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ON FLATNESS AND COMPLETION FOR INFINITELY GENERATED MODULES OVER NOETHERIAN RINGS

Amnon Yekutieli
Department of Mathematics, Ben Gurion University, Be'er Sheva, Israel

Let A be a noetherian commutative ring, and let $\mathfrak{a}$ be an ideal in A. We study questions of flatness and $\mathfrak{a}$-adic completeness for infinitely generated A-modules. This is done using the notions of* decaying function *and* $\mathfrak{a}$-adically free *A-module.

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INTRODUCTION

Let A be a commutative ring, and let $\mathfrak{a}$ be an ideal of A. For $i \geq 0$, we write $A_i := A/\mathfrak{a}^{i+1}$. Given an A-module M, its $\mathfrak{a}$-adic completion is the A-module

$$\widehat{M} := \lim_{\leftarrow i}(A_i \otimes_A M). \tag{0.1}$$

Recall that M is called $\mathfrak{a}$-*adically complete* if the canonical homomorphism $M \to \widehat{M}$ is bijective. If $M \to \widehat{M}$ is injective, then M is called $\mathfrak{a}$-*adically separated*. It is well known that if A is noetherian and complete, then all finitely generated A-modules are complete. But for infinitely generated modules, and for non-noetherian rings, the picture is quite complicated.

We became interested in the adic completion of infinitely generated modules in the course of our work on deformation quantization (see end of Introduction). After a while we realized that this old and apparently simple concept was not treated adequately in the literature. This article contains our contributions.

In Section 1 we discuss the completion operation in general. In Theorem 1.2 we give a useful criterion to tell whether the $\mathfrak{a}$-adic completion $\widehat{M}$ of an A-module M is itself $\mathfrak{a}$-adically complete. We give an example of an $\mathfrak{a}$-adically separated module M whose $\mathfrak{a}$-adic completion $\widehat{M}$ is not complete (Example 1.8). The moral (made precise in Corollary 1.12) is that one should distinguish between the algebraic notion of $\mathfrak{a}$-adic completion of M (i.e. the inverse limit (0.1)), and the topological notion of completion of the metric space M (with respect to its $\mathfrak{a}$-adic metric, see (1.11)).

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Address correspondence to Amnon Yekutieli, Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, Israel; E-mail: amyekut@math.bgu.ac.il

In Section 2 of the article we introduce the notion of *decaying function*. This idea is inspired by functional analysis. Let Z be a set, and let M be an $\mathfrak{a}$-adically separated A-module. A function $f : Z \to M$ is called decaying if for every i the composed function $Z \to A_i \otimes_A M$ has finite support. We denote by $\mathrm{F}_{\mathrm{dec}}(Z, M)$ the set of all decaying functions $f : Z \to M$, and this is an A-module in the obvious way. The submodule of finite support functions is denoted by $\mathrm{F}_{\mathrm{fin}}(Z, M)$. Note that for $M := A$ the module $\mathrm{F}_{\mathrm{fin}}(Z, A)$ is a free A-module, with basis the collection $\{\delta_z\}_{z \in Z}$ of delta functions.

We prove (Corollary 2.9) that, if M is $\mathfrak{a}$-adically complete, then $\mathrm{F}_{\mathrm{dec}}(Z, M)$ is the $\mathfrak{a}$-adic completion of $\mathrm{F}_{\mathrm{fin}}(Z, M)$. (Recall, however, that the completion need not be complete!) We also prove a complete version of the Nakayama Lemma (Theorem 2.11).

In Section 3 we assume A is noetherian. The main result here, Theorem 3.4, says that for any set Z the A-module $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$ is flat and $\mathfrak{a}$-adically complete. Theorem 3.4 implies, among other things, Corollary 3.5, which says that for any A-module M the completion $\widehat{M}$ is $\mathfrak{a}$-adically complete. (Note that the content of Corollary 3.5 is not new; see Remark 3.7 for a bit of history.) We see that the anomalies of completion disappear when A is noetherian.

An A-module P is called $\mathfrak{a}$-*adically free* if it is isomorphic to $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$ for some set Z. We show (Corollary 3.15) that any $\mathfrak{a}$-adically complete A-module M is a quotient of some $\mathfrak{a}$-adically free module P. We also introduce the notion of $\mathfrak{a}$-*adically projective* A-module; and we prove that P is $\mathfrak{a}$-adically projective if and only if it is a direct summand of an $\mathfrak{a}$-adically free module (Corollary 3.18). We give an example (Example 3.20) demonstrating that the completion functor $M \mapsto \widehat{M}$ is not right exact.

In Section 4 we specialize to the case of a complete noetherian local ring A, with maximal ideal $\mathfrak{m}$. Corollary 4.5 says that an A-module P is $\mathfrak{m}$-adically free if and only if it is flat and $\mathfrak{m}$-adically complete. We discuss $\mathfrak{m}$-adic systems of A-modules.

In Section 5 we study the related geometric problem. Namely, X is a topological space, and we are interested in sheaves of A-modules on X that are flat and $\mathfrak{m}$-adically complete. Here some geometric property is needed for things to work well; we call it *locally $\mathcal{N}$-simply connectedness*, where $\mathcal{N}$ is a sheaf of abelian groups on X (see Definition 5.4).

Here are a few words on the connection between completion and deformation quantization. Suppose $\mathbb{K}$ is a field, and A is a complete noetherian local $\mathbb{K}$-algebra, with maximal ideal $\mathfrak{m}$, such that $A/\mathfrak{m} \cong \mathbb{K}$. Let $\bar{B}$ be a $\mathbb{K}$-algebra. An *associative A-deformation* of $\bar{B}$ is an associative unital (but not necessarily commutative) A-algebra B, which is flat and $\mathfrak{m}$-adically complete, together with a $\mathbb{K}$-algebra isomorphism $\mathbb{K} \otimes_A B \cong \bar{B}$. The main example is $\mathbb{K} := \mathbb{R}$; $A := \mathbb{R}[[\hbar]]$, the ring of formal power series in the variable $\hbar$; and $\bar{B} := \mathrm{C}^{\infty}(X)$, the ring of smooth functions on a differentiable manifold X. In our article [14] we consider the algebro-geometric version of deformation quantization, involving sheaves of A-algebras. The results of Sections 4–5 are needed in [14].

A possible use for the results of Section 3 would be to gain a better understanding of the Matlis–Greenlees–May duality (cf. [1, 8, 10]).

1. SOME RESULTS ABOUT COMPLETION

By default, all rings in this article are commutative.

We begin by recalling some facts about completion. Let A be a ring, and let $\mathfrak{a}$ be an ideal of A. For $i \in \mathbb{N}$, we write $A_i := A/\mathfrak{a}^{i+1}$. Given an A-module M, there are canonical isomorphisms $A_i \otimes_A M \cong M/\mathfrak{a}^{i+1}M$. The $\mathfrak{a}$-*adic completion* of M is the A-module

$$\widehat{M} := \lim_{\leftarrow i}(A_i \otimes_A M). \tag{1.1}$$

There is a canonical homomorphism

$$\tau_M : M \to \widehat{M}.$$

The module M is called $\mathfrak{a}$-*adically separated* if τ_M is injective, and it is called $\mathfrak{a}$-*adically complete* if τ_M is bijective. (Some texts, such as [3], would say that A is separated and complete if τ_M is bijective.) Of course, M is $\mathfrak{a}$-adically separated if and only if $\bigcap_{i\geq 0} \mathfrak{a}^i M = 0$. If M is $\mathfrak{a}$-adically complete, then we often identify M with $\widehat{M}$ via τ_M.

The $\mathfrak{a}$-adic completion $\widehat{A}$ of A is a ring, and $\tau_A : A \to \widehat{A}$ is ring homomorphism. Given an A-module M, its completion $\widehat{M}$ is an $\widehat{A}$-module, with action coming from the action of $\widehat{A}$ on the modules $A_i \otimes_A M$ in the inverse system (1.1). In particular this says that a complete A-module M has a canonical $\widehat{A}$-module structure on it.

Given a homomorphism $\phi : M \to N$ of A-modules, there is an induced homomorphism $\hat{\phi} : \widehat{M} \to \widehat{N}$ making the diagram

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\tau_M"'] & N \arrow[d, "\tau_N"] \\ \widehat{M} \arrow[r, "\widehat{\phi}"] & \widehat{N} \end{tikzcd}$$

commutative.

Sometimes we write $\Lambda_{\mathfrak{a}} M := \widehat{M}$ for an A-module M, and $\Lambda_{\mathfrak{a}}(\phi) := \hat{\phi}$ for a homomorphism ϕ, following [1]. This gives a functor

$$\Lambda_{\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

on the category of A-modules. The functor $\Lambda_{\mathfrak{a}}$ is additive. However, it is not exact, nor even right exact; cf. Example 3.20. The functor $\Lambda_{\mathfrak{a}}$ is not idempotent in general (see Example 1.8). Corollary 3.6 says that the functor $\Lambda_{\mathfrak{a}}$ is idempotent if the ideal $\mathfrak{a}$ is finitely generated. All that can be said in general about the functor $\Lambda_{\mathfrak{a}}$ is that it preserves surjections.

Proposition 1.2. *Let $\phi : M \to N$ be a surjective homomorphism of A-modules. Then $\hat{\phi} : \widehat{M} \to \widehat{N}$ is also surjective.*

This result is part of [12, Proposition 2.2.1].

Proof. For every $i \geq 0$, let us write $M_i := A_i \otimes_A M$ and $N_i := A_i \otimes_A N$. Let $\phi_i : M_i \to N_i$ be the homomorphism induced by ϕ, and let $K_i := \mathrm{Ker}(\phi_i)$. So there is an inverse system of exact sequences

$$0 \to K_i \to M_i \xrightarrow{\phi_i} N_i \to 0.$$

Each K_i is a quotient of $\mathrm{Ker}(\phi)$, and therefore, $K_{i+1} \to K_i$ is surjective. By the Mittag–Leffler argument (as in [2, Proposition 10.2]), in the limit we get an exact sequence

$$0 \to \lim_{\leftarrow i} K_i \to \widehat{M} \xrightarrow{\hat{\phi}} \widehat{N} \to 0.$$

In particular, $\hat{\phi}$ is surjective. $\square$

Let M be an A-module. The homomorphism $\tau_M : M \to \widehat{M}$ induces a homomorphism

$$\tau_{M,i} : A_i \otimes_A M \to A_i \otimes_A \widehat{M} \tag{1.3}$$

for every $i \geq 0$. On the other hand, from the inverse limit (1.1), we have surjective homomorphisms

$$\pi_{M,i} : \widehat{M} \to A_i \otimes_A M. \tag{1.4}$$

Here is a useful criterion to tell whether the $\mathfrak{a}$-adic completion is complete.

Theorem 1.2. *Let M be an A-module, with $\mathfrak{a}$-adic completion $\widehat{M}$. The following conditions are equivalent*:

(i) *The A-module $\widehat{M}$ is $\mathfrak{a}$-adically complete*;
(ii) *All the homomorphisms $\tau_{M,i}$ are surjective*;
(iii) *There is equality* $\mathrm{Ker}(\pi_{M,i}) = \mathfrak{a}^{i+1}\widehat{M}$ *for every $i \geq 0$.*

Proof. The proof is based on ideas in [12, Section 2.2]. Let us write $N := \widehat{M}$, and for $i \geq 0$, let $M_i := A_i \otimes_A M$ and $N_i := A_i \otimes_A N$. There is a commutative diagram

$$\begin{tikzcd} M \arrow[r, "\tau_M"] \arrow[d, "\theta_{M,i}"] & N \arrow[r, "\pi_{M,i}"] \arrow[d, "\theta_{N,i}"] & M_i \arrow[d, "="] \\ M_i \arrow[r, "\tau_{M,i}"] & N_i \arrow[r, "\psi_i"] & M_i \end{tikzcd} \tag{1.6}$$

in which $\theta_{M,i}$ and $\theta_{N,i}$ are the surjections induced by the ring homomorphism $A \to A_i$, and ψ_i is the unique homomorphism that makes the diagram commutative. Since $\psi_i \circ \tau_{M,i}$ is the identity on M_i, it follows that $\tau_{M,i}$ is a split injection. Letting $M_i' := \mathrm{Ker}(\psi_i)$, we have a canonical decomposition $N_i = M_i \oplus M_i'$. So $\tau_{M,i}$ is surjective if and only if ψ_i is injective, if and only if $M_i' = 0$.

Note also that $\operatorname{Ker}(\theta_{N,i}) = \mathfrak{a}^{i+1}N$, and it equals $\operatorname{Ker}(\pi_{M,i})$ if and only if ψ_i is injective. This tells us that (ii) $\Leftrightarrow$ (iii).

The diagrams (1.6) form an inverse system. Passing to the inverse limit in the second row, we get a diagram

$$N \xrightarrow{\tau_N} \widehat{N} \xrightarrow{\psi} N,$$

where $\psi := \lim_{\leftarrow i} \psi_i$. Again $\psi \circ \tau_N = \mathrm{id}_N$, so τ_N is a split injection. Writing $N' := \operatorname{Ker}(\psi)$, we have a canonical decomposition $\widehat{N} = N \oplus N'$. So τ_N is bijective (i.e., N is complete) if and only $N' = 0$.

The decompositions $N_i = M_i \oplus M'_i$ are compatible as i varies, and hence

$$N' \cong \lim_{\leftarrow i} M'_i.$$

Now in the inverse system $\{M'_i\}_{i\geq 0}$ the homomorphisms $M'_j \to M'_i$, for $j \geq i$, are surjective. Therefore, in the limit, every homomorphism $N' \to M'_i$ is surjective. It follows that $N' = 0$ if and only if $M'_i = 0$ for all i. We conclude that (i) $\Leftrightarrow$ (ii). $\square$

In the proof above we also showed the following corollary.

Corollary 1.7. *Let M be an A-module. Then its $\mathfrak{a}$-adic completion $\widehat{M}$ is $\mathfrak{a}$-adically separated. Moreover, the homomorphism $\tau_{\widehat{M}} : \widehat{M} \to \Lambda_{\mathfrak{a}}\widehat{M}$ is a split injection.*

Some examples of the bad behavior of completion can be found in the literature. Strooker [12, Subsection 2.2.5] mentions unpublished work of Bartijn. And there is an example in [3], which is very close to the example we now present.

Example 1.8. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[t_1, t_2, \dots]$, the ring of polynomials in countably many variables. In it, consider the maximal ideal $\mathfrak{a} = (t_1, t_2, \dots)$. We will produce *an A-module M whose $\mathfrak{a}$-adic completion $\widehat{M}$ is not $\mathfrak{a}$-adically complete*. In fact we will take $M := A$, the free module of rank 1.

Let $\widehat{A}$ be the $\mathfrak{a}$-adic completion of A, and let $\mathfrak{b} := \operatorname{Ker}(\pi_{A,0} : \widehat{A} \to A_0)$. The ring $\widehat{A}$ is canonically isomorphic to the ring of formal power series $\mathbb{K}[[t_1, t_2, \dots]]$. In [3, Exercise III.2.12] it is shown that the ring $\widehat{A}$ is not $\mathfrak{b}$-adically complete (when $\mathbb{K}$ is finite). As stated in the previous paragraph, we will show something slightly different: the A-module $\widehat{A}$ is not $\mathfrak{a}$-adically complete (with no assumption on the field $\mathbb{K}$). This is done using Theorem 1.5.

In order to utilize the notation of Theorem 1.5 and its proof, let's write $M := A$ and $N := \widehat{M}$. To prove that N is not $\mathfrak{a}$-adically complete it suffices to show that the homomorphism $\tau_{M,0} : M_0 \to N_0$ is not surjective.

Consider an element $b \in N$, with image $\bar{b} := \theta_{N,0}(b) \in N_0$. The element $\bar{b}$ is in the image of $\tau_{M,0}$ if and only if $b \in \mathfrak{a}N + \operatorname{Im}(\tau_M)$. Now any element of $\mathfrak{a}N$ is of the form $\sum_{k=1}^{n} t_k b_k$ for some $n \geq 0$ and $b_k \in N$. And any element of $\operatorname{Im}(\tau_M)$ is a polynomial; so it lies in $\mathbb{K} \oplus \mathfrak{a}N$. Thus $b \in \mathfrak{a}N + \operatorname{Im}(\tau_M)$ if and only if

$$b = \lambda + \sum_{k=1}^{n} t_k b_k \tag{1.9}$$

for some $\lambda \in \mathbb{K}$, $n \geq 0$ and $b_k \in N$.

Let us take $b \in N = \mathbb{K}[[t_1, t_2, \ldots]]$ to be the power series $b := \sum_{k=1}^{\infty} t_k^k$. Then b cannot be written as in (1.9), and hence $\bar{b} \notin \mathrm{Im}(\tau_{M,0})$.

We end this section with a discussion of the topological interpretation of $\mathfrak{a}$-adic completion. Any A-module M has on it the $\mathfrak{a}$-adic topology, in which the collection of submodules $\{\mathfrak{a}^i M\}_{i\geq 0}$ is a basis of open neighborhoods of the element 0. Any homomorphism $\phi : M \to N$ of A-modules is continuous for the $\mathfrak{a}$-adic topologies.

Now consider an $\mathfrak{a}$-adically separated A-module M. Recall that for an element $m \in M$ its *order* (with respect to $\mathfrak{a}$) is

$$\mathrm{ord}_{\mathfrak{a}}(m) := \sup\{i \in \mathbb{N} \mid m \in \mathfrak{a}^i M\} \in \mathbb{N} \cup \{\infty\}. \tag{1.10}$$

Sometimes we shall write $\mathrm{ord}_{\mathfrak{a},M}(m)$, when we need to emphasize the module M (e.g., in the proof of Lemma 3.1). Since $\bigcap_{i\geq 0} \mathfrak{a}^i M = 0$, we see that $\mathrm{ord}_{\mathfrak{a}}(m) = \infty$ if and only if $m = 0$. And $\mathrm{ord}_{\mathfrak{a}}(m) = i \in \mathbb{N}$ if and only if $m \in \mathfrak{a}^i M - \mathfrak{a}^{i+1} M$.

For elements $m, n \in M$ define

$$\mathrm{dist}_{\mathfrak{a}}(m, n) := \left(\tfrac{1}{2}\right)^{\mathrm{ord}_{\mathfrak{a}}(m-n)}. \tag{1.11}$$

The function $\mathrm{dist}_{\mathfrak{a}}$ is a metric on M, which we call the *$\mathfrak{a}$-adic metric*. This metric determines the $\mathfrak{a}$-adic topology on M. The module M is $\mathfrak{a}$-adically complete if and only if it is a complete metric space with respect to the $\mathfrak{a}$-adic metric. See [2, Section 10] or [3, Section III.2.5].

We continue with the assumption that M is $\mathfrak{a}$-adically separated; and we view M as a submodule of $\widehat{M}$ via the homomorphism τ_M. The $\mathfrak{a}$-adically separated A-module $\widehat{M}$ has on it two descending filtrations, defining two possibly distinct metrics:

(a) The filtration $\{\mathrm{F}^i \widehat{M}\}_{i\geq 0}$, where $\mathrm{F}^i \widehat{M} := \mathrm{Ker}(\pi_{M,i-1})$ for $i \geq 1$, and $\mathrm{F}^0 \widehat{M} := \widehat{M}$. Here $\pi_{M,i-1}$ is the homomorphism in (1.4). There is a corresponding order function

$$\mathrm{ord}'(m) := \sup\{i \in \mathbb{N} \mid m \in \mathrm{F}^i \widehat{M}\}$$

for $m \in \widehat{M}$, and the corresponding metric is

$$\mathrm{dist}'(m, n) := \left(\tfrac{1}{2}\right)^{\mathrm{ord}'(m-n)}$$

for $m, n \in \widehat{M}$.

(b) The filtration $\{\mathfrak{a}^i \widehat{M}\}_{i\geq 0}$, namely the $\mathfrak{a}$-adic filtration of the A-module $\widehat{M}$ itself. The corresponding order function $\mathrm{ord}_{\mathfrak{a},\widehat{M}}$ and metric $\mathrm{dist}_{\mathfrak{a},\widehat{M}}$ are given by formulas (1.10) and (1.11), replacing M with $\widehat{M}$.

The standard fact (see [2, Section 10]) is that the metric space $(\widehat{M}, \mathrm{dist}')$ is always the completion of the metric space $(M, \mathrm{dist}_{\mathfrak{a}})$.

Corollary 1.12. *Let M be an A-module, with $\mathfrak{a}$-adic completion $\widehat{M}$. The following conditions are equivalent*:

(i) *The A-module $\widehat{M}$ is $\mathfrak{a}$-adically complete*;
(ii) *The metrics $\mathrm{dist}_{\mathfrak{a},\widehat{M}}$ and dist' on $\widehat{M}$ coincide.*

Proof. This is immediate from the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.5. □

Example 1.13. Consider the module M from Example 1.8. Since its $\mathfrak{a}$-adic completion $\widehat{M}$ is not $\mathfrak{a}$-adically complete, we know that the metrics $\mathrm{dist}_{\mathfrak{a},\widehat{M}}$ and dist' are not the same. Indeed, a little calculation shows that for the element $b = \sum_{k=1}^{\infty} t_k^k$ we have $\mathrm{dist}_{\mathfrak{a},\widehat{M}}(b, 0) = 1$, but $\mathrm{dist}'(b, 0) = \frac{1}{2}$.

2. MODULES OF DECAYING FUNCTIONS

The ideas in this section are inspired by functional analysis. As in Section 1, A is a ring and $\mathfrak{a}$ is an ideal in it.

Let Z be a set, and let M be an A-module. We denote by $\mathrm{F}(Z, M)$ the set of all functions $f : Z \to M$, and by $\mathrm{F}_{\mathrm{fin}}(Z, M)$ the set of functions with finite support. So

$$\mathrm{F}(Z, M) \cong \prod_{z\in Z} M$$

and

$$\mathrm{F}_{\mathrm{fin}}(Z, M) \cong \bigoplus_{z\in Z} M.$$

The set $\mathrm{F}(Z, M)$ is an A-module, and $\mathrm{F}_{\mathrm{fin}}(Z, M)$ is a submodule.

Now let us look at the special case $M = A$. For every $z \in Z$ there is the delta function $\delta_z \in \mathrm{F}_{\mathrm{fin}}(Z, A)$, namely, $\delta_z(z) := 1$, and $\delta_z(z') := 0$ for $z' \neq z$. The A-module $\mathrm{F}_{\mathrm{fin}}(Z, A)$ is free; as basis we can take the collection of elements $\{\delta_z\}_{z\in Z}$.

Suppose M is an $\mathfrak{a}$-adically separated A-module, and $m \in M$. Recall the $\mathfrak{a}$-adic order $\mathrm{ord}_{\mathfrak{a}}(m)$ from formula (1.10).

Definition 2.1. Let Z be a set and M an $\mathfrak{a}$-adically separated A-module. A function $f : Z \to M$ is called *decaying* if for every $i \in \mathbb{N}$ the set

$$\{z \in Z \mid \mathrm{ord}_{\mathfrak{a}}(f(z)) \leq i\}$$

is finite. We denote by $\mathrm{F}_{\mathrm{dec}}(Z, M)$ the set of all decaying functions $f : Z \to M$.

The support of a decaying function is of course countable. Any function with finite support is decaying. Thus we have inclusions

$$\mathrm{F}_{\mathrm{fin}}(Z, M) \subset \mathrm{F}_{\mathrm{dec}}(Z, M) \subset \mathrm{F}(Z, M).$$

It is easy to see that $\mathrm{F}_{\mathrm{dec}}(Z, M)$ is an A-submodule of $\mathrm{F}(Z, M)$.

Example 2.2. Suppose that $\mathfrak{a}^i M = 0$ for some i. Then a decaying function has finite support, and we have

$$\mathrm{F}_{\mathrm{fin}}(Z, M) = \mathrm{F}_{\mathrm{dec}}(Z, M).$$

Example 2.3. Suppose A is complete. Take variables $t_1, \ldots, t_n$, and consider the ring of restricted formal power series $A\{t_1, \ldots, t_n\}$ as in [3, Section III.4.2]. Then as A-modules we have $A\{t_1, \ldots, t_n\} \cong \mathrm{F}_{\mathrm{dec}}(\mathbb{N}^n, A)$.

Proposition 2.4. *Let M be an $\mathfrak{a}$-adically separated module. Then $F_{dec}(Z, M)$ is $\mathfrak{a}$-adically separated.*

Proof. Let $f : Z \to M$ be a decaying function and let $a \in \mathfrak{a}^i$. Then $af(z) \in \mathfrak{a}^i M$ for every $z \in Z$. We see that

$$\mathfrak{a}^i \cdot \mathrm{F}_{\mathrm{dec}}(Z, M) \subset \mathrm{F}_{\mathrm{dec}}(Z, \mathfrak{a}^i M).$$

But M is separated, so

$$\bigcap_{i \geq 0} \mathfrak{a}^i \cdot \mathrm{F}_{\mathrm{dec}}(Z, M) \subset \bigcap_{i \geq 0} \mathrm{F}_{\mathrm{dec}}(Z, \mathfrak{a}^i M) = \mathrm{F}_{\mathrm{dec}}\left(Z, \bigcap_{i \geq 0} \mathfrak{a}^i M\right) = 0. \qquad \square$$

Let $\phi : M \to N$ be a homomorphism between $\mathfrak{a}$-adically separated A-modules. For any $m \in M$, we have $\mathrm{ord}_{\mathfrak{a},N}(\phi(m)) \geq \mathrm{ord}_{\mathfrak{a},M}(m)$. Hence there is an induced A-linear homomorphism

$$\mathrm{F}_{\mathrm{dec}}(Z, M) \to \mathrm{F}_{\mathrm{dec}}(Z, N), \qquad f \mapsto \phi \circ f.$$

Let us denote by $\mathsf{Mod}_{\mathrm{sep}}\, A$ the full subcategory of $\mathsf{Mod}\, A$ consisting of $\mathfrak{a}$-adically separated modules; this is an additive category. We see that for a fixed set Z, there is an additive functor

$$\mathrm{F}_{\mathrm{dec}}(Z, -) : \mathsf{Mod}_{\mathrm{sep}}\, A \to \mathsf{Mod}_{\mathrm{sep}}\, A.$$

Suppose M is an $\mathfrak{a}$-adically separated A-module. Let Z be a set, and let $f : Z \to M$ be a function. One says that the *series* $\sum_{z \in Z} f(z)$ *converges* in the $\mathfrak{a}$-adic topology, to some element $m \in M$, if for any natural number i there is a finite subset $Z_i \subset Z$, such that

$$m - \sum_{z \in Z_i} f(z) \in \mathfrak{a}^{i+1} M,$$

and $f(z) \in \mathfrak{a}^{i+1} M$ for all $z \notin Z_i$. In this case one writes

$$m = \sum_{z \in Z} f(z).$$

Of course, if the series converges, then the sum m is unique. Cf. [3, Section III.2.6].

Proposition 2.5. *Let M be an $\mathfrak{a}$-adically complete A-module, and let $f : Z \to M$ be a function. The following conditions are equivalent*:

(i) *The function f is decaying*;
(ii) *The series $\sum_{z \in Z} f(z)$ converges in M for the $\mathfrak{a}$-adic topology.*

The proof is easy, and we leave it out. An immediate consequence is that for an $\mathfrak{a}$-adically complete module M there is an A-linear homomorphism

$$\mathrm{F}_{\mathrm{dec}}(Z, M) \to M, \qquad f \mapsto \sum_{z \in Z} f(z).$$

Corollary 2.6. *Let Z be a set, let M be an A-module, and let $f : Z \to M$ be any function. Assume A and M are $\mathfrak{a}$-adically complete. Then for any $g \in F_{\mathrm{dec}}(Z, A)$ the series $\sum_{z \in Z} g(z) f(z)$ converges in M. This gives rise to an A-linear homomorphism*

$$\phi : F_{\mathrm{dec}}(Z, A) \to M, \qquad \phi(g) := \sum_{z \in Z} g(z) f(z).$$

Proof. Given $g \in \mathrm{F}_{\mathrm{dec}}(Z, A)$ consider the function $Z \to M$, $z \mapsto g(z) f(z)$. This function is decaying, so by Proposition 2.5 the series $\sum_{z \in Z} g(z) f(z)$ converges. It is easy to check that the resulting function ϕ is A-linear. $\square$

Theorem 2.7. *Let M be an A-module whose $\mathfrak{a}$-adic completion $\widehat{M}$ is $\mathfrak{a}$-adically complete. Then the canonical homomorphism*

$$\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M}) \to \lim_{\leftarrow i} \mathrm{F}_{\mathrm{fin}}(Z, A_i \otimes_A M) \tag{2.8}$$

induced by

$$\pi_{M,i} : \widehat{M} \to A_i \otimes_A M$$

is bijective.

Proof. Suppose $f \in \mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ is nonzero. So $f(z) \neq 0$ for some $z \in Z$. Since $\widehat{M}$ is separated, there is some i such that the image $\pi_{M,i}(f(z))$ of $f(z)$ in $A_i \otimes_A M$ is nonzero. This implies that the homomorphism (2.8) is injective.

Conversely, suppose $\{f_i\}_{i \geq 0}$ is an inverse system of functions $f_i : Z \to A_i \otimes_A M$, each with finite support. For any $z \in Z$, let

$$f(z) := \lim_{\leftarrow i} f_i(z) \in \lim_{\leftarrow i} (A_i \otimes_A M) = \widehat{M}.$$

We get a function $f : Z \to \widehat{M}$ satisfying $\pi_{M,i} \circ f = f_i$. Since each f_i has finite support, and by Theorem 1.5 we know that $\mathrm{Ker}(\pi_{M,i}) = \mathfrak{a}^{i+1} \widehat{M}$, it follows that f is a decaying function. $\square$

Corollary 2.9. *Let M be as in Theorem* 2.7. *Then the homomorphism*

$$\mathrm{F}_{\mathrm{fin}}(Z, M) \to \mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$$

induced by τ_M makes $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ *into an* $\mathfrak{a}$*-adic completion of* $\mathrm{F}_{\mathrm{fin}}(Z, M)$. *More precisely, there is an isomorphism*

$$\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M}) \cong \Lambda_{\mathfrak{a}} \mathrm{F}_{\mathrm{fin}}(Z, M)$$

that commutes with the homomorphisms from $\mathrm{F}_{\mathrm{fin}}(Z, M)$ *and is functorial in* M.

The reason for the careful wording of the corollary is because $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ might fail to be $\mathfrak{a}$-adically complete. Cf. Example 1.8 and Corollary 1.12.

Proof. Since there is a canonical isomorphism

$$A_i \otimes_A \mathrm{F}_{\mathrm{fin}}(Z, M) \cong \mathrm{F}_{\mathrm{fin}}(Z, A_i \otimes_A M)$$

for every i, this follows from Theorem 2.7. $\square$

Definition 2.10. Let M be an A-module, and let $\{m_z\}_{z \in Z}$ be a collection of elements of M. Assume A and M are $\mathfrak{a}$-adically complete. We say the collection $\{m_z\}_{z \in Z}$ *$\mathfrak{a}$-adically generates M* if for every element $m \in M$ there exists some decaying function $g : Z \to A$ such that

$$m = \sum_{z \in Z} g(z) m_z.$$

Here is a version of the Nakayama Lemma.

Theorem 2.11. *Let M be an A-module, and let $\{m_z\}_{z \in Z}$ be a collection of elements of M. Assume A and M are $\mathfrak{a}$-adically complete. Write $M_0 := A_0 \otimes_A M$, and let $\pi_0 : M \to M_0$ be the canonical homomorphism. Then the two conditions below are equivalent*:

(i) *The collection $\{\pi_0(m_z)\}_{z \in Z}$ generates the A_0-module M_0*;
(ii) *The collection $\{m_z\}_{z \in Z}$ $\mathfrak{a}$-adically generates M.*

Proof. Let $\phi : \mathrm{F}_{\mathrm{dec}}(Z, A) \to M$ be the homomorphism corresponding to the function $f : Z \to M$, $f(z) := m_z$, as in Corollary 2.6. Then $\{m_z\}_{z \in Z}$ $\mathfrak{a}$-adically generates M if and only if ϕ is surjective.

For every $i \geq 0$, let $M_i := A_i \otimes_A M$. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{F}_{\mathrm{dec}}(Z, A) & \xrightarrow{\phi} & M \\ \downarrow & & \downarrow \\ \mathrm{F}_{\mathrm{fin}}(Z, A_i) & \xrightarrow{\phi_i} & M_i \end{array} \tag{2.12}$$

in which the vertical arrows are the surjections coming from the ring homomorphisms $A \to A_i$. For $i = 0$, we have $\pi_0(m_z) = \phi_0(\delta_z) \in M_0$ for all $z \in Z$. Hence the collection $\{\pi_0(m_z)\}_{z \in Z}$ generates the A_0-module M_0 if and only if ϕ_0 is surjective. The implication (ii) $\Rightarrow$ (i) is now clear.

Now let us assume (i), namely, that ϕ_0 is surjective. Since for every i the ideal $\mathfrak{a}/\mathfrak{a}^{i+1} = \mathrm{Ker}(A_i \to A_0)$ is nilpotent, the usual Nakayama Lemma (see [3, Corollary II.3.1]) says that ϕ_i is surjective. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{F}_{\mathrm{dec}}(Z, A) \arrow[r, "\phi"] \arrow[d] & M \arrow[d, "\tau_M"] \\
\lim_{\leftarrow i} \mathrm{F}_{\mathrm{fin}}(Z, A_i) \arrow[r, "\lim_{\leftarrow i} \phi_i"] & \lim_{\leftarrow i} M_i
\end{tikzcd}$$

gotten as the inverse limit of the sequences (2.12). As in the proof of Proposition 1.2 one shows that the homomorphism $\lim_{\leftarrow i} \phi_i$ is surjective. By Theorem 2.7 the left vertical arrow is bijective; and by assumption τ_M is bijective. It follows that ϕ is surjective. $\square$

To end this section, here are some remarks.

Remark 2.13. Suppose A is complete. There is a canonical pairing

$$\mathrm{F}(Z, A) \times \mathrm{F}_{\mathrm{dec}}(Z, A) \to A,$$

namely,

$$\langle f, g \rangle := \sum_{z \in Z} f(z) g(z).$$

If we put the discrete topology on $\mathrm{F}_{\mathrm{dec}}(Z, A)$, and a suitable topology on $\mathrm{F}(Z, A)$, then this becomes a perfect pairing (i.e., it identifies each of these A-modules with the continuous dual of the other).

Suppose $h : Y \to Z$ is a function. Then there is a ring homomorphism

$$h^* : \mathrm{F}(Z, A) \to \mathrm{F}(Y, A)$$

called pullback, namely, $h^*(f) = f \circ h$. And there is an $\mathrm{F}(Z, A)$-module homomorphism

$$h_* : \mathrm{F}_{\mathrm{dec}}(Y, A) \to \mathrm{F}_{\mathrm{dec}}(Z, A),$$

which is

$$h_*(g)(z) := \sum_{y \in h^{-1}(z)} g(y) \in A.$$

In this way $\mathrm{F}_{\mathrm{dec}}(Z, A)$ resembles the space $\mathrm{L}^1(Z)$ from functional analysis, and $\mathrm{F}(Z, A)$ resembles the space $\mathrm{L}^\infty(Z)$.

Remark 2.14. Suppose $\{M_z\}_{z \in Z}$ is a collection of $\mathfrak{a}$-adically separated A-modules. By an obvious generalization of Definition 2.1, we can form the decaying direct product $\prod^{\mathrm{dec}}_{z \in Z} M_z$, which is a submodule of $\prod_{z \in Z} M_z$. In case A is noetherian and complete, and all the modules M_z are finitely generated, one can show (just as in Theorem 3.4) that the module $\prod^{\mathrm{dec}}_{z\in Z} M_z$ is $\mathfrak{a}$-adically complete, and it is flat if and only if all the modules M_z are flat.

3. NOETHERIAN RINGS AND THEIR COMPLETIONS

In this section A is a noetherian ring, and $\mathfrak{a}$ is an ideal in it. The $\mathfrak{a}$-adic completion of A is $\widehat{A}$, and we write $\hat{\mathfrak{a}} := \mathfrak{a}\widehat{A}$, which is an ideal in $\widehat{A}$. It is well-known that the ring $\widehat{A}$ is $\hat{\mathfrak{a}}$-adically complete, flat over A, and for every $i \geq 0$ the canonical homomorphism $A_i = A/\mathfrak{a}^{i+1} \to \widehat{A}/\hat{\mathfrak{a}}^{i+1}$ is bijective. It is also well-known that every finitely generated $\widehat{A}$-module is $\hat{\mathfrak{a}}$-adically complete. We are of course allowing the case $A = \widehat{A}$. See [2, Section 10] or [3, Section III.3].

Let M be an $\widehat{A}$-module. Since $\mathfrak{a}^i M = \hat{\mathfrak{a}}^i M$ for every $i \geq 0$, it follows that $\Lambda_{\mathfrak{a}} M = \Lambda_{\hat{\mathfrak{a}}} M$. So M is $\mathfrak{a}$-adically separated (resp., complete) if and only if it is $\hat{\mathfrak{a}}$-adically separated (resp., complete). And when M is separated we have $\mathrm{ord}_{\mathfrak{a},M} = \mathrm{ord}_{\hat{\mathfrak{a}},M}$, so a function $f : Z \to M$ is $\mathfrak{a}$-adically decaying if and only if it is $\hat{\mathfrak{a}}$-adically decaying.

Lemma 3.1. *Suppose M is a finitely generated $\widehat{A}$-module, and N is an $\widehat{A}$-submodule of M. Then*

$$\mathrm{F}_{\mathrm{dec}}(Z, N) = \mathrm{F}_{\mathrm{dec}}(Z, M) \cap \mathrm{F}(Z, N)$$

as submodules of $\mathrm{F}(Z, M)$.

Proof. Since $\mathfrak{a}^i N \subset \mathfrak{a}^i M$ for any $i \geq 0$, it follows that $\mathrm{ord}_{\mathfrak{a},N}(n) \leq \mathrm{ord}_{\mathfrak{a},M}(n)$ for any $n \in N$. By the Artin–Rees Lemma (cf. [3, Corollary III.3.1]), there is some i_0 such that

$$N \cap (\mathfrak{a}^{i_0+i} M) \subset \mathfrak{a}^i N$$

for all $i \geq 0$. Therefore, for $n \in N$ we have

$$\mathrm{ord}_{\mathfrak{a},M}(n) \leq \mathrm{ord}_{\mathfrak{a},N}(n) + i_0.$$

We conclude that the $\mathfrak{a}$-adic decay conditions with respect to M and to N are equivalent, for a function $f : Z \to N$. $\square$

Let us denote by $\mathsf{Mod}_{\mathrm{f}}\, \widehat{A}$ the full subcategory of $\mathsf{Mod}\, \widehat{A}$ consisting of finitely generated $\widehat{A}$-modules. The subcategory $\mathsf{Mod}_{\mathrm{f}}\, \widehat{A}$ is abelian (since $\widehat{A}$ is noetherian). Note that $\mathsf{Mod}_{\mathrm{f}}\, \widehat{A} \subset \mathsf{Mod}_{\mathrm{sep}}\, \widehat{A}$.

Lemma 3.2. *For a given set Z, the functor*

$$\mathrm{F}_{\mathrm{dec}}(Z, -) : \mathsf{Mod}_{\mathrm{f}}\, \widehat{A} \to \mathsf{Mod}\, \widehat{A}$$

is exact.

Proof. Consider an exact sequence

$$0 \to M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \to 0$$

of finitely generated $\widehat{A}$-modules. We want to show that the sequence

$$0 \to \mathrm{F}_{\mathrm{dec}}(Z, M') \xrightarrow{\phi} \mathrm{F}_{\mathrm{dec}}(Z, M) \xrightarrow{\psi} \mathrm{F}_{\mathrm{dec}}(Z, M'') \to 0$$

is also exact.

Since $\psi(M) = M''$, it follows that $\psi(\mathfrak{a}^i M) = \mathfrak{a}^i M''$ for all i. Take any $f \in \mathrm{F}_{\mathrm{dec}}(Z, M'')$. For any $z \in Z$, we can lift $f(z) \in M''$ to some element $g(z) \in M$, such that $\mathrm{ord}_{\mathfrak{a},M}(g(z)) \geq \mathrm{ord}_{\mathfrak{a},M''}(f(z))$. We get a decaying function $g : Z \to M$ lifting f. So we have exactness at $\mathrm{F}_{\mathrm{dec}}(Z, M'')$.

Exactness at $\mathrm{F}_{\mathrm{dec}}(Z, M)$ is by Lemma 3.1, and exactness at $\mathrm{F}_{\mathrm{dec}}(Z, M')$ is trivial. □

Lemma 3.3. *Let M be a finitely generated $\widehat{A}$-module. Then the canonical homomorphism*

$$M \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \to \mathrm{F}_{\mathrm{dec}}(Z, M)$$

is bijective.

Proof. We use the standard trick of finite free presentations. Choose some finite presentation of M; namely, an exact sequence $Q \to P \to M \to 0$, where P and Q are finitely generated free $\widehat{A}$-modules. There is an induced commutative diagram

$$\begin{tikzcd}
Q \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \arrow[r] \arrow[d, "\phi_Q"] & P \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \arrow[r] \arrow[d, "\phi_P"] & M \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \arrow[r] \arrow[d, "\phi_M"] & 0 \\
\mathrm{F}_{\mathrm{dec}}(Z, Q) \arrow[r] & \mathrm{F}_{\mathrm{dec}}(Z, P) \arrow[r] & \mathrm{F}_{\mathrm{dec}}(Z, M) \arrow[r] & 0 .
\end{tikzcd}$$

The top row is exact because of right-exactness of the tensor product; and the bottom row is exact by Lemma 3.2. The homomorphisms ϕ_P and ϕ_Q are bijective since P and Q are finite rank free modules. It follows that ϕ_M is also bijective. □

Here is the main result of this section. Observe that it refers only to the complete ring $\widehat{A}$.

Theorem 3.4. *Let $\widehat{A}$ be a noetherian ring, $\hat{\mathfrak{a}}$-adically complete with respect to some ideal $\hat{\mathfrak{a}}$. Let Z be any set. Then*:

(1) *For any $i \geq 0$ the canonical homomorphism*

$$A_i \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \to \mathrm{F}_{\mathrm{fin}}(Z, A_i)$$

is bijective. Here $A_i := \widehat{A}/\hat{\mathfrak{a}}^{i+1}$.

(2) *The $\widehat{A}$-module $\mathrm{F}_{dec}(Z, \widehat{A})$ is flat and $\hat{\mathfrak{a}}$-adically complete.*

Proof. (1) This is true by Lemma 3.3, with $M := A_i$.

(2) Since $\widehat{A}$ is noetherian, the $\widehat{A}$-module $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$ is flat if and only if the functor $- \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$ is exact on $\mathsf{Mod}_{\mathrm{f}}\, \widehat{A}$. The latter is true by Lemmas 3.2 and 3.3.

As for completeness, combining part (1) above with Theorem 2.7 (for the module $M := \widehat{A}$) we see that the canonical homomorphism

$$\tau_{\mathrm{F}_{\mathrm{dec}}(Z,\widehat{A})} : \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \to \lim_{\leftarrow i} \big(A_i \otimes_{\widehat{A}} \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})\big)$$

is bijective. □

Here are several corollaries to Theorem 3.4.

Corollary 3.5. *Let M be any A-module. Its $\mathfrak{a}$-adic completion $\widehat{M}$ is $\mathfrak{a}$-adically complete.*

Proof. Choose any surjection $\phi : \mathrm{F}_{\mathrm{fin}}(Z, A) \to M$, where Z is some set, and write $Q := \mathrm{F}_{\mathrm{fin}}(Z, A)$. By Proposition 1.2 the homomorphism $\hat{\phi} : \widehat{Q} \to \widehat{M}$ is surjective. Hence for every $i \geq 0$ we get a commutative diagram

$$\begin{tikzcd} A_i \otimes_A Q \arrow[r, "\tau_{Q,i}"] \arrow[d, "\mathrm{id}_{A_i} \otimes \phi"'] & A_i \otimes_A \widehat{Q} \arrow[d, "\mathrm{id}_{A_i} \otimes \hat{\phi}"] \\ A_i \otimes_A M \arrow[r, "\tau_{M,i}"] & A_i \otimes_A \widehat{M} \end{tikzcd}$$

with surjective vertical arrows. By Corollary 2.9 and Theorem 3.4(2) the module $\widehat{Q}$ is $\mathfrak{a}$-adically complete, and hence by Theorem 1.5 the homomorphisms $\tau_{Q,i}$ is surjective. It follows that $\tau_{M,i}$ is also surjective, for every i. Again using Theorem 1.5, we conclude that $\widehat{M}$ is complete. □

Corollary 3.6. *Let B be a ring, and let $\mathfrak{b}$ be a finitely generated ideal in it. Given any B-module M, its $\mathfrak{b}$-adic completion $\Lambda_{\mathfrak{b}} M$ is $\mathfrak{b}$-adically complete.*

Proof. Choose generators $b_1, \ldots, b_n$ of the ideal $\mathfrak{b}$. Consider the polynomial ring $A := \mathbb{Z}[t_1, \ldots, t_n]$, the ideal $\mathfrak{a} := (t_1, \ldots, t_n) \subset A$, and the ring homomorphism $f : A \to B$ defined by $f(t_i) := b_i$. Then for any B-module N there is a canonical isomorphism of B-modules $\Lambda_{\mathfrak{b}} N \cong \Lambda_{\mathfrak{a}} N$, that commutes with the homomorphisms from N. Since A is noetherian, we know from Corollary 3.5 that $N := \Lambda_{\mathfrak{a}} M$ is $\mathfrak{a}$-adically complete. □

Remark 3.7. The assertions of Corollaries 3.5 and 3.6 are not new, yet they seem to be virtually unknown. After we proved Theorem 3.4, A.-M. Simon mentioned to us the book [12], and in Subsection 2.2.5 of that book we located these assertions (in slightly different wording). We then learned that Corollary 3.6 appeared much earlier as [10, Theorem 15]. Note that our proof of Theorem 3.4, involving the concept of decaying functions, is completely new and is not similar to the proofs in these cited works.

Corollary 3.6 resembles [3, Proposition III.14]. However, a close inspection reveals that these two assertions refer to distinct notions of completion. See Example 1.8, Corollary 1.12, and the discussion between them.

Corollary 3.8. *Let M be any A-module. Then the A-module $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ is $\mathfrak{a}$-adically complete.*

Proof. According to Corollary 3.5, the module $\widehat{M}$ is complete. By Corollary 2.9 we know that $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ is (canonically isomorphic to) the $\mathfrak{a}$-adic completion of $\mathrm{F}_{\mathrm{fin}}(Z, M)$. Now use Corollary 3.5 again to conclude that $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{M})$ is complete. □

Corollary 3.9. *Let Z be a set, let M be an $\mathfrak{a}$-adically complete A-module, and let $f : Z \to M$ be any function. Then there is a unique A-linear homomorphism*

$$\phi : \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \to M$$

such that $\phi(\delta_z) = f(z)$ for all $z \in Z$.

Proof. The existence of such a homomorphism was already proved in Corollary 2.6. Recall that the formula is

$$\phi(g) = \sum_{z \in Z} g(z) f(z) \in M$$

for $g \in \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$. Uniqueness is because M is complete, and the image of $\mathrm{F}_{\mathrm{fin}}(Z, A)$ in $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$, which is the A-submodule generated by the collection $\{\delta_z\}_{z \in Z}$, is dense in $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$, by Theorem 3.4(1). □

Example 3.10. Take any set Z. Consider the function $f : Z \to \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$, $f(z) := \delta_z$. The corresponding homomorphism ϕ is the identity of $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$. This says that

$$g = \sum_{z \in Z} g(z) \delta_z$$

for any $g \in \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$.

Definition 3.11. An A-module P is called $\mathfrak{a}$-*adically free* if it isomorphic to the A-module $\mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$ for some set Z.

Corollary 3.12. *Suppose B is another noetherian ring, $\mathfrak{b} \subset B$ is an ideal, and $f : A \to B$ is a ring homomorphism satisfying $f(\mathfrak{a}) \subset \mathfrak{b}$. If P is an $\mathfrak{a}$-adically free A-module, then the B-module*

$$B \mathbin{\widehat{\otimes}}_A P := \Lambda_{\mathfrak{b}}(B \otimes_A P)$$

is $\mathfrak{b}$-adically free.

Proof. Letting $B_i := B/\mathfrak{b}^{i+1}$, we have induced ring homomorphisms $A_i \to B_i$ for all $i \geq 0$. Choose an A-module isomorphism $P \cong \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$. Then by Theorem 3.4 we have canonical isomorphisms

$$B_i \otimes_A P \cong B_i \otimes_{A_i} A_i \otimes_A \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A}) \cong B_i \otimes_{A_i} \mathrm{F}_{\mathrm{fin}}(Z, A_i) \cong \mathrm{F}_{\mathrm{fin}}(Z, B_i).$$

We see that

$$\Lambda_{\mathfrak{b}}(B \otimes_A P) \cong \lim_{\leftarrow i}(B_i \otimes_A P) \cong \lim_{\leftarrow i} \mathrm{F}_{\mathrm{fin}}(Z, B_i) \cong \mathrm{F}_{\mathrm{dec}/\mathfrak{b}}(Z, \Lambda_{\mathfrak{b}} B),$$

where $\mathrm{F}_{\mathrm{dec}/\mathfrak{b}}(Z, -)$ refers to the $\mathfrak{b}$-adic decay condition. □

Proposition 3.13. *The following two conditions are equivalent for an A-module P:*

(i) *P is $\mathfrak{a}$-adically free;*
(ii) *P is isomorphic to $\mathfrak{a}$-adic completion $\widehat{Q}$ of some free A-module Q.*

Proof. First suppose $P \cong \widehat{Q}$ for some free A-module Q. By choosing a basis for Q, indexed by a set Z, we get an isomorphism $Q \cong \mathrm{F}_{\mathrm{fin}}(Z, A)$. According to Corollary 2.9 we get an isomorphism $P \cong \mathrm{F}_{\mathrm{dec}}(Z, \widehat{A})$. The reverse implication is proved similarly. □

Example 3.14. Suppose A is complete, $\mathbb{K}$ is a field, and $\mathbb{K} \to A$ is a ring homomorphism. Let V be a $\mathbb{K}$-module. The A-module $A \otimes_{\mathbb{K}} V$ is free, and therefore, its $\mathfrak{a}$-adic completion $A \widehat{\otimes}_{\mathbb{K}} V := \Lambda_{\mathfrak{a}}(A \otimes_{\mathbb{K}} V)$ is $\mathfrak{a}$-adically free.

Corollary 3.15. *Suppose M is an $\mathfrak{a}$-adically complete A-module. Then there is a surjection $\phi : P \to M$ for some $\mathfrak{a}$-adically free A-module P.*

Proof. Choose a surjection $\psi : Q \to M$, where Q is some free A-module. By Proposition 1.2 the induced homomorphism $\hat{\psi} : \widehat{Q} \to \widehat{M}$ is surjective. We know that $P := \widehat{Q}$ is $\mathfrak{a}$-adically free (see Proposition 3.13), and that $\tau_M : M \to \widehat{M}$ is bijective. So we can take $\phi := \tau_M^{-1} \circ \hat{\psi}$. □

Definition 3.16. An A-module P is called $\mathfrak{a}$-*adically projective* if it satisfies the following two conditions:

(i) P is $\mathfrak{a}$-adically complete;
(ii) Suppose M and N are $\mathfrak{a}$-adically complete A-modules, and $\phi : M \to N$ a surjective homomorphism. Then any homomorphism $\psi : P \to N$ can be lifted to a homomorphism $\tilde{\psi} : P \to M$, such that $\psi = \phi \circ \tilde{\psi}$.

Remark 3.17. It can be shown that condition (ii) above is equivalent to P being *topologically projective*, in the sense of [6, Section 0_{IV}.19.2]

Corollary 3.18. *An A-module P is $\mathfrak{a}$-adically projective if and only if it is a direct summand of an $\mathfrak{a}$-adically free module.*

Proof. First assume that P is a direct summand of an $\mathfrak{a}$-adically free module; say $P \oplus P' = Q$. By Theorem 3.4(2) and Corollary 3.9, the $\mathfrak{a}$-adically free module Q is $\mathfrak{a}$-adically projective. And it is easy to see that a direct summand of an $\mathfrak{a}$-adically projective module is also $\mathfrak{a}$-adically projective.

Conversely, assume that P is $\mathfrak{a}$-adically projective. Because P is complete, by Corollary 3.15 there exists a surjection $\phi : Q \to P$ for some $\mathfrak{a}$-adically free module Q. Since P and Q are both complete, condition (ii) says that ϕ is split. □

To finish this section, here are a couple of examples and a remark. The first example is a bit facile, but instructive.

Example 3.19. Let $\mathbb{K}$ be a field, and let $A := \mathbb{K}[[t]]$, the ring of formal power series in one variable. It is a complete noetherian local ring with maximal ideal $\mathfrak{a} = (t)$. Let $K := \mathbb{K}((t))$, the field of fractions of A. Consider the inclusion $\phi : A \to K$. For any $i \geq 0$, we have $A_i \otimes_A K = 0$. Therefore, $\widehat{K} = 0$, and $\hat{\phi} : \widehat{A} \to \widehat{K}$ is not injective.

We see that the functor $\Lambda_{\mathfrak{a}}$ does not respect injections. Since it does respect surjections (Proposition 1.2), one is tempted to guess that $\Lambda_{\mathfrak{a}}$ is right exact. But here is a counterexample.

Example 3.20. With $A := \mathbb{K}[[t]]$ and $\mathfrak{a} = (t)$ as in the previous example, let $P, Q := \mathrm{F}_{\mathrm{dec}}(\mathbb{N}, A)$. Define a homomorphism $\phi : P \to Q$ by $\phi(\delta_i) := t^i \delta_i$, where $\delta_i \in \mathrm{F}_{\mathrm{dec}}(\mathbb{N}, A)$ are the delta functions. It is easy to see that ϕ is injective.

We claim that the submodule $L := \mathrm{Im}(\phi)$ is not closed in Q. Indeed, consider the element $f := \sum_{i \in \mathbb{N}} t^i \delta_i \in Q$. Clearly, f is in the closure $\bar{L}$ of L. If there were some $g \in P$ such that $f = \phi(g)$, then writing $a_i := g(i) \in A$, we would have $g = \sum_i a_i \delta_i$. Hence

$$f = \phi(g) = \sum_i a_i \phi(\delta_i) = \sum_i a_i t^i \delta_i.$$

By uniqueness of the series expansion, it would follow that $a_i = 1$ for all i. But then the function $g : \mathbb{N} \to A$ would not be decaying; so we arrive at a contradiction.

Let us define $M := Q/L$. So there is an exact sequence of A-modules

$$0 \to P \xrightarrow{\phi} Q \xrightarrow{\psi} M \to 0. \tag{3.21}$$

Now P and Q are complete, so we can identify them with their completions $\widehat{P}$ and $\widehat{Q}$. According to Proposition 1.2 the homomorphism $\hat{\psi} : Q \to \widehat{M}$ is surjective, and by Corollary 3.5 the module $\widehat{M}$ is complete. Therefore, $\mathrm{Ker}(\hat{\psi}) = \bar{L}$. Because $L \subsetneq \bar{L}$ we see that $\tau_M : M \to \widehat{M}$ is surjective but not bijective. Thus M *is not* $\mathfrak{a}$*-adically complete*. Also, since $\hat{\phi} = \phi$, we see that the sequence

$$0 \to \widehat{P} \xrightarrow{\hat{\phi}} \widehat{Q} \xrightarrow{\hat{\psi}} \widehat{M} \to 0$$

that we get by completing (3.21) is not exact at $\widehat{Q}$. This shows that *the functor* $\Lambda_{\mathfrak{a}}$ *is not right exact*.

Remark 3.22. Suppose A is complete and Q is a free A-module of *countable* rank. V. Drinfeld and M. Hochster mentioned to us an alternative proof of the fact that $\widehat{Q}$ is flat and $\mathfrak{a}$-adically complete. In this case Q is isomorphic, as A-module, to the polynomial algebra $A[t]$. Then the completion $\widehat{Q}$ is isomorphic, as A-module, to the algebra $A\{t\}$ of restricted formal power series; see Example 2.3. It is shown in [3] that $A\{t\}$ is $\mathfrak{a}$-adically complete and flat over A.

4. COMPLETE NOETHERIAN LOCAL RINGS

In this section, A is a complete noetherian local commutative ring, with maximal ideal $\mathfrak{m}$. For $i \geq 0$, we write $A_i := A/\mathfrak{m}^{i+1}$.

Definition 4.1. An $\mathfrak{m}$*-adic system of* A*-modules* is a collection $\{M_i\}_{i\in\mathbb{N}}$ of A-modules, together with a collection $\{\psi_i\}_{i\in\mathbb{N}}$ of homomorphisms $\psi_i : M_{i+1} \to M_i$. The conditions are:

(i) For every i, one has $\mathfrak{m}^{i+1}M_i = 0$. Thus M_i is an A_i-module;
(ii) For every i, the A_i-linear homomorphism $A_i \otimes_{A_{i+1}} M_{i+1} \to M_i$ induced by ψ_i is an isomorphism.

Usually, the collection of homomorphisms $\{\psi_i\}_{i\in\mathbb{N}}$ remains implicit.

Example 4.2. Suppose M is an A-module, and let $M_i := A_i \otimes_A M$. Then $\{M_i\}_{i\in\mathbb{N}}$ is an $\mathfrak{m}$-adic system of A-modules.

Theorem 4.3. *Let A be a complete noetherian local ring, with maximal ideal $\mathfrak{m}$, and let $\{M_i\}_{i\in\mathbb{N}}$ be an $\mathfrak{m}$-adic system of A-modules. Assume that M_i is flat over A_i for every i. Define $M := \lim_{\leftarrow i} M_i$. Then the following hold*:

(1) *The A-module M is $\mathfrak{m}$-adically free*;
(2) *For every $i \geq 0$, the canonical homomorphism $A_i \otimes_A M \to M_i$ is bijective.*

We need an auxiliary result.

Lemma 4.4. *In the setup of the theorem, suppose M_i is a free A_i-module with basis $\{\bar{m}_z\}_{z\in Z}$. Let $m_z \in M_{i+1}$ be a lifting of $\bar{m}_z$. Then M_{i+1} is a free A_{i+1}-module with basis $\{m_z\}_{z\in Z}$.*

This result must be well known, but we could not locate a reference in the literature. The closest we got is [11, Proposition 3.G].

Proof. Since the ideal $\mathfrak{m}^{i+1}/\mathfrak{m}^{i+2} = \operatorname{Ker}(A_{i+1} \to A_i)$ is nilpotent, Nakayama's Lemma says that the collection $\{m_z\}_{z\in Z}$ generates M_{i+1}. So there is an exact sequence of A_{i+1}-modules

$$0 \to N \to \mathrm{F}_{\mathrm{fin}}(Z, A_{i+1}) \xrightarrow{\phi} M_{i+1} \to 0,$$

where $\phi(\delta_z) := m_z$ and $N := \mathrm{Ker}(\phi)$. Applying the operation $A_i \otimes_{A_{i+1}} -$ to this sequence we get an exact sequence

$$\mathrm{Tor}_1^{A_{i+1}}(A_i, M_{i+1}) \to A_i \otimes_{A_{i+1}} N \to \mathrm{F}_{\mathrm{fin}}(Z, A_i) \xrightarrow{\bar{\phi}} M_i \to 0.$$

Since M_{i+1} is flat we get $\mathrm{Tor}_1^{A_{i+1}}(A_i, M_{i+1}) = 0$. On the other hand, since $\{\bar{m}_z\}_{z\in Z}$ is a basis, we see that $\bar{\phi} : \mathrm{F}_{\mathrm{fin}}(Z, A_i) \to M_i$ is bijective. It follows that $A_i \otimes_{A_{i+1}} N = 0$. Using the Nakayama Lemma once more we see that $N = 0$. □

Proof of the Theorem. Since A_0 is a field, the A_0-module M_0 is free. Let us choose a basis $\{m_z\}_{z\in Z}$ for M_0. By the lemma above, used recursively, we can lift this basis to a basis of M_i for every $i \geq 0$. Thus we get an inverse system of isomorphisms $M_i \cong \mathrm{F}_{\mathrm{fin}}(Z, A_i)$. In the limit we get $M \cong \mathrm{F}_{\mathrm{dec}}(Z, A)$, by Theorem 3.4(1,2). So M is $\mathfrak{m}$-adically free.

Finally, according to Theorem 3.4(1), we have $A_i \otimes_A M \cong M_i$. □

Corollary 4.5. *The following conditions are equivalent for an A-module M:*

(i) *M is flat and $\mathfrak{m}$-adically complete*;
(ii) *There is an $\mathfrak{m}$-adic system of A-modules $\{M_i\}_{i\in\mathbb{N}}$, such that M_i is flat over A_i for every i, and an isomorphism of A-modules $M \cong \lim_{\leftarrow i} M_i$*;
(iii) *M is $\mathfrak{m}$-adically free.*

Proof. The implication (i) ⇒ (ii) is trivial. The implication (ii) ⇒ (iii) is Theorem 4.3(1). And the implication (iii) ⇒ (i) is Theorem 3.4(2). □

Remark 4.6. A special case of Corollary 4.5, namely, when $A = \mathbb{K}[[t]]$, the ring of formal power series in a variable t over a field $\mathbb{K}$, was proved in [4, Lemma A.1].

Remark 4.7. Assume A is an equal characteristic complete local ring, namely, it contains a field $\mathbb{K}$ such that $\mathbb{K} \cong A/\mathfrak{m}$. Let Q be a free A-module and $P := \widehat{Q}$. In this case there is an alternative way to prove Theorem 3.4(2). First choose an isomorphism $Q \cong A \otimes_{\mathbb{K}} V$ for some $\mathbb{K}$-module V. Next choose a filtered $\mathbb{K}$-basis $\{a_j\}_{j\in\mathbb{N}}$ for A (cf. [13, Definition 6.5]; we may assume $\mathfrak{m}$ is not nilpotent). Then we obtain $\mathbb{K}$-module isomorphisms $A \cong \prod_{j\geq 0} \mathbb{K}$, $P \cong \prod_{j\geq 0} V$ and $\mathfrak{m}^i P \cong \prod_{j\geq j_i} V$, where $0 = j_0 < j_1 < j_2 \cdots$. This implies completeness of P. Flatness is proved similarly, but it is a bit more complicated.

5. FLAT COMPLETE SHEAVES OF MODULES

In this section there is some overlap with material from [9].

Let X be a topological space and A a commutative ring. Recall that given sheaves $\mathcal{M}, \mathcal{N}$ of A-modules on X, the sheaf of A-modules $\mathcal{N} \otimes_A \mathcal{M}$ is the sheaf associated to the presheaf

$$U \mapsto \Gamma(U, \mathcal{N}) \otimes_A \Gamma(U, \mathcal{M}),$$

for open sets $U \subset X$. If N is an A-module, then we can similarly consider the sheaf $N \otimes_A \mathcal{M}$ on X; this is the sheaf associated to the presheaf

$$U \mapsto N \otimes_A \Gamma(U, \mathcal{M}).$$

Given an A-algebra B, the sheaf $B \otimes_A \mathcal{M}$ becomes a sheaf of B-modules. If $\{\mathcal{M}_i\}_{i \in \mathbb{N}}$ is an inverse system of sheaves of modules on X, then $\lim_{\leftarrow i} \mathcal{M}_i$ is the sheaf $U \mapsto \lim_{\leftarrow i} \Gamma(U, \mathcal{M}_i)$. Recall that the sheaf $\mathcal{M}$ is said to be flat over A if for every point $x \in X$ the stalk $\mathcal{M}_x$ is a flat A-module.

Now suppose A is a complete noetherian local ring, with maximal ideal $\mathfrak{m}$. For $i \geq 0$ we write $A_i := A/\mathfrak{m}^{i+1}$.

Definition 5.1. Let $\mathcal{M}$ be a sheaf of A-modules on X.

(1) The $\mathfrak{m}$-adic completion of $\mathcal{M}$ is the sheaf

$$\widehat{\mathcal{M}} := \lim_{\leftarrow i}(A_i \otimes_A \mathcal{M}).$$

(2) The sheaf $\mathcal{M}$ is called $\mathfrak{m}$-*adically complete* if the canonical sheaf homomorphism $\tau_{\mathcal{M}} : \mathcal{M} \to \widehat{\mathcal{M}}$ is an isomorphism.

We sometimes use the notation $\Lambda_{\mathfrak{m}}\mathcal{M} := \widehat{\mathcal{M}}$. With this notation we have an additive functor

$$\Lambda_{\mathfrak{m}} : \mathsf{Mod}\, A_X \to \mathsf{Mod}\, A_X.$$

Here A_X is the constant sheaf A on X, and $\mathsf{Mod}\, A_X$ is the category of sheaves of A_X-modules, which is the same as the category of sheaves of A-modules on X.

Suppose B is another complete noetherian local ring, with maximal ideal $\mathfrak{n}$, and we are given a local homomorphism $A \to B$. For any sheaf of A-modules $\mathcal{M}$ on X, and any B-module N, we write

$$N \widehat{\otimes}_A \mathcal{M} := \Lambda_{\mathfrak{n}}(N \otimes_A \mathcal{M}).$$

The inverse limit in the completion operation does not commute with the direct limit of passing to stalks. Hence the stalk $\mathcal{M}_x$ of an $\mathfrak{m}$-adically complete sheaf of A-modules $\mathcal{M}$, at a point $x \in X$, is usually not an $\mathfrak{m}$-adically complete A-module. This is a well known fact; see [5, Paragraph 10.1.5], or the next example.

Example 5.2. Take $X := \mathbf{A}^1_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[t]$, the affine line over an infinite field $\mathbb{K}$, with coordinate t and structure sheaf $\mathcal{O}_X$. Let $A := \mathbb{K}[[s]]$, the formal power series ring in the variable s. This is a complete noetherian local ring, whose maximal ideal is $\mathfrak{m} = (s)$. Let

$$\mathcal{M} := \mathcal{O}_X[[s]] \cong A \,\widehat{\otimes}_{\mathbb{K}}\, \mathcal{O}_X.$$

The sheaf $\mathcal{M}$ is $\mathfrak{m}$-adically complete; indeed on any open set $U \subset X$ (they are all affine) one has $\Gamma(U, \mathcal{M}) \cong \Gamma(U, \mathcal{O}_X)[[s]]$.

Now let us look at the closed point $x := (t) \in X$. Here the stalk is

$$\mathcal{M}_x \cong \lim_{U \to} \Gamma(U, \mathcal{O}_X)[[s]],$$

where U runs over the open neighborhoods of x. This is a dense submodule of its completion $\widehat{\mathcal{M}_x} \cong \mathcal{O}_{X,x}[[s]] \cong \mathbb{K}[t]_{(t)}[[s]]$. Given an element $a \in \mathcal{M}_x$, there is an open neighborhood U of x, such that $a = \sum_{i \geq 0} a_i s^i$, with $a_i \in \Gamma(U, \mathcal{O}_X)$. Thus if we choose a sequence $\{\lambda_i\}_{i \geq 0}$ of distinct elements of $\mathbb{K}$, all nonzero, then the power series $a := \sum_{i \geq 0} (t - \lambda_i)^{-1} s^i$ is in $\widehat{\mathcal{M}_x}$ but not in $\mathcal{M}_x$.

Even if the ideal $\mathfrak{m}$ is nilpotent, so completion is not an issue, it is not very useful to consider sheaves of A-modules on X that are locally free. This is because such a sheaf must be locally constant. The standard practice is to talk about flat sheaves of A-modules.

Let $\mathcal{M}$ be a sheaf of A-modules on X. For $i \geq 0$ we define $\mathfrak{m}^i \mathcal{M}$ to be the image of the canonical sheaf homomorphism $\mathfrak{m}^i \otimes_A \mathcal{M} \to \mathcal{M}$. Let

$$\mathrm{gr}^i_{\mathfrak{m}} \mathcal{M} := \mathfrak{m}^i \mathcal{M} / \mathfrak{m}^{i+1} \mathcal{M}.$$

The direct sum $\mathrm{gr}_{\mathfrak{m}} \mathcal{M} := \bigoplus_i \mathrm{gr}^i_{\mathfrak{m}} \mathcal{M}$ is a sheaf of graded modules over the graded ring $\mathrm{gr}_{\mathfrak{m}} A$.

Proposition 5.3. *Let $\mathcal{M}$ be a flat sheaf of A-modules on X. Then the canonical sheaf homomorphism*

$$(gr_{\mathfrak{m}} A) \otimes_{A_0} gr^0_{\mathfrak{m}} \mathcal{M} \to gr_{\mathfrak{m}} \mathcal{M}$$

is an isomorphism.

Proof. It is enough to show that this homomorphism becomes an isomorphism at stalks. But at a point $x \in X$ the A-module $\mathcal{M}_x$ is flat, so we can use [3, Theorem III.5.1]. □

Definition 5.4. Let $\mathcal{N}$ be a sheaf of abelian groups on X.

(1) We say that an open set U of X is *$\mathcal{N}$-simply connected* if $\mathrm{H}^1(U, \mathcal{N}) = 0$. Here $\mathrm{H}^1(u, N)$ denotes sheaf cohomology.
(2) The space X is said to be *locally $\mathcal{N}$-simply connected* if it has a basis of the topology consisting of open sets that are $\mathcal{N}$-simply connected.

Example 5.5. Here are a few typical examples of a topological space X, and a sheaf $\mathcal{N}$, such that X is locally $\mathcal{N}$-simply connected:

(1) X is an algebraic variety over a field, with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module. Then any affine open set U is $\mathcal{N}$-simply connected;
(2) X is a complex analytic manifold, with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is a coherent $\mathcal{O}_X$-module. Then any Stein open set U is $\mathcal{N}$-simply connected;
(3) X is a differentiable manifold, with structure sheaf $\mathcal{O}_X$, and $\mathcal{N}$ is any $\mathcal{O}_X$-module. Then any open set U is $\mathcal{N}$-simply connected;
(4) X is a differentiable manifold, and $\mathcal{N}$ is a constant sheaf of abelian groups. Then any contractible open set U is $\mathcal{N}$-simply connected.

Theorem 5.6. *Let A be a complete noetherian local ring, with maximal ideal $\mathfrak{m}$. Let X be a topological space, and let $\mathcal{M}$ be a flat $\mathfrak{m}$-adically complete sheaf of A-modules on X. We write $\mathcal{M}_i := A_i \otimes_A \mathcal{M}$, $M := \Gamma(X, \mathcal{M})$ and $M_i := \Gamma(X, \mathcal{M}_i)$ for $i \geq 0$. Assume that X is $\mathcal{M}_0$-simply connected. Then the following are true*:

(1) *The A-module M is $\mathfrak{m}$-adically free.*
(2) *For every $i \geq 0$ the canonical homomorphism $A_i \otimes_A M \to M_i$ is bijective.*

We need a lemma first.

Lemma 5.7. *In the setup of the theorem, let N be an A_i-module. Then*:

(1) $\mathrm{H}^1(X, N \otimes_{A_i} \mathcal{M}_i) = 0$.
(2) *The canonical homomorphism*

$$N \otimes_{A_i} M_i \to \Gamma(X, N \otimes_{A_i} \mathcal{M}_i)$$

is bijective.

Again this is familiar, but we did not find a reference.

Proof. (1) The proof is by induction on i. For $i = 0$ the ring $\mathbb{K} := A_0$ is a field, so N is a free $\mathbb{K}$-module, and

$$\mathrm{H}^1(X, N \otimes_{\mathbb{K}} \mathcal{M}_0) \cong N \otimes_{\mathbb{K}} \mathrm{H}^1(X, \mathcal{M}_0) = 0.$$

Now assume $i \geq 1$. We have an exact sequence of A_i-modules

$$0 \to V \to N \to A_{i-1} \otimes_{A_i} N \to 0,$$

where V is some $\mathbb{K}$-module. Since the sheaf $\mathcal{M}_i$ is flat over A_i, there is an exact sequence of sheaves

$$0 \to V \otimes_{A_i} \mathcal{M}_i \to N \otimes_{A_i} \mathcal{M}_i \to A_{i-1} \otimes_{A_i} N \otimes_{A_i} \mathcal{M}_i \to 0,$$

which can be rewritten as

$$0 \to V \otimes_{\mathbb{K}} \mathcal{M}_0 \to N \otimes_{A_i} \mathcal{M}_i \to \bar{N} \otimes_{A_{i-1}} \mathcal{M}_{i-1} \to 0,$$

where $\bar{N} := A_{i-1} \otimes_{A_i} N$. In global cohomology we get an an exact sequence

$$\cdots \to \mathrm{H}^1(X, V \otimes_{\mathbb{K}} \mathcal{M}_0) \to \mathrm{H}^1(X, N \otimes_{A_i} \mathcal{M}_i) \to \mathrm{H}^1(X, \bar{N} \otimes_{A_{i-1}} \mathcal{M}_{i-1}) \to \cdots.$$

The induction hypothesis says that the two extremes vanish; and hence so does the middle term.

(2) The proof is like the first part. For $i = 0$ we have $N \otimes_{\mathbb{K}} M_0 \cong \Gamma(X, N \otimes_{\mathbb{K}} \mathcal{M}_0)$ since N is a free $\mathbb{K}$-module. For $i \geq 1$ we have a commutative diagram

$$\begin{tikzcd}
 & V \otimes_{\mathbb{K}} M_0 \arrow[r] \arrow[d] & N \otimes_{A_i} M_i \arrow[r] \arrow[d] & \bar{N} \otimes_{A_{i-1}} M_{i-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X, V \otimes_{\mathbb{K}} \mathcal{M}_0) \arrow[r] & \Gamma(X, N \otimes_{A_i} \mathcal{M}_i) \arrow[r] & \Gamma(X, \bar{N} \otimes_{A_{i-1}} \mathcal{M}_{i-1}) &
\end{tikzcd}$$

with exact rows. By induction the extreme vertical arrows are bijective. Hence so is the middle one. □

Proof of the Theorem. We know that $M \cong \lim_{\leftarrow i} M_i$. In view of Theorem 4.3 it suffices to prove that for each i the module M_i is flat over A_i, and the canonical homomorphism $A_{i-1} \otimes_{A_i} M_i \to M_{i-1}$ is bijective. The second assertion is true by Lemma 5.7(2), taking $N := A_{i-1}$.

As for flatness of M_i, take an exact sequence

$$0 \to N' \to N \to N'' \to 0$$

of A_i-modules. Since the sheaf $\mathcal{M}_i$ is flat over A_i, we get an exact sequence of sheaves

$$0 \to N' \otimes_{A_i} \mathcal{M}_i \to N \otimes_{A_i} \mathcal{M}_i \to N'' \otimes_{A_i} \mathcal{M}_i \to 0.$$

By Lemma 5.7(1) we know that $\mathrm{H}^1(X, N' \otimes_{A_i} \mathcal{M}_i) = 0$, so the sequence

$$0 \to \Gamma(X, N' \otimes_{A_i} \mathcal{M}_i) \to \Gamma(X, N \otimes_{A_i} \mathcal{M}_i) \to \Gamma(X, N'' \otimes_{A_i} \mathcal{M}_i) \to 0$$

is exact. Finally, using Lemma 5.7(2) we see that the sequence

$$0 \to N' \otimes_{A_i} M_i \to N \otimes_{A_i} M_i \to N'' \otimes_{A_i} M_i \to 0$$

is also exact. □

Here is the geometric version of Definition 4.1.

Definition 5.8. An $\mathfrak{m}$-*adic system of sheaves of* A-*modules* on X is a collection $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ of sheaves of A-modules, together with a collection $\{\psi_i\}_{i\in\mathbb{N}}$ of A-linear sheaf homomorphisms $\psi_i : \mathcal{M}_{i+1} \to \mathcal{M}_i$. The conditions are:

(i) For every i one has $\mathfrak{m}^{i+1}\mathcal{M}_i = 0$. Thus $\mathcal{M}_i$ is a sheaf of A_i-modules;
(ii) For every i the A_i-linear sheaf homomorphism $A_i \otimes_{A_{i+1}} \mathcal{M}_{i+1} \to \mathcal{M}_i$ induced by ψ_i is an isomorphism.

Remark 5.9. When $A = \widehat{\mathbb{Z}}_l$, the l-adic completion of $\mathbb{Z}$ for some prime number l, this is called an l-adic sheaf. Cf. [7, Section 12].

Corollary 5.10. *Let A be a complete noetherian local ring, with maximal ideal $\mathfrak{m}$, and let $\{\mathcal{M}_i\}_{i\in\mathbb{N}}$ be an $\mathfrak{m}$-adic system of sheaves of A-modules on X. Assume these conditions hold*:

(i) *For every* $i \geq 0$ *the sheaf of* A_i*-modules* $\mathscr{M}_i$ *is flat*;
(ii) X *is locally* $\mathscr{M}_0$*-simply connected.*

Define the sheaf of A*-modules* $\mathscr{M} := \lim_{\leftarrow i} \mathscr{M}_i$. *Then the following are true*:

(1) $\mathscr{M}$ *is flat and* $\mathfrak{m}$*-adically complete*;
(2) *For every* i *the canonical sheaf homomorphism* $A_i \otimes_A \mathscr{M} \to \mathscr{M}_i$ *is an isomorphism*;
(3) *Let* U *be an open set of* X *that is* $\mathscr{M}_0$*-simply connected. Then the* A*-module* $\Gamma(U, \mathscr{M})$ *is* $\mathfrak{m}$*-adically free.*

Proof. Let U be an $\mathscr{M}_0$-simply connected open set in X. Write $M := \Gamma(U, \mathscr{M})$ and $M_i := \Gamma(U, \mathscr{M}_i)$, so that $M \cong \lim_{\leftarrow i} M_i$. Fix some $j \geq 0$. Theorem 5.6, applied to the artinian ring A_j and the sheaf of A_j-modules $\mathscr{M}_j|_U$, says that M_j is a free A_j-module, and for every $i \leq j$ the canonical homomorphism $A_i \otimes_{A_j} M_j \to M_i$ is bijective. Hence the collection $\{M_i\}_{i \geq 0}$ satisfies the assumptions of Theorem 4.3, and we conclude that M is $\mathfrak{m}$-adically free over A, and $A_i \otimes_A M \cong M_i$ for every i.

We have shown that the canonical homomorphism

$$A_i \otimes_A \Gamma(U, \mathscr{M}) \to \Gamma(U, \mathscr{M}_i)$$

is bijective for every open set U that is $\mathscr{M}_0$-simply connected. Since these open sets form a basis of the topology, it follows that $A_i \otimes_A \mathscr{M} \to \mathscr{M}_i$ is an isomorphism of sheaves for all i. So $\mathscr{M}$ is $\mathfrak{m}$-adically complete.

Finally, we must prove that for any point $x \in X$ the stalk $\mathscr{M}_x$ is a flat A-module. But $\mathscr{M}_x \cong \lim_{U \to} \Gamma(U, \mathscr{M})$, where the limit is over the open neighborhoods of x that are $\mathscr{M}_0$-simply connected. Since each $\Gamma(U, \mathscr{M})$ is flat over A (by Corollary 4.5), so is their direct limit. □

Corollary 5.11. *Suppose* B *is another complete noetherian local ring, with maximal ideal* $\mathfrak{n}$, *and* $A \to B$ *is a local homomorphism. Let* $\mathscr{M}$ *be a flat* $\mathfrak{m}$*-adically complete sheaf of* A*-modules on* X. *Assume that* X *is locally* $\mathscr{M}_0$*-simply connected, where* $\mathscr{M}_0 := A_0 \otimes_A \mathscr{M}$. *Then the sheaf of* B*-modules* $B \widehat{\otimes}_A \mathscr{M}$ *is flat and* $\mathfrak{n}$*-adically complete.*

Proof. Write $\mathscr{M}_i := A_i \otimes_A \mathscr{M}$; so $\{\mathscr{M}_i\}_{i \in \mathbb{N}}$ is an $\mathfrak{m}$-adic system of sheaves A-modules, and $\mathscr{M}_i$ is flat over A_i. Let $\mathscr{N} := B \widehat{\otimes}_A \mathscr{M}$, $B_i := B/\mathfrak{n}^{i+1}$ and $\mathscr{N}_i := B_i \otimes_{A_i} \mathscr{M}_i$. Then $\{\mathscr{N}_i\}_{i \in \mathbb{N}}$ is an $\mathfrak{n}$-adic system of sheaves B-modules, $\mathscr{N}_i$ is flat over B_i, and $\mathscr{N} \cong \lim_{\leftarrow i} \mathscr{N}_i$. Since B_0 is a free module over the field A_0, we have

$$\mathrm{H}^1(U, \mathscr{N}_0) \cong B_0 \otimes_{A_0} \mathrm{H}^1(U, \mathscr{M}_0)$$

for every open set U in X. Therefore X is locally $\mathscr{N}_0$-simply connected. Now we can use Corollary 5.10. □

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