TRACES AND DIFFERENTIAL OPERATORS OVER BEILINSON COMPLETION ALGEBRAS

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0. Introduction

In the short paper [Be] A. Beilinson introduced a generalized version of adeles, with values in any quasi-coherent sheaf on a noetherian scheme X. In particular, taking the structure sheaf $\mathcal{O}_X$ one gets the cosimplicial ring of adeles $\mathbb{A}^{\cdot}(X, \mathcal{O}_X)$. In each degree n, $\mathbb{A}^n(X, \mathcal{O}_X)$ is a subring (a "restricted product") of the product of local factors $\prod_\xi \mathcal{O}_{X,\xi}$. Here $\xi = (x_0, \ldots, x_n)$ runs over all chains of length n of points in X. The Beilinson completion $\mathcal{O}_{X,\xi}$ is gotten by a process of inverse and direct limits. For $n = 0$, $\mathcal{O}_{X,(x_0)}$ is simply the $\mathfrak{m}$-adic completion of the local ring at x_0. For applications to duality theory one is primarily interested in the completion $\mathcal{O}_{X,\xi}$ along a *saturated chain* ξ. As shown in [Ye1], the semi-local ring $\mathcal{O}_{X,\xi}$ carries a natural topology, and its residue fields carry rank n valuations.

In the present paper we isolate the completion $\mathcal{O}_{X,\xi}$ from its geometric environment, and study it as a separate algebraic-topological object, which we call a *Beilinson completion algebra* (BCA). The methods used here belong to commutative algebra, analysis and differential geometry. Our main results have to do with *dual modules* of BCAs, their functorial behavior and their interaction with differential operators. These results, in turn, have some noteworthy applications to algebraic geometry (see §0.3).

One may view our paper partly as a continuation of the work of Lipman, Kunz and others on explicit formulations of duality theory (cf. [Li1], [Li2], [Ku], [Hu1], [Hu2], [HK1], [HK2], [HS], [LS], [Hg]). Their work deals with *linear* aspects of duality theory - construction of dualizing modules, trace maps etc. To that we have little new to add in the present paper. The novelty of our work is in establishing the *nonlinear* properties of duality theory. We show how duality interacts with *differential phenomena*, such as $\mathcal{D}$-modules and De Rham complexes. Such results seem to have been beyond the reach of the methods of commutative algebra used henceforth in this area.

In the remainder of the introduction we outline the content of the paper.

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0.1. Beilinson completion algebras. Let k be a fixed perfect base field. A local BCA A is a quotient of a ring $F((\underline{s}))[[\underline{t}]] = F((s_1, \ldots, s_m))[[t_1, \ldots, t_n]]$, where F is a finitely generated field extension of k, and $F((s_1, \ldots, s_m)) = F((s_m)) \cdots ((s_1))$ is an iterated field of Laurent series. A is a complete noetherian local ring, and a semi-topological (ST) k-algebra. On the residue field $A/\mathfrak{m}$ there is a structure of m-dimensional topological local field (TLF). (These terms are explained briefly in §1-2.) The surjection $F((\underline{s}))[[\underline{t}]] \twoheadrightarrow A$ is not part of the structure of A. A general BCA is a finite product of local ones.

We are interested in two kinds of homomorphisms between BCAs. The first is called a *morphism of BCAs*, and the second is called an *intensification homomorphism.* Rather than defining these notions here (this is done in §2-3), we demonstrate them by examples. Let $A := k(s)[[t]]$ and $B := k(s)((t))$. These local BCAs arise geometrically: take $X := \mathbf{A}^2_k = \operatorname{Spec} k[s,t]$ and $x = (0), y = (t), z = (s,t) \in X$. Then $A \cong \mathcal{O}_{X,(y)}$ and $B \cong \mathcal{O}_{X,(x,y)}$, the Beilinson completions of $\mathcal{O}_X$ along the chains $(y), (x,y)$ respectively. The inclusion $A \to B$ is a morphism, which in "cosimplicial" notation is $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,(x,y)}$. Now let $\hat{A} := k((s))[[t]] \cong \mathcal{O}_{X,(y,z)}$. Then $A \to \hat{A}$ is an intensification homomorphism, which we also write as $\partial^- : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,(y,z)}$.

Whenever $A \to B$ is a morphism and $A \to \hat{A}$ is an intensification, there is a BCA $\hat{B} = B \otimes^{(\wedge)}_A \hat{A}$, a morphism $\hat{A} \to \hat{B}$ and an intensification $B \to \hat{B}$. This situation is called *intensification base change*. In our example, $\hat{B} = k((s))((t)) \cong \mathcal{O}_{X,(x,y,z)}$.

BCAs and morphisms of BCAs constitute a category which is denoted by $\mathsf{BCA}(k)$.

0.2. The Results. There are three main results in the paper. Their precise statement is in the body of the paper, and what follows is only a sketch.

A *finite type ST module* M over a BCA A is a quotient of A^n for some n, with the quotient topology (so if $A/\mathfrak{m}$ is discrete, M has the $\mathfrak{m}$-adic topology.) The *fine topology* on an A-module M is characterized by the property that each finitely generated submodule $M' \subset M$, with the subspace topology, is of finite type. (More on ST modules in §1.) Given a TLF K (i.e. a BCA which is a field), we denote by $\omega(K)$ the top degree component of the separated algebra of differentials $\Omega^{\cdot,\mathrm{sep}}_{K/k}$.

Theorem 6.14: (Dual modules) Let A be a local BCA and M a finite type ST A-module. Then there is a *dual module* $\operatorname{Dual}_A M$, enjoying the following properties. To any morphism $\sigma : K \to A$ in $\mathsf{BCA}(k)$ with K a field, there is a bijection

$$\Psi^M_\sigma : \operatorname{Dual}_A M \xrightarrow{\simeq} \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M, \omega(K)).$$

If $\sigma = \tau \circ f$ for some morphisms $f : K \to L$ and $\tau : L \to A$, then

$$\Psi^M_\sigma(\phi) = \operatorname{Res}_{L/K} \circ \Psi^M_\tau(\phi),$$

where $\operatorname{Res}_{L/K} : \omega(L) \to \omega(K)$ is the residue on TLFs, see [Ye1] §2.4. If $\sigma, \sigma' : K \to A$ are two pseudo coefficients fields (i.e. morphisms such that $[A/\mathfrak{m} : K] < \infty$) which are congruent modulo $\mathfrak{m}$, then the isomorphism

$$\Psi^M_{\sigma,\sigma'} = \Psi^M_{\sigma'} \circ (\Psi^M_\sigma)^{-1} : \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma}(M,\omega(K)) \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma'}(M,\omega(K))$$

has an explicit formula in terms of "Taylor expansions" and differential operators.

In particular for $M = A$ we set $\mathcal{K}(A) := \mathrm{Dual}_A A$, with the fine topology. $\mathcal{K}(A)$ is an injective hull of the residue field $A/\mathfrak{m}$. Note that for a field K, $\mathcal{K}(K) = \omega(K)$. If M is any ST A-module we define

$$\mathrm{Dual}_A M := \mathrm{Hom}^{\mathrm{cont}}_A(M, \mathcal{K}(A))$$

with the Hom topology. (When M is of finite type this is consistent with Thm. 6.14.) We show that given an intensification homomorphism $v : A \to \hat{A}$ there is a continuous homomorphism of ST A-modules

$$q^M_{\hat{A}/A} = q^M_v : \mathrm{Dual}_A M \to \mathrm{Dual}_{\hat{A}}(\hat{A} \otimes_A M).$$

Theorem 7.4: (Traces) Let $A \to B$ be a morphism in $\mathsf{BCA}(k)$. Then there exists a continuous A-linear trace map $\mathrm{Tr}_{B/A} : \mathcal{K}(B) \to \mathcal{K}(A)$. This trace is functorial: $\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B}$. It induces a bijection

$$\mathcal{K}(B) \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_A(B, \mathcal{K}(A)).$$

The trace commutes with intensification base change: given an intensification $A \to \hat{A}$, and letting $\hat{B} := B \otimes^{(\wedge)}_A \hat{A}$, we have

$$q_{\hat{A}/A} \circ \mathrm{Tr}_{B/A} = \mathrm{Tr}_{\hat{B}/\hat{A}} \circ q_{\hat{B}/B}.$$

If $\sigma : K \to A$ is a morphism with K a field, then

$$\mathrm{Tr}_{A/K}(\phi) = \Psi^A_\sigma(\phi)(1) \in \omega(K)$$

for $\phi \in \mathcal{K}(A)$.

Theorem 8.6: (Duals of continuous differential operators) Suppose M, N are ST A-modules with the fine topologies and $D : M \to N$ is a continuous DO. Then there is a continuous DO

$$\mathrm{Dual}_A(D) : \mathrm{Dual}_A N \to \mathrm{Dual}_A M.$$

This operation is transitive in D and compatible with intensification base change $A \to \hat{A}$. $\mathrm{Dual}_A(D)$ is unique, has an explicit description using the isomorphisms $\Psi^M_\sigma, \Psi^N_\sigma$, and is the adjoint of D w.r.t. suitably defined residue pairings.

0.3. Applications. The primary application of our results, and the original motivation of the paper, is the explicit construction of residue complexes on k-schemes. This is carried out in [Ye2]. The construction is extremely simple, and we shall sketch it here. Suppose X is a k-scheme of finite type and (x, y) is a saturated chain of points in it (i.e. y is an immediate specialization of x). There are natural homomorphisms $\partial^- : \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,(x,y)}$ and $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,(x,y)}$, the first being an intensification and the second a morphism (cf. example in §0.1 above). According to Theorems 6.14 and 7.4 we get an $\mathcal{O}_X$-linear homomorphism

$$\delta_{(x,y)} : \mathcal{K}(\mathcal{O}_{X,(x)}) \xrightarrow{q_{\partial^-}} \mathcal{K}(\mathcal{O}_{X,(x,y)}) \xrightarrow{\mathrm{Tr}_{\partial^+}} \mathcal{K}(\mathcal{O}_{X,(y)}).$$

Considering $\mathcal{K}(\mathcal{O}_{X,(x)})$ as a skyscraper sheaf sitting on $\{x\}^-$, we define

$$\begin{aligned} \mathcal{K}^{\cdot}_X &:= \bigoplus_{x \in X} \mathcal{K}(\mathcal{O}_{X,(x)}) \\ \delta_X &:= \sum_{(x,y)} \delta_{(x,y)}. \end{aligned}$$

Then $(\mathcal{K}^{\cdot}_X, \delta_X)$ is the residue complex on X (cf. [RD], [EZ], [Ye1] and [SY]).

A special feature of this particular construction of $\mathcal{K}^{\cdot}_X$ is that given a DO $D : \mathcal{M} \to \mathcal{N}$ between $\mathcal{O}_X$-modules, there is a dual DO

$$\mathrm{Dual}_X(D) : \mathcal{H}om_{\mathcal{O}_X}(\mathcal{N}, \mathcal{K}^{\cdot}_X) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X) \tag{0.1}$$

which is a homomorphism of complexes. This implies that $\mathcal{K}^{\cdot}_X$ is a complex of right $\mathcal{D}_X$-modules. Conversely, $\mathcal{D}_X$ can be recovered from DOs acting on $\mathcal{K}^{\cdot}_X$. Another consequence of (0.1) is that $\mathcal{F}^{\cdot\cdot}_X := \mathcal{H}om_{\mathcal{O}_X}(\Omega^{\cdot}_{X/k}, \mathcal{K}^{\cdot}_X)$ has a natural structure of double complex. Using $\mathcal{F}^{\cdot\cdot}_X$ we are able to analyze the niveau spectral sequence converging to $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$, the algebraic De Rham homology of X.

0.4. Plan of the paper.
Section 1: a quick review of semi-topological rings and modules, as well as new facts on ST Hom modules.

Section 2: definition of BCAs and morphisms, including examples.

Section 3: definition of intensification homomorphisms, base change.

Section 4: general facts on continuous differential operators over ST algebras; the Lie derivative.

Section 5: the structure of the ring of continuous DOs $\mathcal{D}(K)$ over a TLF K; $\omega(K)$ is a right $\mathcal{D}(K)$-module, and the action is by adjunction in a suitable sense.

Section 6: existence of dual modules is proved.

Section 7: contravariance of dual modules w.r.t. morphisms is proved (traces).

Section 8: the interaction between dual modules and DOs is examined, leading to Thm. 8.6 and a few corollaries.

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1. Some Results on Semi-Topological Rings

Let us recall some definitions and results from [Ye1] §1. A semi-topological (ST) ring is a ring A, with a linear topology on its underlying additive group, such that for all $a \in A$, left and right multiplication by a are continuous maps $\lambda_a, \rho_a : A \to A$. A ST left A-module is an A-module M, whose underlying additive group is linearly topologized, and such that for all $a \in A$ and $x \in M$, the multiplication maps they define $\lambda_a : M \to M$ and $\rho_x : A \to M$ are continuous. ST left A-modules and continuous A-linear homomorphisms form a category, denoted $\mathsf{STMod}(A)$. Similarly one defines ST right modules and bimodules.

Assume for simplicity that the ST ring A is commutative. In $\mathsf{STMod}(A)$ there are direct and inverse limits, and a tensor product. Given a ST A-module M, the associated separated module $M^{\mathrm{sep}} = M/\{0\}^-$ is also a ST A-module. The category $\mathsf{STMod}(A)$ is additive, but not abelian. An exact sequence in it is, by definition, a sequence $M' \xrightarrow{\phi} M \xrightarrow{\psi} M''$ which is exact in the untopologized sense (i.e. in $\mathsf{Mod}(A)$), and such that both ϕ and ψ are strict.

On any A-module M there is a finest topology making it into a ST module; it is called the fine A-module topology. If M has the fine topology, then for any ST A-module N, one has $\mathrm{Hom}^{\mathrm{cont}}_A(M, N) = \mathrm{Hom}_A(M, N)$, and this in fact characterizes the fine topology. Trivially, if M has the fine topology, then so does M^{sep}. A free ST A-module is a free A-module with the fine topology. So F is free iff $F \cong \bigoplus A$ with the $\bigoplus$ topology. A ST module M has the fine topology iff it admits a strict surjection $F \twoheadrightarrow M$ with F free.

Definition 1.1. Let M, N be ST A-modules. The (weak) Hom topology on the abelian group $\mathrm{Hom}^{\mathrm{cont}}_A(M, N)$ is the coarsest linear topology such that for every $x \in M$, the map $\rho_x : \mathrm{Hom}^{\mathrm{cont}}_A(M, N) \to N$, $\phi \mapsto \phi(x)$, is continuous.

Unless otherwise specified, this is the topology we consider on $\mathrm{Hom}^{\mathrm{cont}}_A(M, N)$. If M has the fine topology, we shall often drop the superscript "cont".

Remark 1.2. A basis of neighborhoods of 0 for the Hom topology is the collection of open subgroups $\{V(F, U)\}$, where F runs over the finite subsets of M, U runs over the open subgroups of N, and $V(F, U) = \{\phi \mid \phi(F) \subset U\}$. Such a topology is sometimes called the weak topology (cf. [Ko]). Usually, to obtain a duality one needs a finer topology - the strong topology of [Ko], or the compact-open topology of [Mc]. In the present paper duality is defined by indirect means, and for our purposes the weak topology suffices (cf. Remark 8.3).

The next lemma summarizes the properties of the Hom topology. Its easy proof is left to the reader.

Lemma 1.3. *Let A be a commutative ST ring.*

(1) *Let* $\phi : M' \to M$ *and* $\psi : N \to N'$ *be homomorphisms in* $\mathsf{STMod}(A)$. *Then the induced homomorphism* $\mathrm{Hom}_A^{\mathrm{cont}}(M, N) \to \mathrm{Hom}_A^{\mathrm{cont}}(M', N')$ *is continuous.*

(2) *Let* M, N *be ST* A*-modules. Then* $\mathrm{Hom}_A^{\mathrm{cont}}(M, N)$ *is a ST* A*-module.* $\mathrm{End}_A^{\mathrm{cont}}(M) = \mathrm{Hom}_A^{\mathrm{cont}}(M, M)$ *is a ST* A*-algebra, and* M *is a ST left* $\mathrm{End}_A^{\mathrm{cont}}(M)$*-module. The natural bijection* $M \xrightarrow{\simeq} \mathrm{Hom}_A^{\mathrm{cont}}(A, M)$, $x \mapsto \rho_x$, *is an isomorphism of ST* A*-modules.*

(3) *Suppose in (1)* ϕ *is surjective and* ψ *is a strict monomorphism. Then* $\mathrm{Hom}_A^{\mathrm{cont}}(M, N) \to \mathrm{Hom}_A^{\mathrm{cont}}(M', N')$ *is a strict monomorphism.*

(4) *Let* $(M_\alpha)_{\alpha \in I}$ *be a direct system in* $\mathsf{STMod}(A)$, *with* I *a directed set. Then for any ST* A*-module* N *the natural map*

$$\lim_{\leftarrow \alpha} \mathrm{Hom}_A^{\mathrm{cont}}(M_\alpha, N) \to \mathrm{Hom}_A^{\mathrm{cont}}(\lim_{\alpha \to} M_\alpha, N)$$

is an isomorphism of ST A*-modules.*

From parts (1) and (2) of the lemma it follows that $\mathrm{Hom}_A^{\mathrm{cont}}$ is an additive bifunctor $\mathsf{STMod}(A)^{\circ} \times \mathsf{STMod}(A) \to \mathsf{STMod}(A)$.

Tensor products are defined in $\mathsf{STMod}(A)$. The usual tensor product $M \otimes_A N$ is given the finest linear topology s.t. the maps $\rho_y : M \to M \otimes_A N$, $x' \mapsto x' \otimes y$ and $\lambda_x : N \to M \otimes_A N$, $y' \mapsto x \otimes y'$ are all continuous (see [Ye1] Def. 1.2.11).

Lemma 1.4. (Adjunction) *Let* A, B *be ST rings (not necessarily commutative), let* L *be a ST left* A*-module,* N *a ST left* B*-module, and* M *a ST* B*-*A*-bimodule. Then*

$$\mathrm{Hom}_B^{\mathrm{cont}}(M \otimes_A L, N) \cong \mathrm{Hom}_A^{\mathrm{cont}}(L, \mathrm{Hom}_B^{\mathrm{cont}}(M, N))$$

as topological abelian groups.

Proof. Immediate from the definitions of the Hom and $\otimes$ topologies. $\square$

We say a homomorphism $\phi : M \to N$ of topological abelian groups is dense if $\phi(M) \subset N$ is (everywhere) dense.

Lemma 1.5. *Suppose* A *is a ST ring and* $M \to \hat{M}$, $N \to \hat{N}$ *are continuous dense homomorphisms of ST* A*-modules. Then* $M \otimes_A N \to \hat{M} \otimes_A \hat{N}$ *is dense.*

Proof. By transitivity of denseness it suffices to prove that $M \otimes_A N \to \hat{M} \otimes_A N$ is dense. Choose a surjection from a free module $A^{(I)} = \bigoplus A$ onto N. This induces surjections $M^{(I)} \twoheadrightarrow M \otimes_A N$ and $\hat{M}^{(I)} \twoheadrightarrow \hat{M} \otimes_A N$. But according to [Ye1] Prop. 1.1.8 (c), $M^{(I)} \to \hat{M}^{(I)}$ is dense. $\square$

Definition 1.6. Let A be a commutative noetherian ST ring. A ST A-module M is called of *finite type* (resp. *cofinite type*, resp. *torsion type*) if it is finitely generated (resp. it is artinian, resp. $\mathrm{Supp}\, M \subset \mathrm{Spec}\, A$ consists solely of maximal ideals), and if it has the fine topology.

Denote the full subcategories of $\mathsf{STMod}(A)$ consisting of finite type (resp. cofinite type) modules by $\mathsf{STMod}_{\mathrm{f}}(A)$ (resp. $\mathsf{STMod}_{\mathrm{cof}}(A)$).

Generalizing the Zariski and Artin-Rees properties for noetherian rings with adic topologies, we make the following definition. Let us point out that this definition is stronger then [Ye1] Definition 3.2.10.

Definition 1.7. Let A be a noetherian commutative ST ring. A is said to be a *Zariski ST ring* if

(i) Every ST A-module, which is either of finite type or of torsion type, is separated.
(ii) Every (continuous) A-linear homomorphism between two ST A-modules, each either of finite type or of torsion type, is strict.

Proposition 1.8. *Let A be a local Zariski ST ring, with maximal ideal $\mathfrak{m}$. Assume that $A \cong \lim_{\leftarrow i} A/\mathfrak{m}^{i+1}$ as ST rings. Let M, N be ST A-modules.*

(1) *If M, N are both of finite type then so is $\mathrm{Hom}_A^{\mathrm{cont}}(M, N)$.*
(2) *If M is of finite type and N is of cofinite type then $\mathrm{Hom}_A^{\mathrm{cont}}(M, N)$ is of cofinite type.*
(3) *If M, N are both of cofinite type then $\mathrm{Hom}_A^{\mathrm{cont}}(M, N)$ is of finite type.*

Proof. (1) Let $A^r \twoheadrightarrow M$ be a surjection. By Lemma 1.3 (2) and (3), $\mathrm{Hom}_A^{\mathrm{cont}}(M, N) \hookrightarrow N^r$ is a strict monomorphism. Now use the Zariski property to conclude that $\mathrm{Hom}_A(M, N)$ has the fine topology.

(2) Like (1).

(3) Let $M_i := \mathrm{Hom}_A^{\mathrm{cont}}(A/\mathfrak{m}_{i+1}, M)$, so $M_i \hookrightarrow M$ is strict, M_i has the fine topology, and $M = \lim_{i \to} M_i$. Similarly define N_i. By part (4) of Lemma 1.3,

$$\mathrm{Hom}_A^{\mathrm{cont}}(M, N) = \lim_{\leftarrow i} \mathrm{Hom}_A^{\mathrm{cont}}(M_i, N) = \lim_{\leftarrow i} \mathrm{Hom}_A^{\mathrm{cont}}(M_i, N_i).$$

Now M_i and N_i are of finite type, so we can we can use part (1) and [Ye1] Prop. 1.2.20. □

Corollary 1.9. (ST Version of Matlis Duality) *Let A be as in the proposition. Suppose I is an injective hull of $A/\mathfrak{m}$, endowed with the fine topology. Then $\mathrm{Hom}_A^{\mathrm{cont}}(-, I)$ is an equivalence*

$$\mathsf{STMod}_{\mathrm{f}}(A)^{\circ} \leftrightarrow \mathsf{STMod}_{\mathrm{cof}}(A)\ .$$

2. Definitions and Basic Properties of BCAs

In this section k is a fixed perfect field. If A is a ST k-algebra and $\underline{t} = (t_1, \ldots, t_n)$ is a sequence of indeterminates, we denote by $A[[\underline{t}]] = A[[t_1, \ldots, t_n]]$ the ring of formal power series, with the topology given by

$$A[[\underline{t}]] = \lim_{\leftarrow i} A[\underline{t}]/(\underline{t})^i,$$

where for each i, $A[\underline{t}]/(\underline{t})^i$ has the fine A-module topology. The ring of Laurent series $A((t))$ is topologized by

$$A((t)) = \lim_{j \to} t^{-j} A[[t]],$$

and we define recursively

$$A((\underline{t})) = A((t_1, \ldots, t_n)) := A((t_2, \ldots, t_n))((t_1)).$$

According to [Ye1] §1.3, $A[[\underline{t}]]$ and $A((\underline{t}))$ are ST k-algebras.

A *topological local field* (TLF) over k is a field K, together with a topology, and valuation rings $\mathcal{O}_i$, $i = 1, \ldots, n$, such that the residue field κ_i of $\mathcal{O}_i$ is the fraction field of $\mathcal{O}_{i+1}$, and $K = \mathrm{Frac}(\mathcal{O}_1)$. These data are related by the existence of a *parametrization*: an isomorphism $K \cong F((t_1, \ldots, t_n))$ of ST k-algebras, s.t. $\mathcal{O}_i \cong F((t_{i+1}, \ldots, t_n))[[t_i]]$. Here F is a discrete field, and $\Omega^1_{F/k}$ has finite rank. The number n is the dimension of the local field K. Topological local fields constitute a category $\mathsf{TLF}(k)$. For more details see [Ye1] §2.1.

Definition 2.1. A local *Beilinson completion algebra* (BCA) over k is a commutative semi-topological local ring A, together with a structure of topological local field on the residue field $A/\mathfrak{m}$. The following condition must be satisfied: there exists a surjective homomorphism of k-algebras

$$F((\underline{s}))[[\underline{t}]] = F((s_1, \ldots, s_m))[[t_1, \ldots, t_n]] \to A,$$

which is strict (topologically), and induces and isomorphism of TLFs $F((\underline{s})) \cong A/\mathfrak{m}$. Such a surjection is called a parametrization of A.

A Beilinson completion algebra is a finite product of local BCAs.

Remark 2.2. In greater generality one can define a BCA over any noetherian ring R, to be any finite algebra over the R-algebra $\mathbb{A}(\Xi, \mathcal{O}_X) = \prod_{\xi \in \Xi} \mathcal{O}_{X,\xi}$, where Ξ is a finite set of saturated chains in some finite type R-scheme X, and $\mathbb{A}(-,-)$ is Beilinson's scheme theoretical group of adeles. See [Be], [Hr], [Ye1] and [HY] for the definition of adeles, and cf. Examples 2.3 and 2.4 below.

Observe that a Beilinson completion algebra A is necessarily an $\mathfrak{r}$-adically complete, noetherian, semi-local ring, where $\mathfrak{r}$ is the Jacobson radical of A. If A is artinian, then in the terminology of [Ye1], it is a cluster of TLFs (a CTLF).

For any $\mathfrak{m} \in \operatorname{Max} A$ set $\operatorname{res.dim}_{\mathfrak{m}} A := \dim A/\mathfrak{m}$, the local field dimension. We say that A is equidimensional of dimension n if $\operatorname{res.dim}_{\mathfrak{m}} A = n$ for all $\mathfrak{m}$. In this case we set $\operatorname{res.dim} A := n$, and

$$\begin{aligned} \mathcal{O}_i(A) &:= \prod_{\mathfrak{m} \in \operatorname{Max} A} \mathcal{O}_i(A/\mathfrak{m}) \\ \kappa_i(A) &:= \prod_{\mathfrak{m} \in \operatorname{Max} A} \kappa_i(A/\mathfrak{m}) \end{aligned}$$

for $1 \leq i \leq n$. Also we set $\mathcal{O}_0(A) := A$ and $\kappa_0(A) := A/\mathfrak{r} = \prod A/\mathfrak{m}$.

The motivating example is:

Example 2.3. Let X be a scheme of finite type over k, and let $\xi = (x_0, \ldots, x_m)$ be a saturated chain in X. Then the Beilinson completion $\mathcal{O}_{X,\xi}$ of the structure sheaf along ξ is defined; see [Ye1] §3.1. We claim that $\mathcal{O}_{X,\xi}$ is an equidimensional BCA, of dimension m. To see why, first choose a coefficient field $\sigma : k(x_0) \to \hat{\mathcal{O}}_{X,x_0} = \mathcal{O}_{X,(x_0)}$. According to [Ye1] Lemma 3.3.9, σ extends to a lifting $\sigma_\xi : k(\xi) = k(x_0)_\xi \to \mathcal{O}_{X,\xi}$. Sending $t_1, \ldots, t_n$ to generators of the maximal ideal $\mathfrak{m}_{x_0}$, we get a strict surjection $k(\xi)[[t_1, \ldots, t_n]] \twoheadrightarrow \mathcal{O}_{X,\xi}$. Finally, according to [Ye1] Prop. 3.3.6, $k(\xi)$ is a finite product of TLFs, all of dimension m.

Example 2.4. Consider a BCA $A = F((s_1, \ldots, s_m))[[t_1, \ldots, t_n]]$. We claim it is of the form $\mathcal{O}_{X,\xi}$. Choose an integral k-scheme of finite type Y such that $F = k(Y)$. Set $X := \mathbf{A}^{n+m}_Y = \mathbf{A}^{n+m}_k \times_k Y$, and let $\xi = (x_0, \ldots, x_m)$ be the saturated chain $x_i := (t_1, \ldots, t_n, s_1, \ldots, s_i)$, where we write $\mathbf{A}^{n+m}_k = \operatorname{Spec} k[\underline{s}, \underline{t}]$. Then $F((\underline{s}))[[\underline{t}]] \cong \mathcal{O}_{X,\xi}$ (cf. [Ye1] Thm. 3.3.2 (c); it can be assumed that Y is normal).

Let A be a local BCA of res. dim n. For every $1 \leq i \leq n$ there is a subring $\mathcal{O}_{1,\ldots,i}(A) \subset A$ defined by

$$\mathcal{O}_{1,\ldots,i}(A) \cong A \times_{A/\mathfrak{m}} \mathcal{O}_1(A/\mathfrak{m}) \times_{\kappa_1(A/\mathfrak{m})} \cdots \times_{\kappa_{i-1}(A/\mathfrak{m})} \mathcal{O}_i(A/\mathfrak{m}).$$

It is the largest subring of A which projects onto $\kappa_i(A)$, and it is actually the valuation ring of a rank i valuation (hence local). In [Ye1] the notation $\mathcal{O}(A)$ was used for $\mathcal{O}_{1,\ldots,n}(A)$.

Definition 2.5. (Morphisms) Let A and B be Beilinson completion algebras. A morphism $f : A \to B$ is a continuous k-algebra homomorphism, satisfying the following local condition. Given a maximal ideal $\mathfrak{n} \subset B$, let $\mathfrak{m} \subset A$ be the unique maximal ideal such that $f^{-1}(\mathfrak{n}) \subset \mathfrak{m}$. Set $i := \operatorname{res.dim} B_\mathfrak{n} - \operatorname{res.dim} A_\mathfrak{m}$, which is assumed to be non-negative. Then $f(A_\mathfrak{m}) \subset \mathcal{O}_{1,\ldots,i}(B_\mathfrak{n})$, the induced homomorphism $A_\mathfrak{m} \to \kappa_i(B/\mathfrak{n})$ sends $\mathfrak{m}$ to 0, and $A/\mathfrak{m} \to \kappa_i(B/\mathfrak{n})$ is a finite morphism of local fields.

The composition of two morphisms is again a morphism, so we get a category, which is denoted by $\mathsf{BCA}(k)$. The number i in the definition is called the relative residual dimension of f at $\mathfrak{n}$, denoted $\operatorname{res.dim}_\mathfrak{n} f$. If f is equidimensional we shall ommit the subscript $\mathfrak{n}$. We call f finite if B is a finitely generated A-module. Observe that the full subcategory of $\mathsf{BCA}(k)$ consisting of fields coincides with the full subcategory of $\mathsf{TLF}(k)$ consisting of TLFs whose last residue field is finitely generated over k. (In characteristic 0 this is all of $\mathsf{TLF}(k)$.)

Here are some typical examples of morphisms of BCAs.

Example 2.6. Let $A := k[[s]]$, $B := k((s))[[t]]$, and let $f : A \to B$ be the inclusion. Then $\mathfrak{m} = (s)$, $\mathfrak{n} = (t)$, $\operatorname{res.dim}_\mathfrak{m} A = 0$, $\operatorname{res.dim}_\mathfrak{n} B = 1$ and $\operatorname{res.dim}_\mathfrak{n} f = 1$.

Example 2.7. Let X be a finite type k-scheme, $\xi = (x, \ldots, y)$ a saturated chain in X, $A := \mathcal{O}_{X,(y)}$, $B := \mathcal{O}_{X,\xi}$, and $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ the coface map. Now $\operatorname{res.dim} A = 0$, and $\operatorname{res.dim} B = \operatorname{res.dim} \partial^+$ equals the length of ξ.

Example 2.8. Let X, Y be finite type k-schemes, $f : X \to Y$ a k-morphism, $y \in Y$ any point and x a closed point in the fibre $X_y := f^{-1}(y)$. Since $k(y) \to k(x)$ is finite, $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is a morphism of BCAs, with $\operatorname{res.dim} f^* = 0$.

Definition 2.9. Let A be a local BCA over k, with maximal ideal $\mathfrak{m}$. A *coefficient field* (resp. *quasi coefficient field*, resp. *pseudo coefficient field*) for A is a morphism $\sigma : K \to A$ in $\mathsf{BCA}(k)$, with K a field, and such that the induced homomorphism $K \to A/\mathfrak{m}$ is bijective (resp. finite separable, resp. finite).

By definition, every local BCA has a coefficient field.

Lemma 2.10. *Let A be a local BCA over k, with maximal ideal $\mathfrak{m}$. Then:*

(a) *Suppose A is artinian and $K \to A$ is a pseudo coefficient field. Then A has the fine K-module topology.*
(b) *Letting $A_i := A/\mathfrak{m}^{i+1}$, the map $A \to \lim_{\leftarrow i} A_i$ is an isomorphism of ST k-algebras.*
(c) *Let $K \to A$ be a pseudo coefficient field, and let M be a torsion type ST A-module (see Def.* 1.6*). Then M is a free ST K-module.*
(d) *Suppose $\sigma : K \to A$ is a morphism of BCAs, with K a field. Then there exists a finite morphism $f : L[[\underline{t}]] \to A$ extending σ, i.e. $\sigma : K \to L \to L[[\underline{t}]] \xrightarrow{f} A$.*

Proof. (a) By [Ye1] Prop. 2.2.2.

(b) This is true for $F((\underline{s}))[[\underline{t}]]$ (by definition!) and hence, by [Ye1] Prop. 1.2.20, for every quotient A.

(c) Set $M_i := \operatorname{Hom}_A(A_i, M)$, with the fine A-module topology. According to [Ye1] Cor. 1.2.6, $M \cong \lim_{i \to} M_i$. Now M_i is a ST A_i-module with the fine topology. Since A_i has the fine K-module topology, so does M_i. Passing to the limit, M has the fine K-module topology, so it is a free ST K-module.

(d) According to [Ye1] Cor. 2.1.19 we can find a finite morphism $K((\underline{s})) = L \to A/\mathfrak{m}$. As in the proof of ibid. Prop. 2.2.2., this extends to a morphism $L \to \lim_{\leftarrow i} A_i = A$, which we then extend to $f : L[[\underline{t}]] \to A$ by sending the t_i to generators of the maximal ideal ideal $\mathfrak{m}$. $\square$

Proposition 2.11. *Let A be a BCA over k. Then:*

(a) *If $f : A \to B$ is a finite morphism in $\mathsf{BCA}(k)$, then B has the fine A-module topology.*
(b) *Conversely, if B is a finite A-algebra, then B admits a unique structure of BCA s.t. $A \to B$ is a morphism of BCAs.*
(c) *A is a Zariski ST ring. Moreover, every finite type or torsion type ST A-module is complete.*

Proof. (a) Let $\mathfrak{r} \subset A$ and $\mathfrak{s} \subset B$ be the Jacobson radicals. According to [Ye1] Prop. 2.2.2 (b), $B_i := B/\mathfrak{s}^{i+1}$ has the fine $A_i := A/\mathfrak{r}^{i+1}$-module topology, for each $i \geq 0$.

So B_i also has the fine A-module topology. Now use Lemma 2.10 (b) and [Ye1] Prop. 1.2.20.

(b) According to [Ye1] Prop. 2.2.2 (c), this is true for $A_i \to B_i$. Now use $B \cong \lim_{\leftarrow i} B_i$.

(c) It suffices to consider $A = F((\underline{s}))[[\underline{t}]]$. By [Ye1] Thm. 3.3.8, A is a Zariski ST ring in the sense of ibid. Def. 3.2.10. This means that every finite type ST A-module is separated, and every homomorphism between two such modules is strict.

Now consider two torsion type ST A-modules, M and N. We may assume A is local. Choose a pseudo coefficient field $K \to A$. Then M, N are free ST K-modules, and in particular they are separated and complete (cf. [Ye1] Prop. 1.5). To prove that any homomorphism $\phi : M \to N$ is strict, we may assume it is injective. Then any K-linear splitting $M \twoheadrightarrow N$ is continuous, showing that ϕ is strict.

Finally, given a homomorphism $\phi : M \to N$, with M, N either of finite type or of torsion type, then the module $\bar{M} := \phi(M)$, endowed with the fine topology, is a ST module of both types. Therefore $M \twoheadrightarrow \bar{M}$ and $\bar{M} \hookrightarrow N$ are both strict. $\square$

3. Intensification Base Change

The operation of base change to be discussed in this subsection is a generalization of the one in [Ye1] §2.2. The important notion is that of an intensification homomorphism $u : A \to \hat{A}$ between two BCAs (Def. 3.6). Differentially u is "étale": the differential invariants of $\hat{A}$ descend to A. From the point of view of valuations, $\hat{A}$ is like a completion of A. Again k is a fixed perfect field.

Definition 3.1. Let $A, \hat{A} \in \mathsf{BCA}(k)$ have Jacobson radicals $\mathfrak{r}, \hat{\mathfrak{r}}$ respectively, and let $u : A \to \hat{A}$ be a continuous k-algebra homomorphism, with $u(\mathfrak{r}) \subset \hat{\mathfrak{r}}$.

- (a) u is called *radically unramified* if $\hat{\mathfrak{r}} = \hat{A} \cdot u(\mathfrak{r})$.
- (b) u is called *finitely ramified* if $\hat{A}/\hat{A} \cdot u(\mathfrak{r})$ is artinian, and if for every $\hat{\mathfrak{m}} \in \operatorname{Max} \hat{A}$ lying over some $\mathfrak{m} \in \operatorname{Max} A$, letting $n := \operatorname{res.dim} \hat{A}/\hat{\mathfrak{m}}$, the image of $(A/\mathfrak{m})^{\times}$ in the rank n valuation group of $\hat{A}/\hat{\mathfrak{m}}$ has finite index.

Proposition 3.2. (Finitely Ramified Base Change) *Let $K, \hat{K}, A \in \mathsf{BCA}(k)$, with A a local ring and $K, \hat{K}$ fields. Suppose $f : K \to A$ is a morphism in $\mathsf{BCA}(k)$ and $u : K \to \hat{K}$ is a finitely ramified homomorphism. Then there exists a BCA $\hat{A}$, a morphism $\hat{f} : \hat{K} \to \hat{A}$ in $\mathsf{BCA}(k)$, and a finitely ramified homomorphism $v : A \to \hat{A}$, satisfying:*

- (i) *$v \circ f = \hat{f} \circ u$, and moreover the homomorphism $A \otimes_K \hat{K} \to \hat{A}$ is dense.*
- (ii) $\operatorname{res.dim} \hat{f} = \operatorname{res.dim} f$.
- (iii) *Suppose $\hat{g} : \hat{K} \to \hat{C}$ is a morphism in $\mathsf{BCA}(k)$, with $\hat{C}$ local, and let $n := \operatorname{res.dim} \hat{g} - \operatorname{res.dim} \hat{f}$. Suppose also $w : A \to \hat{C}$ is a continuous homomorphism s.t. $w \circ f = \hat{g} \circ u$, $w(A) \subset \mathcal{O}_{1,\ldots,n}(\hat{C})$, and $A \to \kappa_n(\hat{C})$ is finitely ramified. Then there exists a unique morphism $\hat{h} : \hat{A} \to \hat{C}$ (of* $\operatorname{res.dim} n$*) in $\mathsf{BCA}(k)$, such that $\hat{g} = \hat{h} \circ \hat{f}$ and $w = \hat{h} \circ v$.*

Proof. Choose a finite morphism $K((\underline{s}))[[\underline{t}]] \to A$ (cf. Lemma 2.10), and set

$$\hat{A} := A \otimes_{K((\underline{s}))[[\underline{t}]]} \hat{K}((\underline{s}))[[\underline{t}]].$$

$\hat{A}$ is a BCA by Prop. 2.11, and $\hat{f}, v$ are the obvious maps.

Let us prove that $A \otimes_K \hat{K} \to \hat{A}$ is dense. Denoting by $K[\underline{s}, \underline{s}^{-1}]$ the ring of Laurent polynomials, we have $\hat{K} \otimes_K A \cong \hat{K}[\underline{s}, \underline{s}^{-1}, \underline{t}] \otimes_{K[\underline{s}, \underline{s}^{-1}, \underline{t}]} A$. By [Ye1] Lemma 1.3.9 the homomorphism $\hat{K}[\underline{s}, \underline{s}^{-1}] \to \hat{K}((\underline{s}))$ is dense, and a similar argument shows that so is $\hat{K}[\underline{s}, \underline{s}^{-1}, \underline{t}] \to \hat{K}((\underline{s}))[[\underline{t}]]$. Now use Lemma 1.5.

Finally, given $\hat{C}$, the arguments in the proof of [Ye1] Thm. 2.2.4 imply there is a morphism $\hat{K}((\underline{s}))[[\underline{t}]] \to \hat{C}$, and tensoring with A we get $\hat{h} : \hat{A} \to \hat{C}$. Uniqueness follows from the denseness of $A \otimes_K \hat{K} \to \hat{A}$. $\square$

The algebra $\hat{A}$ in the proposition is unique (up to a unique isomorphism). We shall denote it by

$$\hat{A} = A \otimes_K^{(\wedge)} \hat{K} \ . \tag{3.1}$$

In contrast with the usual tensor product, this is not a symmetric expression - we shall always put the algebra which is the range of the finitely ramified homomorphism to the right.

In [Ye1] §1.5 the notion of a topologically étale homomorphism relative to k was defined. A homomorphism $v : A \to \hat{A}$ in $\mathsf{STComAlg}(k)$, the category of commutative ST k-algebras, is called topologically étale relative to k if for any separated ST $\hat{A}$-module $\hat{M}$, any continuous k-linear derivation $\partial : A \to \hat{M}$ has a unique extension to a continuous derivation $\hat{\partial} : \hat{A} \to \hat{M}$. Often we shall suppress the phrase "relative to k"; this should not cause any confusion as we have no notion of absolute topologically étale homomorphism.

Lemma 3.3.

(a) *The homomorphism $v : A \to \hat{A} = A \otimes_K^{(\wedge)} \hat{K}$ is flat.*
(b) *If $u : K \to \hat{K}$ is topologically étale relative to k, then $v : A \to \hat{A}$ is topologically étale and radically unramified.*

Proof. (a) We have $\hat{A} \cong A \otimes_{K((\underline{s}))[[\underline{t}]]} \hat{K}((\underline{s}))[[\underline{t}]]$. According to [CA] Ch. III §5.4 Prop. 4, the homomorphism $K((\underline{s}))[[\underline{t}]] \to \hat{K}((\underline{s}))[[\underline{t}]]$ is flat; hence so is $A \to \hat{A}$.

(b) As in the proof of [Ye1] Thm. 2.4.23, $K((\underline{s}))[[\underline{t}]] \to \hat{K}((\underline{s}))[[\underline{t}]]$ is topologically étale. By [Ye1] Prop. 1.5.9 (b), so is $A \to \hat{A}$. The ring $\hat{A}/\hat{A} \cdot v(\mathfrak{m}) \cong A/\mathfrak{m} \otimes_{K((\underline{s}))} \hat{K}((\underline{s}))$ is reduced, since $K((\underline{s})) \to \hat{K}((\underline{s}))$ is separable (cf. proof of [Ye1] Thm. 2.4.23). This shows that $\hat{A} \cdot v(\mathfrak{m})$ is the Jacobson radical of $\hat{A}$. $\square$

Let $A, \hat{A}$ be two local BCAs, with maximal ideals $\mathfrak{m}, \hat{\mathfrak{m}}$ respectively. Suppose $v : A \to \hat{A}$ is a finitely ramified, radically unramified homomorphism. Let $\sigma : K \to A$ be a pseudo coefficient field, and assume there is some subfield $\hat{K} \subset \hat{A}/\hat{\mathfrak{m}}$ such that $K \to \hat{K}$ is topologically étale relative to k, and $A/\mathfrak{m} \otimes_K \hat{K} \to \hat{A}/\hat{\mathfrak{m}}$ is bijective.

Then $\hat{K} \to \hat{A}/\hat{\mathfrak{m}}$ is finite, and $K \to \hat{K}$ is finitely ramified. Also, this $\hat{K}$ is unique. Since some lifting $\hat{K} \to \hat{A}$ exists, there is a unique pseudo coefficient field

$$\hat{\sigma} : \hat{K} \to \hat{A} \tag{3.2}$$

extending σ (cf. [Ye1] formula (4.1.11)).

Example 3.4. If $v : A \to \hat{A}$ is topologically étale and $K \to A/\mathfrak{m}$ is purely inseparable, then such a subfield $\hat{K}$ exists. Indeed, we have $\hat{A}/\hat{\mathfrak{m}} = \hat{A} \otimes_A A/\mathfrak{m}$, so $A/\mathfrak{m} \to \hat{A}/\hat{\mathfrak{m}}$ is also topologically étale. If σ is a coefficient field, the statement is trivial. Otherwise, see [Ye1] formula (4.1.10).

We make $\mathrm{gr}_{\mathfrak{m}} A = \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$ into a graded ST ring by putting on $\mathfrak{m}^i \subset A$ the subspace topology, and putting on $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ the quotient topology. Similarly, for a ST A-module M, $\mathrm{gr}_{\mathfrak{m}} M$ is a graded ST $\mathrm{gr}_{\mathfrak{m}} A$-module.

Proposition 3.5. *In the situation above, suppose that $v : A \to \hat{A}$ is flat. Then:*

(a) *For any finite type ST A-module M which has finite length, the canonical homomorphism*
$$\hat{K} \otimes_K M \to \hat{A} \otimes_A M$$
is an isomorphism of ST $\hat{K}$-modules.

(b) *The canonical morphism $A \otimes_K^{(\wedge)} \hat{K} \to \hat{A}$ in $\mathsf{BCA}(k)$ is an isomorphism.*

(c) *For any finite type ST A-module M, the canonical homomorphism*
$$\hat{K} \otimes_K \mathrm{gr}_{\mathfrak{m}} M \to \mathrm{gr}_{\mathfrak{m}}(\hat{A} \otimes_A M)$$
is an isomorphism of graded ST $\hat{K}$-modules.

Proof. (a) The proof is by induction on the length of M. For M of length 1, we have by assumption

$$\hat{K} \otimes_K M \cong \hat{K} \otimes_K A/\mathfrak{m} \cong \hat{A}/\hat{\mathfrak{m}} \cong \hat{A} \otimes_A M \ .$$

Otherwise, we can find an exact sequence (of untopologized A-modules)

$$M^{\cdot} = (0 \to M' \to M \to M'' \to 0)$$

which gives rise, by flatness, to a homomorphism of exact sequences $\hat{K} \otimes_K M^{\cdot} \to \hat{A} \otimes_A M^{\cdot}$. By induction and the Five Lemma, we conclude that $\hat{K} \otimes_K M \cong \hat{A} \otimes_A M$. Since both modules have the fine $\hat{K}$-module topologies, this is a homeomorphism.

(b) We have $A \otimes_K^{(\wedge)} \hat{K} \cong \lim_{\leftarrow i} A/\mathfrak{m}^{i+1} \otimes_K \hat{K}$, and by Lemma 2.10 (b), $\hat{A} \cong \lim_{\leftarrow i} \hat{A}/\hat{\mathfrak{m}}^{i+1}$. Now use part (a) above, together with the isomorphism $\hat{A} \otimes_A (A/\mathfrak{m}^{i+1}) \cong \hat{A}/\hat{\mathfrak{m}}^{i+1}$.

(c) By flatness and the fact that A and $\hat{A}$ are Zariski ST rings, it follows that $\hat{A} \otimes_A \mathfrak{m}^i M \cong \mathfrak{m}^i(\hat{A} \otimes_A M) \subset \hat{A} \otimes_A M$ as ST $\hat{A}$-modules. Therefore $\hat{A} \otimes_A (\mathrm{gr}_{\mathfrak{m}} M)_i \cong \mathrm{gr}_{\mathfrak{m}}(\hat{A} \otimes_A M)_i$ as ST $\hat{A}/\hat{\mathfrak{m}}$-modules. Now use part (a). $\square$

Definition 3.6. (Intensification) Let $u : A \to \hat{A}$ be a continuous k-algebra homomorphism between two BCAs. If u is flat, finitely ramified, radically unramified and topologically étale relative to k, then u is called an *intensification* homomorphism.

Example 3.7. Let X be a finite type k-scheme, $\xi = (x, \ldots, y)$ a saturated chain in X, $A := \mathcal{O}_{X,(x)}$, $B := \mathcal{O}_{X,\xi}$, and $\partial^- : \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,\xi}$ the coface map. Then ∂^- is an intensification homomorphism (cf. Example 2.7).

Theorem 3.8. (Intensification Base Change) *Let $A, \hat{A}, B$ be local BCAs, let $f : A \to B$ be a morphism in* $\mathsf{BCA}(k)$*, and let $u : A \to \hat{A}$ be an intensification homomorphism. Then there is a BCA $\hat{B} = B \otimes_A^{(\wedge)} \hat{A}$, a morphism $\hat{f} : \hat{A} \to \hat{B}$ and an intensification homomorphism $v : B \to \hat{B}$, satisfying conditions* (i) - (iii) *of Prop.* 3.2 (*but replacing the letters K, A with A, B*).

$$\begin{tikzcd} B \arrow[r, "v"] & \hat{B} \\ A \arrow[u, "f"] \arrow[r, "u"] & \hat{A} \arrow[u, "\hat{f}"'] \end{tikzcd} \tag{3.3}$$

Proof. Choose a coefficient field $\sigma : K \to A$, and let $\hat{\sigma} : \hat{K} = K \otimes_A \hat{A} \to \hat{A}$ be its unique extension. So $\hat{A} \cong A \otimes_K^{(\wedge)} \hat{K}$. Set $\hat{B} := B \otimes_K^{(\wedge)} \hat{K}$. We can find a surjective morphism $K[[\underline{t}]] \twoheadrightarrow A$, and it gives $\hat{A} \cong A \otimes_{K[[\underline{t}]]} \hat{K}[[\underline{t}]]$. The homomorphism $\hat{K}[\underline{t}] \to B$ extends uniquely to a morphism $\hat{K}[[\underline{t}]] \to B$: define it inductively into $\mathcal{O}_i(B)$, $i = \operatorname{res.dim} f, \ldots, 2, 1$. Hence $\hat{f} : \hat{A} \to \hat{B}$ is also defined. The uniqueness of $\hat{B}$ is clear from its construction. $\square$

Example 3.9. Let $A := k(s)[[t]]$, $\hat{A} := k((s))[[t]]$ and $B := k(s)((t))$, so the inclusion $A \to \hat{A}$ (resp. $A \to B$) is an intensification (resp. a morphism). We then have

$$k(s)((t)) \otimes_{k(s)[[t]]}^{(\wedge)} k((s))[[t]] \cong k((s))((t)).$$

Proposition 3.10. (Associativity) *Say $C \leftarrow B \to \hat{B} \leftarrow \hat{A} \to \hat{\hat{A}}$ are BCAs and homomorphisms, where the "$\leftarrow$" are morphisms, and the "$\to$" are intensifications. Then there is a canonical isomorphism of BCAs*

$$(C \otimes_B^{(\wedge)} \hat{B}) \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}} \cong C \otimes_B^{(\wedge)} (\hat{B} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}}).$$

Proof. Set $\hat{\hat{B}} := \hat{B} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}}$ and $\hat{C} := C \otimes_B^{(\wedge)} \hat{B}$. By construction (cf. Prop. 3.2) we get an intensification homomorphism $\hat{C} \to C \otimes_B^{(\wedge)} \hat{\hat{B}}$, and together with the morphism $\hat{\hat{A}} \to \hat{\hat{B}} \to C \otimes_B^{(\wedge)} \hat{\hat{B}}$ we deduce, using Cor. 3.8, the existence of a morphism $h : \hat{C} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}} \to C \otimes_B^{(\wedge)} \hat{\hat{B}}$. The same corollary says there is a morphism $\hat{\hat{B}} \to \hat{C} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}}$, and together with the intensification $C \to \hat{C} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}}$ we get a morphism $h' : C \otimes_B^{(\wedge)} \hat{\hat{B}} \to \hat{C} \otimes_{\hat{A}}^{(\wedge)} \hat{\hat{A}}$. Since the maps from $C \otimes_B \hat{B} \otimes_{\hat{A}} \hat{\hat{A}}$ to both these BCAs are dense, h and h' must be each other's inverse. $\square$

4. Continuous Differential Operators

We begin with some general results on continuous differential operators (DOs) over ST algebras. Let k be a discrete commutative ring, let A be a commutative, separated, ST k-algebra, and let M be a separated ST A-module. For $n \geq 0$, the separated module of principal parts $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is the ST A-A-bimodule $(A \otimes_k A/I^{n+1})^{\mathrm{sep}}$, where $I := \ker(A \otimes_k A \to A)$. Set $\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M) := (\mathcal{P}^{n,\mathrm{sep}}_{A/k} \otimes_A M)^{\mathrm{sep}}$, which is an A-module by $a \cdot ((1 \otimes 1) \otimes x) = (a \otimes 1) \otimes x$. The universal continuous DO of order n is $\mathrm{d}^n_M : M \to \mathcal{P}^{n,\mathrm{sep}}_{A/k}(M)$, $\mathrm{d}^n_M(x) = (1 \otimes 1) \otimes x$ (see [EGA] Ch. IV §16.8 and [Ye1] §1.5). For any separated ST A-module N, d^n_M induces a bijection

$$\mathrm{Hom}^{\mathrm{cont}}_A(\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M), N) \cong \mathrm{Diff}^{n,\mathrm{cont}}_{A/k}(M, N) .$$

There are inclusions

$$\mathrm{Diff}^{n,\mathrm{cont}}_{A/k}(M, N) \subset \mathrm{Diff}^{\mathrm{cont}}_{A/k}(M, N) \subset \mathrm{Hom}^{\mathrm{cont}}_k(M, N) .$$

$\mathrm{Diff}^{\mathrm{cont}}_{A/k}(M, N)$ is a filtered A-A-bimodule, where for $D \in \mathrm{Diff}^{\mathrm{cont}}_{A/k}(M, N)$ and $a, b \in A$ we have $aDb = a \circ D \circ b : M \to N$. Denote the order of the DO D by $\mathrm{ord}_A(D)$.

Definition 4.1. Given a separated ST A-module M, let $\mathcal{D}(A; M) := \mathrm{Diff}^{\mathrm{cont}}_{A/k}(M, M)$, which is a filtered k-algebra. For $M = A$ we shall write simply $\mathcal{D}(A) := \mathcal{D}(A; A)$.

Denote the left action of $\mathcal{D}(A; M)$ on M by $D * x$, for $D \in \mathcal{D}(A; M)$ and $x \in M$.

Remark 4.2. $\mathcal{D}(A)$ can be made into a ST k-algebra by giving it the subspace topology w.r.t. the embedding $\mathcal{D}(A) \subset \mathrm{End}^{\mathrm{cont}}_k(A)$. However we shall not make use of this topology.

Lemma 4.3. *Assume that for some $n \geq 0$, $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is a finite type ST left A-module. If M is a finite type ST A-module, then so is $\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M)$.*

Proof. First note that $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is a commutative ST ring, admitting two continuous k-algebra homomorphisms $A \to \mathcal{P}^{n,\mathrm{sep}}_{A/k}$. By [Ye1] Cor. 4.5, $\mathcal{P}^{n,\mathrm{sep}}_{A/k} \otimes_A M$ is a finite type ST $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$-module. The left A-module structure on $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ comes from the algebra homomorphism $a \mapsto a \otimes 1$. From [Ye1] Prop. 2.9 and our assumption it follows that $\mathcal{P}^{n,\mathrm{sep}}_{A/k} \otimes_A M$ has the fine A-module topology. But then the same is true for $\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M) = (\mathcal{P}^{n,\mathrm{sep}}_{A/k} \otimes_A M)^{\mathrm{sep}}$. $\square$

Define

$$\mathcal{T}(A) := \mathrm{Der}^{\mathrm{cont}}_k(A, A) = \mathrm{Hom}^{\mathrm{cont}}_A(\Omega^{1,\mathrm{sep}}_{A/k}, A) .$$

Corresponding to the decomposition $\mathcal{P}^{1,\mathrm{sep}}_{A/k} = A \oplus \Omega^{1,\mathrm{sep}}_{A/k}$ we have $A \oplus \mathcal{T}(A) = \mathcal{D}^1(A) \subset \mathcal{D}(A)$, and just like in the discrete case, $\mathcal{T}(A)$ is a Lie algebra over k.

Lemma 4.4. *Suppose $\hat{A}$ is another commutative, separated, ST k-algebra, and $u : A \to \hat{A}$ is a topologically étale homomorphism relative to k. Then there is an induced homomorphism of filtered k-algebras $\mathcal{D}(A) \to \mathcal{D}(\hat{A})$, sending an operator $D : A \to A$ to its unique extension $\hat{D} : \hat{A} \to \hat{A}$. More generally, if M is a ST A-module, there is a homomorphism $\mathcal{D}(A; M^{\mathrm{sep}}) \to \mathcal{D}(\hat{A}; (\hat{A} \otimes_A M)^{\mathrm{sep}})$.*

Proof. The existence and uniqueness of this ring homomorphism are immediate consequences of [Ye1] Thm. 1.5.11 (iv). □

The ring homomorphism $\mathcal{D}(A) \to \mathcal{D}(\hat{A})$ restricts to a Lie algebra homomorphism $\mathcal{T}(A) \to \mathcal{T}(\hat{A})$.

Proposition 4.5. *Let $A, \hat{A}$ be separated ST k-algebras, and let $u : A \to \hat{A}$ be a flat, topologically étale homomorphism relative to k. Assume that for every $n \geq 0$, $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is a finitely presented, finite type ST left A-module. Then the homomorphism $\mathcal{D}(A) \to \mathcal{D}(\hat{A})$ induces an isomorphism of filtered $\hat{A}$-$\mathcal{D}(A)$-bimodules*

$$\hat{A} \otimes_A \mathcal{D}(A) \xrightarrow{\simeq} \mathcal{D}(\hat{A}) .$$

Proof. Since $\otimes$ commutes with $\lim_{\to}$, it suffices to prove that for all $n \geq 0$, $\hat{A} \otimes_A \mathcal{D}^n(A) \to \mathcal{D}^n(\hat{A})$ is bijective. The assumptions imply that

$$\begin{aligned} \hat{A} \otimes_A \mathcal{D}^n(A) &= \hat{A} \otimes_A \mathrm{Hom}^{\mathrm{cont}}_A(\mathcal{P}^{n,\mathrm{sep}}_{A/k}, A) \\ &= \mathrm{Hom}^{\mathrm{cont}}_{\hat{A}}(\hat{A} \otimes_A \mathcal{P}^{n,\mathrm{sep}}_{A/k}, \hat{A}) \\ &\cong \mathrm{Hom}^{\mathrm{cont}}_{\hat{A}}(\mathcal{P}^{n,\mathrm{sep}}_{\hat{A}/k}, \hat{A}) \\ &= \mathcal{D}^n(\hat{A}). \end{aligned}$$

□

Now consider the separated algebra of differentials $\Omega^{\cdot,\mathrm{sep}}_{A/k}$, which is a graded ST k-algebra (see [Ye1] Def. 1.5.3). Then

$$\mathcal{T}(\Omega^{\cdot,\mathrm{sep}}_{A/k}) := \mathrm{Der}^{\mathrm{cont}}_k(\Omega^{\cdot,\mathrm{sep}}_{A/k}, \Omega^{\cdot,\mathrm{sep}}_{A/k})$$

is a graded Lie algebra. For instance, the exterior derivative d is an element of degree 1 in $\mathcal{T}(\Omega^{\cdot,\mathrm{sep}}_{A/k})$.

We shall need a version of the Lie derivative for semi-topological algebras (see [Wa] §2.24 for the differentiable manifold version).

Proposition 4.6. (Lie derivative) *Let A be a separated ST k-algebra and let ∂ be a continuous k-derivation of A. Then there exists a unique continuous, degree 0, k-linear derivation L_∂ of $\Omega^{\cdot,\mathrm{sep}}_{A/k}$, which extends ∂ and commutes with d. The map $\partial \mapsto L_\partial$ is a homomorphism of k-Lie algebras $\mathcal{T}(A) \to \mathcal{T}(\Omega^{\cdot,\mathrm{sep}}_{A/k})$, and is functorial with respect to topologically étale homomorphisms $A \to \hat{A}$ in $\mathsf{STComAlg}(k)$.*

Proof. Let $\partial \in \mathcal{T}(A) = \mathrm{Der}_k^{\mathrm{cont}}(A, A)$ be given. Since A is separated we get a continuous A-linear map $\Omega^{1,\mathrm{sep}}_{A/k} \to A$, which extends by universality to a continuous degree -1 derivation $\iota_\partial : \Omega^{\cdot,\mathrm{sep}}_{A/k} \to \Omega^{\cdot,\mathrm{sep}}_{A/k}$, the interior derivative.

Define $\mathrm{L}_\partial := \iota_\partial \circ \mathrm{d} + \mathrm{d} \circ \iota_\partial$ (i.e. the graded commutator of ι_∂ and d). The properties of L_∂ are easily deduced from its definition and the fact that $\mathrm{d}^2 = 0$. To show uniqueness it suffices to consider $\mathrm{L}_\partial(\alpha)$ for $\alpha = a$ or $\alpha = \mathrm{d}a$, $a \in A$. But $\mathrm{L}_\partial(a) = \partial(a)$ and $\mathrm{L}_\partial(\mathrm{d}a) = \mathrm{d}(\mathrm{L}_\partial(a)) = \mathrm{d} \circ \partial(a)$.

Now let $\partial_1, \partial_2 \in \mathcal{T}(A)$. Then $[\mathrm{L}_{\partial_1}, \mathrm{L}_{\partial_2}]$ is a continuous derivation of $\Omega^{\cdot,\mathrm{sep}}_{A/k}$ commuting with d, and for all $a \in A$,

$$[\mathrm{L}_{\partial_1}, \mathrm{L}_{\partial_2}](a) = [\partial_1, \partial_2](a) = \mathrm{L}_{[\partial_1, \partial_2]}(a) \ ,$$

so $[\mathrm{L}_{\partial_1}, \mathrm{L}_{\partial_2}] = \mathrm{L}_{[\partial_1, \partial_2]}$. The functoriality of L follows from the same functoriality of ι (and d). $\square$

Lemma 4.7. *Suppose $\Omega^{n+1,\mathrm{sep}}_{A/k} = 0$ for some n. Then for any $a \in A$, $\alpha \in \Omega^{n,\mathrm{sep}}_{A/k}$ and $\partial \in \mathcal{T}(A)$, one has $\mathrm{L}_{a\partial}(\alpha) = \mathrm{L}_\partial(a\alpha)$.*

Proof. First note that

$$\partial(a)\alpha - \mathrm{d}(a)\iota_\partial(\alpha) = \iota_\partial(\mathrm{d}(a)\alpha) = 0. \tag{4.1}$$

Since $\mathrm{d}\alpha = 0$, $\iota_{a\partial(\alpha)} = a\iota_\partial(\alpha)$ and $= \partial(a)\alpha + a\,\mathrm{L}_\partial(\alpha)$, it follows that $\mathrm{L}_{a\partial}(\alpha) = \mathrm{L}_\partial(a\alpha)$. $\square$

Now assume k is a perfect field.

Proposition 4.8. *Let $A \in \mathsf{BCA}(k)$, and let M be a finite type ST A-module. Then for any $n \geq 0$, $\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M)$ is a finite type ST left A-module. In particular, so is $\mathcal{P}^{1,\mathrm{sep}}_{A/k} = A \oplus \Omega^{1,\mathrm{sep}}_{A/k}$, so A is differentially of finite type over k (In the sense of* [Ye1] *Def.* 1.5.16*).*

Proof. We may assume A is a local ring. Choose a parametrization of A, i.e. a surjective morphism of BCAs $F((\underline{s}))[[\underline{t}]] \to A$. Let $\underline{u} = (u_1, \ldots, u_l)$ be a separating transcendency basis for F over k. By [Ye1] Cor. 1.5.19, $k[\underline{u}, \underline{s}, \underline{t}] \to F((\underline{s}))[[\underline{t}]]$ is topologically étale (rel. to k). Therefore, using [Ye1] formula (1.4.2) and Theorem 1.5.11, it follows that

$$\mathcal{P}^{n,\mathrm{sep}}_{F((\underline{s}))[[\underline{t}]]/k} \cong F((\underline{s}))[[\underline{t}]] \otimes_{k[\underline{u},\underline{s},\underline{t}]} \mathcal{P}^{n,\mathrm{sep}}_{k[\underline{u},\underline{s},\underline{t}]/k}$$

is a free ST left $F((\underline{s}))[[\underline{t}]]$-module of finite rank. Now in general, if $\phi : M \twoheadrightarrow N$ is a strict surjection of ST k-modules, then so is $\phi \otimes \phi : M \otimes_k M \to N \otimes_k N$; and if $M' \subset M$ and $N' \subset N$ and submodules such that $\phi(M') \subset N'$, then $\bar{\phi} : M/M' \twoheadrightarrow N/N'$ is also strict. This implies that $\mathcal{P}^{n,\mathrm{sep}}_{F((\underline{s}))[[\underline{t}]]/k} \to \mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is a strict surjection. Hence $\mathcal{P}^{n,\mathrm{sep}}_{A/k}$ is a ST module of finite type over A (as a left module, via $a \mapsto a \otimes 1$).

Given a finite type ST A-module M as above, use Lemma 4.3. $\square$

5. $\mathcal{D}$-Modules over TLFs

Henceforth k is a fixed perfect field. Let K be a topological local field(TLF) over k. We need to understand the structure of the ring $\mathcal{D}(K)$ of continuous differential operators. First assume k has characteristic p. Let M be a free ST K-module of finite rank. We know from [Ye1] Theorems 2.1.14 and 1.4.9 that $\mathcal{D}(K;M)$ admits the p-filtration

$$\mathcal{D}(K;M) = \mathrm{Diff}_{K/k}(M,M) = \bigcup_{n=0}^{\infty} \mathrm{End}_{K^{(p^n/k)}}(M) \ .$$

Here $K^{(p^n/k)} = k \otimes_k K$, with $1 \otimes \lambda = \lambda^{p^n} \otimes 1$ for $\lambda \in k$. This filtration is cofinal with the order filtration - see [Ye1] Lemma 1.4.8. According to ibid. Prop. 2.1.13, the relative Frobenius map $K^{(p^n/k)} \to K$, $\lambda \otimes a \mapsto \lambda a^{p^n}$, is a finite morphism in $\mathsf{TLF}(k)$.

In characteristic 0, $\mathcal{D}(K)$ is a "topologically étale localization" of a Weyl algebra. Choose a parametrization $K \cong F((\underline{s}))$ and a separating transcendency basis $\underline{u}$ for F over k. Let $\underline{t} = (t_1, \ldots, t_m) := (\underline{u}, \underline{s}) = (u_1, \ldots, s_1 \ldots)$ be the concatenated sequence. Then $k[\underline{t}] \to K$ is a flat, topologically étale homomorphism in $\mathsf{STComAlg}(k)$. The ring $\mathcal{D}(k[\underline{t}])$ is a Weyl algebra over k: $\mathcal{D}(k[\underline{t}]) \cong k[\underline{t}] \otimes_k k[\partial_1, \ldots, \partial_m]$, where $\partial_i := \frac{\partial}{\partial t_i}$, and the multiplication is determined by $(1 \otimes \partial_i)(t_j \otimes 1) = t_j \otimes \partial_i + (\partial_i * t_j) \otimes 1$. By Prop. 4.5, we have $\mathcal{D}(K) \cong K \otimes_{k[\underline{t}]} \mathcal{D}(k[\underline{t}])$. Considering the faithful action of $\mathcal{D}(K)$ on K, we get a presentation

$$\begin{array}{rcl} \mathcal{D}(K) & \cong & K \otimes_k k[\partial_1, \ldots, \partial_m] \\ (1 \otimes \partial_i)(a \otimes 1) & = & a \otimes \partial_i + (\partial_i * a) \otimes 1 \end{array} \tag{5.1}$$

for $i = 1, \ldots, m$ and $a \in K$ (i.e. $\mathcal{D}(K)$ is a smash product of K and the universal enveloping algebra of the abelian k-Lie algebra spanned by the derivations ∂_i).

Definition 5.1. Let K be a TLF over k. Define $\omega(K)$ to be the top degree component of $\Omega^{\cdot,\mathrm{sep}}_{K/k}$. It is a free ST K-module of rank 1.

At this point we can exhibit the canonical right $\mathcal{D}(K)$-module structure on $\omega(K)$ (cf. [Bo] Ch. VI §3.2).

Proposition 5.2. *For any $K \in \mathsf{TLF}(k)$ there is a unique right $\mathcal{D}(K)$-module structure on $\omega(K)$, written $\alpha * D$, for $\alpha \in \omega(K)$ and $D \in \mathcal{D}(K)$, such that:*

(i) *If $D = a \in K$ then $\alpha * a = a\alpha$.*
(ii) *If $D = \partial \in \mathcal{T}(K)$ then $\alpha * \partial = -\mathrm{L}_\partial(\alpha)$, where L_∂ is the Lie derivative (see Prop. 4.6).*
(iii) *If $\mathrm{char}\, k = p$ and $D \in \mathcal{D}^{p^n-1}(K)$ for some $n \geq 0$, then for every $a \in K$,*

$$\langle D * a, \alpha \rangle_{K/K^{(p^n/k)}} = \langle a, \alpha * D \rangle_{K/K^{(p^n/k)}} \ ,$$

where $\langle -, - \rangle_{K/K^{(p^n/k)}}$ is the trace pairing of [Ye1] *formula* (2.3.8).

Proof. First assume $\operatorname{char} k = 0$. Since $[\mathrm{L}_{\partial_i}, \mathrm{L}_{\partial_j}] = 0$, $\alpha * \partial_i := -\mathrm{L}_{\partial_i}(\alpha)$ is an action of $k[\partial_1, \ldots, \partial_m]$ on $\omega(K)$. According to the presentation (5.1), in order to extend this to a right action of $\mathcal{D}(K)$ it suffices to show that

$$-a\,\mathrm{L}_{\partial_i}(\alpha) = -\mathrm{L}_{\partial_i}(a\alpha) + \partial_i(a)\alpha$$

which is true since L_{∂_i} is an even derivation of $\Omega^{\cdot,\mathrm{sep}}_{K/k}$ and $\mathrm{L}_{\partial_i}(a) = \partial_i(a)$. By Lemma 4.7, condition (ii) holds for an arbitrary derivation $\partial = \sum a_i \partial_i$, $a_i \in K$.

Next consider the case $\operatorname{char} k = p$. Let $D \in \mathcal{D}^n(K)$. By [Ye1] Lemma 1.4.8, D is $K^{(p^n/k)}$-linear. The trace pairing $\langle -, - \rangle_{K/K^{(p^n/k)}}$ is perfect ([Ye1] Prop. 2.3.9), so by adjunction D acts on $\omega(K)$. The functoriality of the trace guarantees that this action is independent of n. We thus get a right action satisfying conditions (i) and (iii). In order to check (ii) it suffices to look at $\partial = \partial_i$. Let $\beta := \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_{i-1} \wedge \mathrm{d}t_{i+1} \wedge \cdots \wedge \mathrm{d}t_m$. We can compute the difference:

$$\begin{aligned} \langle \partial_i(a), b\mathrm{d}t_i \wedge \beta \rangle_{K/K^{(p^n/k)}} &- \langle a, -\mathrm{L}_{\partial_i}(b\mathrm{d}t_i \wedge \beta) \rangle_{K/K^{(p^n/k)}} \\ &= \mathrm{Tr}_{K/K^{(p/k)}}(\partial_i(ab)\mathrm{d}t_i \wedge \beta) \\ &= \mathrm{Tr}_{K/K^{(p/k)}}(\mathrm{d}(ab\beta)) = 0 \end{aligned}$$

since $\mathrm{Tr}_{K/K^{(p/k)}}$ commutes with d and vanishes on $\Omega^{\leq m-1,\mathrm{sep}}_{K/k}$. □

Let $\mathcal{D}(K)^\circ$ denote the opposite ring of $\mathcal{D}(K)$.

Proposition 5.3. *The right $\mathcal{D}(K)$ action on $\omega(K)$ of the previous proposition induces a canonical isomorphism of filtered k-algebras*

$$T_K : \mathcal{D}(K)^\circ \xrightarrow{\simeq} \mathcal{D}(K; \omega(K)).$$

Proof. If $\operatorname{char} k = p$ we have, for every $n \geq 0$, an isomorphism (of K-K-bimodules) $T^n : \mathrm{End}_{K^{(p^n/k)}}(K)^\circ \xrightarrow{\simeq} \mathrm{End}_{K^{(p^n/k)}}(\omega(K))$ induced by adjunction. In the limit we get T_K.

If $\operatorname{char} k = 0$, choose a topologically étale homomorphism $k[\underline{t}] \to K$. Let $T_{\underline{t}} : \mathcal{D}(K) \to \mathcal{D}(K)$ be the involution such that $T_{\underline{t}}|_K$ is the identity and $T_{\underline{t}}(\partial_i) = -\partial_i$ (cf. formula (5.1)). Let $\phi_{\underline{t}} : K \xrightarrow{\simeq} \omega(K)$ be the K-linear isomorphism defined by $\phi_{\underline{t}}(1) = \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_m$. Then for any $D \in \mathcal{D}(K)$,

$$T_K(D) = \phi_{\underline{t}} \circ T_{\underline{t}}(D) \circ \phi_{\underline{t}}^{-1} \in \mathcal{D}(K; \omega(K)).$$

□

Corollary 5.4. *Let $K, \hat{K} \in \mathsf{BCA}(k)$ be fields and let $K \to \hat{K}$ be a topologically étale homomorphism in $\mathsf{STComAlg}(k)$. Then $\omega(K) \to \omega(\hat{K})$ is a homomorphism of right $\mathcal{D}(K)$-modules.*

Proof. In characteristic 0 this follows from the covariance of the Lie derivative. In positive characteristics it follows from the fact that the trace map commutes with base change, cf. [Ye1] Prop. 2.3.11. □

On the category $\mathsf{TLF}(k)$ there is a functorial residue map. To each morphism $f : K \to L$ it assigns a homomorphism of differential graded ST left $\Omega^{\cdot,\mathrm{sep}}_{K/k}$-modules, $\mathrm{Res}_{L/K} = \mathrm{Res}_f : \Omega^{\cdot,\mathrm{sep}}_{L/k} \to \Omega^{\cdot,\mathrm{sep}}_{K/k}$ (cf. [Ye1] Thm. 2.4.3). The residue pairing

$$\begin{gathered} \langle -,- \rangle_{L/K} : L \times \omega(L) \to \omega(K) \\ \langle a, \alpha \rangle_{L/K} = \mathrm{Res}_{L/K}(a\alpha) \end{gathered}$$

is a perfect pairing of ST K-modules, in the sense that the induced map $\omega(L) \to \mathrm{Hom}^{\mathrm{cont}}_K(L, \omega(K))$ is bijective (cf. [Ye1] Thm. 2.4.22 - Topological Duality).

Theorem 5.5. *Let $K \in \mathsf{TLF}(k)$ and assume that $k \to K$ is a morphism in $\mathsf{TLF}(k)$. Given a DO $D \in \mathcal{D}(K)$, let $D^\vee \in \mathrm{End}_k(\omega(K))$ be its adjoint relative to the residue pairing $\langle -,- \rangle_{K/k}$. Then for every $\alpha \in \omega(K)$,*

$$D^\vee(\alpha) = \alpha * D.$$

In other words, the adjoint action of $\mathcal{D}(K)$ on $\omega(K)$ coincides with the canonical right action.

Proof. We must show that for all $a \in K$, $\alpha \in \omega(K)$ and $D \in \mathcal{D}(K)$, $\langle D * a, \alpha \rangle_{K/k} = \langle a, \alpha * D \rangle_{K/k}$. In characteristic p this follows immediately from condition (iii) of Prop. 5.2 and the functoriality of the residue maps ([Ye1] Thm. 2.4.2).

in characteristic 0 first choose a parametrization $K \cong F((\underline{t})) = F((t_1, \ldots, t_n))$. Then $k \to F$ is finite separable and any k-linear DO is also F-linear. Given $\lambda \in F$, $\underline{i} \in \mathbb{Z}^n$ and $1 \leq j \leq n$, write $\mathrm{dlog}(\underline{t}) := \mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_n) = t_1^{-1}\mathrm{d}t_1 \wedge \cdots \wedge t_n^{-1}\mathrm{d}t_n$ and $\partial_j := \frac{\partial}{\partial t_j}$. Then

$$\begin{aligned} &\mathrm{L}_{\partial_j}(\lambda \underline{t}^{\underline{i}} \,\mathrm{dlog}(\underline{t})) = \\ &\qquad (-1)^{j-1}\mathrm{d}(\lambda t_j^{-1} \underline{t}^{\underline{i}} \,\mathrm{dlog}(t_1) \wedge \cdots \wedge \mathrm{dlog}(t_{j-1}) \wedge \mathrm{dlog}(t_{j+1}) \wedge \cdots \wedge \mathrm{dlog}(t_n)) \end{aligned}$$

so $\mathrm{Res}_{K/k}(\mathrm{L}_{\partial_j}(\lambda \underline{t}^{\underline{i}})) = 0$. By continuity we conclude that

$$\mathrm{Res}_{K/k}(\mathrm{L}_{\partial_j}(\alpha)) = 0 \tag{5.2}$$

for all α.

To prove the theorem it suffices to consider either $D = b \in K$ or $D = \partial_j$. For $D = b$ we get

$$\langle a, \alpha * b \rangle_{K/k} = \langle a, b\alpha \rangle_{K/k} = \mathrm{Res}_{K/k}(ab\alpha) = \langle ab, \alpha \rangle_{K/k} = \langle b * a, \alpha \rangle_{K/k}.$$

For $D = \partial_j$ we use "integration by parts":

$$\begin{aligned} &\langle \partial_j * a, \alpha \rangle_{K/k} - \langle a, \alpha * \partial_j \rangle_{K/k} = \\ &\qquad \mathrm{Res}_{K/k}(\mathrm{L}_{\partial_j}(a)\alpha + a\,\mathrm{L}_{\partial_j}(\alpha)) = \mathrm{Res}_{K/k}(\mathrm{L}_{\partial_j}(a\alpha)) = 0 \end{aligned}$$

by (5.2). $\square$

6. Duals of Finite Type Modules

The purpose of this subsection is to establish the existence of a canonical dual module $\operatorname{Dual}_A M$ to every finite type ST A-module M. If $k \to A$ is a morphism in $\mathsf{BCA}(k)$, then we set $\operatorname{Dual}_A M := \operatorname{Hom}_k^{\mathrm{cont}}(M, k)$, endowed with the fine A-module topology. Otherwise we define $\operatorname{Dual}_A M$ using differential operators, and show this definition is independent of choices made by a base change argument, which reduces things to the case when $k \to A$ is a morphism. Recall that k is a fixed perfect field. For a TLF K, $\omega(K)$ is the top degree component of $\Omega^{\cdot,\mathrm{sep}}_{K/k}$, a rank 1 free ST K-module.

Definition 6.1. Let $A, K \in \mathsf{BCA}(k)$ be a local ring and a field, respectively, and let $\sigma : K \to A$ be a morphism in $\mathsf{BCA}(k)$. For any finite type ST A-module M define

$$\operatorname{Dual}_\sigma M := \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M, \omega(K)),$$

the set of continuous K-linear homomorphisms, where M is a K-module via σ. Put on $\operatorname{Dual}_\sigma M$ the *fine A-module topology.*

Remark 6.2. The module $\operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M, \omega(K))$, with the (weak) Hom topology, is a ST A-module. Therefore the identity map $\operatorname{Dual}_\sigma M \to \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M, \omega(K))$ is continuous. However, this will not be a homeomorphism unless $\sigma : K \to A$ is a pseudo coefficient field and M is a finite length module.

Let A be a commutative noetherian local ring, with maximal ideal $\mathfrak{m}$, and let I be an injective hull of $A/\mathfrak{m}$. Then $M \mapsto \operatorname{Hom}_A(M, I)$ is a duality between finite type (i.e. finitely generated) A-modules and cofinite type (i.e. artinian) A-modules. The module $\operatorname{Hom}_A(M, I)$ is called a *Matlis dual* of M (cf. [LC] §4).

Lemma 6.3. *Let $\sigma : K \to A$ and M be as in Def.* 6.1.

(a) *Suppose $\tau : L \to A$ and $f : K \to L$ are morphisms in $\mathsf{BCA}(k)$, with L a field, and $\sigma = \tau \circ f$. Then the map*

$$\begin{array}{rcl} \operatorname{Dual}_\tau M & \to & \operatorname{Dual}_\sigma M \\ \phi & \mapsto & \operatorname{Res}_{L/K} \circ \phi \end{array}$$

is an isomorphism of ST A-modules.

(b) *The (untopologized) A-module* $\operatorname{Dual}_\sigma M$ *is a Matlis dual of M. In particular, Taking $M = A$, it follows that* $\operatorname{Dual}_\sigma A$ *is an injective hull of $A/\mathfrak{m}$. As a ST A-module,* $\operatorname{Dual}_\sigma M$ *is of cofinite type.*

Proof. (a) First consider the case when $\tau : L \to A$ is finite; so A has the fine L-module topology (Prop. 2.11 (a)). Then M is a free ST L-module of finite rank. By Topological Duality ([Ye1] Thm. 2.4.22), $\operatorname{Dual}_\tau M \to \operatorname{Dual}_\sigma M$ is bijective, and it is an isomorphism of ST A-modules since both modules have the fine A-module topologies.

Next assume $\tau : L \to A$ is a pseudo coefficient field. Because $\omega(K)$ (resp. $\omega(L)$) is a simple, separated ST K-module (resp. L-module), and $M \cong \lim_{\leftarrow n} M/\mathfrak{m}^n M$, we can use [Ye1] Prop. 1.2.22 to conclude that

$$\operatorname{Dual}_\sigma M = \bigcup_{n=1}^{\infty} \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M/\mathfrak{m}^n M, \omega(K)) \tag{6.1}$$

and similarly for L. For any $n \geq 1$, $M/\mathfrak{m}^n M$ is a finite type ST $A/\mathfrak{m}^n$-module, so we are back to the first step.

For the general situation, we may factor τ through some pseudo coefficient field $\tau' : L' \to A$ (cf. Lemma 2.10 (d)), and use the functoriality of the residue maps.

(b) By part (a) we can assume that $\sigma : K \to A$ is a pseudo coefficient field. Then in (6.1) we can drop the superscript "cont", in which case the statement is well known (cf. [LC] p. 63 Example 1). $\square$

Let $A, \hat{A}$ be local BCAs, with maximal ideals $\mathfrak{m}, \hat{\mathfrak{m}}$ respectively, and let $v : A \to \hat{A}$ be an intensification homomorphism. Note that v, being a local homomorphism, is faithfully flat. Let $\sigma : K \to A$ be a morphism in $\mathsf{BCA}(k)$, with K a field. Assume that there is an intensification homomorphism $u : K \to \hat{K}$ and a morphism $\hat{\sigma} : \hat{K} \to \hat{A}$ s.t. $\hat{A} \cong A \otimes^{(\wedge)}_K \hat{K}$.

Proposition 6.4. *Let M be a finite type ST A-module, and set $\hat{M} := \hat{A} \otimes_A M$. Then any $\phi \in \operatorname{Dual}_\sigma M$ has a unique extension $\hat{\phi} \in \operatorname{Dual}_{\hat{\sigma}} \hat{M}$. The resulting continuous homomorphism*

$$q^M_{v;\sigma} : \operatorname{Dual}_\sigma M \to \operatorname{Dual}_{\hat{\sigma}} \hat{M}$$

is injective, and induces an isomorphism of ST $\hat{A}$-modules

$$1 \otimes q^M_{v;\sigma} : \hat{A} \otimes_A \operatorname{Dual}_\sigma M \xrightarrow{\simeq} \operatorname{Dual}_{\hat{\sigma}} \hat{M}.$$

Proof. Let $n := \operatorname{res.dim} \sigma$. Then we can extend σ to a a pseudo coefficient field $K((\underline{s})) = K((s_1, \ldots, s_n)) \to A$, and extend $\hat{\sigma}$ to $\hat{K}((\underline{s})) \to \hat{A}$. By replacing $K, \hat{K}$ with $K((\underline{s})), \hat{K}((\underline{s}))$ we can then assume that $\sigma, \hat{\sigma}$ are pseudo coefficient fields. This puts us in the setup of Prop. 3.5. For $i \geq 0$ define

$$H^i := \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M/\mathfrak{m}^{i+1} M, \omega(K)) \subset \operatorname{Dual}_\sigma M$$

and similarly define $\hat{H}^i$. Since $\operatorname{Dual}_\sigma M$ and $\operatorname{Dual}_{\hat{\sigma}} \hat{M}$ both have the fine topologies, it suffices to exhibit an isomorphism $\hat{A} \otimes_A H^i \xrightarrow{\simeq} \hat{H}^i$, with $\hat{\phi} := 1 \otimes \phi$ extending ϕ. By Prop. 3.5 (a),

$$\hat{K} \otimes_K (M/\mathfrak{m}^{i+1} M) \cong \hat{A} \otimes_A (M/\mathfrak{m}^{i+1} M) \cong \hat{M}/\hat{\mathfrak{m}}^{i+1} \hat{M}.$$

Since $K \to \hat{K}$ is topologically étale, $\hat{K} \otimes_K \omega(K) \xrightarrow{\simeq} \omega(\hat{K})$. Therefore $\hat{K} \otimes_K H^i \xrightarrow{\simeq} \hat{H}^i$; and again by Prop. 3.5 (a), $\hat{A} \otimes_A H^i \xrightarrow{\simeq} \hat{H}^i$. $\square$

Let A be a local BCA with maximal ideal $\mathfrak{m}$. Suppose $\sigma, \sigma' : K \to A$ are pseudo coefficient fields, such that $\sigma \equiv \sigma' \pmod{\mathfrak{m}}$. Let M be a finite type ST A-module. Given a nonzero element $x \in M$, its order with respect to $\mathfrak{m}$ is

$$\mathrm{ord}_{\mathfrak{m}}(x) := \max\{n \mid x \in \mathfrak{m}^n M\}.$$

If $\mathrm{ord}_{\mathfrak{m}}(x) = n$, then the symbol of x is its image in $\mathfrak{m}^n/\mathfrak{m}^{n+1} \subset \mathrm{gr}_{\mathfrak{m}} M$.

Definition 6.5. An $\mathfrak{m}$-filtered K-basis of M is a sequence $\underline{x} = (x_0, x_1, \dots)$ of elements of M, such that the symbols of $x_0, x_1, \dots$ form a K-basis of $\mathrm{gr}_{\mathfrak{m}} M$, and such that $\mathrm{ord}_{\mathfrak{m}}(x_i) \leq \mathrm{ord}_{\mathfrak{m}}(x_{i+1})$.

Choose such a basis $\underline{x}$. Then any $x \in M$ is expressed uniquely as a convergent sum

$$x = \sum_i \sigma'(\lambda_i) x_i = \sum_i \sigma(\mu_i) x_i$$

with $\lambda_i, \mu_i \in K$. Define functions $D_{ij} : K \to K$ by the equation

$$\sigma'(\lambda) x_i = \sum_j \sigma(D_{ij}(\lambda)) x_j.$$

Lemma 6.6. *$D_{ij} \in \mathcal{D}(K)$, i.e. it is a continuous differential operator over K relative to k.*

Proof. Pick two indices i_0, i_1, and let $n := \max\{\mathrm{ord}_{\mathfrak{m}}(x_{i_0}), \mathrm{ord}_{\mathfrak{m}}(x_{i_1})\}$. We can compute the function $D_{i_0 i_1}$ for the module $M/\mathfrak{m}^{n+1}M$ instead of M. Define

$$A^- := \sigma(K) \oplus \mathfrak{m} = \sigma'(K) \oplus \mathfrak{m} \subset A.$$

This is a local BCA, with $A^-/\mathfrak{m} \cong K$, and $A^- \to A$ is a finite morphism. Let l be the length of $M/\mathfrak{m}^{n+1}M$ over A^-, and let $E, E' : K^l \xrightarrow{\simeq} M/\mathfrak{m}^{n+1}M$ be the K-linear homeomorphisms

$$\begin{aligned} E(\lambda_0, \dots, \lambda_{l-1}) &:= \sum_{i=0}^{l-1} \sigma(\lambda_i) x_i \\ E'(\lambda_0, \dots, \lambda_{l-1}) &:= \sum_{i=0}^{l-1} \sigma'(\lambda_i) x_i \end{aligned}$$

($M/\mathfrak{m}^{n+1}M$ is a free ST K-module via σ and via σ'). According to [Ye1] Prop. 1.4.4, E, E^{-1}, E' and $(E')^{-1}$ are DOs over A^-, relative to k. Set

$$D := E^{-1} \circ E' : K^l \xrightarrow{\simeq} K^l ,$$

which is a DO over A^-, and hence over K. Expanding D as an $l \times l$ matrix with entries in $\mathcal{D}(K)$, one gets $D = [D_{ij}]$. $\square$

One can easily show that

$$\mathrm{ord}_K(D_{ij}) \leq 2\max\{-1, \mathrm{ord}_{\mathfrak{m}}(x_j) - \mathrm{ord}_{\mathfrak{m}}(x_i)\}$$

and $D_{ii} = 1$. Thus the matrix of DOs looks like this:

$$[D_{ij}] = \begin{bmatrix} 1 & * & * & \\ 0 & 1 & * & \cdots \\ 0 & 0 & 1 & \\ & \vdots & & \end{bmatrix}$$

Definition 6.7. In the situation described above, define a function

$$\Psi^M_{\sigma,\sigma'} : \mathrm{Dual}_\sigma M \to \mathrm{Dual}_{\sigma'} M$$

by the equation

$$\Psi^M_{\sigma,\sigma'}(\phi)(\sum_i \sigma'(\lambda_i)x_i) = \sum_{i,j} \lambda_i(\phi(x_j) * D_{ij}) \tag{6.2}$$

for $\phi \in \mathrm{Dual}_\sigma M$ and $\lambda_i \in K$.

The second sum in (6.2) makes sense, since there are only finitely many nonzero terms in it. At first glance this somewhat strange definition seems to depend on the basis $\underline{x}$. We shall soon see that there is no dependence on the basis, and that in fact $\Psi^M_{\sigma,\sigma'}$ is an isomorphism of ST A-modules. Immediately from the definition we get:

Lemma 6.8. *$\Psi^M_{\sigma,\sigma'}$ is a k-linear bijection, with inverse $\Psi^M_{\sigma',\sigma}$. Given a third pseudo coefficient field $\sigma'' : K \to A$ one has*

$$\Psi^M_{\sigma,\sigma''} = \Psi^M_{\sigma'',\sigma'} \circ \Psi^M_{\sigma,\sigma'}.$$

Further properties of $\Psi^M_{\sigma,\sigma'}$ are less obvious.

Lemma 6.9. *Under the combined assumptions of Prop.* 6.4 *and Def.* 6.7*, one has*

$$\Psi^{\hat{M}}_{\hat{\sigma},\hat{\sigma}'} \circ q^M_{v;\sigma} = q^M_{v;\sigma'} \circ \Psi^M_{\sigma,\sigma'}.$$

Here we are using the $\hat{\mathfrak{m}}$-filtered basis $(1 \otimes x_0, 1 \otimes x_1, \ldots)$ on $\hat{M}$ to define $\Psi^{\hat{M}}_{\hat{\sigma},\hat{\sigma}'}$.

Proof. The DOs $\hat{D}_{ij} \in \mathcal{D}(\hat{K})$ which appear in the definition of $\Psi^{\hat{M}}_{\hat{\sigma},\hat{\sigma}'}$ are precisely the images of the DOs $D_{ij} \in \mathcal{D}(K)$ under the natural ring homomorphism $\mathcal{D}(K) \to \mathcal{D}(\hat{K})$. By Cor. 5.4, $\omega(K) \to \omega(\hat{K})$ is a homomorphism of right $\mathcal{D}(K)$-modules. $\square$

Lemma 6.10. *In the situation of Def.* 6.7*, suppose in addition that $k \to K$ is a morphism in $\mathsf{BCA}(k)$. Then for any $\phi \in \mathrm{Dual}_\sigma M$, one has*

$$\mathrm{Res}_{K/k} \circ \phi = \mathrm{Res}_{K/k} \circ \Psi^M_{\sigma,\sigma'}(\phi).$$

Proof. Say $\phi(x_i) = \alpha_i \in \omega(K)$. Given $x = \sum_i \sigma'(\lambda_i)x_i \in M$, with $\lambda_i \in K$, we have by definition $x = \sum_{i,j} \sigma(D_{ij} * \lambda_i)x_j$. So

$$\mathrm{Res}_{K/k} \circ\phi(x) = \mathrm{Res}_{K/k}(\sum_{i,j}(D_{ij} * \lambda_i)\alpha_j).$$

On the other hand, setting $\phi' := \Psi^M_{\sigma,\sigma'}(\phi)$, one has

$$\mathrm{Res}_{K/k} \circ\phi'(x) = \mathrm{Res}_{K/k}(\sum_{i,j} \lambda_i(\alpha_j * D_{ij})).$$

By linearity and continuity, it suffices to prove that for all $i, j \geq 0$:

$$\mathrm{Res}_{K/k} \left((D_{ij} * \lambda_i)\alpha_j\right) = \mathrm{Res}_{K/k} \left(\lambda_i(\alpha_j * D_{ij})\right) \ ;$$

but this is done in Thm. 5.5. $\square$

Lemma 6.11. *Let $K \in \mathsf{BCA}(k)$ be a field. There exists a field $\hat{K} \in \mathsf{BCA}(k)$ and a homomorphism $u : K \to \hat{K}$ in $\mathsf{STComAlg}(k)$ such that $k \to \hat{K}$ is a morphism in $\mathsf{BCA}(k)$ and u is an intensification. Moreover we can choose u to be dense.*

Proof. Choose a parametrization $K \cong F((\underline{t}))$. F is a finitely generated field extension of k; let $\underline{s} = (s_1, \ldots, s_m)$ be a transcendency basis for F/k. Then $k(\underline{s}) \to F$ is a finite morphism in $\mathsf{BCA}(k)$. The map $k(\underline{s}) \to k((\underline{s}))$ is certainly an intensification homomorphism. Applying finitely ramified base change (Thm. 3.8) we get a dense intensification homomorphism $K \to K \otimes^{(\wedge)}_{k(\underline{s})} k((\underline{s}))$. Thus the BCA $K \otimes^{(\wedge)}_{k(\underline{s})} k((\underline{s}))$ is a reduced cluster of TLFs, and we can take $\hat{K}$ to be any local factor of it. $\square$

Proposition 6.12. *Let A be a local BCA with maximal ideal $\mathfrak{m}$, and let $\sigma, \sigma' : K \to A$ be two pseudo coefficient fields, such that $\sigma \equiv \sigma' \pmod{\mathfrak{m}}$. Let M be a finite type ST A-module. Then the map $\Psi^M_{\sigma,\sigma'}$ is an isomorphism of ST A-modules, independent of the $\mathfrak{m}$-filtered K-basis $\underline{x} = (x_0, x_1, \ldots)$.*

Proof. First we reduce the problem to the case when $K \xrightarrow{\simeq} A/\mathfrak{m}$, i.e. when σ, σ' are coefficient fields. Let A^- be the algebra $\sigma(K) \oplus \mathfrak{m} \subset A$, cf. proof of Lemma 6.6. The map $\Psi^M_{\sigma,\sigma'}$ is the same when restricting M to an A^--module, so we may replace A with A^-.

Choose an intensification homomorphism $u : K \to \hat{K}$ as in Lemma 6.11, and define $\hat{A} := A \otimes^{(\wedge)}_K \hat{K}$, w.r.t. the morphism $\sigma : K \to A$. So the homomorphism $v : A \to \hat{A}$ is also an intensification, $\hat{A}$ is local with maximal ideal $\hat{\mathfrak{m}} = \hat{A} \cdot v(\mathfrak{m})$, and $\hat{A}/\hat{\mathfrak{m}} \cong \hat{K}$. Let $R : \mathrm{Dual}_{\hat{\sigma}} \hat{M} \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_k(\hat{M}, k)$ be the $\hat{A}$-linear isomorphism $\phi \mapsto \mathrm{Res}_{\hat{K}/k} \circ\phi$ of Lemma 6.3, and similarly define R'. According to Lemmas 6.9 and 6.10, the diagram

$$
\begin{tikzcd}
\mathrm{Dual}_\sigma M \arrow[r, "q^M_{v;\sigma}"] \arrow[d, "\Psi^M_{\sigma,\sigma'}"] & \mathrm{Dual}_{\hat\sigma} \hat M \arrow[d, "\Psi^{\hat M}_{\hat\sigma,\hat\sigma'}"] \arrow[dr, "R"] & \\
\mathrm{Dual}_{\sigma'} M \arrow[r, "q^M_{v;\sigma'}"] & \mathrm{Dual}_{\hat\sigma'} \hat M \arrow[r, "R'"] & \mathrm{Hom}^{\mathrm{cont}}_k(M,k)
\end{tikzcd}
\tag{6.3}
$$

is commutative. Since $q^M_{v;\sigma}$ and $q^M_{v;\sigma'}$ are injections, we deduce the independence of $\Psi^M_{\sigma,\sigma'}$ of the basis $\underline{x}$, and that $\Psi^M_{\sigma,\sigma'}$ is an A-linear bijection. Since both $\mathrm{Dual}_\sigma M$ and $\mathrm{Dual}_{\sigma'} M$ have the fine topologies, $\Psi^M_{\sigma,\sigma'}$ is in fact a homeomorphism. □

Proposition 6.13. *Under the hypothesis of Prop. 6.12, suppose $\tau, \tau' : L \to A$ are pseudo coefficient fields, and $f : K \to L$ is a (finite) morphism in $\mathsf{BCA}(k)$, such that $\tau \equiv \tau' \pmod{\mathfrak{m}}$, $\sigma = \tau \circ f$ and $\sigma' = \tau' \circ f$. Then for any $\phi \in \mathrm{Dual}_\tau M$ one has*

$$
\Psi^M_{\sigma,\sigma'}(\mathrm{Tr}_f \circ \phi) = \mathrm{Tr}_f \circ \Psi^M_{\tau,\tau'}(\phi). \tag{6.4}
$$

Proof. After making a reduction as in prop. 6.12, we can assume that $L \xrightarrow{\simeq} A/\mathfrak{m}$. Now set $A^- := \sigma(K) \oplus \mathfrak{m} \subset A$. Choose a homomorphism $u : K \to \hat K$ as in Lemma 6.11, and define BCAs $\hat A^- := A^- \otimes^{(\wedge)}_K \hat K$ and $\hat A := A \otimes^{(\wedge)}_K \hat K$, w.r.t. the morphisms $\sigma : K \to A^- \to A$. Let $v : A \to \hat A$ be the resulting intensification homomorphism. The algebra $\hat A^-$ is local, with maximal ideal $\hat A^- \cdot v(\mathfrak{m})$. Denote by $\hat{\mathfrak{r}}$ the Jacobson radical of $\hat A$; so $\hat{\mathfrak{r}} = \hat A \cdot v(\mathfrak{m})$. Set $\hat L := \hat A / \hat{\mathfrak{r}} \cong L \otimes_K \hat K$. For each $\hat{\mathfrak{m}} \in \operatorname{Max} \hat A$ denote by $f_{\hat{\mathfrak{m}}} : K \to L_{\hat{\mathfrak{m}}}$, $v_{\hat{\mathfrak{m}}} : A \to \hat A_{\hat{\mathfrak{m}}}$ and $u_{\hat{\mathfrak{m}}} : L \to \hat L_{\hat{\mathfrak{m}}}$ the localized homomorphisms. We have $\hat A_{\hat{\mathfrak{m}}} \cong A \otimes^{(\wedge)}_L \hat L_{\hat{\mathfrak{m}}}$, and there are coefficient fields $\hat\tau_{\hat{\mathfrak{m}}}, \hat\tau'_{\hat{\mathfrak{m}}} : \hat L_{\hat{\mathfrak{m}}} \to \hat A_{\hat{\mathfrak{m}}}$ extending τ, τ'. All the claims above follow from Prop. 3.5.

Let $\hat M := \hat A \otimes_A M$. For every $\hat{\mathfrak{m}} \in \operatorname{Max} \hat A$ there is a homomorphism

$$
q^M_{v_{\hat{\mathfrak{m}}};\tau} : \mathrm{Dual}_\tau M \to \mathrm{Dual}_{\hat\tau_{\hat{\mathfrak{m}}}} \hat M_{\hat{\mathfrak{m}}}
$$

and a corresponding homomorphism $q^M_{v_{\hat{\mathfrak{m}}};\tau'}$, which, by Lemma 6.9, intertwine $\Psi^M_{\tau,\tau'}$ with $\Psi^{\hat M_{\hat{\mathfrak{m}}}}_{\hat\tau_{\hat{\mathfrak{m}}},\hat\tau'_{\hat{\mathfrak{m}}}}$. There are also (injective) homomorphisms $q^M_{v;\sigma}$ and $q^M_{v;\sigma'}$. Since the trace maps satisfy

$$
u \circ \mathrm{Tr}_f = \sum_{\hat{\mathfrak{m}}} \mathrm{Tr}_{f_{\hat{\mathfrak{m}}}} \circ u_{\hat{\mathfrak{m}}}
$$

we get

$$
q^M_{v;\sigma}(\mathrm{Tr}_f \circ \phi) = \sum_{\hat{\mathfrak{m}}} \mathrm{Tr}_{f_{\hat{\mathfrak{m}}}} \circ q^M_{v_{\hat{\mathfrak{m}}};\tau}(\phi)
$$

and similarly with σ', τ', so the problem is reduced to the case when $k \to K$ is a morphism.

In this case, using 6.10 twice and the transitivity of residues, we get

$$
\mathrm{Res}_{K/k} \circ \Psi^M_{\sigma,\sigma'}(\mathrm{Tr}_f \circ \phi) = \mathrm{Res}_{K/k} \circ \mathrm{Tr}_f \circ \Psi^M_{\tau,\tau'}(\phi)
$$

which, in virtue of Lemma 6.3 (a), implies formula (6.4). □

We are ready to prove the first main result of this article.

Theorem 6.14. (Dual Modules) *Let A be a local Beilinson completion algebra over k, and let M be a finite type semi topological A-module. Then the following data exist:*

(a) *A ST A-module* $\operatorname{Dual}_A M$*, called the* dual module *of M.*
(b) *For every morphism $\sigma : K \to A$ in $\mathsf{BCA}(k)$, with K a field, an isomorphism of ST A-modules*
$$\Psi^M_\sigma : \operatorname{Dual}_A M \xrightarrow{\simeq} \operatorname{Dual}_\sigma M = \operatorname{Hom}^{\mathrm{cont}}_{K;\sigma}(M, \omega(K)).$$

These data satisfy, and are completely determined by the following conditions:

(i) *Let $f : K \to L$ and $\tau : L \to A$ be morphisms in $\mathsf{BCA}(k)$, with K, L fields, and let $\sigma := \tau \circ f$. Then for any $\phi \in \operatorname{Dual}_A M$,*
$$\Psi^M_\sigma(\phi) = \operatorname{Res}_f \circ \Psi^M_\tau(\phi).$$
Here $\operatorname{Res}_f : \omega(L) \to \omega(K)$ is the residue map in $\mathsf{TLF}(k)$, cf. [Ye1] §2.4.
(ii) *Denote by $\mathfrak{m}$ the maximal ideal of A. If $\sigma, \sigma' : K \to A$ are pseudo coefficient fields such that $\sigma \equiv \sigma' \pmod{\mathfrak{m}}$, then*
$$\Psi^M_{\sigma'} = \Psi^M_{\sigma,\sigma'} \circ \Psi^M_\sigma,$$
where $\Psi^M_{\sigma,\sigma'}$ is the isomorphism defined in Def. 6.7.

Observe that if $A = K$ is a TLF, then there is a canonical isomorphism $\operatorname{Dual}_K M \cong \operatorname{Hom}_K(M, \omega(K))$, corresponding to the identity morphism $K \to K$, thought of as a coefficient field.

Proof. The proof is divided into four steps.

(1) Fix a coefficient field $\tau_0 : L_0 = A/\mathfrak{m} \to A$, and set $\operatorname{Dual} M := \operatorname{Dual}_{\tau_0} M$. Given another coefficient field $\tau : L_0 \to A$, we are forced by condition (ii) to define $\Psi^M_\tau := \Psi^M_{\tau_0,\tau}$. For any other coefficient field $\tau' : L_0 \to A$ this condition is satisfied, on account of Lemma 6.8; condition (i) is irrelevant.

(2) Now let $\sigma : K \to A$ be a pseudo coefficient field which factors through some coefficient field $\tau : L_0 \to A$ (if σ is a quasi coefficient field then there is precisely one such τ). Define $\Psi^M_\sigma : \operatorname{Dual} M \xrightarrow{\simeq} \operatorname{Dual}_\sigma$ to be $\phi \mapsto \operatorname{Tr}_{L_0/K} \circ \Psi^M_{\tau_0,\tau}(\phi)$, as is forced by condition (i). According to Prop. 6.13, this definition is independent of the coefficient field τ.

(3) Let $\sigma : K \to A$ be any pseudo coefficient field. Choose some pseudo coefficient field $\sigma' : K \to A$ such that $\sigma \equiv \sigma' \pmod{\mathfrak{m}}$ and such that σ' factors through some coefficient field. For example, take $\sigma' := \tau_0 \circ \pi \circ \sigma$, where $\pi : A \twoheadrightarrow L_0$ is the natural projection. Define $\Psi^M_\sigma := \Psi^M_{\sigma',\sigma} \circ \Psi^M_{\sigma'}$. Prop. 6.13 shows that this definition is independent of the choice of σ', and furthermore it shows that conditions (i) and (ii) hold for all pseudo coefficient fields.

(4) Finally let $\sigma : K \to A$ be a morphism with $\operatorname{res.dim} \sigma \geq 1$. Choose a factorization $\sigma = \tau \circ f$, with $\tau : L \to A$ a pseudo coefficient field and $f : K \to L$ a morphism. Define $\Psi^M_\sigma(\phi) := \operatorname{Res}_f \circ \Psi^M_\tau(\phi)$, $\phi \in \operatorname{Dual} M$. Now condition (ii) is no

longer relevant. To verify condition (i) it suffices to prove the independence of this definition on τ. So suppose that σ also factors into $\sigma = \tau' \circ f'$.

First assume there exists some finite morphism $g : L \to L'$ such that $\tau = \tau' \circ g$ and $f' = g \circ f$. Then applying condition (i) to $\tau = \tau' \circ g$, we get

$$\operatorname{Res}_f \circ \Psi^M_\tau(\phi) = \operatorname{Res}_f \circ \operatorname{Tr}_g \circ \Psi^M_{\tau'}(\phi) = \operatorname{Res}_{f'} \circ \Psi^M_{\tau'}(\phi) \tag{6.5}$$

for $\phi \in \operatorname{Dual} M$. By taking L' to be the separable closure of L in L_0, and then using formula (6.5), we can assume that $L \to L_0$ is purely inseparable.

It remains to consider the case when $L, L' \subset L_0$, and both $L \to L_0$ and $L' \to L_0$ are purely inseparable. Choose $j >> 0$ such that $L_0^{(p^j/k)} \subset L \cap L'$. Define $L_1 := K L_0^{(p^j/k)} \subset L_0$ and let $\tau_1, \tau_1' : L_1 \to A$ be the restrictions of τ, τ'. To finish the verification use formula (6.5) twice more. □

7. Traces on Dual Modules

As before, k is a fixed perfect field. Suppose A is a local BCA. Then $\operatorname{Dual}_A : M \mapsto \operatorname{Dual}_A M$ is a functor on the category of finite type ST A-modules. Given a finite type ST A-module M and an element $x \in M$, let $\rho_x : A \to M$ be the function $a \mapsto ax$. As in [LC] Lemma 4.1, and by our Lemma 6.3 (b), sending $\phi \in \operatorname{Dual}_A M$ to the homomorphism $x \mapsto \operatorname{Dual}_A(\rho_x)(\phi)$ gives an isomorphism $\operatorname{Dual}_A M \to \operatorname{Hom}_A(M, \mathcal{K}(A))$.

Any BCA A over k decomposes into local factors: $A = \prod_{\mathfrak{m} \in \operatorname{Max} A} A_{\mathfrak{m}}$, as ST k-algebras. Any morphism in $\mathsf{BCA}(k)$ decomposes accordingly.

Definition 7.1. Let A be a BCA over k. Define

$$\mathcal{K}(A) := \bigoplus_{\mathfrak{m} \in \operatorname{Max} A} \operatorname{Dual}_{A_{\mathfrak{m}}} A_{\mathfrak{m}}.$$

Given any ST A-module M, define

$$\operatorname{Dual}_A M := \operatorname{Hom}^{\mathrm{cont}}_A(M, \mathcal{K}(A))$$

with the (weak) Hom topology.

With this definition Dual_A is an additive functor $\mathsf{STMod}(A)^\circ \to \mathsf{STMod}(A)$. In view of the previous discussion and Prop. 1.8 (2), there is no conflict of definitions when A is local and M is a ST A-module of finite type.

Proposition 7.2. (Covariance of dual modules) *Let $v : A \to \hat{A}$ be an intensification homomorphism between two BCAs. Given a ST A-module M, set $\hat{M} := \hat{A} \otimes_A M$. Then there is a unique homomorphism in* $\mathsf{STMod}(A)$*,*

$$q^M_v : \operatorname{Dual}_A M \to \operatorname{Dual}_{\hat{A}} \hat{M},$$

with the following properties:

(i) *If* $\phi : M \to N$ *is a homomorphism in* $\mathsf{STMod}(A)$*, then*

$$q_v^M \circ \mathrm{Dual}_A(\phi) = \mathrm{Dual}_{\hat{A}}(1 \otimes \phi) \circ q_v^N .$$

In other words, $q_v : \mathrm{Dual}_A \to \mathrm{Dual}_{\hat{A}}(\hat{A} \otimes_A -)$ *is a natural transformation of functors.*

(ii) *If* M *is a ST* A*-module of finite type then the induced homomorphism*

$$1 \otimes q_v^M : \hat{A} \otimes_A \mathrm{Dual}_A M \to \mathrm{Dual}_{\hat{A}} \hat{M}$$

is an isomorphism.

(iii) *Let* $\sigma : K \to A$ *be a morphism in* $\mathsf{BCA}(k)$*. Suppose* K *is a field,* A *is local, and there is an intensification homomorphism* $u : K \to \hat{K}$ *s.t.* $\hat{A} \cong A \otimes_K^{(\wedge)} \hat{K}$*. Then for any ST* A*-module of finite type* M*,*

$$q_v^M = (\Psi_{\hat{\sigma}}^{\hat{M}})^{-1} \circ q_{v;\sigma}^M \circ \Psi_\sigma^M .$$

(iv) *If* $w : \hat{A} \to \hat{\hat{A}}$ *is another flat, finitely ramified, radically unramified and topologically étale homomorphism, then* $q_{w \circ v} = q_w \circ q_v$*.*

These properties characterize q_v^M*.*

Proof. We may assume A is local. Let us first check uniqueness. If M is a finite type ST A-module, this follows from condition (iii). If M has the fine topology then $M \cong \lim_{\alpha \to} M_\alpha$ with each M_α a finite type module. By Lemma 1.3 (4) we get $\mathrm{Dual}_A M \cong \lim_{\leftarrow \alpha} \mathrm{Dual}_A M_\alpha$, and we may use condition (i). Finally any ST A-module M is a quotient of a module $\tilde{M}$ which has the fine topology, and $\mathrm{Dual}_A M \hookrightarrow \mathrm{Dual}_A \tilde{M}$.

To define q_v^M for M of finite type amounts, essentially, to repeating the steps of the proof of Thm. 6.14, using Lemma 6.9 at every step. For a general ST A-module M, let q_v^M be the canonical continuous homomorphism

$$\mathrm{Hom}_A^{\mathrm{cont}}(M, \mathcal{K}(A)) \to \mathrm{Hom}_{\hat{A}}^{\mathrm{cont}}(\hat{A} \otimes_A M, \mathcal{K}(\hat{A}))$$

induced by $q_v = q_v^A : \mathcal{K}(A) \to \mathcal{K}(\hat{A})$. $\square$

Definition 7.3. Let $K, A \in \mathsf{BCA}(k)$, with K a field, and let $\sigma : K \to A$ be a morphism. Define

$$\mathrm{Res}_{A/K} = \mathrm{Res}_\sigma : \mathcal{K}(A) \to \omega(K)$$

to be the function sending

$$\phi = \sum_{\mathfrak{m}} \phi_{\mathfrak{m}} \in \mathcal{K}(A) = \bigoplus_{\mathfrak{m}} \mathrm{Dual}_{A_{\mathfrak{m}}} A_{\mathfrak{m}}$$

to $\sum_{\mathfrak{m}} \Psi_\sigma^{A_{\mathfrak{m}}}(\phi_{\mathfrak{m}})(1) \in \omega(K)$. Here $\mathfrak{m}$ runs through the maximal ideals of A.

The residue map $\mathrm{Res}_{A/K}$ is K-linear. It is also continuous: this follows from the adjunction formula, Lemma 1.4 (cf. Remark 6.2). Because of the transitivity of residues, if there is a factorization $\sigma : K \xrightarrow{f} L \xrightarrow{\tau} A$, then $\mathrm{Res}_{A/K} = \mathrm{Res}_{L/K} \circ \mathrm{Res}_{A/L}$.

Here is the second main result of this article:

Theorem 7.4. (Traces) *Let $f : A \to B$ be a morphism in $\mathsf{BCA}(k)$. There is a unique continuous A-linear homomorphism*

$$\mathrm{Tr}_{B/A} = \mathrm{Tr}_f : \mathcal{K}(B) \to \mathcal{K}(A)$$

having the following properties:

(i) *(Transitivity) Given another morphism $g : B \to C$, one has*

$$\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B} .$$

(ii) *(Base Change) Suppose $u : A \to \hat{A}$ is an intensification homomorphism. Let $\hat{B} := B \otimes_A^{(\wedge)} \hat{A}$, $v : B \to \hat{B}$ and $\hat{f} : \hat{A} \to \hat{B}$ be the algebras and homomorphisms gotten by intensification base change (cf. Thm. 3.8). Then*

$$q_u \circ \mathrm{Tr}_{B/A} = \mathrm{Tr}_{\hat{B}/\hat{A}} \circ q_v ,$$

where q_u, q_v are the homomorphisms of Prop. 7.2.

(iii) *If A is a field, then $\mathrm{Tr}_{B/A} = \mathrm{Res}_{B/A} : \mathcal{K}(B) \to \mathcal{K}(A) = \omega(A)$.*

(iv) *The map*

$$\mathcal{K}(B) \to \mathrm{Hom}_A^{\mathrm{cont}}(B, \mathcal{K}(A))$$

induced by $\mathrm{Tr}_{B/A}$ is bijective.

Proof. We may assume both A, B are local, with maximal ideals $\mathfrak{m}, \mathfrak{n}$. Given any morphism $\sigma : K \to A$ with K a field, define $\mathrm{Tr}_{B/A;\sigma} : \mathcal{K}(B) \to \mathcal{K}(A)$ by

$$\mathrm{Tr}_{B/A;\sigma} := (\Psi^A_\sigma)^{-1} \circ \mathrm{Dual}_\sigma(f) \circ \Psi^B_{f\circ\sigma} \tag{7.1}$$

where $\mathrm{Dual}_\sigma(f)(\phi) = \phi \circ f$ for $\phi \in \mathrm{Dual}_{f\circ\sigma} B$.

The claim is that $\mathrm{Tr}_{B/A;\sigma}$ is independent of σ. Let $\tau : L = A/\mathfrak{m} \to A$ be any coefficient field. It suffices to prove that $\mathrm{Tr}_{B/A;\sigma} = \mathrm{Tr}_{B/A;\tau}$. To do so we choose an intensification homomorphism $K \to \hat{K}$ s.t. $k \to \hat{K}$ is a morphism of BCAs, and set $\hat{A} := A \otimes_K^{(\wedge)} \hat{K}$, $\hat{B} := B \otimes_K^{(\wedge)} \hat{K}$ and $\hat{L} := L \otimes_K^{(\wedge)} \hat{K}$. Let $\hat{\tau} : \hat{L} \to \hat{A}$ be the unique extension of τ. Note that by Prop. 3.5, $\hat{A} \cong A \otimes_L^{(\wedge)} \hat{L}$. According to Prop. 7.2 (iii),

$$q_{\hat{A}/A} \circ \mathrm{Tr}_{B/A;\sigma} = \mathrm{Tr}_{\hat{B}/\hat{A};\hat{\sigma}} \circ q_{\hat{B}/B}$$

and similarly for τ. Since $\rho : k \to \hat{A}$ is a morphism, we get (using Thm. 6.14 (i))

$$\mathrm{Tr}_{\hat{B}/\hat{A};\hat{\sigma}} = \mathrm{Tr}_{\hat{B}/\hat{A};\rho} = \mathrm{Tr}_{\hat{B}/\hat{A};\hat{\tau}} .$$

But $q_{\hat{A}/A}$ is injective, so the claim is proved. Our arguments also imply properties (i), (ii) and (iii).

Let us now prove that $\mathrm{Tr}_{B/A}$ is continuous. First assume that $\mathrm{res.dim}\, f \leq 1$. Then $\mathcal{K}(B)$, being a cofinite type ST B-module, actually has the fine A-module topology. (cf. [Ye1] Def. 3.3 or Def. 3.2.1 (b.ii)). Since $\mathrm{Tr}_{B/A} : \mathcal{K}(B) \to \mathcal{K}(A)$ is A-linear, it is continuous. Now assume $\mathrm{res.dim}\, f = n > 1$. Consider the prime ideal $\mathfrak{p} := \mathrm{Ker}(A \to \kappa_{n-1}(B))$. We can assume that $\mathfrak{p} \neq \mathfrak{m}$, by replacing (if necessary) A with $A[[t]]$, and sending t to a parameter of $\mathcal{O}_n(B)$. Thus $A/\mathfrak{p}$ is a DVR and $C := \lim_{\leftarrow i}(A/\mathfrak{p}^i)_\mathfrak{p}$ is a BCA. The morphism $A \to B$ factors into morphisms $A \to$

$C \to B$, both of $\operatorname{res.dim} < n$. By induction $\operatorname{Tr}_{B/C}$ and $\operatorname{Tr}_{C/A}$ are continuous, and $\operatorname{Tr}_{B/A} = \operatorname{Tr}_{B/C} \circ \operatorname{Tr}_{C/A}$.

Finally to prove (iv), take a coefficient field $\sigma : K \to A$. Then

$$\Psi^{B}_{f\circ\sigma} : \mathcal{K}(B) \to \operatorname{Dual}_{f\circ\sigma} B = \operatorname{Hom}^{\mathrm{cont}}_{K}(B, \omega(K))$$

is bijective. On the other hand, one easily sees that

$$\operatorname{Hom}^{\mathrm{cont}}_{A}(B, \mathcal{K}(A)) \to \operatorname{Hom}^{\mathrm{cont}}_{K}(B, \omega(K))$$

is injective, so $\mathcal{K}(B) \xrightarrow{\simeq} \operatorname{Hom}^{\mathrm{cont}}_{A}(B, \mathcal{K}(A))$. $\square$

Remark 7.5. Suppose A, B are BCAs, $f : A \to B$ is a continuous k-algebra homomorphism, and M is a torsion type ST A-module. For instance, A, B could be any complete local k-algebras which are residually finitely generated over k, f could be any local homomorphism, and M any 0-dimensional A-module. If M has finite length, define

$$f_{\#}M := \operatorname{Dual}_B(B \otimes_A \operatorname{Dual}_A M).$$

Otherwise $M = \lim_{\alpha\to} M_\alpha$ where each M_α has finite length, and we set $f_{\#}M := \lim_{\alpha\to} f_{\#}M_\alpha$. This gives a functor $f_{\#} : \mathsf{STMod}_{\mathrm{tors}}(A) \to \mathsf{STMod}_{\mathrm{tors}}(B)$. Note that $f_{\#}\mathcal{K}(A) = \mathcal{K}(B)$. If f is a morphism in $\mathsf{BCA}(k)$, the trace map $\operatorname{Tr}_f : \mathcal{K}(B) \to \mathcal{K}(A)$ defines a trace map $\operatorname{Tr}_f : f_{\#}M \to M$ for any M. The collection of data $(\mathsf{STMod}_{\mathrm{tors}}(A), f_{\#})$ is a realization (and generalization) of Lipman's pseudofunctor on 0-dimensional modules; cf. [Hg].

8. Duals of Continuous Differential Operators

In this section we consider a continuous differential operator $D : M \to N$, and construct a dual operator $\operatorname{Dual}_A(D) : \operatorname{Dual}_A N \to \operatorname{Dual}_A M$. The idea is to use the right $\mathcal{D}(K)$-module structure of $\omega(K)$, for a TLF K.

Let A be a local BCA with maximal ideal $\mathfrak{m}$, and let $\sigma : K \to A$ be a pseudo coefficient field. Given two finite type ST A-modules M, N, choose $\mathfrak{m}$-filtered K-bases $\underline{x} = (x_0, x_1, \ldots)$ and $\underline{y} = (y_0, y_1, \ldots)$ for M and N, respectively (cf. Def. 6.5). Suppose $D : M \to N$ is a continuous DO over A relative to k. For $i, j \geq 0$ let $D_{ij} : K \to K$ be the functions such that, for $\lambda \in K$,

$$D(\sigma(\lambda)x_i) = \sum_j \sigma(D_{ij}(\lambda))y_j.$$

Then, just like in Lemma 6.6, $D_{ij} \in \mathcal{D}(K)$.

Definition 8.1. Let $\operatorname{Dual}_\sigma(D) : \operatorname{Dual}_\sigma N \to \operatorname{Dual}_\sigma M$ be the function taking $\phi \in \operatorname{Dual}_\sigma N$ to

$$\operatorname{Dual}_\sigma(D)(\phi) : \sum_i \sigma(\lambda_i)x_i \mapsto \sum_{i,j} \lambda_i(\phi(y_j) * D_{ij}).$$

There is no reference in the notation "$\operatorname{Dual}_\sigma(D)$" to the bases $\underline{x}, \underline{y}$. This is not an oversight - as we shall see, this function is independent of the bases. First, another definition:

Definition 8.2. Let M be a ST A-module (not necessarily of finite type). Define the residue pairing to be

$$\begin{aligned} \langle -,-\rangle^M_{A/K} &: M \times \operatorname{Dual}_A M \to \omega(K) \\ \langle x,\phi\rangle^M_{A/K} &= \operatorname{Res}_{A/K}(\phi(x)) \end{aligned}$$

where $\operatorname{Res}_{A/K}$ is as in Def. 7.3.

Remark 8.3. Suppose K is discrete (i.e. $\dim K = 0$) and M is a finite type or a cofinite type ST A-module. Then the topology on M is K-linear (cf. [Ye1] Prop. 3.2.5). As a topological vector space over K, M is strongly reflexive, in the sense of [Ko] §13.3. One can show that the strong Hom_K topology on $\operatorname{Dual}_A M \cong \operatorname{Hom}^{\mathrm{cont}}_K(M, \omega(K))$ coincides with the fine A-module topology on it. Hence $\langle -,-\rangle^M_{A/K}$ is a perfect pairing also from the point of view of [Ko].

Lemma 8.4.

(a) *Suppose* $\operatorname{ord}_K(D) = 0$*, i.e.* D *is* K*-linear. Then* $\operatorname{Dual}_\sigma(D)(\phi) = \phi \circ D$ *for all* $\phi \in \operatorname{Dual}_\sigma N$*.*

(b) *Suppose* $k \to K$ *is a morphism in* $\mathsf{BCA}(k)$*. Then for all* $\phi \in \operatorname{Dual}_\sigma N$*,*

$$\operatorname{Res}_{K/k} \circ \operatorname{Dual}_\sigma(D)(\phi) = \operatorname{Res}_{K/k} \circ \phi \circ D.$$

In other words, $\operatorname{Dual}_\sigma(D)$ *is adjoint to* D *with respect to the the residue pairings* $\langle -,-\rangle^M_{A/k}$ *and* $\langle -,-\rangle^N_{A/k}$*.*

Proof. One has $D_{ij} = \mu_{ij} \in K \subset \mathcal{D}(K)$, where $D(x_i) = \sum_j \sigma(\mu_{ij}) y_j$. Now simply plug this into the definition of $\operatorname{Dual}_\sigma(D)$.

(b) Say $\phi(y_j) = \alpha_j \in \omega(K)$. Given $x = \sum_i \sigma(\lambda_i) x_i \in M$, with $\lambda_i \in K$, we have by the definition of the DOs D_{ij}:

$$D(x) = \sum_{i,j} \sigma(D_{ij} * \lambda_i) y_j \ ,$$

so

$$\begin{aligned} \langle D(x), \phi\rangle^N_{A/k} &= \operatorname{Res}_{K/k} \circ \phi \circ D(x) \\ &= \operatorname{Res}_{K/k}\Big(\sum_{i,j} (D_{ij} * \lambda_i)\alpha_j\Big) = \sum_{i,j} \langle D_{ij} * \lambda_i, \alpha_j\rangle_{K/k}. \end{aligned}$$

On the other hand, by the definition of $\operatorname{Dual}_\sigma(D)$,

$$\begin{aligned} \langle x, \operatorname{Dual}_\sigma(D)(\phi)\rangle^M_{A/k} &= \operatorname{Res}_{K/k} \circ (\operatorname{Dual}_\sigma(D)(\phi))(x) \\ &= \operatorname{Res}_{K/k}\Big(\sum_{i,j} \lambda_i (\alpha_j * D_{ij})\Big) = \sum_{i,j} \langle \lambda_i, \alpha_j * D_{ij}\rangle_{K/k}. \end{aligned}$$

Now use Thm. 5.5. $\square$

Lemma 8.5.

(a) $\operatorname{Dual}_\sigma(D) : \operatorname{Dual}_\sigma N \to \operatorname{Dual}_\sigma M$ *is a continuous DO over* A*, relative to* k*, of order* $\leq \operatorname{ord}_A(D)$*. It is independent of the* $\mathfrak{m}$*-filtered* K*-bases* $\underline{x}, \underline{y}$*.*

(b) *Let $\mathfrak{m} \subset A$ be the maximal ideal. Suppose $\sigma' : K \to A$ is another pseudo coefficient field, s.t. $\sigma' \equiv \sigma \pmod{\mathfrak{m}}$. Then*

$$\operatorname{Dual}_{\sigma'}(D) = \Psi^M_{\sigma,\sigma'} \circ \operatorname{Dual}_\sigma(D) \circ \Psi^N_{\sigma',\sigma}.$$

(c) *Suppose $\tau : L \to A$ is another pseudo coefficient field, and $f : K \to L$ is a (finite) morphism in $\mathsf{BCA}(k)$, s.t. $\sigma = \tau \circ f$. Then for each $\phi \in \operatorname{Dual}_\tau N$,*

$$\operatorname{Dual}_\sigma(D)(\operatorname{Tr}_f \circ \phi) = \operatorname{Tr}_f \circ \operatorname{Dual}_\tau(D)(\phi).$$

Proof. The proof resembles that of Prop. 6.12. Choose an intensification homomorphism $u : K \to \hat{K}$ such that $k \to \hat{K}$ is a morphism. Let $\hat{A} := A \otimes_K^{(\wedge)} \hat{K}$ and $v : A \to \hat{A}$. Replacing A with each of the localizations $\hat{A}_{\hat{\mathfrak{m}}}$, $\hat{\mathfrak{m}} \in \operatorname{Max} \hat{A}$, allows us to assume that $k \to K$ is itself a morphism in $\mathsf{BCA}(k)$. By Lemma 8.4 (b) we see that $\operatorname{Dual}_\sigma(D)$ is the adjoint of D w.r.t. the residue pairings $\langle -,-\rangle^M_{A/k}$ and $\langle -,-\rangle^N_{A/k}$, so in particular it is independent of the $\mathfrak{m}$-filtered K-bases $\underline{x}, \underline{y}$. It also follows that for any $a \in A$,

$$[\operatorname{Dual}_\sigma(D), a] = -\operatorname{Dual}_\sigma([D,a]) : \operatorname{Dual}_\sigma M \to \operatorname{Dual}_\sigma N,$$

bounding the order of the operator $\operatorname{Dual}_\sigma(D)$. Here "$[-,-]$" denotes the commutator. Parts (b),(c) of the present lemma are similarly proved, using Lemma 6.9.

As for the continuity of $\operatorname{Dual}_\sigma(D)$, it can be deduced from the fact that it is a linear combination of the continuous operators D_{ij} appearing in its definition. □

The ST A-module $\mathcal{K}(A)$ is separated. Therefore for any ST A-module M, the canonical surjection $M \twoheadrightarrow M^{\mathrm{sep}}$ induces an isomorphism $\operatorname{Dual}_A M^{\mathrm{sep}} \xrightarrow{\simeq} \operatorname{Dual}_A M$. Here is the third main result of the paper:

Theorem 8.6. (Duals of Continuous DOs) *Let A be a BCA over k. Let M and N be ST A-modules with the fine topologies, and let $D : M \to N$ be a continuous DO over A relative to k. Then there is a unique function*

$$\operatorname{Dual}_A(D) : \operatorname{Dual}_A N \to \operatorname{Dual}_A M,$$

satisfying the conditions below:

(i) *$\operatorname{Dual}_A(D) : \operatorname{Dual}_A N \to \operatorname{Dual}_A M$ is a continuous DO over A relative to k, of order $\leq \operatorname{ord}_A(D)$.*

(ii) *(Transitivity) if $E : N \to P$ is another such operator, then $\operatorname{Dual}_A(E \circ D) = \operatorname{Dual}_A(D) \circ \operatorname{Dual}_A(E)$.*

(iii) *(Linearity) if D is A-linear, then $\operatorname{Dual}_A(D)$ is the homomorphism $\phi \mapsto \phi \circ D$, for $\phi \in \operatorname{Dual}_A N = \operatorname{Hom}^{\mathrm{cont}}_A(N, \mathcal{K}(A))$.*

(iv) *(Base change) let $v : A \to \hat{A}$ be an intensification homomorphism, and let $\hat{D} : (\hat{A} \otimes_A M)^{\mathrm{sep}} \to (\hat{A} \otimes_A N)^{\mathrm{sep}}$ be the unique extension of D. Then*

$$\operatorname{Dual}_{\hat{A}}(\hat{D}) \circ q^N_v = q^M_v \circ \operatorname{Dual}_A(D),$$

where q^M_v, q^N_v are the homomorphisms of Prop. 7.2.

(v) *Assume* $\sigma : K \to A$ *is a morphism in* $\mathsf{BCA}(k)$ *s.t.* D *is* K*-linear. Then* $\operatorname{Dual}_A(D)$ *is the adjoint to* D *w.r.t. the residue pairings* $\langle -,- \rangle^M_{A/K}$ *and* $\langle -,- \rangle^N_{A/K}$.

(vi) *Suppose* A *is local and* M, N *are finite type ST* A*-modules. Given a pseudo coefficient field* $\sigma : K \to A$*, one has*

$$\Psi^M_\sigma \circ \operatorname{Dual}_A(D) = \operatorname{Dual}_\sigma(D) \circ \Psi^N_\sigma.$$

Here $\Psi^M_\sigma, \Psi^N_\sigma$ *are the isomorphisms of Thm.* 6.14*, and* $\operatorname{Dual}_\sigma(D)$ *is the function defined in Def.* 8.1.

Remark 8.7. Trivially, the category $\mathsf{Mod}(A)$ of A-modules and A-linear homomorphisms, and the category $\mathsf{STMod}_{\mathrm{fine}}(A)$ of ST A-modules with fine topologies and continuous A-linear homomorphisms, are equivalent (under the functor untop : $\mathsf{STMod}(A) \to \mathsf{Mod}(A)$ which forgets the topology). However, if we take the same classes of objects, but enlarge the set of morphisms between two objects to be DOs and continuous DOs, respectively, these new categories are no longer equivalent. This is so at least when $\operatorname{char} k = 0$ and $\operatorname{res.dim} A \geq 1$. Our results are valid only for continuous DOs.

Proof. Let M, N be finite type ST A-modules. Using Lemma 8.5, and proceeding just like in the proofs of Theorems 6.14 and 7.4, we arrive at a function $\operatorname{Dual}_A(D)$ which satisfies conditions (i)-(iv), (vi). As for condition (v), after a base change $K \to \hat{K}$ we reduce to the case when $k \to A$ is a morphism. Now we can use Lemma 8.4 (b).

Now let M, N be ST A-modules with fine topologies. After possibly applying $(-)^{\mathrm{sep}}$ to these modules, we may assume they are separated. Choose an isomorphism $M \cong \lim_{\alpha \to} M_\alpha$, with the M_α modules of finite type. Let N_α be the A-module $A \cdot D(M_\alpha) \subset N$, endowed with the fine topology. Say $d = \operatorname{ord}_A(D)$. Because $\mathcal{P}^{d,\mathrm{sep}}_{A/k}(M_\alpha)$ is a finite type ST A-module (by Prop. 4.8), N_α is of finite type, and $D_\alpha := D|_{M_\alpha} : M_\alpha \to N_\alpha$ is continuous. Let $\psi : \lim_{\alpha \to} N_\alpha \to N$ be the inclusion, and set

$$\operatorname{Dual}_A(D) := (\lim_{\leftarrow \alpha} \operatorname{Dual}_A(D_\alpha)) \circ \operatorname{Dual}_A(\psi).$$

Since the functor Dual_A sends $\lim_{\to}$ to $\lim_{\leftarrow}$ (cf. Lemma 1.3 (4)), this extended definiton of $\operatorname{Dual}_A(D)$ satisfies all the conditions of the theorem. $\square$

Occasionally we shall abbreviate $\operatorname{Dual}_A M$ to $M^\vee$, and $\operatorname{Dual}_A(D)$ to $D^\vee$.

Corollary 8.8.

(a) *Let* M, N *be each either finite type or cofinite type ST* A*-modules, and let* $D \in \operatorname{Diff}^{\mathrm{cont}}_{A/k}(M, N)$*. Then under the canonical isomorphisms* $M \xrightarrow{\simeq} M^{\vee\vee}$ *and* $N \xrightarrow{\simeq} N^{\vee\vee}$*, one has* $D \mapsto D^{\vee\vee}$.

(b) *With* M, N *as in* (a)*, the map* $\operatorname{Diff}^{\mathrm{cont}}_{A/k}(M, N) \to \operatorname{Diff}^{\mathrm{cont}}_{A/k}(M^\vee, N^\vee)$*,* $D \mapsto D^\vee$*, is an anti-isomorphism of filtered* A*-*A*-bimodules. In particular,* $\mathcal{D}(A; \mathcal{K}(A)) \cong \mathcal{D}(A)^\circ$ *as filtered* k*-algebras.*

Proof. (a) Using base change we can assume that $k \to A$ is a morphism in $\mathsf{BCA}(k)$. Then both D and $D^{\vee\vee}$ are adjoints to $D^{\vee}$ w.r.t. the residue pairings $\langle -,-\rangle^{M}_{A/k}$ and $\langle -,-\rangle^{N}_{A/k}$.

(b) Immediate from part (a). $\square$

Here are a couple of examples to illustrate the scope of our results:

Example 8.9. Suppose A is a noetherian, local, residually finitely generated k-algebra. Let I be an injective hull of the residue field $A/\mathfrak{m}$. Then I is (non-canonically) a right $\mathcal{D}(A)$-module, and moreover $\mathrm{Diff}_{A/k}(I,I) \cong \mathcal{D}(\hat{A})^{\circ}$, where $\hat{A}$ is the $\mathfrak{m}$-adic completion. This is because $\hat{A}$ is a BCA, there exists an isomorphism of $\hat{A}$-modules $I \cong \mathcal{K}(\hat{A})$, and any DO $I \to I$ is automatically continuous for the $\mathfrak{m}$-adic topology.

Example 8.10. Let A be a BCA. Suppose $M^{\cdot}$ is a bounded complex with each M^q a finite type ST A-module, and $D : M^q \to M^{q+1}$ a continuous DO (for instance, $M^{\cdot} = \Omega^{\cdot,\mathrm{sep}}_{A/k}$). Then $\mathrm{Dual}_A M^{\cdot}$ is also a complex (of cofinite type modules), and a standard spectral sequence argument shows that the homomorphism of complexes

$$M^{\cdot} \to \mathrm{Dual}_A \mathrm{Dual}_A M^{\cdot}$$

(in the abelian category of untopologized k-modules) is a quasi-isomorphism.

Question 8.11. In the example above, suppose the complex $M^{\cdot}$ is acyclic. Is the same true of the dual complex $\mathrm{Dual}_A M^{\cdot}$? A slight variation is: suppose $\mathrm{rank}_k \mathrm{H}^q M^{\cdot} < \infty$ for all q. Is the same true for $\mathrm{Dual}_A M^{\cdot}$?

Corollary 8.12. *Let $f : A \to B$ be a morphism in $\mathsf{BCA}(k)$, let M (resp. N) be a ST A-module (resp. B-module) with the fine topology, and let $D \in \mathrm{Diff}^{\mathrm{cont}}_{A/k}(M,N)$. Then there is a DO*

$$\mathrm{Dual}_{B/A}(D) = \mathrm{Dual}_f(D) : \mathrm{Dual}_B N \to \mathrm{Dual}_A M.$$

The asignment $D \mapsto \mathrm{Dual}_f(D)$ satisfies the obvious generalizations of conditions (i)-(v) of Thm. 8.6. For instance (iii): if D is A-linear, then $\mathrm{Dual}_f(D)(\phi) = \mathrm{Tr}_{B/A} \circ \phi \circ D$.

Proof. We may assume that M is a finite type ST A-module, and that N is separated. So D factors into $M \xrightarrow{\mathrm{d}^n_M} \mathcal{P}^{n,\mathrm{sep}}_{A/k}(M) \xrightarrow{\phi} N$, with $\phi \in \mathrm{Hom}^{\mathrm{cont}}_A(\mathcal{P}^{n,\mathrm{sep}}_{A/k}(M), N)$ and $n \geq \mathrm{ord}_A(D)$. Let $\phi^{\vee} : \mathrm{Dual}_B N \to \mathrm{Dual}_A \mathcal{P}^{n,\mathrm{sep}}_{A/k}(M)$ be the homomorphism $\psi \mapsto \mathrm{Tr}_{B/A} \circ \psi \circ \phi$, for $\psi \in \mathrm{Dual}_B N = \mathrm{Hom}^{\mathrm{cont}}_B(N, \mathcal{K}(B))$. Define $\mathrm{Dual}_f(D) := \mathrm{Dual}_A(\mathrm{d}^n_M) \circ \phi^{\vee}$. The transitivity and uniqueness properties follow from base change and the uniqueness of adjoints. $\square$

Example 8.13. If $f : A \to B$ is a morphism of BCAs, the trace map $\mathrm{Tr}_{B/A} : \mathcal{K}(B) \to \mathcal{K}(A)$ and the continuous DGA homomorphism $\Omega^{\cdot,\mathrm{sep}}_{A/k} \to \Omega^{\cdot,\mathrm{sep}}_{B/k}$ induce a map $\mathrm{Tr}_{B/A} : \mathrm{Dual}_B \, \Omega^{\cdot,\mathrm{sep}}_{B/k} \to \mathrm{Dual}_A \, \Omega^{\cdot,\mathrm{sep}}_{A/k}$, which by the corollary is a homomorphism of complexes. This fact is important for the construction of the De Rham - residue double complex in [Ye2].

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