ON THE STRUCTURE OF BEHAVIORS

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ABSTRACT. A behavior is a closed shift invariant subspace of the space of sequences with entries in a field $\Bbbk$. We work out an explicit duality for $\Bbbk$-modules. This duality is then used to derive properties of behaviors, and their noncommutative generalizations.

0. Introduction

Let $\Bbbk$ be a field. Put on $\Bbbk$ the discrete topology, and put on

$$\Bbbk^{\mathbb{N}} = \{\phi : \mathbb{N} \to \Bbbk\} = \prod_{i \in \mathbb{N}} \Bbbk$$

the product topology. Then $\Bbbk^{\mathbb{N}}$ is a topological $\Bbbk$-module. The shift operator $\sigma : \Bbbk^{\mathbb{N}} \to \Bbbk^{\mathbb{N}}$ is

$$\sigma(\lambda_0, \lambda_1, \lambda_2, \ldots) := (\lambda_1, \lambda_2, \ldots).$$

A *behavior* (or *discrete linear system*) is a closed shift invariant $\Bbbk$-submodule $M \subset (\Bbbk^{\mathbb{N}})^r$ for some natural number r. To be precise, this is a 1-*dimensional behavior*. For any $n \geq 1$ there is a corresponding definition of n-*dimensional behavior*, which is a closed shift invariant $\Bbbk$-submodule of $(\Bbbk^{\mathbb{N}^n})^r$. But in the introduction we will stick to the 1-dimensional case. The notion of behavior is attributed to J.C. Willems. See [Fu1] and its references for more background information.

A behavior is naturally a module over the polynomial ring $\Bbbk[z]$; the variable z acts by the shift operator σ. Early on it was realized that behaviors are intimately related to finitely generated $\Bbbk[z]$-modules, and that in fact these are dual mathematical objects. Yet some of the more subtle aspects were not fully understood (cf. [Fu2]).

There does exist a detailed treatment of behaviors by Oberst in [Ob]. However we feel that this treatment is unduly complicated; perhaps because the author wanted to consider discrete and continuous systems in a unified fashion. Thus Oberst showed that the $\Bbbk[z]$-module $\Bbbk^{\mathbb{N}}$ is an injective cogenerator of the category $\mathsf{Mod}\,\Bbbk[z]$ of $\Bbbk[z]$-modules, and then he considered behaviors as modules over the endomorphism ring $\mathrm{End}_{\Bbbk[z]}(\Bbbk^{\mathbb{N}})$. A similar approach was taken in [Lo]. In doing so some interesting features of the theory were missed (such as the counterexample in Section 5, or the noncommutative generalization in Section 6).

The aim of our paper is to clarify the structure of behaviors and to point out possible generalizations.

Date: 28 May 2004.

Key words and phrases. Behaviors; duality; topological modules.

Mathematics Subject Classification 2000. Primary: 93C05; Secondary: 93B25, 16W80, 13J10, 16P40.

Our basic observation is that the duality underlying behaviors has nothing to do with polynomial rings – it is a duality for $\mathbb{k}$-modules (i.e. vector spaces). We establish a duality $\mathrm{D} : M \mapsto M^*$ between the category $\mathsf{Mod}\, \mathbb{k}$ of $\mathbb{k}$-modules, and the category $\mathsf{TopMod}_{\mathrm{pf}}\, \mathbb{k}$ of *profinite topological* $\mathbb{k}$*-modules*. By definition a profinite topological $\mathbb{k}$-module is a topological $\mathbb{k}$-module that is an inverse limit of finitely generated $\mathbb{k}$-modules (with discrete topologies). A morphism in $\mathsf{TopMod}_{\mathrm{pf}}\, \mathbb{k}$ is a continuous $\mathbb{k}$-linear homomorphism. We prove that a closed $\mathbb{k}$-submodule of a profinite topological $\mathbb{k}$-module is also profinite. It turns out (as a consequence of the duality) that the profinite topological $\mathbb{k}$-modules are precisely the linearly compact topological vector spaces, in the sense of [Ko, Section 10.9].

Of course this duality is just a very easy instance of Gabriel-Matlis duality, valid for noncommutative noetherian rings and suitable module categories over them. But instead of quoting from classical (and complicated) work in ring theory, or alternatively dressing up the more naive duality results of [Ko], we simply work out the proofs explicitly. This provides us with a lot of information that is particular to $\mathbb{k}$-modules.

Given a commutative $\mathbb{k}$-algebra A, we consider the category $\mathsf{TopMod}_{\mathrm{pf}/\mathbb{k}}\, A$ consisting of topological A-modules that are profinite as $\mathbb{k}$-modules. Then the duality D restricts to a duality $\mathrm{D} : \mathsf{Mod}\, A \to \mathsf{TopMod}_{\mathrm{pf}/\mathbb{k}}\, A$.

In this framework the behaviors can be described as follows. Take $A := \mathbb{k}[z]$. The shift module $\mathbb{k}^{\mathbb{N}}$ coincides with the dual $\mathbb{k}[z]^*$ as topological $\mathbb{k}[z]$-modules, and it is profinite over $\mathbb{k}$. A behavior M is then a closed $\mathbb{k}[z]$-submodule of $(\mathbb{k}[z]^*)^r$ for some natural number r; and so it is profinite as $\mathbb{k}$-module. By duality M^* is a quotient of $\mathbb{k}[z]^r$, and hence it is a finitely generated $\mathbb{k}[z]$-module. This argument can be reversed. See Theorems 4.7 and 4.9 for precise statements (for n-dimensional behaviors).

The paper is organized as follows. In Section 1 we study profinite topological $\mathbb{k}$-modules. Section 2 is devoted to the duality between the category $\mathsf{Mod}\, \mathbb{k}$ of $\mathbb{k}$-modules and the category $\mathsf{TopMod}_{\mathrm{pf}}\, \mathbb{k}$ of profinite topological $\mathbb{k}$-modules. In Section 3 we consider the category $\mathsf{TopMod}_{\mathrm{pf}/\mathbb{k}}\, A$ for any $\mathbb{k}$-algebra (not necessarily commutative). In Section 4 we concentrate on n-dimensional behaviors, which are modules over the polynomial ring $\mathbb{k}[z_1, \ldots, z_n]$. In Section 5 we exhibit $\mathbb{k}[z]$-linear homomorphisms between behaviors that are not continuous. Finally in Section 6 we study noncommutative generalizations of behaviors.

Acknowledgements. I wish to thank Paul Fuhrmann for getting me interested in behaviors. Also thanks to Sverre Smalø for reading the paper and suggesting some improvements. Finally thanks to the referees for their remarks.

1. Profinite Topological $\mathbb{k}$-Modules

In this section $\mathbb{k}$ is a noetherian commutative ring (e.g. a field or the ring of integers $\mathbb{Z}$). Denote by $\mathsf{Mod}\, \mathbb{k}$ the category of $\mathbb{k}$-modules.

By a topological $\mathbb{k}$-module we mean a $\mathbb{k}$-module V endowed with a topology (of any sort) such that addition $V \times V \to V$ is continuous, and for any $\lambda \in \mathbb{k}$ the multiplication map $\lambda : V \to V$ is continuous. Let $\mathsf{TopMod}\, \mathbb{k}$ be the category whose objects are the topological $\mathbb{k}$-modules and whose morphisms are the continuous $\mathbb{k}$-linear homomorphisms. So

$$\mathrm{Hom}_{\mathsf{TopMod}\, \mathbb{k}}(W_1, W_2) = \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{k}}(W_1, W_2).$$

The category $\mathsf{TopMod}\,\Bbbk$ is additive, but it is not abelian. See [HS, Chapter II] for background material on categories and functors.

Let $\phi : W_1 \to W_2$ be a morphism in $\mathsf{TopMod}\,\Bbbk$. The morphism ϕ is called a *strict monomorphism* if ϕ is injective and W_1 has the subspace topology induced from W_2. The morphism ϕ is called a *strict epimorphism* if ϕ is surjective and W_2 has the quotient topology induced from W_1. And ϕ is called a *strict morphism* if it factors into $\phi = \phi_2 \circ \phi_1$, where $\phi_1 : W_1 \to W$ is a strict epimorphism and $\phi_2 : W \to W_2$ is a strict monomorphism. A sequence of homomorphisms (possibly infinite on either side)

$$\cdots \to W_0 \xrightarrow{\phi_0} W_1 \xrightarrow{\phi_1} W_2 \to \cdots$$

in $\mathsf{TopMod}\,\Bbbk$ is called *strict-exact* if for all i one has $\mathrm{Im}(\phi_{i-1}) = \mathrm{Ker}(\phi_i)$, and ϕ_i is strict. See [Bo] for more details on strict homomorphisms.

Recall that a *quasi-ordered* set is a pair $I = (I_0, I_1)$ consisting of a set I_0, and a set of arrows $I_1 = \{i \to j\} \subset I_0 \times I_0$, such that I_1 is a reflexive and transitive relation on I_0. Thus I is a category with at most one morphism between any two objects. We note that the pair $I^{\mathrm{op}} := (I_0, I_1^{\mathrm{op}})$, in which all the arrows are reversed, is also a quasi-ordered set. A quasi-ordered set I is called a *directed set* if for any two elements $i, j \in I_0$ there exist arrows $i \to k$ and $j \to k$ in I_1.

Suppose I is a quasi-ordered set and C is any category. A *direct system* in C indexed by I is a functor $F : I \to \mathsf{C}$. Usually one refers to this by saying that $\{C_i\}_{i\in I}$ is a direct system, where for any $i \in I_0$ one writes $C_i := Fi \in \mathsf{C}$, and the morphisms $F(i \to j) : C_i \to C_j$ are implicit. A *direct limit* of the system $\{C_i\}_{i\in I}$ is an object $C \in \mathsf{C}$, equipped with a compatible system of morphisms $\phi_i : C_i \to C$, which is universal for this property. By 'compatible system of morphisms' we mean that for every arrow $\alpha : i \to j$ in I one has $\phi_j \circ \alpha = \phi_i$. And by 'universal' we mean that given any $D \in \mathsf{C}$ with a compatible system of morphisms $\psi_i : C_i \to D$ there is exactly one morphism $\psi : C \to D$ such that $\psi_i = \psi \circ \phi_i$. A direct limit is unique if it exists; and in many cases it does exist, e.g. when C is the category of sets Sets or the category $\mathsf{Mod}\,\Bbbk$. The direct limit C is denoted by $\varinjlim F$ or $\varinjlim_{i\in I} C_i$.

An *inverse system* in C indexed by I is a functor $F : I^{\mathrm{op}} \to \mathsf{C}$. So there is a set $\{C_i\}_{i\in I}$ of objects of C, and for each arrow $\alpha : i \to j$ in I we get an arrow $F(\alpha) : C_j \to C_i$ in C. An *inverse limit* is an object $C \in \mathsf{C}$, with a compatible system of morphisms $C \to C_i$, which is universal for this property. It is denoted by $\varprojlim F$ or $\varprojlim_{i\in I} C_i$.

Observe that $F : I^{\mathrm{op}} \to \mathsf{C}$ can be viewed as a functor $F : I \to \mathsf{C}^{\mathrm{op}}$, where C^{op} is the opposite category (same objects but reversed morphisms). So an inverse system in C is the same as a direct system in C^{op}, and $\varprojlim F$ in C coincides with $\varinjlim F$ in C^{op}.

Refer to [Ro, Chapter 2] for more on limits in categories.

We shall be mainly interested in inverse limits in $\mathsf{TopMod}\,\Bbbk$. Given an inverse system $\{W_i\}_{i\in I}$ of topological $\Bbbk$-modules indexed by a quasi-ordered set I, the inverse limit $W := \varprojlim_{i\in I} W_i$ is constructed like this. Inside the product $\prod_{i\in I} W_i$, endowed with the product topology, one takes the submodule consisting of sequences (w_i) such that for any arrow $i \to j$ in I the homomorphism $W_j \to W_i$ sends $w_j \mapsto w_i$.

The limit module W is equipped with a system of continuous $\mathbb{k}$-linear homomorphisms $\pi_i : W \to W_i$. Given any topological $\mathbb{k}$-module U the system of morphisms $\{\pi_i\}$ induces a bijection of sets

$$\operatorname{Hom}^{\mathrm{cont}}_{\mathbb{k}}(U, W) \to \lim_{\substack{\longleftarrow \\ i\in I}} \operatorname{Hom}^{\mathrm{cont}}_{\mathbb{k}}(U, W_i),$$

where this last limit is taken in $\mathsf{Mod}\,\mathbb{k}$. Cf. [Ye1, Section 1.1].

Definition 1.1. Given a topological $\mathbb{k}$-module W let us denote by $\mathsf{Cofin}\, W$ the set of open cofinite $\mathbb{k}$-submodules of W, namely those submodules $W' \subset W$ such that the quotient W/W' is a discrete finitely generated $\mathbb{k}$-module.

The set $\mathsf{Cofin}\, W$ is partially ordered by inclusion, and hence it is a quasi-ordered set.

Lemma 1.2. *The quasi-ordered sets* $\mathsf{Cofin}\, W$ *and* $(\mathsf{Cofin}\, W)^{\mathrm{op}}$ *are both directed.*

Proof. First consider $\mathsf{Cofin}\, W$. Let $W_1, W_2 \in \mathsf{Cofin}\, W$. We have to show that there is some $W_3 \in \mathsf{Cofin}\, W$ such that $W_1 \subset W_3$ and $W_2 \subset W_3$. For this we can take $W_3 := W$.

Next consider $(\mathsf{Cofin}\, W)^{\mathrm{op}}$. We have to show that there is some $W_3 \in \mathsf{Cofin}\, W$ such that $W_3 \subset W_1$ and $W_3 \subset W_2$. Take the intersection $W_3 := W_1 \cap W_2$. Clearly W_3 is open in W. It remains to show it is cofinite. We know that W/W_1 and W/W_2 are finitely generated. Because $\mathbb{k}$ is noetherian the submodule $\frac{W_1}{W_1 \cap W_2} \subset \frac{W}{W_2}$ is finitely generated. By the exact sequence

$$0 \to \frac{W_1}{W_1 \cap W_2} \to \frac{W}{W_1 \cap W_2} \to \frac{W}{W_1} \to 0$$

it follows that $\frac{W}{W_1 \cap W_2}$ is finitely generated. □

Definition 1.3. Let W be a topological $\mathbb{k}$-module. For any $W' \in \mathsf{Cofin}\, W$ the quotient W/W' has the discrete topology. The inverse limit $\lim_{\leftarrow} W/W'$, as W' runs over the quasi-ordered set $\mathsf{Cofin}\, W$, is endowed with the $\lim_{\leftarrow}$ topology. The topological $\mathbb{k}$-module W is called *profinite* if the canonical homomorphism

$$W \to \lim_{\substack{\longleftarrow \\ W' \in \mathsf{Cofin}\, W}} W/W'$$

is an isomorphism in $\mathsf{TopMod}\,\mathbb{k}$.

We denote by $\mathsf{TopMod}_{\mathrm{pf}}\,\mathbb{k}$ the full subcategory of $\mathsf{TopMod}\,\mathbb{k}$ consisting of profinite modules.

Lemma 1.4. *Let W be a profinite topological $\mathbb{k}$-module. Then the set $\mathsf{Cofin}\, W$ is a basis of the topology at 0; namely any open neighborhood U of 0 contains some open cofinite submodule $W' \subset W$.*

Proof. For convenience let us write $I := \mathsf{Cofin}\, W$ and $\mathsf{Cofin}\, W = \{W'_i\}_{i\in I}$. By the definition of the product topology U contains $W \cap \prod_{i\in I} U_i$ where $U_i \subset W/W'_i$ is open and $U_i = W/W'_i$ for all but finitely many i. Since W/W'_i is discrete we can assume that $U_i = 0$ for all i in some finite subset $I' \subset I$, and $U_i = W/W'_i$ for all $i \notin I'$. Now

$$W \cap \prod_{i\in I} U_i = \bigcap_{i\in I'} \operatorname{Ker}(W \to W/W'_i) = \bigcap_{i\in I'} W'_i.$$

But by Lemma 1.2 the submodule $W' := \bigcap_{i\in I'} W'_i \subset W$ is open and cofinite. □

Lemma 1.5. *Suppose W is a profinite topological $\Bbbk$-module and V is a discrete topological $\Bbbk$-module. Then the canonical map of sets*

$$\Psi : \varinjlim_{W' \in \mathsf{Cofin}\, W} \mathrm{Hom}_{\Bbbk}(W/W', V) \to \mathrm{Hom}^{\mathrm{cont}}_{\Bbbk}(W, V)$$

is bijective. In words, any continuous $\Bbbk$-linear homomorphism $\phi : W \to V$ factors via some discrete finitely generated quotient W/W'.

Proof. Clearly Ψ is injective. Suppose we are given a continuous $\Bbbk$-linear homomorphism $\phi : W \to V$. Because $\{0\} \subset V$ is open the kernel $\mathrm{Ker}(\phi)$ is open in W. According to Lemma 1.4 there is some $W' \in \mathsf{Cofin}\, W$ such that $W' \subset \mathrm{Ker}(\phi)$. Then denoting by $\bar{\phi} : W/W' \to V$ the induced homomorphism, we see that $\phi = \Psi(\bar{\phi})$. □

Lemma 1.6. *Let W be a profinite topological $\Bbbk$-module. Then W is separated (i.e. Hausdorff).*

Proof. This is because the product $\prod_{W' \in \mathsf{Cofin}\, W} W/W'$ is separated, and W has the subspace topology. □

Remark 1.7. Actually a profinite topological $\Bbbk$-module W is also complete (in the sense of Cauchy filters); but we do not need this fact.

Proposition 1.8. *Suppose $\{W_i\}_{i \in I}$ is an inverse system of finitely generated discrete $\Bbbk$-modules indexed by a quasi-ordered set I, and let $W := \varprojlim_{i \in I} W_i$ in $\mathsf{TopMod}\, \Bbbk$. Assume I^{op} is directed. Then W is a profinite topological $\Bbbk$-module.*

Proof. Replacing W_i by the discrete finitely generated module $\mathrm{Im}(W \to W_i)$ we can assume the inverse system has surjections $W \twoheadrightarrow W_i$ for all i. For each i let $W'_i := \mathrm{Ker}(W \to W_i)$, which is an open cofinite submodule. It suffices to prove that the inverse system $\{W'_i\}_{i \in I}$ is cofinal in the quasi-ordered set $(\mathsf{Cofin}\, W)^{\mathrm{op}}$. So pick any $W' \in \mathsf{Cofin}\, W$. Since W' is open we have $\bigcap_{i \in I'} W'_i \subset W'$ for some finite subset $I' \subset I$. Now I^{op} is directed, so there is some $j \in I$ with arrows $j \to i$ for all $i \in I'$. Hence $W'_j \subset \bigcap_{i \in I'} W'_i$. □

Theorem 1.9. *Let $\Bbbk$ be a noetherian commutative ring, let W be a profinite topological $\Bbbk$-module and let U be a closed submodule. Then U, with the induced topology, is also a profinite topological $\Bbbk$-module.*

Proof. As before let I denote the quasi-ordered set $\mathsf{Cofin}\, W$ and write W'_i for the submodule labelled $i \in I$. Define $W_i := W/W'_i$ with the discrete topology. By definition $W \cong \varprojlim_{i \in I} W_i$ in $\mathsf{TopMod}\, \Bbbk$. Let $U_i := \mathrm{Im}(U \to W_i)$, with the discrete topology. Since $\Bbbk$ is noetherian the module U_i is finitely generated. By Proposition 1.8 the inverse limit $\bar{U} := \varprojlim_{i \in I} U_i$ is a profinite topological $\Bbbk$-module. Since $U_i \subset W_i$ for all i we get an injection $\bar{U} \hookrightarrow W$. By Lemma 1.4 and because $U \to U_i$ is surjective for every i it follows that U is dense in $\bar{U}$. But U is closed in W, so we get $U = \bar{U}$. □

Remark 1.10. If $W \cong \varprojlim_{i \in \mathbb{N}} W_i$ for some inverse system $\{W_i\}_{i \in \mathbb{N}}$ of discrete modules, and $U \subset W$ is a closed submodule, then the quotient W/U is also profinite; see [Ye1, Proposition 1.1.6]. (By the way in this case W is a metrizable topological space.) We do not know if W/U is profinite when W is not a countable inverse limit of discrete modules.

2. Duality for Topological $\Bbbk$-Modules

From here on $\Bbbk$ is a field.

Given a $\Bbbk$-module V let $\mathsf{Fin}\, V$ be the set of finitely generated $\Bbbk$-submodules of V. Inclusion makes it a partially ordered set. It is directed, since given $V_1, V_2 \in \mathsf{Fin}\, V$ the sum $V_1 + V_2$ is also in $\mathsf{Fin}\, V$. And V is the direct limit

$$V = \lim_{\substack{\longrightarrow \\ V' \in \mathsf{Fin}\, V}} V' .$$

Lemma 2.1. *Given a $\Bbbk$-module V the inclusions $V' \to V$ induce a bijection*

$$\Psi : \mathrm{Hom}_{\Bbbk}(V, \Bbbk) \xrightarrow{\simeq} \lim_{\substack{\longleftarrow \\ V' \in \mathsf{Fin}\, V}} \mathrm{Hom}_{\Bbbk}(V', \Bbbk).$$

Proof. Say $\phi : V \to \Bbbk$ is nonzero. Then $\phi|_{V'} \neq 0$ for some $V' \in \mathsf{Fin}\, V$. This shows Ψ is injective. Conversely, suppose $\{\phi|_{V'} : V' \to \Bbbk\}$ is a compatible system of homomorphisms. Then we can patch the homomorphisms $\phi_{V'}$ to a "global" homomorphism $\phi : V \to \Bbbk$. So Ψ is surjective. □

Definition 2.2. Given a $\Bbbk$-module V let $\mathrm{D}V := \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ be its dual module. We shall make $\mathrm{D}V$ into a topological $\Bbbk$-module as follows.

(1) If V is finitely generated then $\mathrm{D}V$ has the discrete topology.
(2) For any V we put on

$$\mathrm{D}V \cong \lim_{\substack{\longleftarrow \\ V' \in \mathsf{Fin}\, V}} \mathrm{D}V'$$

the inverse limit topology.

Lemma 2.3. *The assignment $V \mapsto \mathrm{D}V$ gives rise to a contravariant functor*

$$\mathrm{D} : \mathsf{Mod}\, \Bbbk \to \mathsf{TopMod}\, \Bbbk .$$

Proof. We must show that for any $\Bbbk$-linear homomorphism $\phi : V_1 \to V_2$ there is an induced continuous homomorphism $\mathrm{D}(\phi) : \mathrm{D}V_2 \to \mathrm{D}V_1$. Now let $V_1' \in \mathsf{Fin}\, V_1$. Then $V_2' := \phi(V_1') \in \mathsf{Fin}\, V_2$, and we get a continuous homomorphism

$$\mathrm{D}(\phi|_{V_1'}) : \mathrm{D}V_2 \to \mathrm{D}V_2' \to \mathrm{D}V_1' .$$

Passing to the inverse limit in V_1' we get continuous homomorphism $\mathrm{D}(\phi) : \mathrm{D}V_2 \to \mathrm{D}V_1$. □

Proposition 2.4. *Let $V \in \mathsf{Mod}\, \Bbbk$.*

(1) *The topological $\Bbbk$-module $\mathrm{D}V$ is profinite.*
(2) *The topology on $\Bbbk$-module $\mathrm{D}V$ coincides with the weak* topology.*

Proof. (1) See Proposition 1.8.

(2) Recall that in the weak* topology on $W := \mathrm{Hom}_{\Bbbk}(V, \Bbbk) = \mathrm{D}V$ a fundamental system of neighborhoods of 0 is the set

$$\{ W(v_1, \ldots, v_n) \mid n \geq 0,\ v_1, \ldots, v_n \in V \},$$

where

$$W(v_1, \ldots, v_n) := \{ \phi \in W \mid \phi(v_1) = \cdots = \phi(v_n) = 0 \}.$$

On the other hand a fundamental system of neighborhoods of 0 in the topology of Definition 2.2 consists of the set

$$\{\mathrm{Ker}(W \to \mathrm{D}V') \mid V' \in \mathsf{Fin}\, W\}.$$

Now letting $V' := \sum_{i=1}^{n} \Bbbk v_i \subset V$ we have

$$W(v_1, \ldots, v_n) = \mathrm{Ker}(W \to \mathrm{D}V').$$

Since any finitely generated submodule V' arises in this way, the two topologies coincide. □

Take an object $W \in \mathsf{TopMod}\,\Bbbk$. Denote by

$$\mathrm{D}^{\mathrm{c}} W := \mathrm{Hom}_{\Bbbk}^{\mathrm{cont}}(W, \Bbbk) = \mathrm{Hom}_{\mathsf{TopMod}\,\Bbbk}(W, \Bbbk)$$

the continuous dual, which we consider as a $\Bbbk$-module (without any topology). Thus we get a contravariant functor

$$\mathrm{D}^{\mathrm{c}} : \mathsf{TopMod}\,\Bbbk \to \mathsf{Mod}\,\Bbbk.$$

Given $V \in \mathsf{Mod}\,\Bbbk$ there is an adjunction (or evaluation) homomorphism $\alpha_V : V \to \mathrm{D}^{\mathrm{c}}\mathrm{D}V$ whose formula is $\alpha_V(v)(\phi) := \phi(v)$. Likewise given $W \in \mathsf{TopMod}\,\Bbbk$ there is an adjunction homomorphism $\beta_W : W \to \mathrm{D}\mathrm{D}^{\mathrm{c}} W$ whose formula is $\beta_W(w)(\psi) := \psi(w)$.

Observe that a finitely generated $\Bbbk$-module V is an object of $\mathsf{Mod}\,\Bbbk$, and at the same time, when endowed with the discrete topology, V is an object of $\mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. Such V is clearly reflexive, in the sense that $\alpha_V : V \to \mathrm{D}^{\mathrm{c}}\mathrm{D}V$ is an isomorphism.

It turns out that *all* $\Bbbk$*-modules are reflexive*:

Theorem 2.5. (1) *Let* $V \in \mathsf{Mod}\,\Bbbk$. *Then the adjunction homomorphism* $\alpha_V : V \to \mathrm{D}^{\mathrm{c}}\mathrm{D}V$ *is an isomorphism in* $\mathsf{Mod}\,\Bbbk$.
(2) *Let* $W \in \mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. *Then the adjunction homomorphism* $\beta_W : W \to \mathrm{D}\mathrm{D}^{\mathrm{c}} W$ *is an isomorphism in* $\mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$.
(3) *The functor* $\mathrm{D} : \mathsf{Mod}\,\Bbbk \to \mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$ *is a duality* (*i.e. a contravariant equivalence*), *with adjoint* D^{c}.

Proof. (1) Define $I := \mathsf{Fin}\, V$, and rewrite $\mathsf{Fin}\, V = \{V_i'\}_{i \in I}$. Also let $W := \mathrm{D}V$. So $V \cong \varinjlim_{i \in I} V_i'$ and $W = \varprojlim_{i \in I} \mathrm{D}V_i'$. For any index i let $W_i' := \mathrm{Ker}(W \to \mathrm{D}V_i')$, so $\mathrm{D}V_i' \cong W/W_i'$. The inverse system $\{W_i'\}_{i \in I}$ is cofinal in $(\mathsf{Cofin}\, W)^{\mathrm{op}}$, hence by Lemma 1.5 we have

$$\mathrm{D}^{\mathrm{c}} W \cong \varinjlim_{W' \in \mathsf{Cofin}\, W} \mathrm{D}^{\mathrm{c}}(W/W') \cong \varinjlim_{i \in I} \mathrm{D}^{\mathrm{c}}(W/W_i') \cong \varinjlim_{i \in I} \mathrm{D}^{\mathrm{c}}\mathrm{D}V_i' \cong \varinjlim_{i \in I} V_i' \cong V.$$

Denote by $\psi : \mathrm{D}^{\mathrm{c}} W \xrightarrow{\simeq} V$ the composition of this chain of isomorphisms, going from left to right. Then $\psi \circ \alpha_V = \mathbf{1}_V : V \to V$, and therefore α_V is an isomorphism.

(2) Define $I := \mathsf{Cofin}\, W$, and rewrite $\mathsf{Cofin}\, W = \{W_i'\}_{i \in I}$. Let $V := \mathrm{D}^{\mathrm{c}} W$, and for any index i let $V_i' := \mathrm{D}^{\mathrm{c}}(W/W_i')$. From Lemma 1.5 we know that $V \cong \varinjlim_{i \in I} V_i'$ in $\mathsf{Mod}\,\Bbbk$. It follows that the direct system $\{V_i'\}_{i \in I}$ is cofinal in $\mathsf{Fin}\, V$. Also $\mathrm{D}V_i' \cong W/W_i'$. Hence

$$\mathrm{D}\mathrm{D}^{\mathrm{c}} W = \mathrm{D}V = \varprojlim_{V' \in \mathsf{Fin}\, V} \mathrm{D}V' \cong \varprojlim_{i \in I} \mathrm{D}V_i' \cong \varprojlim_{i \in I} W/W_i' \cong W$$

in $\mathsf{TopMod}\,\Bbbk$. Denote by $\phi : \mathrm{D}\mathrm{D}^{\mathrm{c}} W \xrightarrow{\simeq} W$ the composition of this chain of isomorphisms, going from left to right. Then $\phi \circ \beta_W = \mathbf{1}_W : W \to W$, and therefore β_W is an isomorphism.

(3) This follows from parts (1) and (2). □

In particular the theorem tells us that:

Corollary 2.6. *For any $V_1, V_2 \in \mathsf{Mod}\,\Bbbk$ there is a canonical $\Bbbk$-linear isomorphism*

$$\mathrm{Hom}_{\Bbbk}(V_1, V_1) \xrightarrow{\simeq} \mathrm{Hom}^{\mathrm{cont}}_{\Bbbk}(\mathrm{D}V_2, \mathrm{D}V_1),\ \phi \mapsto \mathrm{D}(\phi).$$

For the sake of convenience, and when no confusion may arise, we shall write $V^* := \mathrm{D}V$ for $V \in \mathsf{Mod}\,\Bbbk$, and also $W^* := \mathrm{D}^{\mathrm{c}} W$ for $W \in \mathsf{TopMod}\,\Bbbk$.

Corollary 2.7. *Let $\phi : W_1 \to W_2$ be a morphism in $\mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. Then:*

1. *The morphism ϕ is strict.*
2. *The $\Bbbk$-modules $\mathrm{Ker}(\phi)$, $\mathrm{Im}(\phi)$ and $\mathrm{Coker}(\phi)$, with their induced topologies, are profinite.*
3. *The module $\mathrm{Im}(\phi)$ is closed in W_2.*

Proof. (1) Applying D^{c} we get $\phi^* : W_2^* \to W_1^*$ in $\mathsf{Mod}\,\Bbbk$. Define $V_{2,2} := \mathrm{Ker}(\phi^*)$, and choose a complement $W_2^* = V_{2,1} \oplus V_{2,2}$. Similarly let $V_{1,1} := \mathrm{Im}(\phi^*)$, and choose a complement $W_1^* = V_{1,1} \oplus V_{1,2}$. Thus ϕ^* has a matrix representation

$$\phi^* : W_2^* = V_{2,1} \oplus V_{2,2} \xrightarrow{\cdot \left[\begin{smallmatrix} \psi_{1,1} & 0 \\ 0 & 0 \end{smallmatrix}\right]} V_{1,1} \oplus V_{1,2} = W_1^*$$

where $\psi_{1,1}$ is an isomorphism. Dualizing back we get a decomposition

$$\phi : W_1 = V_{1,1}^* \oplus V_{1,2}^* \xrightarrow{\left[\begin{smallmatrix} \psi_{1,1}^* & 0 \\ 0 & 0 \end{smallmatrix}\right] \cdot} V_{2,1}^* \oplus V_{2,2}^* = W_2$$

in $\mathsf{TopMod}\,\Bbbk$. Now $W_1 \to V_{1,1}^*$ is a strict epimorphism, and $V_{1,1}^* \to W_2$ is a strict monomorphism.

(2) By the proof of part (1) we have isomorphisms $\mathrm{Ker}(\phi) \cong V_{1,2}^*$, $\mathrm{Im}(\phi) \cong V_{1,1}^* \cong V_{2,1}^*$ and $\mathrm{Coker}(\phi) \cong V_{2,2}^*$ in $\mathsf{TopMod}\,\Bbbk$.

(3) This is because $\mathrm{Im}(\phi) = \mathrm{Ker}(W_2 \to V_{2,2}^*)$, and $V_{2,2}^*$ is separated. □

Corollary 2.8. *Let*

$$S = \left(0 \to W_0 \xrightarrow{\phi_0} W_1 \xrightarrow{\phi_1} W_2 \to 0\right)$$

be a sequence of morphisms in $\mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. The following are equivalent:

- (i) *The sequence S is exact (neglecting topologies).*
- (ii) *The sequence S is strict-exact in $\mathsf{TopMod}\,\Bbbk$.*
- (iii) *The sequence S is split-exact in $\mathsf{TopMod}\,\Bbbk$.*
- (iv) *The dual sequence*

$$S^* := \left(0 \to W_2^* \xrightarrow{\phi_1^*} W_1^* \xrightarrow{\phi_0^*} W_0^* \to 0\right)$$

is exact in $\mathsf{Mod}\,\Bbbk$.

Condition (iii) says that there exists a continuous $\Bbbk$-linear homomorphism $\sigma : W_2 \to W_1$ such that $\phi_1 \circ \sigma = \mathbf{1}_{W_2}$; i.e. $W_1 \cong W_0 \oplus W_2$ in $\mathsf{TopMod}\,\Bbbk$.

Proof. Any exact sequence in $\mathsf{Mod}\,\Bbbk$ splits; and by duality we deduce (iv) $\Rightarrow$ (iii). The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. It remains to verify (i) $\Rightarrow$ (iv).

Suppose $\psi : W \to W_1$ is some morphism in $\mathsf{TopMod}\,\Bbbk$ such that $\phi_1 \circ \psi = 0$. Because ϕ_0 is a strict monomorphism it follows that ψ factors through ϕ_0. So ϕ_0 is the kernel of ϕ_1 in $\mathsf{TopMod}\,\Bbbk$ (in the categorical sense, see [HS, Sections II.6 and II.9]).

Next suppose $\psi : W_1 \to W$ is some morphism in $\mathsf{TopMod}\,\Bbbk$ such that $\psi \circ \phi_0 = 0$. Because ϕ_1 is a strict epimorphism it follows that ψ factors through ϕ_1. So ϕ_1 is the cokernel of ϕ_0 in $\mathsf{TopMod}\,\Bbbk$

Since the categories $\mathsf{TopMod}\,\Bbbk$ and $\mathsf{Mod}\,\Bbbk$ are dual it follows that $\phi_1^* : W_2^* \to W_1^*$ is the kernel of $\phi_0^* : W_1^* \to W_0^*$, and vice versa. Therefore the sequence S^* is exact. □

The classical notion of *linearly compact* topological $\Bbbk$-module is due to Lefchetz. See [Ko, Sections 10.10 - 10.11].

Corollary 2.9. *Let W be a topological $\Bbbk$-module. W is profinite if and only if it is linearly compact.*

Proof. Say $W \in \mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. Let $V := \mathrm{D}^{\mathrm{c}} W \in \mathsf{Mod}\,\Bbbk$. By Theorem 2.5(2) we know that $W \cong \mathrm{D}V$ in $\mathsf{TopMod}\,\Bbbk$. According to Proposition 2.4(2), the topology on $\mathrm{D}V$ is the weak* topology. Hence by [Ko, Section 10.10 item (1)] it follows that $\mathrm{D}V$ is linearly compact.

Conversely, suppose W is linearly compact. Let V be as above. By [Ko, Section 10.10 item (3)] we get $W \cong \mathrm{D}V$ in $\mathsf{TopMod}\,\Bbbk$, and by Proposition 2.4(1) we see that $\mathrm{D}V$ is profinite. □

Given a set X let

$$\Bbbk^{(X)} := \{\phi : X \to \Bbbk \mid \phi \text{ has finite support}\}.$$

This is a free $\Bbbk$-module with basis $\{\delta_x\}_{x \in X}$, where $\delta_x : X \to \Bbbk$ is the "delta function" defined by

$$\delta_x(y) := \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise.} \end{cases}$$

If V is a $\Bbbk$-module with basis X then we get an isomorphism $\Bbbk^{(X)} \xrightarrow{\simeq} V$ by sending $\delta_x \mapsto x$.

As usual $\Bbbk^X$ denotes the set of all functions $\phi : X \to \Bbbk$. We give it the product topology using the isomorphism $\Bbbk^X \cong \prod_{x \in X} \Bbbk$, where each copy of $\Bbbk$ has the discrete topology.

There are evaluation homomorphisms $\alpha_X : \Bbbk^{(X)} \to \mathrm{D}^{\mathrm{c}}\,\Bbbk^X$ and $\beta_X : \Bbbk^X \to \mathrm{D}\,\Bbbk^{(X)}$ with formulas

$$\beta_X(\phi)(\psi) = \alpha_X(\psi)(\phi) := \sum_{x \in X} \phi(x)\psi(x) \in \Bbbk$$

for $\phi \in \Bbbk^X$ and $\psi \in \Bbbk^{(X)}$.

Suppose X and Y are sets and $f : X \to Y$ is a function. Define a continuous $\Bbbk$-linear homomorphism $F^*(f) : \Bbbk^Y \to \Bbbk^X$ by

$$F^*(f)(\psi)(x) := \psi(f(x))$$

for $\psi \in \Bbbk^Y$ and $x \in X$. Thus $F^*(f)(\psi)$ is the pullback of ψ. This is a functor $F^* : \mathsf{Sets}^{\mathrm{op}} \to \mathsf{TopMod}\,\Bbbk$. Also define a $\Bbbk$-linear homomorphism $F(f) : \Bbbk^{(X)} \to \Bbbk^{(Y)}$ by

$$F(f)(\phi)(y) := \sum_{x \in f^{-1}(y)} \phi(x)$$

for $y \in Y$ and $\phi \in \Bbbk^{(X)}$. The function $F(f)(\phi)$ is the trace of ϕ, or "integration on the fibers of f". We get a functor $F : \mathsf{Sets} \to \mathsf{Mod}\,\Bbbk$.

Lemma 2.10. *Let $f : X \to Y$ be any function of sets.*

(1) *The diagram*

$$\begin{array}{ccc} \Bbbk^Y & \xrightarrow{F^*(f)} & \Bbbk^X \\ {\scriptstyle \beta_Y}\big\downarrow & & {\scriptstyle \beta_X}\big\downarrow \\ \mathrm{D}\,\Bbbk^{(Y)} & \xrightarrow{\mathrm{D}(F(f))} & \mathrm{D}\,\Bbbk^{(X)} \end{array}$$

in $\mathsf{TopMod}\,\Bbbk$ *is commutative.*

(2) *The diagram*

$$\begin{array}{ccc} \Bbbk^{(X)} & \xrightarrow{F(f)} & \Bbbk^{(Y)} \\ {\scriptstyle \alpha_X}\big\downarrow & & {\scriptstyle \alpha_Y}\big\downarrow \\ \mathrm{D}^{\mathrm{c}}\,\Bbbk^X & \xrightarrow{\mathrm{D}^{\mathrm{c}}(F^*(f))} & \mathrm{D}^{\mathrm{c}}\,\Bbbk^Y \end{array}$$

in $\mathsf{Mod}\,\Bbbk$ *is commutative.*

Proof. Direct calculation. $\square$

Proposition 2.11. *Let X be any set.*

(1) $\alpha_X : \Bbbk^{(X)} \to \mathrm{D}^{\mathrm{c}}\,\Bbbk^X$ *is an isomorphism in* $\mathsf{Mod}\,\Bbbk$.
(2) $\beta_X : \Bbbk^X \to \mathrm{D}\,\Bbbk^{(X)}$ *is an isomorphism in* $\mathsf{TopMod}\,\Bbbk$.

Proof. Denote by $\mathsf{Fin}\,X$ the set of finite subsets of X, which we rename I, and write $\mathsf{Fin}\,X = \{X_i\}_{i \in I}$. I is quasi-ordered by inclusion, and it is a directed set. The functor $F : \mathsf{Sets} \to \mathsf{Mod}\,\Bbbk$, $FY = \Bbbk^{(Y)}$, makes $\{\Bbbk^{(X_i)}\}_{i \in I}$ into a direct system, and $\Bbbk^{(X)} \cong \varinjlim_{i \in I} \Bbbk^{(X_i)}$ in $\mathsf{Mod}\,\Bbbk$. Moreover the set of finitely generated submodules $\{\Bbbk^{(X_i)}\}_{i \in I}$ is cofinal in $\mathsf{Fin}\,\Bbbk^{(X)}$.

The opposite quasi-ordered set I^{op} is also directed. The functor $F^* : \mathsf{Sets}^{\mathrm{op}} \to \mathsf{TopMod}\,\Bbbk$, $F^*Y = \Bbbk^Y$, makes $\{\Bbbk^{X_i}\}_{i \in I}$ into an inverse system, and $\Bbbk^X \cong \varprojlim_{i \in I} \Bbbk^{X_i}$ in $\mathsf{TopMod}\,\Bbbk$. The inverse system of open cofinite submodules $\{\mathrm{Ker}(\Bbbk^X \to \Bbbk^{X_i})\}_{i \in I^{\mathrm{op}}}$ is cofinal in $(\mathsf{Cofin}\,\Bbbk^X)^{\mathrm{op}}$.

By Proposition 1.8, $\Bbbk^X$ is a profinite topological $\Bbbk$-module; hence by Lemma 1.5 we have isomorphisms

$$\mathrm{D}^{\mathrm{c}}\,\Bbbk^X = \mathrm{Hom}^{\mathrm{cont}}_{\Bbbk}(\Bbbk^X, \Bbbk) \cong \varinjlim_{W' \in \mathsf{Cofin}\,\Bbbk^X} \mathrm{D}^{\mathrm{c}}\,(\Bbbk^X / W') \cong \varinjlim_{i \in I} \mathrm{D}^{\mathrm{c}}\,\Bbbk^{X_i}.$$

Now for the finite set X_i one has $\Bbbk^{(X_i)} = \Bbbk^{X_i}$, and the adjunction map $\alpha_{X_i} : \Bbbk^{X_i} \to \mathrm{D}^{\mathrm{c}}\,\Bbbk^{X_i}$ is bijective. So by Lemma 2.10(2) we can switch the limits from the functor F^* to the functor F:

$$\varinjlim_{i \in I} \mathrm{D}^{\mathrm{c}}\,\Bbbk^{X_i} \cong \varinjlim_{i \in I} \Bbbk^{(X_i)} \cong \Bbbk^{(X)}.$$

We get an isomorphism $\mathrm{D}^{\mathrm{c}}\,\Bbbk^X \cong \Bbbk^{(X)}$ that is compatible with the evaluation pairing, so α_X is an isomorphism.

Finally we look at β_X. We know that $\{\Bbbk^{(X_i)}\}_{i\in I}$ is cofinal in $\mathsf{Fin}\,\Bbbk^{(X)}$, so

$$\mathrm{D}\,\Bbbk^{(X)} = \varprojlim_{V' \in \mathsf{Fin}\,\Bbbk^{(X)}} \mathrm{D}\,V' \cong \varprojlim_{i\in I} \mathrm{D}\,\Bbbk^{(X_i)}$$

in $\mathsf{TopMod}\,\Bbbk$. But using the isomorphisms $\beta_{X_i} : \Bbbk^{X_i} \xrightarrow{\simeq} \mathrm{D}\,\Bbbk^{(X_i)}$ and Lemma 2.10(1) we can switch limits to obtain an isomorphism

$$\varprojlim_{i\in I} \mathrm{D}\,\Bbbk^{(X_i)} \cong \varprojlim_{i\in I} \Bbbk^{X_i} \cong \Bbbk^X$$

in $\mathsf{TopMod}\,\Bbbk$. These isomorphisms are compatible with evaluation. Therefore β_X is an isomorphism. □

Corollary 2.12. *Let $W \in \mathsf{TopMod}_{\mathrm{pf}}\,\Bbbk$. Then $W \cong \Bbbk^X$ in $\mathsf{TopMod}\,\Bbbk$ for some set X.*

Proof. Choose a basis X for $\mathrm{D}^{\mathrm{c}}W$. Then $\mathrm{D}^{\mathrm{c}}W \cong \Bbbk^{(X)}$ in $\mathsf{Mod}\,\Bbbk$, and according to Proposition 2.11 and Theorem 2.5 we get

$$W \cong \mathrm{D}\mathrm{D}^{\mathrm{c}}W \cong \mathrm{D}\,\Bbbk^{(X)} \cong \Bbbk^X.$$

□

Remark 2.13. Not all cofinite submodules of a profinite topological $\Bbbk$-module are open. Take $\Bbbk := \mathbb{Q}$ and $W := \Bbbk^{\mathbb{N}}$. Since the open cofinite submodules of W are parameterized by the submodules of $\Bbbk^{\{0,\ldots,n\}}$ for $n \in \mathbb{N}$ it follows that $|\mathsf{Cofin}\,W| = \aleph_0$. But on the other hand $\mathrm{rank}_{\Bbbk}\,W = |W| = 2^{\aleph_0}$, so there are at least this many cofinite submodules $W' \subset W$.

Remark 2.14. It is curious to note that some $\Bbbk$-modules W do not admit any topology with which they are profinite. Indeed, when $\Bbbk = \mathbb{Q}$ the module $W := \Bbbk^{(\mathbb{N})}$ is such an example. Its rank is $\aleph_0$. Now for a $\Bbbk$-module V one has either $\mathrm{rank}_{\Bbbk}\,V < \aleph_0$, and then $\mathrm{rank}_{\Bbbk}\,\mathrm{D}V = \mathrm{rank}_{\Bbbk}\,V < \aleph_0$; or $\mathrm{rank}_{\Bbbk}\,V \geq \aleph_0$, in which case $\mathrm{rank}_{\Bbbk}\,\mathrm{D}V \geq 2^{\aleph_0}$. So $W \cong \mathrm{D}V$ is impossible for any V.

Remark 2.15. The duality here is really the Lefchetz duality for linearly compact topological vector spaces. See [Ko, Section 10.10]. The way it is presented here lends itself easily to generalizations. Indeed, all definitions and results in this section, up to and including Corollary 2.6, are valid for any artinian commutative local ring $\Bbbk$. The sole modification needed is that the dualities should be $\mathrm{D} := \mathrm{Hom}_{\Bbbk}(-, J)$ and $\mathrm{D}^{\mathrm{c}} := \mathrm{Hom}^{\mathrm{cont}}_{\Bbbk}(-, J)$, where J is some fixed injective hull of the residue field of $\Bbbk$. Likewise all results on function modules (except Corollary 2.12) hold, once we replace $\Bbbk^{(X)}$ with $J^{(X)}$ everywhere.

In even greater generality we arrive at the Gabriel-Matlis theory of pseudo-compact modules. Another variant is the duality theory of Beilinson completion algebras in [Ye2].

3. Topological A-Modules

Now let A be an associative unital algebra over the field $\Bbbk$. We do not assume A is commutative nor (left or right) noetherian. All A-modules are by default left modules. We have the category $\mathsf{TopMod}\,A$ consisting of topological A-modules (any sort of topology) and continuous A-linear homomorphisms. Like $\mathsf{TopMod}\,\Bbbk$, the category

$\mathsf{TopMod}\, A$ is additive, and it has exact sequences (in which the homomorphisms are required to be strict). There are forgetful functors $\mathsf{TopMod}\, A \to \mathsf{TopMod}\, \Bbbk$ and $\mathsf{TopMod}\, A \to \mathsf{Mod}\, A$.

Definition 3.1. A topological A-module M is called *profinite over* $\Bbbk$ if it is profinite when considered as topological $\Bbbk$-module. We denote by $\mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, A$ the full subcategory of $\mathsf{TopMod}\, A$ consisting of topological A-modules profinite over $\Bbbk$.

We denote by A^{op} the opposite algebra, i.e. the same $\Bbbk$-module but with reversed multiplication. A right A-module is then the same as a left A^{op}-module. When A is commutative of course $A^{\mathrm{op}} = A$.

Theorem 3.2. *Let A be a $\Bbbk$-algebra. Given $M \in \mathsf{Mod}\, A$ the dual $M^* = \mathrm{D} M$ is a topological A^{op}-module profinite over $\Bbbk$. The functor*

$$\mathrm{D} : \mathsf{Mod}\, A \to \mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, A^{\mathrm{op}}$$

is an equivalence, with adjoint D^{c}.

Proof. The A^{op}-module structure on $\mathrm{D}M$ is defined as follows. Given an element $a \in A$ the function $\phi_a : M \to M$, $\phi_a(m) := am$, is $\Bbbk$-linear, and so we get a continuous $\Bbbk$-linear homomorphism $\mathrm{D}(\phi_a) : \mathrm{D}M \to \mathrm{D}M$. Because D is a contravariant functor we have

$$\mathrm{D}(\phi_{a_1 a_2}) = \mathrm{D}(\phi_{a_1} \circ \phi_{a_2}) = \mathrm{D}(\phi_{a_2}) \circ \mathrm{D}(\phi_{a_1})$$

and

$$\mathrm{D}(\phi_1) = \mathrm{D}(\mathbf{1}_M) = \mathbf{1}_{\mathrm{D}M}$$

for the element $1 \in A$. Likewise we get an A-module structure on $\mathrm{D}^{\mathrm{c}} N$ for any $N \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, A^{\mathrm{op}}$. One sees easily that the adjunction homomorphism $\beta_N : N \to \mathrm{D}\mathrm{D}^{\mathrm{c}} N$ is A^{op}-linear, and $\alpha_M : M \to \mathrm{D}^{\mathrm{c}}\mathrm{D} M$ is A-linear. According to Theorem 2.5 the homomorphisms β_N and α_M are isomorphisms in the respective categories. □

Corollary 3.3. *Let*

$$S = \Big(\cdots \to N_0 \xrightarrow{\phi_0} N_1 \xrightarrow{\phi_1} N_2 \to \cdots \Big)$$

be a sequence of morphisms in $\mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, A$. The following are equivalent:

(i) *The sequence S is exact (neglecting topologies).*
(ii) *The sequence S is strict-exact.*
(iii) *The dual sequence*

$$S^* := \Big(\cdots \to N_2^* \xrightarrow{\phi_1^*} N_1^* \xrightarrow{\phi_0^*} N_0^* \to \cdots \Big)$$

is exact in $\mathsf{Mod}\, A^{\mathrm{op}}$.

Proof. Combine Theorem 3.2 and Corollary 2.8. □

Corollary 3.4. *Let N be a topological A-module. Then the following are equivalent:*

(i) *N is isomorphic in $\mathsf{TopMod}\, A$ to a closed submodule of $(A^*)^r = (\mathrm{D}A)^r$ for some natural number r.*
(ii) *N is a profinite topological $\Bbbk$-module and $N^* = \mathrm{D}^{\mathrm{c}} N$ is a finitely generated A^{op}-module.*

Proof. (i) $\Rightarrow$ (ii): Let $\psi : N \to (A^*)^r$ be a continuous A-linear homomorphism such that $\mathrm{Im}(\psi)$ is a closed submodule of $(A^*)^r$ and $\psi : N \to \mathrm{Im}(\psi)$ is an isomorphism. By Theorem 1.9 $\mathrm{Im}(\psi)$ is a profinite topological $\Bbbk$-module, and hence so is N. The homomorphism $\psi : N \to (A^*)^r$ is a monomorphism in $\mathsf{TopMod}_{\mathrm{pf}/\Bbbk} A$, so according to Corollary 3.3 the dual $\psi^* : A^r \to N^*$ is surjective. Thus N^* is a finitely generated A^{op}-module.

(ii) $\Rightarrow$ (i): Let $M := N^*$. There is an epimorphism $\phi : A^r \to M$ for some r. Dualizing, and using Corollary 3.3, we get a monomorphism $\phi^* : M^* \to (A^*)^r$. Because N is profinite over $\Bbbk$ there is an isomorphism $N \cong M^*$ in $\mathsf{TopMod}_{\mathrm{pf}/\Bbbk} A$. $\square$

Let A be any $\Bbbk$-algebra. Then $A^* = \mathrm{D}A$ is a topological A-bimodule, i.e. a topological module over $A \otimes_\Bbbk A^{\mathrm{op}}$. For a natural number r we consider elements of $(A^*)^r$ as rows of size r. Thus given $r_0, r_1 \in \mathbb{N}$ and a matrix $G \in \mathrm{M}_{r_0 \times r_1}(A)$, right multiplication by G is a continuous A-linear homomorphism $(A^*)^{r_0} \xrightarrow{\cdot G} (A^*)^{r_1}$.

Corollary 3.5. *Suppose A is a noetherian $\Bbbk$-algebra, $M \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk} A$, $r_0 \in \mathbb{N}$, and $\psi : M \to (A^*)^{r_0}$ is an injective continuous A-linear homomorphism. Then there exists some $r_1 \in \mathbb{N}$, a matrix $G \in \mathrm{M}_{r_0 \times r_1}(A)$, and an exact sequence*

$$0 \to M \xrightarrow{\psi} (A^*)^{r_0} \xrightarrow{\cdot G} (A^*)^{r_1} \tag{3.6}$$

in $\mathsf{TopMod}\, A$.

Proof. By Corollary 3.3 we get an epimorphism $\psi^* : A^{r_0} \to M^*$ in $\mathsf{Mod}\, A^{\mathrm{op}}$. Because A^{op} is noetherian the kernel of ψ^* is a finitely generated A^{op}-module, and therefore there is an exact sequence

$$A^{r_1} \to A^{r_0} \xrightarrow{\psi^*} M^* \to 0 \tag{3.7}$$

in $\mathsf{Mod}\, A^{\mathrm{op}}$. If we think of A^{r_i} as column vectors then the homomorphism $A^{r_1} \to A^{r_0}$ is given by left multiplication with some matrix $G \in \mathrm{M}_{r_0 \times r_1}(A)$. To finish we apply the duality functor D to the sequence (3.7). $\square$

Definition 3.8. Let $M \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk} A$. An exact sequence

$$0 \to M \to (A^*)^{r_0} \xrightarrow{\cdot G} (A^*)^{r_1}$$

in $\mathsf{TopMod}\, A$ (cf. Corollary 3.3), for some $r_0, r_1 \in \mathbb{N}$ and some matrix $G \in \mathrm{M}_{r_0 \times r_1}(A)$, is called a *kernel representation* of M.

Corollary 3.9. *Let A be a commutative $\Bbbk$-algebra, let $M, N \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk} A$, and suppose we are given kernel representations*

$$0 \to M \to (A^*)^{r_0} \xrightarrow{\cdot G} (A^*)^{r_1}$$

and

$$0 \to N \to (A^*)^{s_0} \xrightarrow{\cdot F} (A^*)^{s_1},$$

where $G \in \mathrm{M}_{r_0 \times r_1}(A)$ and $F \in \mathrm{M}_{s_0 \times s_1}(A)$. Let $\phi : M \to N$ be a continuous A-linear homomorphism. Then there exist matrices H_0 and H_1 of appropriate sizes with entries in A, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\phi"'] & (A^*)^{r_0} \arrow[r, "\cdot G"] \arrow[d, "\cdot H_0"] & (A^*)^{r_1} \arrow[d, "\cdot H_1"] \\
0 \arrow[r] & N \arrow[r] & (A^*)^{s_0} \arrow[r, "\cdot F"] & (A^*)^{s_1}
\end{tikzcd}$$

is commutative.

Proof. By duality we obtain exact sequences

$$A^{r_1} \xrightarrow{G\cdot} A^{r_0} \to M^* \to 0$$

and

$$A^{s_1} \xrightarrow{F\cdot} A^{s_0} \to N^* \to 0$$

in $\mathsf{Mod}\, A$. Because the modules A^{r_i} and A^{s_i} are free the homomorphism $\phi^* : N^* \to M^*$ extends to a commutative diagram

$$\begin{tikzcd} A^{s_1} \arrow[r, "F\cdot"] \arrow[d] & A^{s_0} \arrow[r] \arrow[d] & N^* \arrow[r] \arrow[d, "\phi^*"'] & 0 \\ A^{r_1} \arrow[r, "G\cdot"] & A^{r_0} \arrow[r] & M^* \arrow[r] & 0 \end{tikzcd}$$

with exact rows (see [Ro, Theorem 6.9]). The homomorphism $A^{s_i} \to A^{r_i}$ is left multiplication by some matrix $H_i \in \mathrm{M}_{r_i \times s_i}(A)$. Now apply the functor D to this diagram. □

4. Behaviors

Let n be a positive integer. For any $1 \leq j \leq n$ the shift $s_j : \mathbb{N}^n \to \mathbb{N}^n$ is defined by

$$s_j(i_1, \ldots, i_n) := (i_1, \ldots, i_j + 1, \ldots, i_n).$$

Recall that $\Bbbk$ is a field and $\Bbbk^{\mathbb{N}^n} = \{\phi : \mathbb{N}^n \to \Bbbk\}$.

Definition 4.1. The *n-dimensional shift module* is the topological $\Bbbk$-module $\Bbbk^{\mathbb{N}^n} = \prod_{\boldsymbol{i} \in \mathbb{N}^n} \Bbbk$, with the product topology. The shift operators $\sigma_1, \ldots, \sigma_n : \Bbbk^{\mathbb{N}^n} \to \Bbbk^{\mathbb{N}^n}$ are defined as follows: $\sigma_j(\phi) := \phi \circ s_j$ for $\phi \in \Bbbk^{\mathbb{N}^n}$.

Definition 4.2. Let n be a positive integer. A *n-dimensional behavior* is a closed $\Bbbk$-submodule $M \subset (\Bbbk^{\mathbb{N}^n})^r$ for some nonnegative integer r that is invariant under the shift operators $\sigma_1, \ldots, \sigma_n$.

Let $\Bbbk[\boldsymbol{z}] = \Bbbk[z_1, \ldots, z_n]$ be the polynomial algebra in n indeterminates. Since the shift operators commute, the shift module $\Bbbk^{\mathbb{N}^n}$ is a topological $\Bbbk[\boldsymbol{z}]$-module, where z_j acts by the operator σ_j. So an n-dimensional behavior is precisely a closed $\Bbbk[\boldsymbol{z}]$-submodule of $(\Bbbk^{\mathbb{N}^n})^r$.

Proposition 4.3. *One has* $\Bbbk^{\mathbb{N}^n} \cong \mathrm{D}\, \Bbbk[\boldsymbol{z}] = \Bbbk[\boldsymbol{z}]^*$ *in* $\mathsf{TopMod}\, \Bbbk[\boldsymbol{z}]$. *Thus in particular* $\Bbbk^{\mathbb{N}^n} \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, \Bbbk[\boldsymbol{z}]$.

Proof. The set of monomials $\{\boldsymbol{z}^{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{N}^n} \cong \mathbb{N}^n$ is a basis of the $\Bbbk$-module $\Bbbk[\boldsymbol{z}]$, and checking the shift action we see that $\Bbbk[\boldsymbol{z}] \cong \Bbbk^{(\mathbb{N}^n)}$ as $\Bbbk[\boldsymbol{z}]$-modules. Now use Proposition 2.11(2). □

Let $\Bbbk[[x]]$ be the ring of formal power series and $\Bbbk((x))$ the field of Laurent series, with their usual topologies (cf. [Ye1, Section 1.3]). Let us write $x := z^{-1}$. Since $\Bbbk[z] \subset \Bbbk((x)) = \Bbbk((z^{-1}))$ is a subring we get an exact sequence of $\Bbbk[z]$-modules

$$0 \to \Bbbk[z] \to \Bbbk((z^{-1})) \to z^{-1}\, \Bbbk[[z^{-1}]] \to 0. \tag{4.4}$$

Put the discrete topology on $\Bbbk[z]$ and the product topology on $z^{-1}\, \Bbbk[[z^{-1}]] \cong \prod_{i \leq -1} \Bbbk$. Then the sequence (4.4) is split exact in $\mathsf{TopMod}\, \Bbbk$, see [Ye1, Proposition 1.3.5]. (We shall not need this fact.) The next lemma is clear.

Lemma 4.5. *The map* $z^{-1}\,\Bbbk[[z^{-1}]] \to \Bbbk^{\mathbb{N}}$ *sending*

$$\sum_{i\in\mathbb{N}} \lambda_i z^{-1-i} \mapsto (\lambda_0, \lambda_1, \ldots)$$

is an isomorphism in $\mathsf{TopMod}\,\Bbbk[z]$.

Traditionally the shift module is defined as $z^{-1}\,\Bbbk[[z^{-1}]]$, cf. [Fu2]. In view of Lemma 4.5 a behavior in the sense of [Fu2] is precisely a 1-dimensional behavior in the sense of Definition 4.2.

Remark 4.6. Not all $\Bbbk[z]$-submodules of $\Bbbk^{\mathbb{N}}$ are closed. Take

$$w := (1,0,1,0,0,1,0,0,0,1,\ldots).$$

Then the submodule $N := \Bbbk[z]\,w \subset \Bbbk^{\mathbb{N}}$ is dense in $\Bbbk^{\mathbb{N}}$ but not equal to it.

Here is a classification of behaviors.

Theorem 4.7. *Let* M *be an* n*-dimensional behavior and let* $\psi : M \to (\Bbbk^{\mathbb{N}^n})^{r_0}$ *be an injective continuous* $\Bbbk[\boldsymbol{z}]$*-linear homomorphism.*

(1) *There exist natural numbers* $r_1, \ldots, r_n$ *and matrices* $G_i(\boldsymbol{z}) \in \mathrm{M}_{r_{i-1}\times r_i}(\Bbbk[\boldsymbol{z}])$ *such that the sequence of homomorphisms*

$$0 \to M \xrightarrow{\psi} (\Bbbk^{\mathbb{N}^n})^{r_0} \xrightarrow{\cdot G_1(\boldsymbol{z})} (\Bbbk^{\mathbb{N}^n})^{r_1} \xrightarrow{\cdot G_2(\boldsymbol{z})} \cdots \xrightarrow{\cdot G_n(\boldsymbol{z})} (\Bbbk^{\mathbb{N}^n})^{r_n} \to 0$$

is exact, and in fact splits in $\mathsf{TopMod}\,\Bbbk$.

(2) *If* $n = 1$ *there is an isomorphism*

$$M \cong (\Bbbk^{\mathbb{N}})^r \oplus N$$

of topological $\Bbbk[z]$*-modules, where* $r := r_0 - r_1$ *and* N *is finitely generated as* $\Bbbk$*-module.*

Proof. (1) From Corollary 3.5 we get an epimorphism $\psi^* : \Bbbk[\boldsymbol{z}]^{r_0} \to M^*$ in $\mathsf{Mod}\,\Bbbk[\boldsymbol{z}]$. The Hilbert Syzygy Theorem (cf. [Ro, Corollary 9.3.5]) says we can extend it to an exact sequence

$$0 \to \Bbbk[\boldsymbol{z}]^{r_n} \xrightarrow{G_n(\boldsymbol{z})\cdot} \cdots \xrightarrow{G_2(\boldsymbol{z})\cdot} \Bbbk[\boldsymbol{z}]^{r_1} \xrightarrow{G_1(\boldsymbol{z})\cdot} \Bbbk[\boldsymbol{z}]^{r_0} \xrightarrow{\psi^*} M^* \to 0 \tag{4.8}$$

for some r_i and some matrices $G_i(\boldsymbol{z})$. This sequence splits in $\mathsf{Mod}\,\Bbbk$. Now apply the functor D.

(2) In the case $n = 1$, by the theory of finitely generated modules over a PID we know that $M^* \cong \Bbbk[z]^r \oplus L$ for some r and some torsion $\Bbbk[z]$-module L. Moreover tensoring the sequence (4.8) with the field $\Bbbk(z)$ we see that $r = r_0 - r_1$. Applying the functor D to the isomorphism $M^* \cong \Bbbk[z]^r \oplus L$ we get $M \cong (\Bbbk^{\mathbb{N}})^r \oplus L^*$. $\square$

Theorem 4.9. *Let* M *and* N *be two* n*-dimensional behaviors, and let* $\phi : M \to N$ *be a continuous* $\Bbbk[\boldsymbol{z}]$*-linear homomorphism. Suppose we are given kernel representations*

$$0 \to M \to (\Bbbk^{\mathbb{N}^n})^{r_0} \xrightarrow{\cdot G_1(\boldsymbol{z})} (\Bbbk^{\mathbb{N}^n})^{r_1}$$

and

$$0 \to N \to (\Bbbk^{\mathbb{N}^n})^{s_0} \xrightarrow{\cdot F_1(\boldsymbol{z})} (\Bbbk^{\mathbb{N}^n})^{s_1}$$

of these behaviors. Then there exist matrices $H_0(\boldsymbol{z})$ and $H_1(\boldsymbol{z})$ of appropriate sizes with entries in $\Bbbk[\boldsymbol{z}]$ such that the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & (\Bbbk^{\mathbb{N}^n})^{r_0} & \xrightarrow{\cdot G_1(\boldsymbol{z})} & (\Bbbk^{\mathbb{N}^n})^{r_1} \\ & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle \cdot H_0(\boldsymbol{z})} & & \downarrow{\scriptstyle \cdot H_1(\boldsymbol{z})} \\ 0 & \longrightarrow & N & \longrightarrow & (\Bbbk^{\mathbb{N}^n})^{s_0} & \xrightarrow{\cdot F_1(\boldsymbol{z})} & (\Bbbk^{\mathbb{N}^n})^{s_1} \end{array}$$

is commutative.

When $n = 1$ this is [Fu2, Theorem 3.4].

Proof. This is a special case of Corollary 3.9 for the ring $A := \Bbbk[\boldsymbol{z}]$. □

5. Noncontinuous Homomorphisms

In this section A is a finitely generated commutative algebra over the field $\Bbbk$. We analyze the algebraic structure of the A-module A^*, namely we forget the topology. This part requires more difficult ring theory.

As usual we shall denote by $\operatorname{Spec} A$ the set of prime ideals of the ring A, and by $\operatorname{Max} A$ the subset of maximal ideals. For any prime ideal $\mathfrak{p}$ let $J(\mathfrak{p})$ be an injective hull of $A/\mathfrak{p}$ considered as A-module. Cf. [Ro, Section 3] or [St, Section V.4].

Proposition 5.1. *There is a (noncanonical) decomposition of A-modules*

$$A^* \cong \Big(\bigoplus_{\mathfrak{m} \in \operatorname{Max} A} J(\mathfrak{m}) \Big) \oplus N.$$

Here N is some injective A-module that contains no nonzero finite length submodules.

Proof. Since for any A-module M there is a functorial isomorphism

$$\operatorname{Hom}_A(M, A^*) \cong M^*$$

it follows that A^* is an injective A-module. Because A is a noetherian ring we know that

$$A^* \cong \bigoplus_{\mathfrak{p} \in \operatorname{Spec} A} J(\mathfrak{p})^{(\mu_{\mathfrak{p}})},$$

where each $\mu_{\mathfrak{p}}$ is a cardinal number, and $J(\mathfrak{p})^{(\mu_{\mathfrak{p}})}$ is a direct sum of $\mu_{\mathfrak{p}}$ copies of $J(\mathfrak{p})$. See [St, Propositions V.4.5 and V.4.6].

Let $\mathfrak{m}$ be some maximal ideal and let $\boldsymbol{k}(\mathfrak{m}) := A/\mathfrak{m}$, the residue field. Then

$$\mu_{\mathfrak{m}} = \operatorname{rank}_{\boldsymbol{k}(\mathfrak{m})} \operatorname{Hom}_A(\boldsymbol{k}(\mathfrak{m}), A^*).$$

On the other hand

$$\operatorname{Hom}_A(\boldsymbol{k}(\mathfrak{m}), A^*) \cong \boldsymbol{k}(\mathfrak{m})^* \cong \boldsymbol{k}(\mathfrak{m})$$

as $\boldsymbol{k}(\mathfrak{m})$-modules. Hence $\mu_{\mathfrak{m}} = 1$.

Finally $N := \bigoplus_{\mathfrak{p}} J(\mathfrak{p})^{(\mu_{\mathfrak{p}})}$, the sum going over all prime ideals that are not maximal. □

Corollary 5.2. *Let n be any positive integer and $\Bbbk[\boldsymbol{z}] := \Bbbk[z_1, \ldots, z_n]$. There exist $\Bbbk[\boldsymbol{z}]$-linear homomorphisms $\phi : \Bbbk^{\mathbb{N}^n} \to \Bbbk^{\mathbb{N}^n}$ that are not continuous.*

Proof. From Theorem 3.2 and Proposition 4.3 we know that

$$\mathrm{Hom}^{\mathrm{cont}}_{\Bbbk[\boldsymbol{z}]}(\Bbbk^{\mathbb{N}^n}, \Bbbk^{\mathbb{N}^n}) \cong \mathrm{Hom}_{\Bbbk[\boldsymbol{z}]}(\Bbbk[\boldsymbol{z}], \Bbbk[\boldsymbol{z}]) \cong \Bbbk[\boldsymbol{z}].$$

Thus a continuous homomorphism $\phi : \Bbbk^{\mathbb{N}^n} \to \Bbbk^{\mathbb{N}^n}$ is multiplication by some polynomial $f(\boldsymbol{z}) \in \Bbbk[\boldsymbol{z}]$. It follows that ϕ must preserve the decomposition in Proposition 5.1. Moreover, because

$$\mathrm{Hom}_{\Bbbk[\boldsymbol{z}]}(J(\mathfrak{m}), J(\mathfrak{m})) \cong \widehat{\Bbbk[\boldsymbol{z}]}_{\mathfrak{m}},$$

where the latter is the $\mathfrak{m}$-adic completion of $\Bbbk[\boldsymbol{z}]$, we see that $\phi = 0$ iff the restriction $\phi|_{J(\mathfrak{m})} = 0$, where $\mathfrak{m}$ is any maximal ideal.

We now go about constructing our counterexample. Choose a particular maximal ideal $\mathfrak{m}_0$; say $\mathfrak{m}_0 := (z_1, \ldots, z_n)$. Using the decomposition of Proposition 5.1 define

$$\phi|_{J(\mathfrak{m}_0)} : J(\mathfrak{m}_0) \to \Bbbk^{\mathbb{N}^n}$$

to be the inclusion; define

$$\phi|_{J(\mathfrak{m})} : J(\mathfrak{m}) \to \Bbbk^{\mathbb{N}^n}$$

to be the zero homomorphism for any other maximal ideal $\mathfrak{m}$; and define

$$\phi|_N : N \to \Bbbk^{\mathbb{N}^n}$$

also to be the zero homomorphism. This homomorphism cannot be continuous. $\square$

Remark 5.3. When $A := \Bbbk[z]$, i.e. the one dimensional case, the module N appearing in Proposition 5.1 is of the form $N \cong \Bbbk(z)^{(\mu_0)}$. Regarding the cardinality μ_0, we know it if $\Bbbk$ is a countable field. In this case we have

$$\mu_0 = \mathrm{rank}_{\Bbbk(z)} N = |N| = |\Bbbk^{\mathbb{N}}| = 2^{\aleph_0} = \aleph.$$

6. Noncommutative n-Dimensional Behaviors

Fix a positive integer n. Let S be the free monoid (i.e. a semigroup with 1) on the generators $s_1, \ldots, s_n$. So the elements of S are the words $s_{i_1} \cdots s_{i_k}$ with $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in \{1, \ldots, n\}$.

Suppose X is a set equipped with functions $f_1, \ldots, f_n : X \to X$. This data determines a left action of the monoid S on X, by letting $s_i(x) := f_i(x)$ for the generators $s_1, \ldots, s_n$ of S. A set X with a left action of S is called a *left S-set.*

Definition 6.1. A *finitely generated left S-set* is a left S-set X, with a finite subset $X_0 \subset X$, such that $X = \{s(x) \mid s \in S \text{ and } x \in X_0\}$.

Recall that $\Bbbk$ is a field. Given a left S-set X there is a right action of S on $\Bbbk^X$ by continuous $\Bbbk$-linear homomorphisms, which we call *shift operators.*

Definition 6.2. A *noncommutative n-dimensional behavior* is a closed shift invariant $\Bbbk$-submodule of $\Bbbk^X$, for some finitely generated left S-set X.

Let $\Bbbk\langle\boldsymbol{z}\rangle = \Bbbk\langle z_1, \ldots, z_n\rangle$ be the free associative algebra on the variables $z_1, \ldots, z_n$. We can identify the monoid S with the multiplicative monoid of monomials in $\Bbbk\langle\boldsymbol{z}\rangle$. In this way any n-dimensional behavior M becomes a right $\Bbbk\langle\boldsymbol{z}\rangle$-module, i.e. $M \in \mathsf{TopMod}_{\mathrm{pf}/\Bbbk}\, \Bbbk\langle\boldsymbol{z}\rangle^{\mathrm{op}}$.

If r is an infinite cardinal number and M is a $\Bbbk$-module, then $M^{(r)}$ denotes direct sum of r copies of M, whereas M^r is the direct product of r copies of M.

Recall that for $M \in \mathsf{TopMod}\, \Bbbk$ we write $M^* := \mathrm{D}^{\mathrm{c}} M = \mathrm{Hom}^{\mathrm{cont}}_{\Bbbk}(M, \Bbbk)$; whereas for $M \in \mathsf{Mod}\, \Bbbk$ we write $M^* := \mathrm{D} M = \mathrm{Hom}_{\Bbbk}(M, \Bbbk)$.

Theorem 6.3. *Let M be a noncommutative n-dimensional behavior. Then there exist $r_0 \in \mathbb{N}$, $r_1 \in \mathbb{N} \cup \{\aleph_0\}$, $G(\boldsymbol{z}) \in \mathrm{M}_{r_0 \times r_1}(\Bbbk\langle \boldsymbol{z} \rangle)$ and a homomorphism $\phi : M \to (\Bbbk^S)^{r_0}$ in $\mathsf{TopMod}\,\Bbbk\langle \boldsymbol{z} \rangle^{\mathrm{op}}$ such that*

$$0 \to M \xrightarrow{\phi} (\Bbbk^S)^{r_0} \xrightarrow{\cdot G(\boldsymbol{z})} (\Bbbk^S)^{r_1} \to 0 \tag{6.4}$$

is an exact sequence.

Proof. By definition $M \subset \Bbbk^X$ for some finitely generated S-set X. Dualizing we obtain a surjection $\Bbbk^{(X)} \twoheadrightarrow M^*$ in $\mathsf{Mod}\,\Bbbk\langle \boldsymbol{z} \rangle$. Since X is finitely generated there is a surjection of sets $\underbrace{S \sqcup \cdots \sqcup S}_{r_0} \twoheadrightarrow X$ for some $r_0 \in \mathbb{N}$. This gives rise to a surjection $(\Bbbk^{(S)})^{r_0} \twoheadrightarrow \Bbbk^{(X)}$. Composing we obtain a $\Bbbk\langle \boldsymbol{z} \rangle$-linear surjection $\phi^* : (\Bbbk^{(S)})^{r_0} \twoheadrightarrow M^*$.

Now $(\Bbbk^{(S)})^{r_0}$ is a free $\Bbbk\langle \boldsymbol{z} \rangle$-module of finite rank, so according to [Co, Section 1.2 Theorem 2.1 and Section 2.4 Corollary 4.3], the submodule $\mathrm{Ker}(\phi^*)$ is free. Let $r_1 \in \mathbb{N} \cup \{\aleph_0\}$ be the rank of $\mathrm{Ker}(\phi^*)$. Hence there is an exact sequence

$$0 \to (\Bbbk^{(S)})^{(r_1)} \to (\Bbbk^{(S)})^{r_0} \to M^* \to 0$$

in $\mathsf{Mod}\,\Bbbk\langle \boldsymbol{z} \rangle$. Finally apply duality. □

Example 6.5. Let M be an n-dimensional (commutative) behavior. By definition M is a closed shift invariant $\Bbbk$-submodule of $\Bbbk^X$, where $X := \underbrace{\mathbb{N}^n \sqcup \cdots \sqcup \mathbb{N}^n}_{r_0}$. We see that M is also a noncommutative n-dimensional behavior.

Sometimes the exponent r_1 in a kernel representation of a noncommutative behavior must be infinite. Here is an example.

Example 6.6. Take $n = 2$, and let $M := \Bbbk^{\mathbb{N}^2}$. The dual module M^* is isomorphic, as $\Bbbk\langle z_1, z_2 \rangle$-module, to the commutative polynomial ring $\Bbbk[z_1, z_2]$. Consider the surjection $\phi^* : \Bbbk\langle z_1, z_2 \rangle \twoheadrightarrow \Bbbk[z_1, z_2]$. Then $N := \mathrm{Ker}(\phi^*)$ is a free left $\Bbbk\langle z_1, z_2 \rangle$-module with basis $\{cs \mid s \in S\}$, where $c := z_1 z_2 - z_2 z_2$ and S is the set of monomials. It is known that the category of finitely presented $\Bbbk\langle z_1, z_2 \rangle$-modules (sometimes called coherent modules) is an abelian subcategory of $\mathsf{Mod}\,\Bbbk\langle \boldsymbol{z} \rangle$; see [Co, Appendix, Theorem A.9]. Therefore from the exact sequence

$$0 \to N \to \Bbbk\langle z_1, z_2 \rangle \xrightarrow{\phi^*} \Bbbk[z_1, z_2] \to 0$$

we see that $\Bbbk[z_1, z_2]$ is not a finitely presented $\Bbbk\langle z_1, z_2 \rangle$-module. It follows that there does not exists any exact sequence

$$\Bbbk\langle z_1, z_2 \rangle^{r_1} \to \Bbbk\langle z_1, z_2 \rangle^{r_0} \to \Bbbk[z_1, z_2] \to 0$$

with $r_0, r_1 \in \mathbb{N}$.

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