eta).$$

$\square$

Lemma 1.13. *Let $U \subset X$ be an open set, $x \in U$ a point, $\phi_x \in \mathcal{K}_X(x)$ an element and $a = (a_\xi) \in \mathbb{A}(S(U)_q^{\mathrm{red}}, \mathcal{O}_X)$ an adele. Then for all but finitely many chains $\xi \in S(U)_q^{\mathrm{red}}$ one has $\phi_x \cdot a_\xi = 0$.*

Proof. The proof is by induction on q. For $q = 0$ there is nothing to prove, so take $q \geq 1$. Because U is covered by finitely many affine open sets, and we are only interested in establishing finiteness, we might as well assume U itself is affine. Moreover by the definition of the product we can neglect those chains in the combinatorial support of a that do not begin with x. Thus we can assume $a \in \mathbb{A}(T, \mathcal{O}_X)$, where T is the set defined in Lemma 1.11. According to this lemma we can find adeles $b \in \mathcal{O}_{X,(x)}$ and $c = (c_\eta) \in \mathbb{A}(\partial_0(T), \mathcal{O}_X)$ such that

$$\phi_x \cdot a_\xi = \phi_x \cdot (b_{(x,y)} \cdot c_\eta) \in \mathcal{K}_X(z)$$

for all $\xi = (x) \vee \eta = (x, y, \ldots, z) \in T$.

The only way to get a nonzero product $\phi_x \cdot a_\xi$ is when ξ is a saturated chain. Consider such a chain ξ. According to Lemma 1.10 we have

$$\phi_x \cdot (b_{(x,y)} \cdot c_\eta) = (\phi_x \cdot b_{(x,y)}) \cdot c_\eta.$$

For any point y occurring one has

$$\phi_x \cdot b_{(x,y)} = \delta_{(x,y)}(b\phi_x) \in \mathcal{K}_X(y).$$

It follows that the product $\psi_y := \phi_x \cdot b_{(x,y)}$ vanishes for all but finitely many points y. Fixing y, the induction hypothesis applied to the element $\psi_y \in \mathcal{K}_X(y)$ says that $\psi_y \cdot c_\eta = 0$ for all but finitely many chains $\eta \in S(U)_{q-1}^{\mathrm{red}}$. $\square$

Theorem 1.14. *$\mathcal{K}_X^{\cdot}$ is a right DG $\underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{O}_X)$-module, with product*

$$\phi \cdot a = \sum_{x,\xi} \phi_x \cdot a_\xi$$

for local sections $\phi = \sum_x \phi_x \in \mathcal{K}_X^{\cdot}$ and $a = (a_\xi) \in \underline{\mathbb{A}}_{\mathrm{red}}^{\cdot}(\mathcal{O}_X)$.

Proof. According to Lemmas 1.13 and 1.10 this is a well defined associative product. It remains to verify that

$$\delta(\phi \cdot a) = \delta(\phi) \cdot a + (-1)^q \phi \cdot \partial(a) \tag{1.15}$$

for $\phi \in \mathcal{K}_X^{-q}$ and $a \in \underline{\mathbb{A}}_{\mathrm{red}}^{q'}(\mathcal{O}_X)$. We may assume $\phi = \phi_x$ and $a = a_\xi$ for some point x of dimension q and a chain $\xi = (x_0, \ldots, x_{q'})$.

Now there are only 3 ways to get any nonzero term in equation (1.15): (i) ξ is saturated and $x_0 = x$; (ii) ξ is saturated and (x, x_0) is saturated; or (iii) $x_0 = x$, and for some index $0 \leq i < q'$ and some point $y \in X$, the chain

$$\eta := (x_0, \ldots, x_i, y, x_{i+1}, \ldots, x_{q'})$$

is saturated.

In case (i), $\delta(\phi) \cdot a = 0$ and

$$\begin{aligned} \phi \cdot \partial(a) &= \phi_x \cdot (-1)^{q'+1} \partial^{q'+1}(a_\xi) \\ &= (-1)^{q'+1} \sum_y \delta_{(x_{q'}, y)}(\phi_x \cdot a_\xi) \\ &= (-1)^q \delta(\phi \cdot a). \end{aligned}$$

In case (ii), $\phi \cdot a = 0$ and

$$\begin{aligned} \phi \cdot \partial(a) &= \phi_x \cdot \partial^0(a)_{(x) \vee \xi} \\ &= \delta_{(x, x_0)}(\phi_x) \cdot a_\xi \\ &= (-1)^{q+1} \delta(\phi) \cdot a. \end{aligned}$$

In this equation $(x) \vee \xi$ is the concatenated chain $(x, x_0, \ldots, x_{q'})$.

Finally in case (iii), $\delta(\phi) \cdot a = 0$, $\delta(\phi \cdot a) = 0$, and it remains to show that also $\phi \cdot \partial(a) = 0$. We note that

$$\phi \cdot \partial(a) = \sum_y \phi_x \cdot (-1)^{i+1} \partial^{i+1}(a)_\eta$$

where y runs over the points such that (x_i, y, x_{i+1}) is a saturated chain, and η is as above (and depends on y).

For any index $0 \leq j \leq q'$ let us write $\xi_j := (x_j, \ldots, x_{q'})$. We shall use an approximation trick of Lemma 1.11 to define recursively elements $\phi_{x_j} \in \mathcal{K}_X(x_j)$ and $b_{(x_j)} \in \mathcal{O}_{X,(x_j)}$ for $0 \leq j \leq i$, and $a_{\xi_j} \in \mathcal{O}_{X,\xi_j}$ for $0 \leq j \leq i+1$. Pick some affine open set $U \subset X$ containing $x_{q'}$ (U does not play an essential role, since our computation is local anyhow; but it appears in Lemma 1.11). For $j = 0$ we note that $x_0 = x$ and $\xi_0 = \xi$, and the elements ϕ_{x_0} and a_{ξ_0} are already defined. Now suppose $j \leq i$ and ϕ_{x_j} and a_{ξ_j} have been defined. By Lemma 1.11 we can find elements $b_{(x_j)} \in \mathcal{O}_{X,(x_j)}$ and $a_{\xi_{j+1}} \in \mathcal{O}_{X,\xi_{j+1}}$ such that

$$\phi_{x_j} \cdot a_{\xi_j} = \phi_{x_j} \cdot (\partial^1(b_{(x_j)})_{(x_j, x_{j+1})} \cdot a_{\xi_{j+1}}) \in \mathcal{K}(x_{q'}).$$

If $j < i$ we also define

$$\phi_{x_{j+1}} := \phi_{x_j} \cdot \partial^1(b_{(x_j)})_{(x_j, x_{j+1})} \in \mathcal{K}_X(x_{j+1}).$$

We thus have for any of the points y under consideration:

$$\phi_{x_j} \cdot \partial^{(i+1)-j}(a_{\xi_j})_{\eta_j} = \phi_{x_{j+1}} \cdot \partial^{(i+1)-(j+1)}(a_{\xi_{j+1}})_{\eta_{j+1}},$$

where

$$\eta_j := (x_j, \ldots, x_i, y, x_{i+1}, \ldots, x_{q'}).$$

Putting it all together we obtain

$$\begin{aligned}
\phi_{x_0} \cdot \partial^{i+1}(a_{\xi_0})_{\eta_0} &= \phi_{x_i} \cdot \partial^1(a_{\xi_i})_{\eta_i} \\
&= (\phi_{x_i} \cdot b_{(x_i)}) \cdot (\partial^0 \circ \partial^0)(a_{\xi_{i+1}})_{\eta_i} \\
&= \delta_{(x_i, y, x_{i+1})}(\phi_{x_i} \cdot b_{(x_i)}) \cdot a_{\xi_{i+1}},
\end{aligned}$$

and hence

$$\phi \cdot \partial(a) = (-1)^{i+1} \sum_y \delta_{(x_i, y, x_{i+1})}(\phi_{x_i} \cdot b_{(x_i)}) \cdot a_{\xi_{i+1}}.$$

But according to [Ye3] Lemma 2.15(3), which is a variant of the Parshin-Lomadze Residue Theorem, we have

$$\sum_y \delta_{(x_i, y, x_{i+1})}(\phi_{x_i} \cdot b_{(x_i)}) = 0.$$

□

Example 1.16. The global section $a := \partial^1(1) \in \Gamma(X, \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{O}_X))$ acts on $\mathcal{K}^q_X$ like $(-1)^{q+1}\delta$, namely for any $\phi \in \mathcal{K}^q_X$ we have

$$\phi \cdot a = \phi \cdot \partial^1(1) = (-1)^{q+1}\delta(\phi).$$

Example 1.17. If X is a nonsingular curve then the theorem takes on a very simple form. Here $\mathcal{K}^{-1}_X = \Omega^1_{K/\Bbbk}$, where K is the function field of X; and $\mathcal{K}^0_X = \mathrm{Coker}(\Omega^1_{X/\Bbbk} \to \Omega^1_{K/\Bbbk})$. And there is actually an isomorphism of complexes

$$\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{K}^{\cdot}_X, \mathcal{K}^{\cdot}_X)$$

(cf. Example 1.4).

If X is integral of dimension n, let $\boldsymbol{\omega}_X$ be the coherent sheaf $\mathrm{H}^{-n}\mathcal{K}^{\cdot}_X$. This is a subsheaf of the constant sheaf $\mathcal{K}^{-n}_X = \Omega^n_{K/\Bbbk}$, where K is the function field of X. In [Li] and [KW] $\boldsymbol{\omega}_X$ is called the sheaf of regular differential forms (cf. [Ye1] Theorem 4.4.16). Since $\mathcal{K}^{\cdot}_X$ is a complex of injectives and $\boldsymbol{\omega}_X \to \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\boldsymbol{\omega}_X)$ is a quasi-isomorphism, there is a map of complexes $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\boldsymbol{\omega}_X) \to \mathcal{K}^{\cdot}_X[-n]$ inducing the identity in H^0. Lipman asked for an explicit formula for such a homomorphism of complexes. Producing such a formula was the main result of [HY1]. The following corollary gives essentially the same formula but in terms of DG modules.

Corollary 1.18. *Suppose X is integral of dimension n. Then the map*

$$\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\boldsymbol{\omega}_X) \cong \boldsymbol{\omega}_X \otimes_{\mathcal{O}_X} \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X) \to \mathcal{K}^{\cdot}_X[-n],$$

sending $\alpha \otimes a \mapsto \alpha \cdot a$, is a homomorphism of complexes, which induces the identity on $\boldsymbol{\omega}_X = \mathrm{H}^0 \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\boldsymbol{\omega}_X) = \mathrm{H}^0 \mathcal{K}^{\cdot}_X[-n]$.

Proof. In $\boldsymbol{\omega}_X \otimes_{\mathcal{O}_X} \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ one has $\partial(\alpha \otimes a) = \alpha \otimes \partial(a)$. On the other hand in $\mathcal{K}^{\cdot}_X[-n]$ the section α has degree 0, and $\delta(\alpha) = 0$, so

$$\delta(\alpha \cdot a) = \delta(\alpha) \cdot a + \alpha \cdot \partial(a) = \alpha \cdot \partial(a).$$

□

Given a morphism $f : X \to Y$ there is a natural DGA homomorphism $f^* : \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_Y) \to f_* \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$. There is also a map of graded $\mathcal{O}_Y$-modules $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$, and this is a map of complexes if f is proper (see [Ye3] Definition 2.11 and Theorem 3.4).

Theorem 1.19. *If $f : X \to Y$ is proper then $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ is a homomorphisms of DG $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_Y)$-modules.*

Proof. We have to show that $\mathrm{Tr}_f(\phi \cdot a) = \mathrm{Tr}_f(\phi) \cdot a$ for local sections $\phi \in \mathcal{K}^{\cdot}_X$ and $a \in \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_Y)$. For this we may as well assume $\phi = \phi_{x_0} \in \mathcal{K}_X(x_0)$ and $a = a_\eta \in \mathcal{O}_{Y,\eta}$ for a saturated chain $\eta = (y_0, \ldots, y_q)$ in Y and a point x_0 which is closed in the fiber $f^{-1}(y_0)$. According to [Ye3] Proposition 2.1 we have an isomorphism of BCAs

$$\prod_{\xi} \mathcal{O}_{X,\xi} \cong \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,\eta}$$

where $\xi = (x_0, \ldots, x_q)$ runs over the finitely many chains in X satisfying $f(x_i) = y_i$. Therefore by [Ye2] Theorem 7.4 the left square in the diagram

$$\begin{array}{ccccc}
\mathcal{K}(\mathcal{O}_{X,(x_0)}) & \xrightarrow{\mathrm{q}} & \bigoplus_{\xi} \mathcal{K}(\mathcal{O}_{X,\xi}) & \xrightarrow{\mathrm{Tr}} & \bigoplus_{x_q} \mathcal{K}(\mathcal{O}_{X,(x_q)}) \\
{\scriptstyle \mathrm{Tr}} \downarrow & & {\scriptstyle \mathrm{Tr}} \downarrow & & {\scriptstyle \mathrm{Tr}} \downarrow \\
\mathcal{K}(\mathcal{O}_{Y,(y_0)}) & \xrightarrow{\mathrm{q}} & \mathcal{K}(\mathcal{O}_{Y,\eta}) & \xrightarrow{\mathrm{Tr}} & \mathcal{K}(\mathcal{O}_{Y,(y_q)})
\end{array}$$

is commutative. The functoriality of Tr with respect to morphisms of BCAs implies the commutativity of the right square. □

Question 1.20. It is plausible to assume that an action as in Theorem 1.14 exists even without the explicit construction of the residue complex. Suppose X is a finite dimensional noetherian scheme endowed with a dimension function $d : X \to \mathbb{Z}$ (e.g. if X has a dualizing complex). Let $\mathcal{L}$ be a quasi-coherent Cousin complex on X, so that $\mathcal{L}^{-q} \cong \bigoplus_{d(x)=q} \mathrm{H}^{-q}_x \mathcal{L}$. Is $\mathcal{L}$ a right DG $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$-module?

2 de Rham Complexes, Residues and Adeles

Let us continue with the setup of Section 1. Consider the algebraic de Rham complex $(\Omega^{\cdot}_{X/\Bbbk}, \mathrm{d})$ of X. For any integers p, q define

$$\mathcal{A}^{p,q}_X := \underline{\mathbb{A}}^q_{\mathrm{red}}(\Omega^p_{X/\Bbbk}).$$

Since $\mathcal{M} \mapsto \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{M})$ is functorial with respect to differential operators $\mathcal{M} \to \mathcal{N}$ (cf. [HY1]), the sheaves $\mathcal{A}^{p,q}_X$ make up a double complex, with commuting operators d and ∂. Let

$$\mathcal{A}^n_X := \bigoplus_{p+q=n} \mathcal{A}^{p,q}_X$$

with coboundary operators

$$\begin{aligned} \mathrm{D}' &:= \mathrm{d} : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p+1,q}_X, \\ \mathrm{D}'' &:= (-1)^p \partial : \mathcal{A}^{p,q}_X \to \mathcal{A}^{p,q+1}_X \end{aligned}$$

and $\mathrm{D} := \mathrm{D}' + \mathrm{D}''$. The Alexander-Whitney product $\mathcal{A}^{p,q}_X \otimes \mathcal{A}^{p',q'}_X \to \mathcal{A}^{p+p',q+q'}_X$ makes $(\mathcal{A}^{\cdot}_X, \mathrm{D})$ into a DGA. To be explicit, write $\mathcal{A}^{p,q}_X \cong \Omega^p_{X/\Bbbk} \otimes_{\mathcal{O}_X} \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$. Then taking local sections $a \in \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$, $b \in \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$, $\alpha \in \Omega^p_{X/\Bbbk}$ and $\beta \in \Omega^{p'}_{X/\Bbbk}$, one has

$$(\alpha \otimes a) \cdot (\beta \otimes b) = (-1)^{qp'} \alpha \wedge \beta \otimes \partial^-(a) \cdot \partial^+(b) \in \mathcal{A}^{p+p',q+q'}_X$$

and

$$\begin{aligned} \mathrm{D}(\alpha \otimes a) &= \mathrm{D}(\alpha) \cdot a + (-1)^p \alpha \cdot \mathrm{D}(a) \\ &= \mathrm{d}(\alpha) \cdot a + (-1)^p \alpha \cdot \mathrm{d}(a) + (-1)^p \alpha \cdot \partial(a). \end{aligned}$$

Each $\mathcal{A}^{p,q}_X$ is a flasque sheaf, and the natural homomorphism of DGAs $\Omega^{\cdot}_{X/\Bbbk} \to \mathcal{A}^{\cdot}_X$ is a quasi-isomorphism.

In [Ye3] it is proved that any differential operator $D : \mathcal{M} \to \mathcal{N}$ between $\mathcal{O}_X$-modules induces a dual operator

$$\mathrm{Dual}(D) : \mathrm{Dual}\,\mathcal{N} \to \mathrm{Dual}\,\mathcal{M},$$

where by definition $\mathrm{Dual}\,\mathcal{M}$ is the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X)$. The operator $\mathrm{Dual}(D)$ is defined locally, in terms of Beilinson completion algebras. Let

$$\mathcal{F}^{p,q}_X := \mathcal{H}om_X(\Omega^{-p}_{X/\Bbbk}, \mathcal{K}^q_X).$$

We see that $\mathcal{F}^{\cdot,\cdot}_X$ is a double complex, with two commuting operators δ and

Dual(d). Set

$$
\begin{aligned}
\mathcal{F}_X^n &:= \bigoplus_{p+q=n} \mathcal{F}_X^{p,q}, \\
\mathrm{D}' &:= (-1)^{p+q+1}\,\mathrm{Dual}(\mathrm{d}) : \mathcal{F}_X^{p,q} \to \mathcal{F}_X^{p+1,q}, \\
\mathrm{D}'' &:= \delta : \mathcal{F}_X^{p,q} \to \mathcal{F}_X^{p,q+1} \quad \text{and} \\
\mathrm{D} &:= \mathrm{D}' + \mathrm{D}'' : \mathcal{F}_X^n \to \mathcal{F}_X^{n+1}.
\end{aligned}
\tag{2.1}
$$

Proof of Theorem 0.2. The left action of $\Omega^{\cdot}_{X/\Bbbk}$ on itself makes $\mathcal{F}_X^{\cdot}$ into a graded right $\Omega^{\cdot}_{X/\Bbbk}$-module. By Theorem 1.14, $\mathcal{F}_X^{\cdot}$ is a graded right $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$-module. The formula for the product $\phi \cdot a$ which makes the signs correct is

$$(\phi \cdot a)(\beta) := (-1)^{pq'}\phi(\beta) \cdot a$$

for local sections $\phi \in \mathcal{F}_X^{p,q}$, $a \in \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$ and $\beta \in \Omega^{-p}_{X/\Bbbk}$. Since $\mathcal{A}_X^{\cdot} \cong \Omega^{\cdot}_{X/\Bbbk} \otimes_{\mathcal{O}_X} \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ as graded algebras, we obtain a structure of graded right $\mathcal{A}_X^{\cdot}$-module on $\mathcal{F}_X^{\cdot}$.

It remains to check the coboundaries, which we break up into four steps, calculating $\mathrm{D}'(\phi \cdot a)$, $\mathrm{D}'(\phi \cdot \alpha)$, $\mathrm{D}''(\phi \cdot a)$ and $\mathrm{D}''(\phi \cdot \alpha)$ separately, with a, ϕ be as above, and $\alpha \in \Omega^{p'}_{X/\Bbbk}$.

Since $\mathrm{D}''(\alpha) = 0$ we get

$$
\begin{aligned}
\mathrm{D}''(\phi \cdot \alpha)(\beta) &= \delta((\phi \cdot \alpha)(\beta)) = \delta(\phi(\alpha \wedge \beta)) \\
&= \mathrm{D}''(\phi)(\alpha \wedge \beta) = (\mathrm{D}''(\phi) \cdot \alpha)(\beta) \\
&= (\mathrm{D}''(\phi) \cdot \alpha + (-1)^{p+q}\phi \cdot \mathrm{D}''(\alpha))(\beta)
\end{aligned}
$$

for every $\beta \in \Omega^{-p}_{X/\Bbbk}$.

Next, by Thm. 1.14

$$
\begin{aligned}
(\mathrm{D}''(\phi \cdot a))(\beta) &= \delta((\phi \cdot a)(\beta)) = \delta((-1)^{pq'}\phi(\beta) \cdot a) \\
&= (-1)^{pq'}\mathrm{D}''(\phi(\beta) \cdot a) \\
&= (-1)^{pq'}\left(\mathrm{D}''(\phi(\beta)) \cdot a + (-1)^{q}\phi(\beta) \cdot \mathrm{D}''(a)\right) \\
&= \left(\mathrm{D}''(\phi) \cdot a + (-1)^{p+q}\phi \cdot \mathrm{D}''(a)\right)(\beta).
\end{aligned}
$$

This takes care of D''.

As for D', by [Ye3] Proposition 5.2 we have

$$\mathrm{D}'(\phi \cdot \alpha) = \mathrm{D}'(\phi) \cdot \alpha + (-1)^{p+q}\phi \cdot \mathrm{D}'(\alpha). \tag{2.2}$$

Finally we will prove that

$$\mathrm{D}'(\phi \cdot a) = \mathrm{D}'(\phi) \cdot a + (-1)^{p+q}\phi \cdot \mathrm{D}'(a)$$

by induction on q'. As before we may assume

$$\phi = \phi_x \in \mathcal{H}om_{\mathcal{O}_X}(\Omega^{-p}_{X/\Bbbk}, \mathcal{K}_X(x)) \subset \mathcal{F}^{p,q}_X$$

for some point $x = x_0$, and

$$a = a_\xi \in \mathcal{O}_{X,\xi} \subset \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$$

for some saturated chain $\xi = (x_0, \ldots, x_{q'})$. Choose an integer n such that $\mathfrak{m}_x^n \cdot \phi_x = 0$. Then we also have $\mathfrak{m}_x^{n+1} \cdot \mathrm{D}'(\phi_x) = 0$.

For $q' = 0$ we have $a_{(x)} \in \mathcal{O}_{X,(x)}$. Choose some local section $b \in \mathcal{O}_X$ near x such that $b \equiv a_{(x)} \bmod \mathfrak{m}_x^{n+1}$. So by Lemma 1.11 one has $\phi \cdot a = \phi \cdot b$, $\mathrm{D}'(\phi) \cdot a = \mathrm{D}'(\phi) \cdot b$ and $\phi \cdot \mathrm{D}'(a) = \phi \cdot \mathrm{D}'(b)$. But by equation (2.2), with $\alpha = b \in \Omega^0_{X/\Bbbk}$, we know that

$$\mathrm{D}'(\phi \cdot b) = \mathrm{D}'(\phi) \cdot b + (-1)^{p+q} \phi \cdot \mathrm{D}'(b). \tag{2.3}$$

Now let us handle the case $q' = 1$, i.e. $\xi = (x_0, x_1)$. By Lemma 1.11 there exist adeles $b \in \mathcal{O}_{X,(x_0)}$ and $c \in \mathcal{O}_{X,(x_1)}$ such that

$$a_\xi = b_{(x_0,x_1)} \cdot c_{(x_1)} \in \mathcal{O}_{X,\xi} / \mathfrak{m}_{x_0}^{n+1} \mathcal{O}_{X,\xi},$$

where $b_{(x_0,x_1)}$ is the (x_0, x_1) component of $\partial^1(b)$. Let $d := 1_{(x_0,x_1)} \in \mathcal{O}_{X,(x_0,x_1)} \subset \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{O}_X)$. We then get $\partial^1(b) = b \cdot d$,

$$a - b \cdot d \cdot c \in \mathfrak{m}_{x_0}^{n+1} \mathcal{O}_{X,\xi},$$

$\phi \cdot a = \phi \cdot (b \cdot d \cdot c)$, $\mathrm{D}'(\phi) \cdot a = \mathrm{D}'(\phi) \cdot (b \cdot d \cdot c)$ and $\phi \cdot \mathrm{D}'(a) = \phi \cdot \mathrm{D}'(b \cdot d \cdot c)$. By the $q' = 0$ case we know that equation (2.3) holds, and also that

$$\mathrm{D}'((\phi \cdot b \cdot d) \cdot c) = \mathrm{D}'(\phi \cdot b \cdot d) \cdot c + (-1)^{p+q+1} (\phi \cdot b \cdot d) \cdot \mathrm{D}'(c).$$

It remains to verify that

$$\mathrm{D}'(\psi \cdot d) = \mathrm{D}'(\psi) \cdot d + (-1)^{p+q} \psi \cdot \mathrm{D}'(d),$$

where we define $\psi := \phi \cdot b$. But for every $\beta \in \Omega^{-p}_{X/\Bbbk}$ we have

$$(\psi \cdot d)(\beta) = (-1)^p \psi(\beta) \cdot d = (-1)^p \delta_{(x_0,x_1)}(\psi(\beta)) = (-1)^{p+q+1} \mathrm{D}''(\psi)(\beta).$$

Therefore $\psi \cdot d = (-1)^{p+q+1} \mathrm{D}''(\psi)$. Since $\mathrm{D}' \circ \mathrm{D}'' = -\mathrm{D}'' \circ \mathrm{D}'$ and $\mathrm{D}'(d) = 0$ we arrive at

$$\begin{aligned}
\mathrm{D}'(\psi \cdot d) &= (-1)^{p+q+1} (\mathrm{D}' \circ \mathrm{D}'')(\psi) = (-1)^{p+q} (\mathrm{D}'' \circ \mathrm{D}')(\psi) \\
&= (-1)^{p+q} \mathrm{D}''(\mathrm{D}'(\psi)) = (-1)^{(p+q)+(p+q+2)} \mathrm{D}'(\psi) \cdot d \\
&= \mathrm{D}'(\psi) \cdot d + (-1)^{p+q} \psi \cdot \mathrm{D}'(d).
\end{aligned}$$

To conclude we consider $q' \geq 2$. Since we are working locally, we can assume by the approximation trick (Lemma 1.11) that $a = b \cdot c$ with $b \in \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{O}_X)$ and $c \in \underline{\mathbb{A}}^{q'-1}_{\mathrm{red}}(\mathcal{O}_X)$. The induction hypothesis applies to b and c, and we have

$$\begin{aligned}
\mathrm{D}'(\phi \cdot a) &= \mathrm{D}'(\phi \cdot b \cdot c) \\
&= \mathrm{D}'(\phi \cdot b) \cdot c + (-1)^{p+q+1} \phi \cdot b \cdot \mathrm{D}'(c) \\
&= \mathrm{D}'(\phi) \cdot b \cdot c + (-1)^{p+q} \phi \cdot \mathrm{D}'(b) \cdot c + (-1)^{p+q+1} \phi \cdot b \cdot \mathrm{D}'(c) \\
&= \mathrm{D}'(\phi) \cdot b \cdot c + (-1)^{p+q} \phi \cdot \mathrm{D}'(b \cdot c) \\
&= \mathrm{D}'(\phi) \cdot a + (-1)^{p+q} \phi \cdot \mathrm{D}'(a).
\end{aligned}$$

□

Proof of Corollary 0.3. By [Ye3] Proposition 5.4 the trace homomorphisms $\mathrm{Tr} : \mathcal{K}^{\cdot}_{X_n} \to \mathcal{K}^{\cdot}_{X_{n+1}}$ give rise to DG $\Omega^{\cdot}_{X_{n+1}/\Bbbk}$-linear homomorphisms $\mathrm{Tr} : \mathcal{F}^{\cdot}_{X_n} \to \mathcal{F}^{\cdot}_{X_{n+1}}$. Therefore $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is a right DG $\mathcal{A}^{\cdot}_{\mathfrak{X}}$-module.

Let $\widehat{\Omega}^{\cdot}_{\mathfrak{X}/\Bbbk}$ be the formal completion of $\Omega^{\cdot}_{Y/\Bbbk}$ along X. According to [HY1] the DGA quasi-isomorphism $\widehat{\Omega}^{\cdot}_{\mathfrak{X}/\Bbbk} \to \mathcal{A}^{\cdot}_{\mathfrak{X}}$ is a flasque resolution, so

$$\mathrm{H}^i(X, \widehat{\Omega}^{\cdot}_{\mathfrak{X}/\Bbbk}) = \mathrm{H}^i\Gamma(X, \mathcal{A}^{\cdot}_{\mathfrak{X}}).$$

By [Ye3] Proposition 6.8 we get $\mathcal{F}^{\cdot}_{\mathfrak{X}} \cong \mathrm{R}\Gamma_X \Omega^{\cdot}_{Y/\Bbbk}[2m]$, where $m := \dim Y$, and hence

$$\mathrm{H}^{2m-i}_X(Y, \Omega^{\cdot}_{Y/\Bbbk}) = \mathrm{H}^{-i}\Gamma(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}),$$

compatible with the cap product. But when $\operatorname{char} \Bbbk = 0$ one has by definition $\mathrm{H}^i_{\mathrm{DR}}(X) := \mathrm{H}^i(X, \widehat{\Omega}^{\cdot}_{\mathfrak{X}/\Bbbk})$ and $\mathrm{H}_i^{\mathrm{DR}}(X) := \mathrm{H}^{2m-i}_X(Y, \Omega^{\cdot}_{Y/\Bbbk})$, see [Ha]. □

Remark 2.4. The role of the residue complex as a canonical flasque resolution of $\Omega^{\cdot}_{X/\Bbbk}$ (for X smooth) appears already in [Ha] and in [EZ]. In order to define the operator dual to d these authors use the fact that $\mathcal{F}^{\cdot}_X \cong \Omega^{\cdot}_{X/\Bbbk} \otimes_{\mathcal{O}_X} \mathcal{K}^{\cdot}_X$ is the Cousin resolution of $\Omega^{\cdot}_{X/\Bbbk}$ in the category of abelian sheaves on X. This should be compared to our approach (in [Ye3] and here) where the dual operator Dual(d) is defined in terms of BCAs, i.e. by algebraic-analytic methods.

Remark 2.5. Suppose X is smooth irreducible of dimension n. One can regard the sheaf $\mathcal{A}^{p,q}_X$ as an analog of the Dolbeault sheaf of smooth (p,q) forms on a complex manifold (cf. [GH] Ch. 0). The adelic resolution

$$0 \to \Omega^p_{X/\Bbbk} \to \mathcal{A}^{p,0}_X \xrightarrow{\mathrm{D}''} \mathcal{A}^{p,1}_X \to \cdots \to \mathcal{A}^{p,n}_X \to 0$$

corresponds to the $\bar{\partial}$ resolution on the manifold. Any section $\phi \in \Gamma(X, \mathcal{F}^{-p,-q}_X)$ determines a functional on $\Gamma(X, \mathcal{A}^{p,q}_X)$, namely $\alpha \mapsto \mathrm{Tr}_{X/\Bbbk}(\phi \cdot \alpha) \in \Bbbk$. Here

$$\mathrm{Tr}_{X/\Bbbk} : \Gamma(X, \mathcal{F}^{0,0}_X) = \Gamma(X, \mathcal{K}^0_X) \to \Bbbk$$

is the "sum of residues" (cf. [Ye3] Def. 1.16). In this way we can think of $\mathcal{F}_X^{-p,-q}$ as an analog of the sheaf of (p,q) currents on a manifold (cf. [GH] Ch. 3).

Remark 2.6. Suppose X is smooth irreducible and $\operatorname{char} \Bbbk = 0$. Let $Z \subset X$ be an irreducible closed subset of codimension d, and let

$$0 \to \mathcal{E}_l \to \cdots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$$

be a finite locally free resolution. Using the adelic Chern-Weil theory of [HY2] it is possible to construct a Chern character form $\operatorname{ch}(\mathcal{E};\nabla) \in \mathcal{A}_X^{\cdot}$, depending on adelic connections ∇_i on the $\mathcal{E}_i$, whose component $\operatorname{ch}(\mathcal{E};\nabla)_{2d} \in \mathcal{A}_X^{2d}$ satisfies

$$\mathrm{C}_X \cdot \operatorname{ch}(\mathcal{E};\nabla)_{2d} = \mathrm{C}_Z \in \mathcal{F}_X^{\cdot}.$$

Here $\mathrm{C}_X \in \mathcal{F}_X^{-2n}$ and $\mathrm{C}_Z \in \mathcal{F}_X^{-2(n-d)}$ are the fundamental classes. This generalizes [HY2] Theorem 6.5.

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