AN AVERAGING PROCESS FOR UNIPOTENT GROUP ACTIONS

AMNON YEKUTIELI

ABSTRACT. We present an averaging process for sections of a torsor under a unipotent group. This process allows one to integrate local sections of such a torsor into a global simplicial section. The results of this paper have applications to deformation quantization of algebraic varieties.

0. INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic 0. For any natural number q let $\mathbb{K}[t_0, \ldots, t_q]$ be the polynomial algebra, and let $\mathbf{\Delta}^q_{\mathbb{K}}$ be the *geometric q-dimensional simplex*

$$\mathbf{\Delta}^q_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1).$$

Our main result is the following theorem.

Theorem 0.1. *Let G be a unipotent algebraic group over $\mathbb{K}$, let X be a $\mathbb{K}$-scheme, let Z be a G-torsor over X, and let Y be any X-scheme. Suppose $\boldsymbol{f} = (f_0, \ldots, f_q)$ is a sequence of X-morphisms $f_i : Y \to Z$. Then there is an X-morphism*

$$\operatorname{wav}_G(\boldsymbol{f}) : \mathbf{\Delta}^q_{\mathbb{K}} \times Y \to Z$$

called the weighted average*, such that the operation* wav_G *is symmetric, simplicial, functorial in the data (G, X, Y, Z), and is the identity for $q = 0$.*

"Symmetric" means that wav_G is equivariant for the action of the permutation group of $\{0, \ldots, q\}$ on the sequence $\boldsymbol{f}$ and the scheme $\mathbf{\Delta}^q_{\mathbb{K}}$. "Simplicial" says that as q varies

$$\operatorname{wav}_G : \operatorname{Hom}_X(Y, Z)^{q+1} \to \operatorname{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)$$

is a map of simplicial sets. "Functorial in the data (G, X, Y, Z)" means that e.g. given a morphism $h : Z \to Z'$ of G-torsors over X, one has

$$h \circ \operatorname{wav}_G(\boldsymbol{f}) = \operatorname{wav}_G(h \circ \boldsymbol{f}),$$

where $h \circ \boldsymbol{f}$ is the sequence $(h \circ f_0, \ldots, h \circ f_q)$. Theorem 0.1 is repeated, in full detail, as Theorem 1.11.

Observe that when we restrict the morphism $\operatorname{wav}_G(\boldsymbol{f})$ to each of the vertices of $\mathbf{\Delta}^q_{\mathbb{K}}$ we recover the original morphisms $f_0, \ldots, f_q$. This is due to the simplicial property of wav_G. Thus $\operatorname{wav}_G(\boldsymbol{f})$ interpolates between $f_0, \ldots, f_q$. This is illustrated in Figure 1.

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Here is a naive corollary of Theorem 0.1, which can further explain the result. Let us write $G(\mathbb{K})$ for the group of $\mathbb{K}$-rational points of G. By a *weight sequence* in $\mathbb{K}$ we mean a sequence $\boldsymbol{w} = (w_0, \ldots, w_q)$ of elements of $\mathbb{K}$ such that $\sum w_i = 1$.

Corollary 0.2. *Let G be a unipotent group over $\mathbb{K}$. Suppose Z is a set with $G(\mathbb{K})$-action which is transitive and has trivial stabilizers. Let $\boldsymbol{z} = (z_0, \ldots, z_q)$ be a sequence of points in Z and let $\boldsymbol{w}$ be a weight sequence in $\mathbb{K}$. Then there is a point $\operatorname{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) \in Z$ called the weighted average. The operation wav_G is symmetric, functorial, simplicial, and is the identity for $q = 0$.*

The corollary is proved in Section 1. The idea is of course to take $Y = X := \operatorname{Spec} \mathbb{K}$ in the theorem, and to note that $\boldsymbol{w}$ is a $\mathbb{K}$-rational point of $\boldsymbol{\Delta}^q_{\mathbb{K}}$.

Another consequence of Theorem 0.1 is a new proof of the fact that a unipotent group in characteristic 0 is special. See Remark 1.12. (This observation is due to Reichstein.)

Theorem 0.1 was discovered in the course of research on deformation quantization in algebraic geometry [Ye], in which we tried to apply ideas of Kontsevich [Ko] to the algebraic context. Here is a brief outline. Suppose X is a smooth n-dimensional $\mathbb{K}$-scheme. The coordinate bundle of X [GK, Ko] is an infinite dimensional bundle $Z \to X$ which parameterizes formal coordinate systems on X. The bundle Z is a torsor under an affine group scheme $\mathrm{GL}_{n,\mathbb{K}} \ltimes G$, where G is pro-unipotent. One is interested in sections of the quotient bundle $\bar{Z} := Z/\mathrm{GL}_{n,\mathbb{K}}$. If we are in the differentiable setup (i.e. $\mathbb{K} = \mathbb{R}$ and X is a C^∞ manifold) then the fibers of $\bar{Z}$ are contractible (since they are isomorphic to $G(\mathbb{R})$), and therefore global C^∞ sections exist. However in the setup of algebraic geometry such an argument is invalid, and we were forced to seek an alternative approach.

Our solution was to use *simplicial sections* of $\bar{Z}$ (see Section 2, and especially Corollary 2.7). This approach was inspired by work of Bott [Bo] on simplicial connections (cf. also [HY]).

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1. The Averaging Process

Throughout the paper $\mathbb{K}$ denotes a fixed base field of characteristic 0. All schemes and all morphisms are over $\mathbb{K}$.

We begin by recalling some standard facts about the combinatorics of simplicial objects. Let $\boldsymbol{\Delta}$ denote the category with objects the ordered sets $[q] := \{0, 1, \ldots, q\}$, $q \in \mathbb{N}$. The morphisms $[p] \to [q]$ are the order preserving functions, and we write $\boldsymbol{\Delta}^q_p := \operatorname{Hom}_{\boldsymbol{\Delta}}([p], [q])$. The i-th co-face map $\partial^i : [p] \to [p+1]$ is the injective function that does not take the value i; and the i-th co-degeneracy map $\mathrm{s}^i : [p] \to [p-1]$ is the surjective function that takes the value i twice. All morphisms in $\boldsymbol{\Delta}$ are compositions of various ∂^i and s^i.

An element of $\boldsymbol{\Delta}^q_p$ may be thought of as a sequence $\boldsymbol{i} = (i_0, \ldots, i_p)$ of integers with $0 \leq i_0 \leq \cdots \leq i_p \leq q$. Given $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$, $\boldsymbol{j} \in \boldsymbol{\Delta}^p_m$ and $\alpha \in \boldsymbol{\Delta}^q_p$, we sometimes write $\alpha_*(\boldsymbol{i}) := \boldsymbol{i} \circ \alpha \in \boldsymbol{\Delta}^m_p$ and $\alpha^*(\boldsymbol{j}) := \alpha \circ \boldsymbol{j} \in \boldsymbol{\Delta}^q_m$.

Let C be some category. A *cosimplicial object* in C is a functor $C : \boldsymbol{\Delta} \to \mathsf{C}$. We shall usually refer to the cosimplicial object as $C = \{C^p\}_{p \in \mathbb{N}}$, and for any $\alpha \in \boldsymbol{\Delta}^q_p$ the corresponding morphism in C will be denoted by $\alpha^* : C^p \to C^q$. A *simplicial* *object* in C is a functor $C : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$. The notation for a simplicial object will be $C = \{C_p\}_{p\in\mathbb{N}}$ and $\alpha_* : C_q \to C_p$.

An important example is the cosimplicial scheme $\{\mathbf{\Delta}^p_{\mathbb{K}}\}_{p\in\mathbb{N}}$. The morphisms are defined as follows. For any p we identify the ordered set $[p]$ with the set of vertices of $\mathbf{\Delta}^p_{\mathbb{K}}$. Given $\alpha \in \mathbf{\Delta}^q_p$ the morphism $\alpha^* : \mathbf{\Delta}^p_{\mathbb{K}} \to \mathbf{\Delta}^q_{\mathbb{K}}$ is then the unique linear morphism extending $\alpha : [p] \to [q]$.

Let G be a unipotent (affine finite type) algebraic group over $\mathbb{K}$, with (nilpotent) Lie algebra $\mathfrak{g}$. We write $d(G)$ for the minimal number d such that there exists a chain of closed normal subgroups $G = G_0 \supset G_1 \cdots \supset G_d = 1$ with G_k/G_{k+1} abelian for all $k \in \{0, \dots, d-1\}$. The exponential map $\exp_G : \mathfrak{g} \to G$ is an isomorphism of schemes, with inverse $\log_G$; see [Ho, Theorem VIII.1.1].

Given a $\mathbb{K}$-scheme X there is a Lie algebra $\mathfrak{g}\times X$ (in the category of X-schemes), and a group-scheme $G \times X$. There is also an induced exponential map

$$\exp_{G\times X} := \exp_G \times \mathbf{1}_X : \mathfrak{g} \times X \to G \times X.$$

We will need the following result.

Lemma 1.1. *Let G, G' be two unipotent groups, with Lie algebras $\mathfrak{g}, \mathfrak{g}'$. Let X, X' be schemes, let $X \to X'$ be a morphism of schemes, and let $\phi : G \times X \to G' \times X'$ be a morphism of group-schemes over X'. Denote by $\mathrm{d}\phi : \mathfrak{g} \times X \to \mathfrak{g}' \times X'$ the induced Lie algebra morphism (the differential of ϕ). Then the diagram*

$$\begin{array}{ccc} \mathfrak{g} \times X & \xrightarrow{\exp_{G\times X}} & G \times X \\ {\scriptstyle \mathrm{d}\phi}\big\downarrow & & {\scriptstyle \phi}\big\downarrow \\ \mathfrak{g}' \times X' & \xrightarrow{\exp_{G'\times X'}} & G' \times X' \end{array} \tag{1.2}$$

commutes.

Proof. For the case $X = X' = \operatorname{Spec}\mathbb{K}$ this is contained in the proof of [Ho, Theorem VIII.1.2].

In order to handle the general case we first recall the Campbell-Baker-Hausdorff formula:

$$\exp_G(\gamma_1) \cdot \exp_G(\gamma_2) = \exp_G(F(\gamma_1, \gamma_2)),$$

where

$$F(\gamma_1, \gamma_2) = \gamma_1 + \gamma_2 + \tfrac{1}{2}[\gamma_1, \gamma_2] + \cdots$$

is a universal power series (see [Ho, Section XVI.2]). Hence if we define $\gamma_1 * \gamma_2 := F(\gamma_1, \gamma_2)$, then $(\mathfrak{g}, *)$ becomes an algebraic group (with unit element 0), and $\exp_G : (\mathfrak{g}, *) \to (G, \cdot)$ is a group isomorphism. In this way we may eliminate G altogether, and just look at the scheme $\mathfrak{g}$ with its two structures: a Lie algebra and a group-scheme. Note that now $\mathfrak{g}$ is its own Lie algebra, as can be seen from the 2-nd order term in the series $F(\gamma_1, \gamma_2)$.

Consider a morphism $\phi : \mathfrak{g} \times X \to \mathfrak{g}' \times X'$ of X'-schemes. Then ϕ is a morphism of Lie algebras over X' iff it is a morphism of group-schemes (for the multiplications $*$). And moreover $\mathrm{d}\phi = \phi$. Therefore the diagram (1.2) is commutative. □

From now on we shall write $\exp_G$ instead of $\exp_{G\times X}$, for the sake of brevity.

Consider the following setup: X is a $\mathbb{K}$-scheme, and Y, Z are two X-schemes. Suppose Z is a torsor under the group scheme $G \times X$. We denote the action of G on Z by $(g, z) \mapsto g \cdot z$.

Let $f_0, \ldots, f_q : \mathbf{\Delta}^q_{\mathbb{K}} \times Y \to Z$ be X-morphisms. We are going to define a new sequence of X-morphisms $f'_0, \ldots, f'_q : \mathbf{\Delta}^q_{\mathbb{K}} \times Y \to Z$. Because Z is a torsor under $G \times X$, for any $i, j \in \{0, \ldots, q\}$ there exists a unique morphism $g_{i,j} : \mathbf{\Delta}^q_{\mathbb{K}} \times Y \to G$ such that $f_j(w, y) = g_{i,j}(w, y) \cdot f_i(w, y)$ for all points $w \in \mathbf{\Delta}^q_{\mathbb{K}}$ and $y \in Y$. Here w and y are scheme-theoretic points, i.e. $w \in \mathbf{\Delta}^q_{\mathbb{K}}(U) = \mathrm{Hom}_{\mathbb{K}}(U, \mathbf{\Delta}^q_{\mathbb{K}})$ and $y \in Y(U) = \mathrm{Hom}_{\mathbb{K}}(U, Y)$ for some $\mathbb{K}$-scheme U. Define

$$f'_i(w, y) := \exp_G \Big(\sum_{j=0}^{q} t_j(w) \cdot \log_G(g_{i,j}(w, y)) \Big) \cdot f_i(w, y). \tag{1.3}$$

In this formula we view t_j as a morphism $t_j : \mathbf{\Delta}^q_{\mathbb{K}} \to \mathbf{A}^1_{\mathbb{K}}$, and we use the fact that $\mathfrak{g}$ is a vector space (in the category of $\mathbb{K}$-schemes).

For any set S let us write $S^{\mathbf{\Delta}^q_0}$ for the set of functions $\mathbf{\Delta}^q_0 \to S$, which is the same as the set of sequences $(s_0, \ldots, s_q)$ in S. As usual $\mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)$ is the set of X-morphisms $\mathbf{\Delta}^q_{\mathbb{K}} \times Y \to Z$. In this notation the sequences $(f_0, \ldots, f_q)$ and $(f'_0, \ldots, f'_q)$ are elements of $\mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$. We denote by

$$\mathrm{wsym}_G : \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0} \to \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$$

the operation $(f_0, \ldots, f_q) \mapsto (f'_0, \ldots, f'_q)$ given by the formula (1.3).

We will also need a similar operation

$$\mathrm{w}_G : \mathrm{Hom}_X(Y, Z)^{\mathbf{\Delta}^q_0} \to \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0},$$

defined by

$$\mathrm{w}_G(f_0, \ldots, f_q) := (f'_0, \ldots, f'_q)$$

with

$$f'_i(w, y) := \exp_G \Big(\sum_{j=0}^{q} t_j(w) \cdot \log_G(g_{i,j}(y)) \Big) \cdot f_i(y). \tag{1.4}$$

It is clear that for $q = 0$ both operations w_G and wsym_G act as the identity, i.e. $\mathrm{w}_G(f_0) = \mathrm{wsym}_G(f_0) = f_0$ for all $f_0 \in \mathrm{Hom}_X(Y, Z)$. Both operations w_G and wsym_G are symmetric, namely they are equivariant for the simultaneous action of the permutation group of $\{0, \ldots, q\}$ on $\mathbf{\Delta}^q_0$ and $\mathbf{\Delta}^q_{\mathbb{K}}$. Also if $\boldsymbol{f} = (f_0, \ldots, f_q) \in \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$ is a constant sequence, i.e. $f_0 = \cdots = f_q$, then $\mathrm{wsym}_G(\boldsymbol{f}) = \boldsymbol{f}$.

Lemma 1.5. *Both operations* w_G *and* wsym_G *are simplicial. Namely, given* $\alpha \in \mathbf{\Delta}^q_p$ *the diagram*

$$\begin{CD}
\mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0} @>{\mathrm{wsym}_G}>> \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0} \\
@V{\alpha_*}VV @V{\alpha_*}VV \\
\mathrm{Hom}_X(\mathbf{\Delta}^p_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^p_0} @>{\mathrm{wsym}_G}>> \mathrm{Hom}_X(\mathbf{\Delta}^p_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^p_0}
\end{CD}$$

is commutative, and likewise for w_G*.*

Proof. It suffices to consider $\alpha = \partial^i$ or $\alpha = \mathrm{s}^i$. Since wsym_G is symmetric, we may assume that $\alpha = \partial^q : [q-1] \to [q]$ or $\alpha = \mathrm{s}^q : [q+1] \to [q]$. Fix a sequence $\boldsymbol{f} = (f_0, \ldots, f_q) \in \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$. Let $g_{i,j} \in \mathrm{Hom}_{\mathbb{K}}(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, G)$ be such that $f_j = g_{i,j} \cdot f_i$, and let $\boldsymbol{f}' = (f'_0, \ldots, f'_q) := \mathrm{wsym}_G(\boldsymbol{f})$.

First let's look at the case $\alpha = \partial^q$. Take $w \in \mathbf{\Delta}^{q-1}_{\mathbb{K}}$ and $y \in Y$, and let $v := \alpha^*(w) \in \mathbf{\Delta}^q_{\mathbb{K}}$. The coordinates of v are $t_j(v) = t_j(w)$ for $j \leq q-1$, and

$t_q(v) = 0$. Then for any i the i-th term in the sequence $\alpha_*(\boldsymbol{f}')$, evaluated at (w, y), equals

$$
\begin{aligned}
f'_i(v,y) &= \exp_G\Big(\sum_{j=0}^{q} t_j(v)\cdot \log_G(g_{i,j}(v,y))\Big)\cdot f_i(v,y) \\
&= \exp_G\Big(\sum_{j=0}^{q-1} t_j(w)\cdot \log_G(g_{i,j}(v,y))\Big)\cdot f_i(v,y).
\end{aligned} \tag{1.6}
$$

On the other hand, the i-th term of the sequence $\mathrm{wsym}_G(\alpha_*(\boldsymbol{f}))$ is

$$
\begin{aligned}
&\exp_G\Big(\sum_{j=0}^{q-1} t_j(w)\cdot \log_G(\alpha_*(g_{i,j})(w,y))\Big)\cdot \alpha_*(f_i)(w,y) \\
&\qquad = \exp_G\Big(\sum_{j=0}^{q-1} t_j(w)\cdot \log_G(g_{i,j}(v,y))\Big)\cdot f_i(v,y).
\end{aligned}
$$

So indeed $\alpha_* \circ \mathrm{wsym}_G = \mathrm{wsym}_G \circ \alpha_*$ in this case.

Next consider the case $\alpha = \mathrm{s}^q$. Take $w \in \boldsymbol{\Delta}^{q+1}_{\mathbb{K}}$ and $y \in Y$, and let $v := \alpha^*(w) \in \boldsymbol{\Delta}^q_{\mathbb{K}}$. The coordinates of v are $t_j(v) = t_j(w)$ for $j \le q-1$, and $t_q(v) = t_q(w) + t_{q+1}(w)$. For any $i \le q$ the i-th term in the sequence $\alpha_*(\boldsymbol{f})$, evaluated at (w, y), is $f'_i(v, y)$, which was calculated in (1.6). The $(q+1)$-st term is also $f'_q(v, y)$. On the other hand, for any $i \le q$ the i-th term in the sequence $\mathrm{wsym}_G(\alpha_*(\boldsymbol{f}'))$, evaluated at (w, y), is

$$
z_i := \exp_G\Big(\sum_{j=0}^{q+1} t_j(w)\cdot \log_G(\alpha_*(g_{i,j})(w,y))\Big)\cdot \alpha_*(f_i)(w,y).
$$

But $t_q(w) + t_{q+1}(w) = t_q(v)$, $\alpha_*(g_{i,j})(w,y) = g_{i,j}(v,y)$ for $j \le q$, and $\alpha_*(g_{i,q+1})(w,y) = g_{i,q}(v,y)$. Therefore

$$
z_i = \exp_G\Big(\sum_{j=0}^{q} t_j(v)\cdot \log_G(g_{i,j}(v,y))\Big)\cdot f_i(v,y).
$$

For $i = q+1$ one has $z_{q+1} = z_q$. We conclude that $\alpha_* \circ \mathrm{wsym}_G = \mathrm{wsym}_G \circ \alpha_*$ in this case too.

The proof for w_G is the same. □

Lemma 1.7. *Both operations w_G and wsym_G are functorial in the data (G, X, Y, Z). Namely, given another such quadruple (G', X', Y', Z'), a morphism of schemes $X \to X'$, a morphism of schemes $e : Y' \to Y$ over X', a morphism of group-schemes $\phi : G \times X \to G' \times X'$ over X', and a $G \times X$ -equivariant morphism of schemes $f : Z \to Z'$ over X', the diagram*

$$
\begin{array}{ccc}
\mathrm{Hom}_X(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\boldsymbol{\Delta}^q_0} & \xrightarrow{\mathrm{wsym}_G} & \mathrm{Hom}_X(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\boldsymbol{\Delta}^q_0} \\
{\scriptstyle (e,f)}\big\downarrow & & {\scriptstyle (e,f)}\big\downarrow \\
\mathrm{Hom}_{X'}(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y', Z')^{\boldsymbol{\Delta}^q_0} & \xrightarrow{\mathrm{wsym}_{G'}} & \mathrm{Hom}_{X'}(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y', Z')^{\boldsymbol{\Delta}^q_0}
\end{array}
$$

is commutative, and likewise for w_G.

Proof. This is due to the functoriality of the exponential map, see Lemma 1.1. □

Lemma 1.8. *Assume G is abelian. Then for any $(f_0, \ldots, f_q) \in \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$ the sequence $\mathrm{wsym}_G(f_0, \ldots, f_q)$ is constant.*

Proof. In this case $\exp : \mathfrak{g} \to G$ is an isomorphism of algebraic groups, where $\mathfrak{g}$ is viewed as an additive group. So we may assume that Z is a torsor under $\mathfrak{g} \times X$. Let $(f'_0, \ldots, f'_q) := \mathrm{wsym}_G(f_0, \ldots, f_q)$, and let $\gamma_{i,j} : \mathbf{\Delta}^q_{\mathbb{K}} \times Y \to \mathfrak{g}$ be morphisms such that $f_j = \gamma_{i,j} + f_i$. Take $(w, y) \in \mathbf{\Delta}^q_{\mathbb{K}} \times Y$. Then

$$f'_i(w, y) = \Big(\sum_{j=0}^{q} t_j(w) \cdot \gamma_{i,j}(w, y) \Big) + f_i(w, y)$$

for any i. Because $\gamma_{i,j} = -\gamma_{j,i} = \gamma_{0,j} - \gamma_{0,i}$, $f_i = f_0 + \gamma_{0,i}$ and $\sum_{j=0}^{q} t_j(w) = 1$, it follows that $f'_i(w, y) = f'_0(w, y)$. $\square$

Let's write wsym^d_G for the d-th iteration of the operation wsym_G.

Lemma 1.9. *For any $\boldsymbol{f} = (f_0, \ldots, f_q) \in \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)^{\mathbf{\Delta}^q_0}$ the sequence $\mathrm{wsym}^{d(G)}_G(\boldsymbol{f})$ is constant. For any $d \geq d(G)$ one has $\mathrm{wsym}^d_G(\boldsymbol{f}) = \mathrm{wsym}^{d(G)}_G(\boldsymbol{f})$.*

Proof. For any k, the orbit of $f_0 \in \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)$ under the action of the group $G_k(\mathbf{\Delta}^q_{\mathbb{K}} \times Y)$ will be denoted by $G_k(\mathbf{\Delta}^q_{\mathbb{K}} \times Y) \cdot f_0$. Let $(f'_0, \ldots, f'_q) := \mathrm{wsym}_G(f_0, \ldots, f_q)$. We will prove that if $f_1, \ldots, f_q \in G_k(\mathbf{\Delta}^q_{\mathbb{K}} \times Y) \cdot f_0$ then $f'_1, \ldots, f'_q \in G_{k+1}(\mathbf{\Delta}^q_{\mathbb{K}} \times Y) \cdot f'_0$. The assertions of the lemma will then follow.

Let $\tilde{Y} = \tilde{X} := \mathbf{\Delta}^q_{\mathbb{K}} \times Y$ and $\tilde{Z} := \tilde{X} \times_X Z$. So $\tilde{Z}$ is a torsor under $G \times \tilde{X}$, and f_0 induces a morphism $\tilde{f}_0 \in \mathrm{Hom}_{\tilde{X}}(\tilde{Y}, \tilde{Z})$. The morphism $\tau : G \times \tilde{X} \to \tilde{Z}$, $(g, \tilde{x}) \mapsto g \cdot \tilde{f}_0(\tilde{x})$, is an isomorphism of $\tilde{X}$-schemes. Define $\tilde{W} := \tau(G_k \times \tilde{X}) \subset \tilde{Z}$. Then $\tilde{W}$ is the "geometric orbit" of $\tilde{f}_0$ under $G_k \times \tilde{X}$; and in particular $\tilde{W}$ is a torsor under $G_k \times \tilde{X}$. By assumption $\tilde{f}_1, \ldots, \tilde{f}_q \in \mathrm{Hom}_{\tilde{X}}(\tilde{Y}, \tilde{W})$. Define $(\tilde{f}'_0, \ldots, \tilde{f}'_q) := \mathrm{wsym}_{G_k}(\tilde{f}_0, \ldots, \tilde{f}_q)$. By Lemma 1.7 it suffices to prove that $\tilde{f}'_1, \ldots, \tilde{f}'_q \in G_{k+1}(\tilde{Y}) \cdot \tilde{f}'_0$.

Define $\bar{W} := \tilde{W}/G_{k+1}$. This is a torsor under the group scheme $(G_k/G_{k+1}) \times \tilde{X}$. Let $\bar{f}_0, \ldots, \bar{f}_q \in \mathrm{Hom}_{\tilde{X}}(\tilde{Y}, \bar{W})$ be the images of $(\tilde{f}_0, \ldots, \tilde{f}_q)$. Because the group G_k/G_{k+1} is abelian, Lemma 1.8 says that $\mathrm{wsym}_{G_k/G_{k+1}}(\bar{f}_0, \ldots, \bar{f}_q)$ is a constant sequence. Again using Lemma 1.7, we see that in fact $\tilde{f}'_1, \ldots, \tilde{f}'_q \in G_{k+1}(\tilde{Y}) \cdot \tilde{f}'_0$. $\square$

Given an X-scheme Y the collections $\big\{\mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)\big\}_{q \in \mathbb{N}}$ and $\big\{\mathrm{Hom}_X(Y, Z)^{\mathbf{\Delta}^q_0}\big\}_{q \in \mathbb{N}}$ are simplicial sets. For $q = 0$ there are equalities

$$\mathrm{Hom}_X(\mathbf{\Delta}^0_{\mathbb{K}} \times Y, Z) = \mathrm{Hom}_X(Y, Z) = \mathrm{Hom}_X(Y, Z)^{\mathbf{\Delta}^0_0}. \tag{1.10}$$

Theorem 1.11. *Let G be a unipotent algebraic group over $\mathbb{K}$, let X be a $\mathbb{K}$-scheme, and let $Z \to X$ be a G-torsor over X. For any X-scheme Y and natural number q there is a function*

$$\mathrm{wav}_G : \mathrm{Hom}_X(Y, Z)^{\mathbf{\Delta}^q_0} \to \mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)$$

called the weighted average*. The function wav_G enjoys the following properties.*

1. *Symmetric: wav_G is equivariant for the action of the permutation group of $\{0, \ldots, q\}$ on $\mathbf{\Delta}^q_0$ and on $\mathbf{\Delta}^q_{\mathbb{K}}$.*
2. *Simplicial: wav_G is a map of simplicial sets*

$$\big\{\mathrm{Hom}_X(Y, Z)^{\mathbf{\Delta}^q_0}\big\}_{q \in \mathbb{N}} \to \big\{\mathrm{Hom}_X(\mathbf{\Delta}^q_{\mathbb{K}} \times Y, Z)\big\}_{q \in \mathbb{N}}.$$

(3) *Functorial: given another such quadruple (G', X', Y', Z'), a morphism of schemes $X \to X'$, a morphism of X'-group-schemes $G \times X \to G' \times X'$, a $G \times X$ -equivariant morphism of X'-schemes $f : Z \to Z'$ and a morphism of X'-schemes $e : Y' \to Y$, the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_X(Y, Z)^{\boldsymbol{\Delta}^q_0} & \xrightarrow{\mathrm{wav}_G} & \mathrm{Hom}_X(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y, Z) \\ {\scriptstyle (e,f)}\downarrow & & {\scriptstyle (e,f)}\downarrow \\ \mathrm{Hom}_{X'}(Y', Z')^{\boldsymbol{\Delta}^q_0} & \xrightarrow{\mathrm{wsym}_{G'}} & \mathrm{Hom}_{X'}(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y', Z') \end{array}$$

is commutative.

(4) *If $q = 0$ then* wav_G *is the identity map of* $\mathrm{Hom}_X(Y, Z)$.

Proof. Given a sequence $\boldsymbol{f} = (f_0, \ldots, f_q) \in \mathrm{Hom}_X(Y, Z)^{\boldsymbol{\Delta}^q_0}$ define $\mathrm{wav}_G(\boldsymbol{f}) := f' \in \mathrm{Hom}_X(\boldsymbol{\Delta}^q_{\mathbb{K}} \times Y, Z)$ to be the morphism such that

$$(\mathrm{wsym}_G^{d(G)} \circ \mathrm{w}_G)(f_0, \ldots, f_q) = (f', \ldots, f');$$

see Lemma 1.9. Properties (1)-(4) follow from the corresponding properties of w_G and wsym_G. □

Proof of Corollary 0.2. Take $X = Y := \mathrm{Spec}\,\mathbb{K}$ in Theorem 1.11, and consider the G-torsor $\underline{Z} := G$. Choose any base point $z \in Z$; this defines an isomorphism of left $G(\mathbb{K})$-sets $\underline{Z}(\mathbb{K}) \cong Z$. The weight sequence $\boldsymbol{w}$ can be considered as a $\mathbb{K}$-rational point of $\boldsymbol{\Delta}^q_{\mathbb{K}}$, and we define

$$\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z}) := \mathrm{wav}_G(\boldsymbol{z})(\boldsymbol{w}) \in Z.$$

If we were to choose another base point $z' \in Z$ this would amount to applying an automorphism of the torsor $\underline{Z}$, namely right multiplication by some element of $G(\mathbb{K})$. Due to the functoriality of wav_G the point $\mathrm{wav}_{G,\boldsymbol{w}}(\boldsymbol{z})$ will be unchanged.

The properties of this set-theoretical averaging process are now immediate consequences of the corresponding properties of the geometric average. □

Remark 1.12. Z. Reichstein observed that our averaging process provides a new proof (in characteristic 0) of the fact that a unipotent group G is special, namely any G-torsor Z over $\mathbb{K}$ has a $\mathbb{K}$-rational point. Let us explain the idea.

Let $z_0 \in Z$ be some closed point. Choose a finite Galois extension L of $\mathbb{K}$ containing the residue field $\boldsymbol{k}(z_0)$. Let Γ be the Galois group of L over $\mathbb{K}$, which acts on the set $Z(L)$. Let $z_0, \ldots, z_q \in Z(L)$ be the Γ-conjugates of z_0. The group Γ acts on the sequence $\boldsymbol{z} := (z_0, \ldots, z_q)$ by permutations. Thus the simultaneous action of Γ on

$$Z(L)^{\boldsymbol{\Delta}^q_0} = \mathrm{Hom}_{\mathrm{Spec}\,\mathbb{K}}(\mathrm{Spec}\,L, Z)^{\boldsymbol{\Delta}^q_0}$$

fixes $\boldsymbol{z}$.

We know that the operator wav_G is symmetric. And functoriality says that the action of the Galois group on $\mathrm{Spec}\,L$ is also respected. Since $\boldsymbol{z}$ is fixed by the simultaneous action of Γ, so is $\mathrm{wav}_G(\boldsymbol{z})$. Take the uniform weight sequence $\boldsymbol{w} := (\frac{1}{q+1}, \ldots, \frac{1}{q+1})$ and define $z' := \mathrm{wav}_G(\boldsymbol{z})(\boldsymbol{w}) \in Z(L)$. Because $\boldsymbol{w}$ is fixed by the permutation group we conclude that z' is Γ-invariant, and hence $z' \in Z(\mathbb{K})$.

Remark 1.13. Theorem 1.11 has a rather obvious parallel in differential geometry. Indeed, a simply connected nilpotent Lie group is the same as the group $G(\mathbb{R})$ of rational points of a unipotent algebraic group G over $\mathbb{R}$.

2. Simplicial Sections

In this section we show how the averaging process is used to obtain simplicial sections of certain bundles.

Suppose H and G are affine group schemes over $\mathbb{K}$, and H acts on G by automorphisms. Namely there is a morphism of schemes $H \times G \to G$ which for every $\mathbb{K}$-scheme Y induces a group homomorphism $H(Y) \to \operatorname{Aut}_{\mathsf{Groups}}(G(Y))$. Then $H \times G$ has a structure of a group scheme, and we denote this group by $H \ltimes G$; it is a geometric semi-direct product.

Recall that an affine group scheme G is called *pro-unipotent* if it is isomorphic to an inverse limit $\lim_{\leftarrow i} G_i$ of an inverse system $\{G_i\}_{i \geq 0}$ of (finite type affine) unipotent groups. One may assume that each of the morphisms $G \to G_i \to G_{i-1}$ is surjective. Thus $G_i \cong G/N_i$ where N_i is a normal closed subgroup of G.

We will be concerned with the following geometric situation.

Scenario 2.1. Let $H \ltimes G$ be an affine group scheme over $\mathbb{K}$. Assume G is pro-unipotent, and moreover there exists a sequence $\{N_i\}_{i \geq 0}$ of H-invariant closed normal subgroups of G such that $G \cong \lim_{\leftarrow i} G/N_i$ and each G/N_i is unipotent. Let $\pi : Z \to X$ be an $H \ltimes G$ -torsor over X which is locally trivial for the Zariski topology of X. Define $\bar{Z} := Z/H$ and let $\bar{\pi} : \bar{Z} \to X$ be the projection.

Theorem 2.2. *Assume Scenario* 2.1. *Suppose $U \subset X$ is an open set and $\sigma_0, \ldots, \sigma_q : U \to \bar{Z}$ are sections of $\bar{\pi}$. Then there exists a morphism*

$$\sigma : \mathbf{\Delta}^q_{\mathbb{K}} \times U \to \bar{Z}$$

such that the diagram

$$\begin{array}{ccc} \mathbf{\Delta}^q_{\mathbb{K}} \times U & \xrightarrow{\sigma} & \bar{Z} \\ {\scriptstyle \mathrm{p}_2}\downarrow & & \downarrow{\scriptstyle \bar{\pi}} \\ U & \longrightarrow & X \end{array}$$

is commutative. The morphism σ depends functorially on U and simplicially on the sequence $(\sigma_0, \ldots, \sigma_q)$. If $q = 0$ then $\sigma = \sigma_0$.

Proof. We might as well assume that $U = X$. Consider the quotient Z/G. Since G is normal in $H \ltimes G$ it follows that Z/G is a torsor under $H \times X$. Let's write $\pi_H : Z \to \bar{Z} = Z/H$ and $\pi_G : Z \to Z/G$ for the projections.

Pick an open set $V \subset X$ which trivializes $\pi : Z \to X$. Let's write $Z|_V := \pi^{-1}(V)$. Because $\pi_H|_V : Z|_V \to \bar{Z}|_V$ is a trivial torsor under $H \times \bar{Z}|_V$, we can lift the sections $\sigma_0, \ldots, \sigma_q$ to sections $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_q : V \to Z$ such that $\pi_H \circ \tilde{\sigma}_j = \sigma_j$. Furthermore, since $\pi_G : Z \to Z/G$ is H-equivariant and Z/G is a torsor under $H \times X$, it follows that we can choose $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_q$ such that $\pi_G \circ \tilde{\sigma}_j = \tau$ for some section $\tau : V \to Z/G$.

Let $F \subset Z|_V$ be the fiber over τ, i.e. $F := V \times_{Z/G} Z$ via the morphisms $\pi_G : Z \to Z/G$ and $\tau : V \to Z/G$. Then F is a torsor under $G \times V$, and $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_q \in$

$\operatorname{Hom}_X(V, F)$. See diagram below.

$$\begin{tikzcd}
 & Z \arrow[dl, "\pi_H"'] \arrow[dr, "\pi_G"] \arrow[dd, "\pi"] & & F \arrow[ll, "\supset"'] \\
\bar{Z} = Z/H \arrow[dr, "\bar{\pi}"] & & Z/G \arrow[dl] & \\
 & X & & \\
 & & & V \arrow[ull, "\supset"] \arrow[uul, dashed, "\tau"] \arrow[uuu, dashed, "\tilde{\sigma}_i"'] \arrow[uulll, bend left, "\sigma_i"]
\end{tikzcd}$$

For any i define $F_i := F/N_i$, which is a torsor under $(G/N_i) \times V$. Let $\alpha_i : F \to F_i$ be the projection, so $\alpha_i \circ \tilde{\sigma}_j \in \operatorname{Hom}_X(V, F_i)$. By Theorem 1.11 we get an average

$$\rho_i := \operatorname{wav}_{G/N_i}(\alpha_i \circ \tilde{\sigma}_0, \ldots, \alpha_i \circ \tilde{\sigma}_q) : \mathbf{\Delta}^q_{\mathbb{K}} \times V \to F_i. \tag{2.3}$$

The functoriality of wav says that the ρ_i form an inverse system, and we let

$$\rho := \lim_{\leftarrow i} \rho_i : \mathbf{\Delta}^q_{\mathbb{K}} \times V \to F \tag{2.4}$$

and

$$\sigma := \pi_H \circ \rho : \mathbf{\Delta}^q_{\mathbb{K}} \times V \to \bar{Z}. \tag{2.5}$$

We claim that the morphism σ does not depend on the choice of the section $\tau : V \to Z/G$. Suppose $\tau' : V \to Z/G$ is another such section. Let F' be the fiber over τ', and let $\rho' : \mathbf{\Delta}^q_{\mathbb{K}} \times V \to F'$ be the corresponding morphism as in (2.4). Now $\tau' = h \cdot \tau$ for some morphism $h : V \to H$. Then $F' = h \cdot F$, and $h : F \to F'$ is a $G \times V$ -equivariant morphism of torsors, with respect to the group-scheme automorphism $\operatorname{Ad}(h) : G \times V \to G \times V$. The new lift of σ_j is $\tilde{\sigma}'_j := h \cdot \tilde{\sigma}_j : V \to F'$. Define $F'_i := F'/N_i$, and let $\rho'_i : \mathbf{\Delta}^q_{\mathbb{K}} \times V \to F'_i$ be the morphism as in (2.3). Since $N_i \times V = \operatorname{Ad}(h)(N_i \times V)$, we get a group-scheme automorphism $\operatorname{Ad}(h) : (G/N_i) \times V \to (G/N_i) \times V$, and a $(G/N_i) \times V$ -equivariant morphism of torsors $h : F_i \to h \cdot F'_i$. By functoriality of wav (property 3 in Theorem 1.11) it follows that $\rho'_i = h \cdot \rho_i$. Therefore $\rho' = h \cdot \rho$, and $\pi_H \circ \rho' = \pi_H \circ \rho = \sigma$.

Property 2 in Theorem 1.11 implies that σ depends simplicially on $(\sigma_0, \ldots, \sigma_q)$.

Finally take an open covering $X = \bigcup V_j$ such that each V_j trivializes $\pi : Z \to X$, and let $\sigma_j : \mathbf{\Delta}^q_{\mathbb{K}} \times V_j \to \bar{Z}|_{V_j}$ be the morphism constructed in (2.5). Since no choices were made we have $\sigma_j|_{V_j \cap V_k} = \sigma_k|_{V_j \cap V_k}$ for any two indices. Therefore these sections can be glued to a morphism $\sigma : \mathbf{\Delta}^q_{\mathbb{K}} \times X \to \bar{Z}$. The functorial and simplicial properties of σ are clear from its construction. □

Let X be a $\mathbb{K}$-scheme, and let $X = \bigcup_{i=0}^m U_{(i)}$ be an open covering, with inclusions $g_{(i)} : U_{(i)} \to X$. We denote this covering by $\boldsymbol{U}$. For any multi-index $\boldsymbol{i} = (i_0, \ldots, i_q) \in \mathbf{\Delta}^m_q$ we write $U_{\boldsymbol{i}} := \bigcap_{j=0}^q U_{(i_j)}$, and we define the scheme $U_q := \coprod_{\boldsymbol{i} \in \mathbf{\Delta}^m_q} U_{\boldsymbol{i}}$. Given $\alpha \in \mathbf{\Delta}^q_p$ and $\boldsymbol{i} \in \mathbf{\Delta}^m_q$ there is an inclusion of open sets $\alpha_* : U_{\boldsymbol{i}} \to U_{\alpha_*(\boldsymbol{i})}$. These patch to a morphism of schemes $\alpha_* : U_q \to U_p$, making $\{U_q\}_{q \in \mathbb{N}}$ into a simplicial scheme. The inclusions $g_{(i)} : U_{(i)} \to X$ induce inclusions $g_{\boldsymbol{i}} : U_{\boldsymbol{i}} \to X$ and morphisms $g_q : U_q \to X$; and one has the relations $g_p \circ \alpha_* = g_q$ for any $\alpha \in \mathbf{\Delta}^q_p$.

Definition 2.6. Let $\pi : Z \to X$ be a morphism of $\mathbb{K}$-schemes. A *simplicial section of* π based on the covering $\boldsymbol{U}$ is a sequence of morphisms

$$\boldsymbol{\sigma} = \{\sigma_q : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \to Z\}_{q \in \mathbb{N}}$$

satisfying the following conditions.

(i) For any q the diagram

$$\begin{tikzcd} \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \arrow[r, "\sigma_q"] \arrow[d, "\mathrm{p}_2"'] & Z \arrow[d, "\pi"'] \\ U_q \arrow[r, "g_q"] & X \end{tikzcd}$$

is commutative.

(ii) For any $\alpha \in \boldsymbol{\Delta}^q_p$ the diagram

$$\begin{tikzcd} & \boldsymbol{\Delta}^p_{\mathbb{K}} \times U_p \arrow[dr, "\sigma_p"] & \\ \boldsymbol{\Delta}^p_{\mathbb{K}} \times U_q \arrow[ur, "\mathbf{1} \times \alpha^*"] \arrow[dr, "\alpha_* \times \mathbf{1}"'] & & Z \\ & \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \arrow[ur, "\sigma_q"'] & \end{tikzcd}$$

is commutative.

Corollary 2.7. *Assume Scenario* 2.1. *Let* $\boldsymbol{U} = \{U_{(i)}\}_{i=0}^m$ *be an open covering of* X. *Suppose that for any* $i \in \{0, \ldots, m\}$ *we are given some section* $\sigma_{(i)} : U_{(i)} \to \bar{Z}$ *of* $\bar{\pi}$. *Then there exists a simplicial section*

$$\boldsymbol{\sigma} = \{\sigma_q : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \to \bar{Z}\}_{q \in \mathbb{N}}$$

based on $\boldsymbol{U}$, *such that* $\sigma_0|_{U_{(i)}} = \sigma_{(i)}$ *for all* $i \in \{0, \ldots, m\}$.

Proof. For any multi-index $\boldsymbol{i} = (i_0, \ldots, i_q)$ we have sections $\sigma_{(i_0)}, \ldots, \sigma_{(i_q)} : U_{\boldsymbol{i}} \to \bar{Z}$. Let $\sigma_{\boldsymbol{i}} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to \bar{Z}$ be the morphism provided by Theorem 2.2. For fixed q these patch to a morphism $\sigma_q : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \to \bar{Z}$. The functorial and simplicial properties in Theorem 2.2 imply that this is a simplicial section. $\square$

This result (with H trivial) is illustrated in Figure 1.

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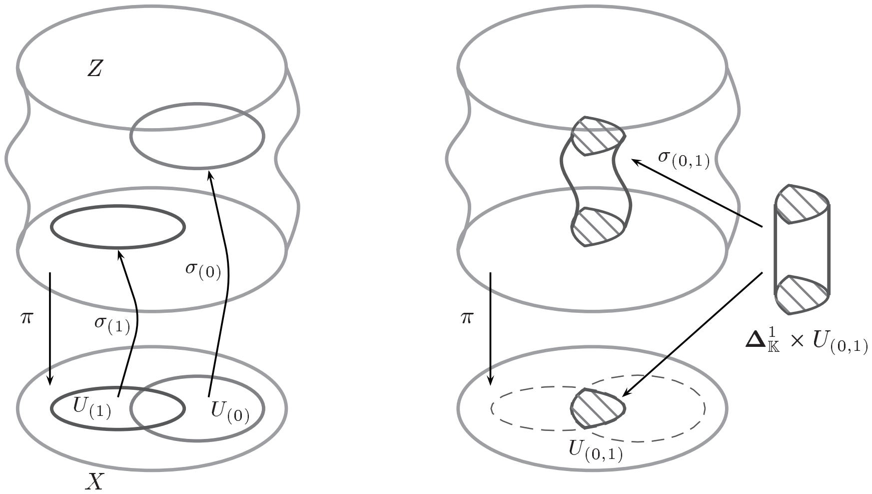

FIGURE 1. Simplicial sections, $q = 1$. We start with sections over two open sets $U_{(0)}$ and $U_{(1)}$ in the left diagram; and we pass to a simplicial section $\sigma_{(0,1)}$ on the right. As can be seen, $\sigma_{(0,1)}$ interpolates between $\sigma_{(0)}$ and $\sigma_{(1)}$.

DEPARTMENT OF MATHEMATICS, BEN GURION UNIVERSITY, BE'ER SHEVA 84105, ISRAEL
E-mail address: amyekut@math.bgu.ac.il