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Adeles and differential forms

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In algebraic number theory global fields and their rings of integers are studied by looking at the associated adele ring, a restricted direct product of the completions of the global fields at its – finite and infinite – primes. In this situation the underlying algebraic scheme is a curve. A. N. Parshin, in [Pa_1], [Pa_2], was able to generalize the definition of adeles to algebraic surfaces over a field and to develop a satisfactory theory for them. In his short note "residues and adeles" A. A. Beilinson introduced adeles for arbitrary noetherian schemes. Beilinson notes that adeles define in a canonical way for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ a flasque resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathbb{A}^0(\mathcal{F}) \xrightarrow{\partial^1} \mathbb{A}^1(\mathcal{F}) \xrightarrow{\partial^2} \cdots$ hence can be used to calculate the cohomology of $\mathcal{F}$ ([Be], see also [Hr_1], [Hr_2] for detailed proofs).

It turns out to be extremely useful to extend the definition of adeles to the category of quasi-coherent $\mathcal{O}_X$-modules with the morphisms being locally differential operators (instead of $\mathcal{O}_X$-linear morphisms). This allows to apply the adelic construction to the de Rham complex $\Omega^{\bullet}_X$ of a variety, thus obtaining a canonical and functorial resolution of $\Omega^{\bullet}_X$. It is very well suited to study the de Rham cohomology, and it has many formal similarities to the Dolbeault double complex $\mathfrak{A}^{\bullet,\bullet}_X$ studied in complex algebraic geometry. In fact in [HY] it will be used as an algebraic substitute for $\mathfrak{A}^{\bullet,\bullet}_X$ in our construction of Chern classes and our proof of the Bott residue formula.

Our main interest in adeles and locally differential operators arose however from an attempt to understand Beilinson's work [Be] on residues and adeles. Beilinson uses the Parshin-Lomadze theory of residues on higher-dimensional local fields (cf. [Pa_1], [Lo]) to obtain an adelic definition of the integral of Grothendieck duality theory for proper smooth morphisms over perfect fields. As the local factors of adeles and higher dimensional local fields and their residues play an important role in higher dimensional local class field theory ([Pa_3], [Kao]), it seems reasonable to assume that global adeles and their integrals

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will figure prominently in any higher dimensional global class field theory (though such a theory is not available yet). An extension of this theory to $\mathbb{Z}$-varieties should be of considerable interest also for arithmetic schemes (Soule, in the introduction to [SABK], suggests to try an adelic approach to a dynamical version of arithmetic algebraic geometry).

Though having many functorial advantages, the global traces defined via adeles are often hard to calculate in concrete situations. Much easier to handle are the traces defined via generalized fractions ([Li], [HK]) and the approach developed by the second author in [Ye_1]. In [$\mathrm{Hü}_1$], [SY] it was shown that – at least up to a sign depending on $\dim(X)$ only – these two traces agree, and in this note we will complete this program by examining the relations to Beilinson's construction. In particular we are able to extend Beilinson's results form the case of smooth k-varieties to reduced k-varieties (using regular differential forms in the singular case instead of holomorphic ones), and we show that – again up to a sign – the trace obtained this way is the trace of [Li] resp. [Ye_1]. Hence all three construction lead essentially to the same result.

We are very grateful to A. N. Parshin for explaining [Be], theorem to us.

§ 1. The Parshin-Beilinson adeles

In this section we will extend the definition of higher dimensional adeles (cf. [Be], [Hr_1], [Hr_2]) to the category of quasi-coherent sheaves with the morphisms being locally differential operators. Many facts about the local theory are contained in [Ye_1], some of which we will generalize to the global case. Furthermore we will prove some new results about local and global adeles which will be useful in this paper and also in [HY].

Throughout this section X will be a noetherian scheme, and all sheaves will be quasi-coherent $\mathcal{O}_X$-modules.

Let $P(X)$ be the set of points of X, and for $x \in P(X)$ denote by $\overline{\{x\}}$ the closure of $\{x\}$ in X. For $m \in \mathbb{N}$ set

$$S(X)^m := \{\xi = (x_0, x_1, \dots, x_m) : x_i \in P(X), x_{i+1} \in \overline{\{x_i\}}\},$$

$$S(X)^m_{\mathrm{red}} := \{\xi = (x_0, x_1, \dots, x_m) : \xi \in S(X)^m \text{ and } x_i \neq x_{i+1}\}$$

and for $K \subseteq S(X)^m$ and $x \in P(X)$ set

$$\hat{x}K = \{\xi = (y_1, \dots, y_m) \in S(X)^{m-1} : (x, y_1, \dots, y_m) \in K\} \subseteq S(X)^{m-1}.$$

An element $\xi \in K$ is called an m-chain. If $\xi = \{y_0, \dots, y_m)$ then ξ is called saturated if $\mathrm{codim}_{\overline{y_i}}(\overline{y_{i-1}}) = 1$ for all $i = 1, \dots, m$. For a chain $\xi = (x_0, \dots, x_n)$, and for $0 \leqq i \leqq n$ set $d_i^n \xi := (x_0, \dots, \hat{x}_i, \dots, x_n)$. Recall

1.1. Proposition/Definition ([Hr_1], (2.4.1), [Hr_2], (2.1.1)). *For each $n \in \mathbb{N}$ and each $K \subseteq S(X)^n$ there exists an additive exact functor*

$$\mathbb{A}(K, \_) : Qco(X) \to (ab)$$

satisfying and being uniquely determined by:

(1) $\mathbb{A}(K, \_)$ *commutes with direct limits.*

(2) *For* $n = 0$ *and* $\mathscr{F}$ *coherent* $\mathbb{A}(K, \mathscr{F}) = \prod_{x \in K} \varprojlim_{l \in \mathbb{N}} \mathscr{F}_x / \mathfrak{m}_x^l \mathscr{F}_x$.

(3) *For* $n > 0$ *and* $\mathscr{F}$ *coherent* $\mathbb{A}(K, \mathscr{F}) = \prod_{x \in P(X)} \varprojlim_{l \in \mathbb{N}} \mathbb{A}({}_xK, [\mathscr{F}_x / \mathfrak{m}_x^l \mathscr{F}_x]_x)$

where for each $\mathcal{O}_{X,x}$*-module* M *we denote by* $[M]_x$ *the associated skyscraper sheaf on* X *with support in* $\overline{\{x\}}$.

1.2. Remark. (i) If $K = \{\xi\}$, then $\mathbb{A}(K, \mathscr{F}) = \mathscr{F}_\xi$ is the Beilinson completion of $\mathscr{F}$ along ξ as defined in [Ye$_1$], (3.2). In this case $\mathcal{O}_{X,\xi}$ is a semi-topological k-algebra (in the sense of [Ye$_1$], (1.2)), and for a quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ the completion $\mathscr{F}_\xi$ is a semi-topological $\mathcal{O}_{X,\xi}$-module with the fine $\mathcal{O}_{X,\xi}$-module topology. Note that for $\xi = (x_0)$ and $\mathscr{F}$ coherent $\mathscr{F}_\xi = \widehat{\mathscr{F}_{x_0}}$ is the completion of $\mathscr{F}_{x_0}$ with respect to the $\mathfrak{m}_{x_0}$-adic topology.

(ii) By [Hr$_2$], (2.1.4) we have a functorial inclusion

$$\mathbb{A}(K, \mathscr{F}) \hookrightarrow \prod_{\xi \in K} \mathscr{F}_\xi$$

which is an isomorphism in case $K \subseteqq S(X)^0$ and $\mathscr{F}$ is coherent. Via this inclusion the k-algebra structure on the local factors $\mathcal{O}_{X,\xi}$ induces a k-algebra structure on $\mathbb{A}(K, \mathcal{O}_X)$ by [Hr$_1$], (3.3.7). Thus in analogy to the classical case of adeles in number theory, the global groups may be viewed as restricted direct product of their local factors (see also [Pa$_2$], § 2). In particular this means that all the explicit calculations can be done in the local factors.

(iii) If $X = \mathrm{Spec}(R)$ is affine we will not distinguish between quasicoherent $\mathcal{O}_X$-modules and associated R-modules. In particular we frequently will write $\mathbb{A}(K, R)$ instead of $\mathbb{A}(K, \mathcal{O}_X)$ etc.

1.3. Remark. If X is a noetherian scheme and if $\mathscr{F}$ is a quasi-coherent $\mathcal{O}_X$-module we write $\mathbb{A}^n(X, \mathscr{F}) := \mathbb{A}(S(X)^n, \mathscr{F})$ and $\mathbb{A}^n_{\mathrm{red}}(X, \mathscr{F}) := \mathbb{A}(S(X)^n_{\mathrm{red}}, \mathscr{F})$. Then

$$U \mapsto \mathbb{A}^n(U, \mathscr{F} \,|\, U) \quad \text{resp.} \quad U \mapsto \mathbb{A}^n_{\mathrm{red}}(U, \mathscr{F}) \quad (U \subseteqq X \text{ open})$$

define flasque sheaves $\underline{\mathbb{A}}^n(\mathscr{F})$ resp. $\underline{\mathbb{A}}^n_{\mathrm{red}}(\mathscr{F})$ on X ([Hr$_2$], (4.2.2), [Be]).

Let $d_i^n : S(X)^n \to S(X)^{n-1}$, $(x_0, \ldots, x_n) \mapsto (x_0, \ldots, \widehat{x_i}, \ldots, x_n)$ be the canonical boundary maps of the simplicial set $\{S(X)^n\}_{n \in \mathbb{N}}$. Then the d_i^n induce for each quasi-coherent $\mathscr{F}$ well defined morphisms

$$\partial_i^n : \underline{\mathbb{A}}^{n-1}(\mathscr{F}) \to \underline{\mathbb{A}}^n(\mathscr{F})$$

by [Hr$_1$], (2.5.3), (4.3.2). Setting $\partial^n := \sum_{j=0}^{n} (-1)^j \partial_j^n$ makes $(\underline{\mathbb{A}}^\bullet(\mathscr{F}), \partial^\bullet)$ a complex of sheaves of abelian groups, and we have

1.4. Theorem ([Be], Cor, [Hr_2], (4.2.3)). *$(\underline{\mathbb{A}}^{\bullet}(\mathscr{F}), \partial^{\bullet})$ and $(\underline{\mathbb{A}}^{\bullet}_{\mathrm{red}}(\mathscr{F}), \partial^{\bullet})$ are flasque resolutions of $\mathscr{F}$.*

Before dealing with the adelization of locally differential operators we will prove some more facts about global and local adeles that will be useful later as well.

Let $X = \operatorname{Spec}(R)$ be an affine noetherian scheme, let M be an R-module, $\mathscr{M} = \tilde{M}$, and let $K \subseteqq S(X)^n$ be a subset. For each $\xi \in K$ the canonical map $M \to \mathscr{M}_\xi$ induces a functorial isomorphism on the local factors

$$\mathcal{O}_{X,\xi} \otimes_R M \to \mathscr{M}_\xi$$

by [Hr_2], (3.2.1). Similarly the diagonal homomorphism $M \to \mathbb{A}(K, \mathscr{M})$ induces an $\mathbb{A}(K, \mathcal{O}_X)$-linear morphism

$$\alpha_{\mathscr{M}} : \mathbb{A}(K, \mathcal{O}_X) \otimes_R M \to \mathbb{A}(K, \mathscr{M})$$

and we get a commutative diagram

$$\begin{array}{ccc}
\prod\limits_{\xi \in K} \mathcal{O}_{X,\xi} \otimes_R M & \xrightarrow{\simeq} & \prod\limits_{\xi \in K} \mathscr{M}_\xi \\
\uparrow & & \\
(\prod\limits_{\xi \in K} \mathcal{O}_{X,\xi}) \otimes_R M & & \uparrow \\
\uparrow & & \\
\mathbb{A}(K, \mathcal{O}_X) \otimes_R M & \xrightarrow{\alpha_{\mathscr{M}}} & \mathbb{A}(K, \mathscr{M}) .
\end{array}$$

1.5. Lemma. *$\alpha_{\mathscr{M}}$ is an isomorphism.*

Proof. First assume that M is finitely generated. Then the proposition is obvious if $M = R^e$ as $\mathbb{A}(K, \_)$ is additive and as $\mathbb{A}(K, \_) \to \prod_{\xi \in K} (\_)_\xi$ is a transformation of additive functors. In the general case choose a presentation

$$R^l \xrightarrow{\psi} R^k \xrightarrow{\varphi} M \longrightarrow 0 .$$

Then the claim follows easily in this case from the (right-)exactness of $\mathbb{A}(K, \_)$ and the five-lemma.

If $\mathscr{M}$ is quasi-coherent we obtain that $\alpha_{\mathscr{M}}$ is an isomorphism by a direct limit argument.

1.6. Proposition ([Be], Lemma). *For any quasi-coherent sheaf $\mathscr{F}$ on X and any $p \in \mathbb{N}$ the canonical map*

$$\alpha_{\mathscr{F}} : \underline{\mathbb{A}}^p_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathscr{F} \to \underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{F})$$

is an isomorphism. The same statement holds true for $\underline{\mathbb{A}}^p(\_)$.

Proof. First note that 1.6 is not an immediate consequence of 1.5 as $\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{F})$ will not be quasi-coherent in general. However in case $X = \operatorname{Spec}(R)$ is affine and $\mathscr{F} = \tilde{M}$ we have for any quasi-coherent $\mathcal{O}_X$-module that the canonical map

$$\mathbb{A}^{\bullet}_{\mathrm{red}}(X, \mathcal{O}_X) \otimes_R \mathcal{F}(X) \to (\underline{\mathbb{A}}^{\bullet}_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})(X)$$

is an isomorphism. (Again this is obvious in case that M is a finitely generated and free R-module, and from this it follows in general by the five lemma and direct limit arguments.)

Now let X be arbitrary, let $U \subseteqq X$ be an open subset and let $\{V_i\}_{i \in I}$ be an open affine cover of U, $V_i = \mathrm{Spec}(R_i)$. Then for each $i \in I$ we have canonical isomorphisms

$$(\underline{\mathbb{A}}^{\bullet}_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})(V_i) \to \mathbb{A}^{\bullet}_{\mathrm{red}}(V_i, \mathcal{O}_{V_i}) \otimes_{R_i} \mathcal{F}(V_i) \to \mathbb{A}^{\bullet}_{\mathrm{red}}(V_i, \mathcal{F})$$

which are obviously compatible with each other (as they are induced by canonical morphisms on the local factors), hence they glue to give a global isomorphism

$$(\underline{\mathbb{A}}^{\bullet}_{\mathrm{red}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})(U) \to \mathbb{A}^{\bullet}_{\mathrm{red}}(U, \mathcal{F}).$$

As $U \subseteqq X$ was arbitrary, the claim follows.

1.7. Corollary. *If $X = \mathrm{Spec}(R)$ is affine, then $\mathbb{A}(K, \mathcal{O}_X)$ is flat as an R-algebra. Hence, for a general X, $\underline{\mathbb{A}}^{\bullet}(\mathcal{O}_X)$ and $\underline{\mathbb{A}}^{\bullet}_{\mathrm{red}}(\mathcal{O}_X)$ are sheaves of flat $\mathcal{O}_X$-algebras.*

Proof. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R-modules and set $\mathcal{M}_i := \tilde{M}_i$ for $i = 1, 2, 3$. From the exactness of $\mathbb{A}(K, \_)$ we deduce a commutative diagram with exact top row

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{A}(K, \mathcal{M}_1) & \to & \mathbb{A}(K, \mathcal{M}_2) & \to & \mathbb{A}(K, \mathcal{M}_3) & \to & 0 \\ & & \wr\uparrow \alpha_{\mathcal{M}_1} & & \wr\uparrow \alpha_{\mathcal{M}_2} & & \wr\uparrow \alpha_{\mathcal{M}_3} & & \\ 0 & \to & \mathbb{A}(K, \mathcal{O}_X) \otimes_R M_1 & \to & \mathbb{A}(K, \mathcal{O}_X) \otimes_R M_2 & \to & \mathbb{A}(K, \mathcal{O}_X) \otimes_R M_3 & \to & 0 \end{array}$$

with the isomorphisms $\alpha_{\mathcal{M}_i}$ of 1.5. Thus the bottom row is exact as well.

As was pointed out in the introduction, it will be crucial to extend the definition of adeles to the category of quasi-coherent $\mathcal{O}_X$-modules and locally differential operators in the sense of [Ye$_1$], (3.1.8). The most interesting example (at least for us) of a locally differential operator is given by the residue map:

1.8. Remark. Let k be a perfect field and let X/k be a scheme of finite type. Suppose $\xi = (x, \ldots, y)$ is a saturated chain in X and $\sigma : k(y) \to \hat{\mathcal{O}}_{X,y}$ is a coefficient field for $\hat{\mathcal{O}}_{X,y}$. In this case a residue map

$$\mathrm{Res}_{\xi,\sigma} : \Omega^{\bullet}_{k(x)/k} \to \Omega^{\bullet}_{k(y)/k}$$

(depending on ξ and σ) is defined ([Lo], p. 516, [Ye$_1$], (4.1.3)) and it is a locally differential operator over $\mathcal{O}_X$ relative k by [Ye$_1$], (4.1.4).

It was shown in [Ye$_1$], (3.1.10) that differential operators respectively locally differential operators behave well with respect to completion along a single chain ξ. This result generalizes as follows:

1.9. Proposition. *Let $K \subseteq S(X)_n$, let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent $\mathscr{O}_X$-modules and suppose $D: \mathscr{M} \to \mathscr{N}$ is a locally differential operator (resp. a differential operator of order $\leqq d$). Then D can be extended to a locally differential operator (resp. a differential operator of order $\leqq d$)*

$$\mathbb{A}(K, D): \mathbb{A}(K, \mathscr{M}) \to \mathbb{A}(K, \mathscr{N})$$

such that for each $\xi \in K$ the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}(K, \mathscr{M}) & \xrightarrow[\mathbb{A}(K,D)]{} & \mathbb{A}(K, \mathscr{N}) \\ \downarrow \text{can} & & \text{can} \downarrow \\ \mathscr{M}_\xi & \xrightarrow{D_\xi} & \mathscr{N}_\xi \end{array}$$

where D_ξ is the locally differential operator of [Ye$_1$], (3.1.10).

Proof (cf. [Ye$_1$], (3.1.10)). By induction on n:

For $n = -1$ there is nothing to show, so assume $n \geqq 0$ and let $\mathscr{M}$ be coherent. By [Ye$_1$], (3.1.9) we may assume that $\mathscr{N}$ is coherent as well. Let d be the order of D. Then we have for each $x \in X$ that $D(\mathfrak{m}_x^{j+d+1}\mathscr{M}_x) \subseteq \mathfrak{m}_x^{j+1}\mathscr{N}_x$, hence we get well defined differential operators (of order $\leqq d$)

$$D_{j,x}: \mathscr{M}_x/\mathfrak{m}_x^{d+j+1}\mathscr{M}_x \to \mathscr{N}_x/\mathfrak{m}_x^{j+1}\mathscr{N}_x$$

which induce by induction differential operators

$$\mathbb{A}({}_xK, D_x): \varprojlim_{j \in \mathbb{N}} \mathbb{A}({}_xK, \mathscr{M}_x/\mathfrak{m}_x^{d+j+1}\mathscr{M}_x) \to \varprojlim_{j \in \mathbb{N}} \mathbb{A}({}_xK, \mathscr{N}_x/\mathfrak{m}_x^{j+1}\mathscr{N}_x)$$

compatible with the maps D_ξ. Taking direct products we obtain

$$\begin{array}{ccc} \mathbb{A}(K, D): \prod_{x \in X} \varprojlim_{j \in \mathbb{N}} \mathbb{A}({}_xK, \mathscr{M}_x/\mathfrak{m}_x^{d+j+1}\mathscr{M}_x) & \to & \prod_{x \in X} \varprojlim_{j \in \mathbb{N}} \mathbb{A}({}_xK, \mathscr{N}_x/\mathfrak{m}_x^{j+1}\mathscr{N}_x) \\ \| & & \| \\ \mathbb{A}(K, \mathscr{M}) & & \mathbb{A}(K, \mathscr{N}) \end{array}$$

and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{A}(K, \mathscr{M}) & \xrightarrow[\mathbb{A}(K,D)]{} & \mathbb{A}(K, \mathscr{N}) \\ \downarrow & & \downarrow \\ \prod_{\xi \in K} \mathscr{M}_\xi & \xrightarrow{\prod D_\xi} & \prod_{\xi \in K} \mathscr{N}_\xi . \end{array}$$

As $\prod_{\xi \in K} D_\xi$ is a differential operator of order $\leqq d$ over $\prod_{\xi \in K} \mathscr{O}_{X,\xi}$, we conclude that $\mathbb{A}(K, D)$ is a differential operator of order $\leqq d$ over $\mathbb{A}(K, \mathscr{O}_X)$.

If $\mathscr{M}$ is a quasi-coherent sheaf we obtain $\mathbb{A}(K, D)$ by a direct limit argument.

The topological structure on Beilinson completions allows to study continuous morphisms on dense subsets. Of particular interest in this context is the following

1.10. Proposition (Approximation Theorem). *Let $\xi = (x, \dots, y, \dots, z)$ be a (not necessarily saturated) chain in X, let $\eta = (x, \dots, y)$ and let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then the face map $\partial : \mathcal{M}_\eta \to \mathcal{M}_\xi$ has dense image.*

1.11. Remark. If ξ is a saturated chain and $\mathcal{O}_{X,\zeta}$ is a Zariski semi-topological ring for all saturated chains ζ of length at most 1, then 1.10 follows from [Ye$_1$], (3.2.12).

Proof. We may assume that $X = \operatorname{Spec}(R)$ is affine. Clearly we also may assume that $\eta = (x) = (\mathfrak{p})$ for some prime $\mathfrak{p} \in \operatorname{Spec}(R)$. First suppose that $\xi = (\mathfrak{p}, \mathfrak{q})$ and $\mathcal{M}_x = k(x)$, hence $\mathcal{M}_{(x)} = k(x)$ (with the discrete topology). Then by [Hr$_1$], (3.2.1) we have

$$\begin{aligned}\mathcal{M}_\xi &= k(x)_{(\mathfrak{p},\mathfrak{q})} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R ((\hat{R}_\mathfrak{q})_\mathfrak{p})\hat{} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R \hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q} \\ &= \varinjlim_{f \notin \mathfrak{p}} (\hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q}) \frac{1}{f}.\end{aligned}$$

Recall the construction of the topology on $\mathcal{M}_\xi$:

$\mathcal{M}_{(x)} = k(x)$ comes equipped with the discrete topology.

$$k(\mathfrak{p}) = \varinjlim_{f \notin \mathfrak{p}} R/\mathfrak{p} \cdot \frac{1}{f}.$$

$$\left(R/\mathfrak{p} \cdot \frac{1}{f}\right)_\xi = \varprojlim_{l \in \mathbb{N}} \left(R/\mathfrak{p} \cdot \frac{1}{f}\right)_\mathfrak{q} \Big/ \mathfrak{q}^l \left(R/\mathfrak{p} \cdot \frac{1}{f}\right)_\mathfrak{q} = (\hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q}) \cdot \frac{1}{f} \quad (\text{inside } (\widehat{R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}})_\mathfrak{p})$$

carries the $\mathfrak{q}$-adic topology, and finally

$$(k(\mathfrak{p}))_\xi = \varinjlim_{f \notin \mathfrak{p}} \left(R/\mathfrak{p} \cdot \frac{1}{f}\right)_\xi = \varinjlim_{f \notin \mathfrak{p}} \hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q} \cdot \frac{1}{f}$$

carries the direct limit topology. Clearly the face map

$$R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q} \cdot \frac{1}{f} \to \hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q} \cdot \frac{1}{f}$$

has dense image. Passing to the direct limit we obtain that also

$$\partial : k(x) \to (\hat{R}_\mathfrak{q}/\mathfrak{p}\hat{R}_\mathfrak{q})_\mathfrak{p} = \mathcal{M}_\xi$$

has dense image ([Ye$_1$], (1.1.8) c)). Starting with this the points (2) (3) and (4) of the proof of [Ye$_1$], (3.2.11) carry over immediately to the generalized situation to give the desired result.

§ 2. Adeles and de Rham cohomology

The adelic construction gives a canonical flasque resolution of a sheaf $\mathscr{F}$ on a noetherian scheme X. Using the results of § 1 we can extend this result to complexes of quasi-coherent $\mathcal{O}_X$-modules and locally differential operators. Applied to de Rham cohomology we obtain an explicit description of the de Rham cohomology and its cup product. The case of smooth morphisms (in characteristic 0) is particularly easy. However it is also the most interesting one, and it may serve as a motivation for the way we proceed in the singular case. Thus we will spend some time on it. We note that the adelic approach is completely functorial and canonical, however it also has many of the nice features of the (non-canonical) Čech resolutions, a fact we will illustrate by giving an explicit description of the Gauß-Manin connection (cf. [Kaz]) and a construction of the cohomology class of a divisor.

First assume that we have a complex of quasi-coherent $\mathcal{O}_X$-modules, $\mathscr{M}^p$,

$$\mathscr{M}^\bullet : 0 \longrightarrow \mathscr{M}^0 \xrightarrow{d^0} \mathscr{M}^1 \xrightarrow{d^1} \mathscr{M}^2 \xrightarrow{d^2} \cdots$$

with the d^p being locally differential operators. By 1.4 we have flasque resolutions

$$0 \longrightarrow \mathscr{M}^p \longrightarrow \underline{\mathbb{A}}^0_{\mathrm{red}}(\mathscr{M}^p) \xrightarrow{\partial} \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathscr{M}^p) \xrightarrow{\partial} \underline{\mathbb{A}}^2_{\mathrm{red}}(\mathscr{M}^p) \longrightarrow \cdots.$$

Setting $\mathscr{A}^{p,q}(\mathscr{M}^\bullet) := \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathscr{M}^p)$ we therefore obtain a double complex with exact rows:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathscr{M}^2 & \longrightarrow & \mathscr{A}^{2,0}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \mathscr{A}^{2,1}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \mathscr{A}^{2,2}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \cdots \\
 & & \uparrow d & & \uparrow \underline{\mathbb{A}}^0_{\mathrm{red}}(d) & & \uparrow \underline{\mathbb{A}}^1_{\mathrm{red}}(d) & & \uparrow \underline{\mathbb{A}}^2_{\mathrm{red}}(d) & & \\
0 & \longrightarrow & \mathscr{M}^1 & \longrightarrow & \mathscr{A}^{1,0}(\mathscr{M}^\bullet) & \xrightarrow{-\partial} & \mathscr{A}^{1,1}(\mathscr{M}^\bullet) & \xrightarrow{-\partial} & \mathscr{A}^{1,2}(\mathscr{M}^\bullet) & \xrightarrow{-\partial} & \cdots \\
 & & \uparrow d & & \uparrow \underline{\mathbb{A}}^0_{\mathrm{red}}(d) & & \uparrow \underline{\mathbb{A}}^1_{\mathrm{red}}(d) & & \uparrow \underline{\mathbb{A}}^2_{\mathrm{red}}(d) & & \\
0 & \longrightarrow & \mathscr{M}^0 & \longrightarrow & \mathscr{A}^{0,0}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \mathscr{A}^{0,1}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \mathscr{A}^{0,2}(\mathscr{M}^\bullet) & \xrightarrow{\partial} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & 0. & &
\end{array}$$

Thus the associated total complex is acyclic, hence defines a quasi-isomorphism

$$\mathscr{M}^\bullet \to \mathrm{Tot}(\mathscr{A}^{\bullet,\bullet}(\mathscr{M}^\bullet), \underline{\mathbb{A}}^\bullet_{\mathrm{red}}(d), \pm\partial) =: (\mathscr{A}^\bullet(\mathscr{M}^\bullet), D^\bullet)$$

i.e. $\mathscr{A}^\bullet(\mathscr{M}^\bullet)$ is a flasque resolution of $\mathscr{M}^\bullet$, and we conclude for its hypercohomology

2.1. Proposition. $\mathbb{H}^\bullet(X, \mathscr{M}^\bullet) \cong H^\bullet(\Gamma(X, \mathscr{A}^\bullet(\mathscr{M}^\bullet)))$.

Proposition 2.1 can be applied directly to a smooth morphism $f: X \to Y$ of noetherian schemes, its relative de Rham complex $(\Omega^\bullet_{X/Y}, d_{X/Y})$ and its hypercohomology, the relative de Rham cohomology of X/Y. For each subset $I = \{i_0, \ldots, i_p\}$ of $\{0, \ldots, n\}$ the canonical projection

$$\mathrm{pr}_I : S(X)^n_{\mathrm{red}} \to S(X)^p_{\mathrm{red}}, \quad (x_0, \ldots, x_n) \mapsto (x_{i_0}, \ldots, x_{i_p})$$

defines for each $\mathcal{O}_X$-module $\mathcal{F}$ a functorial morphism

$$\partial^{(0,\ldots,n)}_{(i_0,\ldots,i_p)} : \underline{\mathbb{A}}^p_{\mathrm{red}}(\mathcal{F}) \to \underline{\mathbb{A}}^n_{\mathrm{red}}(\mathcal{F})$$

which is induced by the obvious morphism $\mathcal{O}_{X,(x_{i_0},\ldots,x_{i_p})} \to \mathcal{O}_{X,(x_0,\ldots,x_n)}$ on the local factors. Crucial for [HY] will be

2.2. Proposition. *In the above situation the relative de Rham cohomology $H^\bullet_{\mathrm{DR}}(X/Y)$ is the sheaf associated to the presheaf*

$$U \mapsto H^\bullet(\Gamma(f^{-1}(U), \mathcal{A}^\bullet(\Omega^\bullet_{X/Y}))) \quad (U \subseteq Y \text{ open})$$

on Y. In particular in case Y is affine

$$H^\bullet_{\mathrm{DR}}(X/Y) \cong H^\bullet(\Gamma(X, \mathcal{A}^\bullet(\Omega^\bullet_{X/Y})))^\sim$$

is the quasi-coherent $\mathcal{O}_Y$-module associated to $H^\bullet(\Gamma(X, \mathcal{A}^\bullet(\Omega^\bullet_{X/Y})))$. The cup product in $H^\bullet_{\mathrm{DR}}(X/Y)$ is induced by the Alexander-Whitney product on $\mathcal{A}^\bullet(\Omega^\bullet_{X/Y})$ which can be described explicitly as follows:

Writing $\mathcal{A}^{p,q}(\Omega^\bullet_{X/Y}) = \Omega^p_{X/Y} \otimes_{\mathcal{O}_X} \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$ and taking local sections $a \in \underline{\mathbb{A}}^q_{\mathrm{red}}(\mathcal{O}_X)$, $b \in \underline{\mathbb{A}}^{q'}_{\mathrm{red}}(\mathcal{O}_X)$, $\omega \in \Omega^p_{X/Y}$ and $\eta \in \Omega^{p'}_{X/Y}$, the product is given by

$$(\omega \otimes a) \cdot (\eta \otimes b) = (-1)^{pq'} \omega \wedge \eta \otimes \partial^{(0,\ldots,q+q')}_{(0,\ldots,q)}(a) \cdot \partial^{(0,\ldots,q+q')}_{(q,\ldots,q+q')}(b)$$

where the product on the right hand side is the obvious product of $\underline{\mathbb{A}}^{q+q'}_{\mathrm{red}}(\mathcal{O}_X)$.

2.3. Remark. In case $Y = \mathrm{Spec}(k)$, k a field of characteristic 0, Parshin has given a similar construction ([Pa$_2$], (1.2)). He uses however a different sign convention.

The proposition is clear by the cosimplicial structure of the adelic resolution. Let us however note that it easily can be seen directly as well. The formula given in 2.2 is easily seen to define a product on $\mathcal{A}^\bullet(\Omega^\bullet_{X/Y})$ such that

$$D(a \cdot b) = D(a) \cdot b + (-1)^n a \cdot D(b)$$

for local sections $a \in \mathcal{A}^n(\Omega^\bullet_{X/Y})$, $b \in \mathcal{A}^\bullet(\Omega^\bullet_{X/Y})$. From [God], II, thm. (6.6.1) it follows that it induces the cup product in de Rham cohomology.

2.4. Remark. The adelic construction provides additional insight into the de Rham cohomology, in particular its functorial behaviour (cf. [Ye$_3$]): Given a morphism $g : X \to Z$ of (smooth) Y-schemes, the obvious morphisms $\Omega^\bullet_{Z/Y,g(x)} \to \Omega^\bullet_{X/Y,x}$ $(x \in X)$ induce a canonical map $g^* : \mathcal{A}^\bullet(\Omega^\bullet_{Z/Y}) \to g_* \mathcal{A}^\bullet(\Omega^\bullet_{X/Y})$ which is compatible with the product defined above and which in cohomology induces the functorial morphism

$$g^* : H^\bullet_{\mathrm{DR}}(Z/Y) \to H^\bullet_{\mathrm{DR}}(X/Y).$$

For the next few remarks let $Y = \mathrm{Spec}(k)$ for some perfect field k. When studying cycles on a variety X the de Rham homology $H_{\bullet}^{\mathrm{DR}}(X/Y)$ together with the cap product on it is of particular interest (cf. [EZ], (2.3), [Har$_1$], II, (7.6)). In case X/Y is smooth the de Rham homology can be calculated as the homology of the total complex associated to the de Rham-adele complex $\mathscr{F}_X^{\bullet,\bullet}$ of El Zein [EZ] (cf. [Ye$_3$]). Recall that $\mathscr{F}_X^{p,q} := \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/Y}^{-p}, \mathscr{K}_X^q)$, where $(\mathscr{K}_X^{\bullet}, \delta_X^{\bullet})$ is a residual complex (which we always assume to be given as in [Ye$_1$], § 4; see also § 3 below), and where the boundary operators of $\mathscr{F}_X^{\bullet,\bullet}$ are given by

$$d := (-1)^{p+1} d^{\vee} : \mathscr{F}_X^{p,q} \to \mathscr{F}_X^{p+1,q}, \quad \delta := (-1)^{p+q+1} \bar{\delta} : \mathscr{F}_X^{p,q} \to \mathscr{F}_X^{p,q+1}$$

where $d^{\vee} = \mathrm{dual}(d_{X/Y})$ is the differential operator dual to $d_{X/Y}$ and where $\bar{\delta}$ is induced by $\delta_X^{\bullet}$ (cf. [EZ], III, [Ye$_3$]). Note however that Yekutieli's $\mathscr{F}_X^{\bullet,\bullet}$ differs from El Zein's by a shift in indices and signs. To make the homology classes of cycles more lucid it will be of advantage to find explicit representatives of these classes and their cap products. To this end we have

2.5. Proposition. *Let X/Y be an algebraic scheme. Then the de Rham-residue complex $\mathscr{F}_X^{\bullet,\bullet}$ comes equipped with a canonical $\mathscr{A}^{\bullet}(\Omega_{X/Y}^{\bullet})$-module structure which in terms of local sections may be described as follows: Write*

$$\mathscr{K}_X^{-q} = \bigoplus_{\dim(\overline{\{x\}}) = q} \mathscr{K}(x) \quad \text{with} \quad \mathscr{K}(x) \cong \mathrm{Hom}_{k(x)}^{\mathrm{cont}}(\mathscr{O}_{X,(x)}, \Omega_{k(x)/k}^q).$$

For a chain $\xi = (z_0, \ldots, z_{q'}) \in S(X)_{q'}^{\mathrm{red}}$, $a \in \mathscr{O}_{X,\xi}$, a local section $\omega \in \Omega_{X/Y}^{p'}$ and

$$\Phi \in \mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/Y}^p, \mathscr{K}(x))$$

we define the product $\Phi \cdot (\omega \otimes a)$ in $\mathscr{H}om_{\mathscr{O}_X}(\Omega_{X/Y}^{-p'+p}, \mathscr{K}(z_{q'}))$ by the formula

$$\Phi \cdot (\omega \otimes a)(\eta) = \begin{cases} (-1)^{qp'} \mathrm{Res}_{\xi}(\Phi(\omega \cdot \eta) \cdot a) & \text{if } z_0 = x \text{ and } \xi \text{ saturated}, \\ 0 & \text{otherwise}. \end{cases}$$

In case X is smooth (so that $\mathrm{Tot}(\mathscr{F}_X^{\bullet,\bullet})$ can be used to calculate the de Rham homology of X), the above product induces the cap product

$$H_{\bullet}^{\mathrm{DR}}(X/Y) \times H_{\mathrm{DR}}^{\bullet}(X/Y) \to H_{\bullet}^{\mathrm{DR}}(X/Y)$$

on the de Rham homology (cf. [Ye$_3$] and [HY] for more details and applications).

Proof. It may be checked directly that the above formula induces a right $\mathscr{A}^{\bullet}(\Omega_{X/Y}^{\bullet})$-module structure on $\mathrm{Tot}(\mathscr{F}_X^{\bullet,\bullet})$. As a complete account of these results (including a generalization to the singular case) can be found in [Ye$_3$], we omit details here. For applications we refer to [HY], § 4.

In [HY] we will construct Chern classes of vector bundles. In view of the splitting principle it will be crucial to understand in particular the case of line bundles and their cohomology classes. So suppose that $Y = \mathrm{Spec}(k)$ for some field k with $\mathrm{char}(k) = 0$ and that X/Y is smooth. Let $A \in Z^r(X)$ be a cycle of dimension r on X. Then we can associate to it a cohomology class $\eta(A) \in H_{\mathrm{DR}}^{2d-2r}(X/k)$, which only depends on the rational equivalence class of Y (cf. [Har_1], II, (7.8)). (Via the identification $H_{\mathrm{DR}}^{2d-2r}(X/Y) \cong H_{2r}^{\mathrm{DR}}(X/Y)$ the cohomology class $\eta(A)$ of an irreducible cycle $i : A \hookrightarrow X$ is the class of $\mathrm{tr}_i(C_A^r)$ in $H_{2r}^{\mathrm{DR}}(X/Y)$, where $C_A^r \in \mathscr{F}_A^{-r,-r}$ is the element defined by the fundamental class $c_{A/Y}^r : \Omega_{A/Y}^r \to \omega_{A/Y}^r \subseteqq \Omega_{Q(A)/k}^r$ of A as introduced in [KW], § 4, and where $\mathrm{tr}_i : i_* \mathscr{F}_A^{\bullet,\bullet} \to \mathscr{F}_X^{\bullet,\bullet}$ is the trace of El Zein [EZ].) For two cycles $A \in Z^r(X)$ and $B \in Z^s(X)$ we have

$$\eta(A \cdot B) = \eta(A) \cup \eta(B)$$

by [Har_1], II, (7.8.2), where $A \cdot B$ denotes the rational equivalence class intersection. Suppose now that $\{U_i\}$ is an open affine cover of X, and that $C = \{(U_i, f_i)\}$ is a Cartier divisor on X. Then $\eta(C)$ can be represented by the Čech-cocycle

$$\left\{ d\log\left(\frac{f_i}{f_j}\right)\right\} \in C^1(\{U_i\}, \Omega_{X/Y}^1)$$

(where for a unit $f \in \mathcal{O}_X^*(U)$ we set $d\log(f) := \dfrac{df}{f}$).

2.6. Proposition. *In the above situation let f_x be a local equation for C at $x \in X$. Then $\eta(C)$ can be represented by the adele*

$$\left\{ d\log\left(\frac{f_x}{f_y}\right)\right\}_{(x,y)\in S(X)_{\mathrm{red}}^1} \in \mathbb{A}_{\mathrm{red}}^1(X, \Omega_{X/Y}^1).$$

In case $C_1, \ldots, C_d$ are d divisors on the (d-dimensional) variety X, intersecting in a finite number of points only, and $f_{i,x}$ is a local equation of C_i at $x \in X$, the cohomology class $\eta(C_1 \cdots C_d)$ is represented by the adele

$$\left\{ (-1)^{\binom{d}{2}} d\log\left(\frac{f_{1,x_0}}{f_{1,x_1}}\right) \wedge \cdots \wedge d\log\left(\frac{f_{d,x_{d-1}}}{f_{d,x_d}}\right)\right\}_{(x_0,\ldots,x_d)\in S(X)_{\mathrm{red}}^d} \in \mathbb{A}_{\mathrm{red}}^d(X, \Omega_{X/Y}^d)$$

and

$$\int_X \eta(C_1 \cdots C_d) = (C_1, \ldots, C_d) \cdot 1_k$$

is the intersection index of $C_1, \ldots, C_d$ in the sense of Parshin [Pa_2]. *Here*

$$\int_X : H_{\mathrm{DR}}^{2d}(X/Y) = H^d(X, \Omega_{X/Y}^d) \to k$$

is the integral constructed by Beilinson [Be].

Proof. The first part can be obtained from the double complex relating Čech- and adelic resolutions. Alternatively this can be done by going through El Zein's construction of the fundamental class of a complete intersection ([EZ], III) in terms of adeles. From this the product formula is immediate in view of 2.2, and 3.3 below in connection with [$\mathrm{Hü}_1$], 3.6 implies the integral formula for the intersection product.

Assume now that $f : X \to Y$ is smooth and that $\Omega_Y^\bullet$ is an exterior differential algebra of Y such that Ω_Y^1 is locally free of rank r. Of particular interest in this context are the Gauß-

Manin connections of X/Y (being, in case Y is a smooth $\mathbb{C}$-variety and $\Omega_Y^\bullet = \Omega_{Y/\mathbb{C}}^\bullet$, the unique integrable connections associated to the local systems $R^i f_* \mathbb{C}$). The Gauß-Manin is the boundary operator $\nabla^{\mathrm{GM}} = d_1^{0q}: R^q f_*(\Omega_{X/Y}^\bullet) \to \Omega_Y^1 \otimes R^q f_*(\Omega_{X/Y}^\bullet)$ of the spectral sequence $R^q f_*(f^*\Omega_Y^p \otimes \Omega_{X/Y}^\bullet) \Rightarrow R^n f_*(\Omega_X^\bullet)$ associated to the filtration $F^p = \operatorname{im}(f^*\Omega_Y^p \otimes \Omega_X^{\bullet - p} \to \Omega_X^\bullet)$. Using adelic resolutions it may be abtained as the connecting morphism of the long exact cohomology sequence associated to

$$(*)\quad 0 \to \Gamma(X, \mathscr{A}^\bullet(F^1/F^2)) \to \Gamma(X, \mathscr{A}^\bullet(F^0/F^2)) \to \Gamma(X, \mathscr{A}^\bullet(F^0/F^1)) \to 0.$$

In the context of Čech resolutions the following construction goes back to Katz and Oda [KO]. It is local in Y, so assume that $Y = \operatorname{Spec}(R)$ is affine and that there exist $r_1, \dots, r_n \in R$ such that $dr_1, \dots, dr_n$ is a basis of Ω_Y^1. For each $x \in X$ choose elements $s_{x,1}, \dots, s_{x,d} \in \mathscr{O}_{X,x}$ such that $ds_{x,1}, \dots, ds_{x,d}$ is a basis of $\Omega_{X/Y,x}^1$. Denoting by $\partial_{x,1}, \dots, \partial_{x,n}, \frac{\partial}{\partial s_{x,1}}, \dots, \frac{\partial}{\partial s_{x,d}}$ the basis of $\operatorname{Hom}_{\mathscr{O}_{X,x}}(\Omega_{X,x}^1, \mathscr{O}_{X,x})$ dual to $dr_1, \dots, dr_n$, $ds_{x,1}, \dots, ds_{x,n}$, we define "partial" derivatives on $\Omega_{X,x}^\bullet$ by

$$d_Y^x(h dr_{i_1} \cdots dr_{i_a} ds_{x,j_1} \cdots ds_{x,j_b}) := \sum_{i=1}^n \partial_{x,i}(h) dr_i dr_{i_1} \cdots dr_{i_a} ds_{x,j_1} \cdots ds_{x,j_b},$$

$$d_{X/Y}^x(h dr_{i_1} \cdots dr_{i_a} ds_{x,j_1} \cdots ds_{x,j_b}) := \sum_{j=1}^n \frac{\partial}{\partial s_{x,j}}(h) ds_{x,j} dr_{i_1} \cdots dr_{i_a} ds_{x,j_1} \cdots ds_{x,j_b}$$

so that $d_{X,x} = d_Y^x + d_{X/Y}^x$. Next define a local section $\varphi_x : \Omega_{X/Y,x}^\bullet \to \Omega_{X,x}^\bullet$ by $\varphi_x(h \cdot dg_1 \cdots dg_p) := h \cdot d_{X/Y}^x(g_1) \cdots d_{X/Y}^x(g_p)$. From 1.7 it follows that the family $\{\varphi_x\}_{x \in X}$ defines a global section

$$\varphi : \mathscr{A}^\bullet(\Omega_{X/Y}^\bullet) \to \mathscr{A}^\bullet(\Omega_X^\bullet)$$

with $\varphi(\beta)_{(x_0, \dots, x_q)} = \varphi_{x_0}(\beta_{(x_0, \dots, x_q)})$. Finally we define $L_Y : \mathscr{A}^{p,q}(\Omega_X^\bullet) \to \mathscr{A}^{p+1,q}(\Omega_X^\bullet)$ by $L_Y(\beta)_{(x_0, \dots, x_q)} = d_Y^{x_0}(\beta_{(x_0, \dots, x_q)})$ and a total interior product on $\Omega_{X,x}^\bullet$ by the formula

$$I^x(h \cdot dg_1 \cdots dg_p) = \sum_{i=1}^p h \cdot dg_1 \cdots d_Y^x(g_i) \cdots dg_p.$$

Set $\lambda : \mathscr{A}^{p,q}(\Omega_X^\bullet) \to \mathscr{A}^{p,q+1}(\Omega_X^\bullet)$, $\lambda(\beta)_{(x_0, \dots, x_{q+1})} = (-1)^p (I^{x_0} - I^{x_1})(\beta_{(x_1, \dots, x_{q+1})})$ and denote by ψ the composition

$$\Gamma(X, \mathscr{A}^\bullet(F^0/F^1)) \xrightarrow{\varphi} \Gamma(X, \mathscr{A}^\bullet(F^0)) \xrightarrow{L_Y + \lambda} \Gamma(X, \mathscr{A}^\bullet(F^1)) \xrightarrow{\mathrm{can}} \Gamma(X, \mathscr{A}^\bullet(F^1/F^2)).$$

One easily checks that all maps are well defined, and that ψ is the connecting morphism of $(*)$. As φ is a section of $\mathscr{A}^\bullet(F^0) \to \mathscr{A}^\bullet(gr^0)$ and $(L_Y + \lambda)(\mathscr{A}^\bullet(F^1)) \subseteq \mathscr{A}^\bullet(F^2)$, this connecting morphism is deduced from

$$\Psi : \mathscr{A}^\bullet(gr^0) = \mathscr{A}^\bullet(F^0)/\mathscr{A}^\bullet(F^1) \xrightarrow{L_Y + \lambda} \mathscr{A}^\bullet(F^1)/\mathscr{A}^\bullet(F^2) = \mathscr{A}^\bullet(gr^1)$$

and we get (cf. [KO], § 3):

2.7. Theorem. *The map $\Psi : \Gamma(X, \mathscr{A}^\bullet(F^0/F^1)) \to \Gamma(X, \mathscr{A}^\bullet(F^1/F^2))$ induces the Gauß-Manin connection on the relative de Rham cohomology*

$$\nabla^{\mathrm{GM}}: H^{\bullet}_{\mathrm{DR}}(X/Y) \to \Omega^1_Y \otimes H^{\bullet}_{\mathrm{DR}}(X/Y).$$

2.8. Remark. In the situation of 2.7 suppose we are given an algebraic differential equation $(\mathscr{M}, \nabla)$ on X (in the sense of [Kaz]), i.e. a locally free $\mathcal{O}_X$-module $\mathscr{M}$ together with an integrable connection ∇ on it. Then the relative de Rham cohomology of $(\mathscr{M}, \nabla)$ on Y is defined and it comes equipped with a canonical integrable connection. This connection can be described similarly in terms of adeles ([Hü$_2$], § 10). By the monodromy theorem this connection will define a regular singular differential equation on Y if $(\mathscr{M}, \nabla)$ defines one on X.

In the singular case the situation is more complicated. To simplify matters we will only treat the case of an embeddable morphism $f: X \to S$ of $\mathbb{Q}$-schemes, i.e. we assume that there exists a closed immersion $i: X \to Y$ into a smooth S-scheme Y, and we furthermore restrict ourselves to the case that S is affine.

Let $\mathfrak{J}$ be the ideal of X in Y, let X_n be the n^{th} infinitesimal neighbourhood of X in Y, i.e. $X_n = \mathfrak{V}(\mathfrak{J}^n)$, let $\mathcal{O}_n := \mathcal{O}_Y/\mathfrak{J}^n$ be its structure sheaf, and let $\mathfrak{X}$ be the formal completion of Y along X, i.e. as a topological space $\mathfrak{X} = X$ with structure sheaf $\mathcal{O}_{\mathfrak{X}} = \varprojlim_{n \in \mathbb{N}} \mathcal{O}_n | X$. Given a coherent sheaf $\mathscr{M}$ on Y we define the sheaf $\mathscr{M}^{\Delta}$ on $\mathfrak{X}$ to be the $\mathfrak{J}$-adic completion of $\mathscr{M}$, i.e. if $\mathscr{M}_n := \mathscr{M}/\mathfrak{J}^n\mathscr{M}$ (viewed as a sheaf on X), then $\mathscr{M}^{\Delta} = \varprojlim_{n \in \mathbb{N}} \mathscr{M}_n$. More generally, given $\mathcal{O}_n$-modules $\mathscr{F}_n$ and $\mathcal{O}_m$-morphisms $\delta_{n,m}: \mathscr{F}_m \to \mathscr{F}_n$ for $m \geqq n$ with $\delta_{n,m} \circ \delta_{m,l} = \delta_{n,l}$ for $l \geqq m \geqq n$, then $\mathscr{F}^{\Delta} := \varprojlim_{n \in \mathbb{N}} \mathscr{F}_n$, the projective limit of the $\mathscr{F}_n$, exists as a sheaf of abelian groups and comes equipped with a canonical structure of an $\mathcal{O}_{\mathfrak{X}}$-module (EGA I, (10.6.6)). If $V \subseteqq Y$ is an open subset, if $U := V \cap X$, $U_n := V \cap X_n$ $(n \in \mathbb{N})$, and if $\mathfrak{U}$ is the formal completion of V along U, then $\mathscr{F}^{\Delta}(\mathfrak{U}) = \varprojlim_{n \in \mathbb{N}} \mathscr{F}_n(U_n)$ by [Har$_2$], II, ex. (1.12).

Assume now that $\{\mathscr{M}_n\}_{n \in \mathbb{N}}$ is a projective system of quasi-coherent $\mathcal{O}_n$-modules, viewed as sheaves on X, and let $\mathscr{M}^{\Delta} := \varprojlim \mathscr{M}_n | X$. Then $\{\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}_n)\}_{n \in \mathbb{N}}$ is a projective system of (not necessarily quasi-coherent) $\mathcal{O}_n$-modules, hence

$$\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}^{\Delta}) := \varprojlim_{n \in \mathbb{N}} \underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}_n)$$

is an $\mathcal{O}_{\mathfrak{X}}$-module, and $\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}^{\Delta})(\mathfrak{U}) = \mathbb{A}^p_{\mathrm{red}}(\mathfrak{U}, (\mathscr{M}|\mathfrak{U})^{\Delta}) = \varprojlim_{n \in \mathbb{N}} \mathbb{A}^p_{\mathrm{red}}(U_n, \mathscr{M}_n | U_n)$ for any open $\mathfrak{U} \subseteqq \mathfrak{X}$ as above.

2.9. Lemma. *The sheaves $\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}^{\Delta})$ are flasque.*

Proof. Let $V \subseteqq U \subseteqq X = \mathfrak{X}$ be open subsets. Then by [Hr$_1$], (2.4.7) we have a canonical isomorphism

$$\mathbb{A}^p_{\mathrm{red}}(U_n, \mathscr{M}_n) = \mathbb{A}^p_{\mathrm{red}}(V_n, \mathscr{M}_n) \oplus \mathbb{A}(L, \mathscr{M}_n)$$

where $L = S(U)_p \setminus S(V)_p$, and such that the restriction $\varrho_{U,V}$ is the projection on the first factor. Thus $\mathbb{A}^p_{\mathrm{red}}(\mathfrak{U}, (\mathscr{M}|\mathfrak{U})^{\Delta}) = \mathbb{A}^p_{\mathrm{red}}(\mathfrak{V}, (\mathscr{M}|\mathfrak{V})^{\Delta}) \oplus \varprojlim_{n \in \mathbb{N}} \mathbb{A}(L, \mathscr{M}_n)$ hence $\underline{\mathbb{A}}^p_{\mathrm{red}}(\mathscr{M}^{\Delta})$ is flasque.

Suppose now that $\{\mathcal{M}_n\}_{n\in\mathbb{N}}$ is a surjective projective system, i.e. the morphisms $\mathcal{M}_n \to \mathcal{M}_m$ are surjective. In this situation we have

2.10. Proposition. *The exact sequences*

$$(*)_n \qquad 0 \to \mathcal{M}_n \xrightarrow{i_n} \underline{\mathbb{A}}^0_{\mathrm{red}}(\mathcal{M}_n) \xrightarrow{\partial^0_n} \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{M}_n) \xrightarrow{\partial^1_n} \cdots$$

induce an exact sequence

$$0 \longrightarrow \mathcal{M}^\Delta \xrightarrow{i} \underline{\mathbb{A}}^0_{\mathrm{red}}(\mathcal{M}^\Delta) \xrightarrow{\partial^0} \underline{\mathbb{A}}^1_{\mathrm{red}}(\mathcal{M}^\Delta) \xrightarrow{\partial^1} \cdots$$

i.e. $\underline{\mathbb{A}}^\bullet(\mathcal{M}^\Delta)$ *is a flasque resolution of* $\mathcal{M}^\Delta$.

Proof. Let $V = \mathrm{Spec}(S) \subseteqq Y$ be an open affine subset, let $U = X \cap V \subseteqq X$ be the corresponding open affine subset of X, and let $\mathfrak{U}$ be the formal completion of V along U. As the morphisms $\mathcal{M}_{n+1} \to \mathcal{M}_n$ are surjective for all $n \in \mathbb{N}$, and as U is affine, also the induced morphisms on global sections $\Gamma(U_{n+1}, \mathcal{M}_{n+1}) \to \Gamma(U_{n+1}, \mathcal{M}_n) = \Gamma(U_n, \mathcal{M}_n)$ are surjective for all $n \in \mathbb{N}$. In particular the system $\{\Gamma(U, \mathcal{M}_n)\}_{n\in\mathbb{N}}$ satisfies the Mittag-Leffler condition (ML) of [EGA 0_{III}], (13.1). By the exactness of $\mathbb{A}^r_{\mathrm{red}}(U_{n+1}, \_)$ also the maps $\mathbb{A}^i_{\mathrm{red}}(U_{n+1}, \mathcal{M}_{n+1}) \to \mathbb{A}^i_{\mathrm{red}}(U_{n+1}, \mathcal{M}_n) = \mathbb{A}^i_{\mathrm{red}}(U_n, \mathcal{M}_n)$ are surjective for $n \in \mathbb{N}$, hence the projective system $\{\mathbb{A}^i_{\mathrm{red}}(U_n, \mathcal{M}_n)\}_{n\in\mathbb{N}}$ satisfies (ML). The sequences $(*)_n$ induce exact sequences of global sections over U by [Hr_2], (4.2.1), hence split into short exact sequences $0 \to \Gamma(U_n, \mathcal{M}_n) \to \mathbb{A}^0(U_n, \mathcal{M}_n) \to C^0_n \to 0$ respectively, for $n > 0$,

$$0 \to C^{i-1}_n \to \mathbb{A}^i_{\mathrm{red}}(U_n, \mathcal{M}_n) \to C^i_n \to 0$$

with $C^i_n := \mathrm{coker}(\Gamma(U, \partial^i_n))$. Thus the projective systems $\{C^i_n\}_{n\in\mathbb{N}}$ satisfy (ML) by [EGA 0_{III}], (13.2.1), and we conclude from [EGA 0_{III}], (13.2.2) that the following sequence is exact

$$0 \longrightarrow \Gamma(\mathfrak{U}, \mathcal{M}^\Delta) \xrightarrow{\mathrm{can}} \mathbb{A}^0_{\mathrm{red}}(\mathfrak{U}, \mathcal{M}^\Delta) \xrightarrow{\partial^0} \mathbb{A}^1_{\mathrm{red}}(\mathfrak{U}, \mathcal{M}^\Delta) \xrightarrow{\partial^1} \cdots.$$

As $V \subseteqq Y$ was an arbitrary open affine subset, and as the open subsets $\mathfrak{U} \subseteqq \mathfrak{X}$ obtained this way form a basis of the topology of $\mathfrak{X}$, the assertion of 2.10 follows.

Hence, if we denote by $\Omega^i_{\mathfrak{X}/S}$ the formal completion of $\Omega^i_{Y/S}$ along X, and by $\mathfrak{A}^{p,q}_{\mathfrak{X}/S} := \underline{\mathbb{A}}^q_{\mathrm{red}}(\Omega^p_{\mathfrak{X}/S})$ we therefore obtain a double complex with exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^2_{\mathfrak{X}/S} & \longrightarrow & \mathfrak{A}^{2,0}_{\mathfrak{X}/S} & \xrightarrow{\partial} & \mathfrak{A}^{2,1}_{\mathfrak{X}/S} & \xrightarrow{\partial} & \cdots \\
 & & \uparrow d & & \uparrow \underline{\mathbb{A}}^0(d) & & \uparrow \underline{\mathbb{A}}^1(d) & & \\
0 & \longrightarrow & \Omega^1_{\mathfrak{X}/S} & \longrightarrow & \mathfrak{A}^{1,0}_{\mathfrak{X}/S} & \xrightarrow{-\partial} & \mathfrak{A}^{1,1}_{\mathfrak{X}/S} & \xrightarrow{-\partial} & \cdots \\
 & & \uparrow d & & \uparrow \underline{\mathbb{A}}^0(d) & & \uparrow \underline{\mathbb{A}}^1(d) & & \\
0 & \longrightarrow & \mathcal{O}_{\mathfrak{X}} & \longrightarrow & \mathfrak{A}^{0,0}_{\mathfrak{X}/S} & \xrightarrow{\partial} & \mathfrak{A}^{0,1}_{\mathfrak{X}/S} & \xrightarrow{\partial} & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

so that again the associated double complex defines a quasi-isomorphism

$$\Omega^{\bullet}_{\mathfrak{X}/S} \to \left(\operatorname{Tot}(\mathfrak{A}^{p,q}_{\mathfrak{X}/S}),\ \mathbb{A}^{\bullet}_{\mathrm{red}}(d),\ \pm\partial\right) =: (\mathfrak{A}^{\bullet}_{\mathfrak{X}/S}, D^{\bullet})$$

from $\Omega^{\bullet}_{\mathfrak{X}/S}$ to the complex $\mathfrak{A}^{\bullet}_{\mathfrak{X}/S}$ of flasque sheaves. The wedge product on $\Omega^{\bullet}_{\mathfrak{X}/S}$ induces a wedge product $\wedge$ on the adeles $\mathfrak{A}^{\bullet}_{\mathfrak{X}/S}$ as follows easily from the proof of [Hr_1], (3.3.7) (see also the discussion in the smooth case), which again passes to cohomology. We may summarize our results as follows:

2.11. Theorem. *In the above situation, i.e. S is affine and $f: X \to S$ is embeddable, we have a canonical isomorphism*

$$H^{\bullet}_{\mathrm{DR}}(X/S) = H^{\bullet}(\Gamma(\mathfrak{X}, \mathfrak{A}^{\bullet}_{\mathfrak{X}/S}))$$

and the adelic wedge product makes the following diagram commutative:

$$\begin{array}{ccc} H^{\bullet}(\Gamma(\mathfrak{X}, \mathfrak{A}^{\bullet}_{\mathfrak{X}/S})) \times H^{\bullet}(\Gamma(\mathfrak{X}, \mathfrak{A}^{\bullet}_{\mathfrak{X}/S})) & \xrightarrow{\ \wedge\ } & H^{\bullet}(\Gamma(\mathfrak{X}, \mathfrak{A}^{\bullet}_{\mathfrak{X}/S})) \\ \wr\downarrow \mathrm{can} & & \wr\downarrow \mathrm{can} \\ H^{\bullet}_{\mathrm{DR}}(X/S) \times H^{\bullet}_{\mathrm{DR}}(X/S) & \xrightarrow{\ \cup\ } & H^{\bullet}_{\mathrm{DR}}(X/S) \end{array}$$

i.e. the cup product in the de Rham cohomology can be computed by the wedge product on the adeles of $\Omega^{\bullet}_{\mathfrak{X}/S}$.

2.12. Remark. For singular morphisms $f: X \to S$, equidimensional of relative dimension d, we have Gauß-Manin connections as well, which are of particular interest in this case (cf. [Har_1], (III.5)). Assume for simplicity that $S = \operatorname{Spec}(R)$ is a smooth and affine k-variety, and that $i: X \hookrightarrow Y$ is a closed immersion into a smooth S-scheme. Denote by $\mathfrak{F}$ the filtration on $\Omega^{\bullet}_{\mathfrak{X}/k}$ deduced from the exact sequence

$$0 \to f_* \Omega^1_{S/k} \to \Omega^1_{X/k} \to \Omega^1_{X/S} \to 0 .$$

The Gauß-Manin connections $\nabla^{\mathrm{GM}}: R^q f_* \Omega^{\bullet}_{\mathfrak{X}/S} \to R^q f_* \Omega^{\bullet}_{\mathfrak{X}/S} \otimes \Omega^1_S$ can again be described as the connecting morphism of the long exact cohomology sequence associated to

$$(*) \qquad 0 \to \mathfrak{A}^{\bullet}(\mathfrak{X}, \mathfrak{F}^1/\mathfrak{F}^2) \to \mathfrak{A}^{\bullet}(\mathfrak{X}, \mathfrak{F}^0/\mathfrak{F}^2) \to \mathfrak{A}^{\bullet}(\mathfrak{X}, \mathfrak{F}^0/\mathfrak{F}^1) \to 0$$

(where $\mathfrak{A}^{\bullet}(\mathfrak{X}, \mathfrak{M})$ denotes the total complex associated to $\{\mathbb{A}^q_{\mathrm{red}}(\mathfrak{X}, \mathfrak{M}^p)\}$). Here the exactness of $(*)$ follows as in the proof of 2.10. The arguments of 2.7 now carry over almost verbatim to give an explicit adelic representative of ∇^{GM}. In the context of singular morphisms it is of great interest as it gives (in many cases) an easy and purely algebraic description of the (purely topological) Picard-Lefschetz monodromy (at least for isolated complete intersection singularities) which in turn greatly determines the topology of these singularities by the work of Brieskorn, Greuel, Hamm and Saito (cf. [Gre] and the references given there).

2.13. Remark. The above description of the de Rham cohomology of singular varieties can be extended to the case of non-embeddable varieties X, cf. [Ye_2].

§ 3. Residues and adeles

Suppose k is a perfect field and X/k is a reduced and irreducible scheme of finite type of dimension d. By [Be], theorem, residues on higher dimensional local fields define a morphism

$$\operatorname{res} : \mathbb{A}^d(X, \Omega^d_{X/k}) \to k$$

such that res in case X/k is smooth and proper induces the trace

$$\operatorname{tr}_{X/k} : H^d(X, \Omega^d_{X/k}) \to k$$

of Grothendieck duality theory. Other constructions of traces have been given by J. Lipman in [Li] and by the second author in [Ye$_1$]. In this section we will show that – up to a sign – all these constructions agree, thus completing the program begun in [Hü$_1$], [SY].

In [Ye$_1$] the construction of the trace is based on an explicit construction of a residual complex $\mathscr{K}^\bullet_X$ for X. Recall that $\mathscr{K}^\bullet_X$ was constructed as follows: Given a point $x \in X$ with $\operatorname{codim}_X(\overline{\{x\}}) = n$, let $\sigma : k(x) \to \hat{\mathcal{O}}_{X,x}$ be a coefficient field $\hat{\mathcal{O}}_{X,x}$ over k and set $\mathscr{K}(x) := \operatorname{Hom}^{\text{cont}}_{k(x)}(\hat{\mathcal{O}}_{X,x}, \Omega^{d-n}_{k(x)/k})$, which up to a canonical isomorphism is independent of the choice of the coefficient field (cf. [Ye$_1$], (4.3.12)) and let

$$\mathscr{K}_X^{-d+n} = \bigoplus_{\operatorname{codim}_X(x) = n} [\mathscr{K}(x)]_x .$$

Given points $x, y \in X$ with $y \in \overline{\{x\}}$ and $\operatorname{codim}_{\bar{x}}(\bar{y}) = 1$ and compatible coefficient fields $\sigma : k(x) \to \hat{\mathcal{O}}_{X,x}$ and $\tau : k(y) \to \hat{\mathcal{O}}_{X,y}$ the residue map

$$\operatorname{Res}_{(x,y),\tau} : \det(\Omega^1_{k(x)/k}) \to \det(\Omega^1_{k(y)/k})$$

of [Ye$_1$], § 4 defines a map $\delta_{(x,y)} : \mathscr{K}(x) \to \mathscr{K}(\bar{y})$, which is independent of σ and τ. The collection of all these maps gives the boundary operator of the complex $\mathscr{K}^\bullet_X$ (see [Ye$_1$], § 4 for more details).

Now let $\omega^d_{X/k}$ be the sheaf of regular d-forms on X as defined in [KW]. According to [Ye$_1$], (4.4.16) $\omega^d_{X/k} = H^{-d}(\mathscr{K}^\bullet_X)$. Note that $\omega^d_{X/k} \otimes_{\mathcal{O}_X} K(X) = \Omega^d_{K(X)/k}$. Let $\underline{\mathbb{A}}^\bullet(\omega^d_{X/k})$ be the complex of adeles with values in $\omega^d_{X/k}$. Then

$$\Gamma(X, \underline{\mathbb{A}}^n(\omega^d_{X/k})) = \mathbb{A}^n(X, \omega^d_{X/k}) \subseteqq \prod_{\xi \in S(X)^n} \omega^d_{X/k,\xi} .$$

For $\xi = (x_0, \ldots, x_n)$ a saturated chain with x_0 the generic point of X we get

$$\omega^d_{X/k,\xi} = \Omega^d_{K(X)/k} \otimes_{K(X)} \mathcal{O}_{X,\xi} = \Omega^{d,\text{sep}}_{k(\xi)/k}$$

where $K(X)$ is the field of rational functions on X, where $k(\xi)$ is the completion of $K(X)$ along the chain ξ, and where $\Omega^{d,\text{sep}}_{k(\xi)/k}$ denotes the module of separated d-differentials of $k(\xi)/k$ of [Ye$_1$], (1.5.3). Given a coefficient field $\sigma : k(x_n) \to \hat{\mathcal{O}}_{X,x_n}$ we get a morphism $\bar{\sigma} : k(x_n) \to k(\xi)$ of topological local fields, and thus we have a well defined residue map

$$\mathrm{Res}_{k(\xi)/k(x_n),\sigma} : \omega^d_{X/k,\xi} \to \Omega^{d,\mathrm{sep}}_{k(\xi)/k} \to \Omega^{d-n}_{k(x_n)/k}$$

inducing an $\mathcal{O}_{X,x_n}$-linear morphism

$$\varphi_\xi : \omega^d_{X/k,\xi} \to \mathscr{K}(x_n) \subseteqq \Gamma(X, \mathscr{K}^{n-d}_X)$$

defined by $\varphi_\xi(\omega)(r) = (-1)^{nd}\mathrm{Res}_{k(\xi)/k(x_n),\sigma}(r\omega)$. By [Ye$_1$], (4.3.12) the map is independent of the choice of σ. If ξ is not a saturated chain, or if x_0 is not the generic point of X, we define

$$\varphi_\xi : \omega^d_{X/k,\xi} \to \Gamma(X, \mathscr{K}^{n-d}_X)$$

by $\varphi_\xi = 0$, and we obtain

3.1. Theorem. *The family* $(\varphi_\xi)_{\xi\in S(X)}$ *defines a morphism of complexes*

$$\varphi^\bullet : \underline{\mathbb{A}}^\bullet(\omega^d_{X/k}) \to \mathscr{K}^\bullet_X[-d]$$

such that the following diagram commutes:

$$\begin{array}{ccc} \omega^d_{X/k} & \xrightarrow[\mathrm{id}]{} & \omega^d_{X/k} \\ {\scriptstyle\mathrm{can}}\downarrow & & \downarrow{\scriptstyle\mathrm{can}} \\ \underline{\mathbb{A}}^\bullet(\omega^d_{X/k}) & \xrightarrow{\varphi^\bullet} & \mathscr{K}^\bullet_X[-d]\,. \end{array}$$

Suppose now in addition that the variety X/k is proper. Then by [Be], theorem, the morphism $\mathrm{res} : \mathbb{A}^d(X, \Omega^d_{X/k}) \to k$ induced by the residues on the local factors passes to cohomology to give a morphism

$$\mathrm{tr}_{X/k} : H^d(X, \Omega^d_{X/k}) \to k$$

(which in case X/k is smooth is the trace of Grothendieck duality theory). Similarly the canonical traces define a map $\mathrm{Tr} : \Gamma(X, \mathscr{K}^0_X) \to k$ which also induces a trace

$$\theta_{X/k} : H^d(X, \omega^d_{X/k}) \to k$$

(cf. [Ye$_1$], (4.4.13), (4.4.16)). The canonical inclusion $\Omega^d_{X/k} \subseteqq \mathscr{K}^{-d}_X$ induces, by [Li], § 3, a map $c^d_{X/k} : \Omega^d_{X/k} \to \omega^d_{X/k}$, the so called fundamental class, and 3.1 implies

3.2. Corollary. *The residue map* $\mathrm{res} : \mathbb{A}^d(X, \omega^d_{X/k}) \to k$ *factors to give a trace morphism* $\mathrm{tr}'_{X/k} : H^d(X, \omega^d_{X/k}) \to k$ *such that*

$$\mathrm{tr}'_{X/k} = (-1)^d \cdot \theta_{X/k}$$

and such that the following diagram commutes:

$$\begin{array}{ccc} H^d(X, \Omega^d_{X/k}) & \xrightarrow{H^d(X, c^d_{X/k})} & H^d(X, \omega^d_{X/k}) \\ {\scriptstyle\mathrm{tr}_{X/k}}\searrow & & \swarrow{\scriptstyle\mathrm{tr}'_{X/k}} \\ & k\,. & \end{array}$$

Proof of the corollary. For each closed point $x \in X$ there is a unique coefficient field $k(x) \to \hat{\mathcal{O}}_{X,x}$. Then Tr is induced by

$$\text{“evaluation at 1”}: \mathscr{K}(x) = \operatorname{Hom}^{\text{cont}}_{k(x)}(\hat{\mathcal{O}}_{X,x}, k(x)) \to k(x)$$

followed by the canonical trace $\operatorname{Tr}_{k(x)/k}$. Thus the claim follows from the (obvious) commutativity of the diagram

$$\begin{array}{ccc} \mathbb{A}^d(X, \omega^d_{X/k}) & \xrightarrow{\Gamma(X,\varphi^d)} & \Gamma(X, \mathscr{K}^0_X) \\ \text{res} \searrow & & \swarrow (-1)^{d^2}\operatorname{Tr} \\ & k. & \end{array}$$

3.3. Remark. By [Hü$_1$] resp. [SY] we have $\theta_{X/k} = (-1)^{d^2 + \frac{(d-1)d}{2}} \cdot \int_{X/k}$ for X/k proper, where $\int_{X/k}$ is the trace map of [Li]. Hence $\operatorname{tr}'_{X/k} = (-1)^{\frac{(d-1)d}{2}} \cdot \int_{X/k}$ by 3.2.

Obviously it suffices to prove the theorem on the level of global sections. Set

$$K := \{\xi = (x_0, \dots, x_n) \in S(X)_n : \operatorname{codim}_X(\overline{x_i}) = i \text{ for all } i = 0, \dots, n\}.$$

If $L = S(X)_n \backslash K$ then we have canonically

$$\mathbb{A}^n(X, \omega^d_{X/k}) = \mathbb{A}(K, \omega^d_{X/k}) \oplus \mathbb{A}(L, \omega^d_{X/k}).$$

Thus to prove that $\varphi^{\bullet}$ is well defined it suffices to show

3.4. Proposition. *Let $\alpha = (\alpha_\xi)_{\xi \in K} \subseteqq \mathbb{A}(K, \omega^d_{X/k})$ be an adele. Then for all but finitely many chains $\xi \in K$ the component $\alpha_\xi \in \omega^d_{X/k,\xi}$ is holomorphic along ξ in the sense of* [Ye$_1$], (4.2.3).

Remark. Note that for each $\xi \in K$ we have $\omega^d_{X/k,\xi} = \Omega^{d,\text{sep}}_{k(\xi)/k}$, so it makes sense to talk of holomorphicity.

By [Hr$_1$], (3.3.2) we may assume for the proof of Proposition 3.4 that $X = \operatorname{Spec}(R)$ is affine. We will proceed by induction on d, the case $d = 1$, i.e. X is a curve, being well known. Recall that for $\xi = (x_0, \dots, x_n)$ we denote $d_0 \xi := (x_1, \dots, x_n)$.

3.5. Lemma. *In the above situation assume that we have for some chain $\xi \in K$ that $\alpha_\xi \in \operatorname{im}(\omega^d_{X/k, d_0\xi} \to \omega^d_{X/k,\xi})$. Then*

$$\varphi_\xi(\alpha_\xi) = 0.$$

Proof. We have $X = \operatorname{Spec}(R)$ is affine, so write $\xi = (\mathfrak{p}_0, \dots, \mathfrak{p}_n)$ with $\operatorname{ht}(\mathfrak{p}_i) = i$. Let $\zeta = (\mathfrak{p}_0, \mathfrak{p}_1)$ and $\eta = (\mathfrak{p}_1, \dots, \mathfrak{p}_n)$, and let $\tau : k(\mathfrak{p}_1) \to \hat{R}_{\mathfrak{p}_1}$ be a coefficient field, compatible with σ. Then

$$\begin{array}{ccccc}
k(\mathfrak{p}_1) & \to & k(\zeta) & \to & k(\zeta \vee d_0\eta) = k(\xi) \\
 & & \uparrow \bar{\tau} & & \uparrow \bar{\tau}_n \\
 & & k(\mathfrak{p}_1) & \to & k(\eta) \\
 & & & & \uparrow \bar{\sigma} \\
 & & & & k(\mathfrak{p}_n)
\end{array}$$

is a finitely ramified base change diagram by [Ye$_1$], (4.1.11). Thus the following diagram commutes:

$$\begin{array}{ccc}
\tilde{\omega}^d_{\hat{R}_{\mathfrak{p}_1}/k} = \omega^d_{R/k, d_0\zeta} & \xrightarrow{\partial} & \omega^d_{R/k, d_0\xi} \\
\downarrow \operatorname{Res}_{k(\zeta)/k(d_0\zeta)} & & \downarrow \operatorname{Res}_{k(\xi)/k(\eta)} \\
\Omega^{d-1}_{k(d_0\zeta)/k} & \longrightarrow & \Omega^{d-1, \mathrm{sep}}_{k(\eta)/k} .
\end{array}$$

Since $\omega^d_{X/k} = \ker(\mathscr{K}_X^{-d} \to \mathscr{K}_X^{-d+1})$, we get $\operatorname{Res}_{k(\zeta)/k(d_0\zeta)}(\tilde{\omega}^d_{\hat{R}_\mathfrak{p}/k}) = 0$. As $\operatorname{im}(\partial)$ is dense in $\omega^d_{R/k, d_0\xi}$ by 1.10, we obtain by the continuity of the residue map

$$\operatorname{Res}_{k(\xi)/k(\eta)}(r\omega_\xi) = 0 \quad \text{for all } r \in \hat{R}_{\mathfrak{p}_n}, \quad \omega_\xi \in \omega^d_{R/k, d_0\xi},$$

hence for all $r \in \hat{R}_{\mathfrak{p}_n}$ we get

$$\varphi_\xi(\alpha_\xi)(r) = \operatorname{Res}_{k(\xi)/k(\mathfrak{p}_n)}(r\alpha_\xi) = \operatorname{Res}_{k(\eta)/k(\mathfrak{p}_n)}(\operatorname{Res}_{k(\xi)/k(\eta)}(r\alpha_\xi)) = 0 .$$

Proof of 3.4. We proceed by induction on n:

If $n = 1$, the proposition is well known.

Let $n > 1$, and for a prime $\mathfrak{p} \in X$ with $\operatorname{ht}(\mathfrak{p}) = 1$ define

$$K_\mathfrak{p} := \{(\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n) \in K : \mathfrak{p}_1 = \mathfrak{p}\} .$$

(Note that $\mathfrak{p}_0$ is the generic point of X.) By [Hr$_1$], (3.3.4), (2.4.3) there exists an $f \in R$ such that

$$(f \cdot \alpha_\xi)_{\xi \in K} \in \mathbb{A}(\widehat{\mathfrak{p}_0} K, \omega^d_{X/k}) = \prod_{\operatorname{ht}(\mathfrak{p}) = 1} \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \omega^d_{X/k}) .$$

In particular $f \cdot \alpha_\xi \in \omega^d_{X/k, d_0\xi}$ for all $\xi \in K$. There is a finite set $E = \{\mathfrak{p}^{(1)}, \ldots, \mathfrak{p}^{(l)}\}$ of primes with $\operatorname{ht}(\mathfrak{p}^{(j)}) = 1$ such that for all $\mathfrak{p} \in X$ with $\operatorname{ht}(\mathfrak{p}) = 1$, $\mathfrak{p} \notin E$ we will have $f \notin \mathfrak{p}$, hence already $\alpha_\xi \in \omega^d_{X/k, d_0\xi}$ for all $\xi \in K_\mathfrak{p}$, $\mathfrak{p} \notin E$. Thus by 3.5 all these α_ξ will be holomorphic along ξ.

Now we restrict ourselves to the primes of E. Fix some $\mathfrak{p} \in E$ and choose a coefficient field $\sigma : k(\mathfrak{p}) \to \widehat{R_\mathfrak{p}}$. Then the associated residue map

$$D_\sigma := \operatorname{Res}_{(\mathfrak{p}_0, \mathfrak{p}), \sigma} : \Omega^d_{K(X)/k} \to \Omega^{d-1}_{k(\mathfrak{p})/k}$$

is a locally differential operator over $\mathscr{O}_X$ relative k, hence induces by 1.9 a locally differential operator

$$D'_\sigma = \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, D_\sigma) : \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^d_{K(X)/k}) \to \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^{d-1}_{k(\mathfrak{p})/k})$$

such that for each $\xi \in K_\mathfrak{p}$ the following diagram commutes:

$$\begin{array}{ccc} \Omega^d_{K(X)/k} & \xrightarrow[D_\sigma]{} & \Omega^{d-1}_{k(\mathfrak{p})/k} \\ \downarrow & & \downarrow \\ \mathbb{A}(K_\mathfrak{p}, \omega^d_{X/k}) = \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^d_{K(X)/k}) & \xrightarrow[D'_\sigma]{} & \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^{d-1}_{k(\mathfrak{p})/k}) \\ \downarrow & & \downarrow \\ \omega^d_{X/k,\xi} = \Omega^d_{K(X)/k, d_0\xi} & \xrightarrow[D_{d_0\xi}]{} & \Omega^{d-1}_{k(\mathfrak{p})/k, d_0\xi}\,. \end{array}$$

Note that $D_{d_0\xi}$ is the residue $\mathrm{Res}_{k(\xi)/k(d_0\xi),\bar\sigma}$ arising from D_σ by the finitely ramified base change $k(\mathfrak{p}) \to k(\mathfrak{p})_{d_0\xi}$.

As $D'_\sigma | R \cdot (\alpha_\xi)_{\xi \in K_\mathfrak{p}}$ is a differential operator over R relative k we get ([Ye$_1$], (3.1.9)):

$$D'_\sigma\big(R \cdot (\alpha_\xi)_{\xi \in K_\mathfrak{p}}\big) \subseteqq \sum_{j=1}^{m} R \cdot \beta_j \subseteqq \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^{d-1}_{k(\mathfrak{p})/k})$$

for suitable $\beta_1 = (\beta_{1,\delta}), \ldots, \beta_m = (\beta_{m,\delta}) \in \mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^{d-1}_{k(\mathfrak{p})/k})$. Now let $Y := \overline{\{\mathfrak{p}\}}$ with its reduced induced subscheme structure, let $\bar R := R/\mathfrak{p}$ and set

$$L = \{(\mathfrak{q}_0, \ldots, \mathfrak{q}_{n-1}) \in S(Y)_{n-1} : \mathrm{ht}(\mathfrak{q}_i) = i\}\,.$$

Then $\mathbb{A}(\widehat{\mathfrak{p}_0} K_\mathfrak{p}, \Omega^{d-1}_{k(\mathfrak{p})/k}) = \mathbb{A}(L, \Omega^{d-1}_{k(\mathfrak{p})/k}) = \mathbb{A}(L, \omega^{d-1}_{Y/k})$, hence by the inductive assumption there ist a finite subset $F \subseteqq L$ such that for all $\delta \in L \backslash F$, $\beta_{1,\delta}, \ldots, \beta_{m,\delta}$ are holomorphic along δ, i.e. if $\delta = (\mathfrak{q}_0, \ldots, \mathfrak{q}_{n-1}) \in L \backslash F$ and if $\mathfrak{q}_n, \ldots, \mathfrak{q}_{d-1} \in Y$ are such that $\eta := (\mathfrak{q}_0, \ldots, \mathfrak{q}_{d-1})$ is a maximal saturated chain of Y, then

$$\mathrm{Res}_{k(\eta)/k}(\hat{\bar{R}}_{\mathfrak{q}_{n-1}} \cdot \beta_{l,\delta}) = 0 \quad \text{for all } l = 1, \ldots, m\,.$$

Now fix such a chain $\delta = (\mathfrak{q}_0, \ldots, \mathfrak{q}_{n-1}) \in L \backslash F$, let $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ be its preimage in $\widehat{\mathfrak{p}_0} K_\mathfrak{p}$ and set $\xi = (\mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_n)$. As $\hat R_{\mathfrak{p}_n}/R$ is topologically étale we obtain from the commutativity of the above diagram

$$\mathrm{Res}_{k(\xi)/k(\delta),\bar\sigma}(\hat R_{\mathfrak{p}_n} \cdot \alpha_\xi) \subseteqq \sum_{j=1}^{m} \hat R_{\mathfrak{p}_n} \cdot \beta_{l,\delta} = \sum_{j=1}^{m} \hat{\bar{R}}_{\mathfrak{q}_{n-1}} \cdot \beta_{j,\delta}\,.$$

Hence if $\zeta = (\mathfrak{p}_{n+1}, \ldots, \mathfrak{p}_d)$ is a chain of primes in R such that, for some given chain $\xi \in K$, $\xi = (\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$, the concatenation $\xi \vee \zeta := (\mathfrak{p}_0, \ldots, \mathfrak{p}_n, \mathfrak{p}_{n+1}, \ldots, \mathfrak{p}_d)$ is a maximal saturated chain, and if η is the image of $d_0(\xi \vee \zeta)$ in $S(Y)$, we deduce from the transitivity of residues that

$$\mathrm{Res}_{k(\xi \vee \zeta)/k}(\hat R_{\mathfrak{p}_n} \cdot \alpha_\xi) = \mathrm{Res}_{k(\eta)/k}\big(\mathrm{Res}_{k(\xi \vee \zeta)/k(\eta),\bar\sigma}(\hat R_{\mathfrak{p}_n} \cdot \alpha_\xi)\big) = 0$$

for all $\xi \in K_\mathfrak{p}$ except the finitely many chains that arise as preimages of the $\delta \in F$, completing the proof of Proposition 3.4.

To complete the proof of Theorem 3.2, it remains to show that $\varphi^\bullet$ is a morphism of complexes. Again we may assume that $X = \mathrm{Spec}(R)$ is affine.

First let $\xi = (\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ with $\mathrm{ht}(\mathfrak{p}_i) = i$ and let $\xi' = (\mathfrak{p}_0, \ldots, \mathfrak{p}_n, \mathfrak{p}_{n+1})$ with $\mathrm{ht}(\mathfrak{p}_{n+1}) = n+1$. Then by the definition of residue data ([Ye$_1$], (4.3.2)) the following diagram commutes:

$$\begin{array}{ccc} \omega^d_{R/k,\xi} & \xrightarrow{\partial} & \omega^d_{R/k,\xi'} \\ \downarrow{\scriptstyle \varphi_\xi} & & {\scriptstyle \varphi_{\xi'}}\downarrow \\ \mathscr{K}(\mathfrak{p}_n) & \xrightarrow{(-1)^d \delta_{(\mathfrak{p}_n, \mathfrak{p}_{n+1})}} & \mathscr{K}(\mathfrak{p}_{n+1}). \end{array}$$

Thus to show that $\varphi^\bullet$ is a morphism of complexes it remains to show

3.6. Proposition. *Let* $i \in \{0, \ldots, n-1\}$, *let* $\zeta = (\mathfrak{p}_0, \ldots, \mathfrak{p}_{i-1}, \mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_n\}$ *with* $\mathrm{ht}(\mathfrak{p}_j) = j$ *for all* j, *and set*

$$K_\zeta = \{\xi = (\mathfrak{p}_0, \ldots, \mathfrak{p}_{i-1}, \mathfrak{q}, \mathfrak{p}_{i+1}, \ldots, \mathfrak{p}_n) \in S(X)_n : \mathfrak{p}_{i-1} \subsetneqq \mathfrak{q} \subsetneqq \mathfrak{p}_{i+1}\}.$$

Then for each $\omega \in \omega^d_{R/k,\zeta}$:

$$\sum_{\xi \in K_\zeta} \mathrm{Res}_{k(\xi)/k(\mathfrak{p}_n),\tau}(\omega) = 0$$

where $\tau : k(\mathfrak{p}_n) \to \hat{R}_{\mathfrak{p}_n}$ *is a fixed coefficient field.*

Proof. Again we may assume that $X = \mathrm{Spec}(R)$ is affine. If $i = 0$ then $\mathrm{Res}_{\xi,\tau}(\omega) = 0$ for each $\xi \in K_\zeta$ by Lemma 3.4, so we may assume that $i > 0$. In this situation the face map $\partial : \Omega^d_{K(X)/k} \to \omega^d_{R/k,\zeta}$ has dense image by 1.10, hence it suffices to show

$$\sum_{\xi \in K_\zeta} \mathrm{Res}_{k(\xi)/k(\mathfrak{p}_n),\tau}(\omega) = 0 \quad \text{for all } \omega \in \Omega^d_{K(X)/k}.$$

As $i > 0$ this is just the residue theorem of [Ye$_1$], (4.2.15)a), resp. [Lo], thm. 3.

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