Rigid Dualizing Complexes Over Commutative Rings

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Abstract In this paper we present a new approach to Grothendieck duality over commutative rings. Our approach is based on the idea of rigid dualizing complexes, which was introduced by Van den Bergh in the context of noncommutative algebraic geometry. The method of rigidity was modified to work over general commutative base rings in our paper (Yekutieli and Zhang, Trans AMS 360:3211–3248, 2008). In the present paper we obtain many of the important local features of Grothendieck duality, yet manage to avoid lengthy and difficult compatibility verifications. Our results apply to essentially finite type algebras over a regular noetherian finite dimensional base ring, and hence are suitable for arithmetic rings. In the sequel paper (Yekutieli, Rigid dualizing complexes on schemes, in preparation) these results will be used to construct and study rigid dualizing complexes on schemes.

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1 Introduction

Grothendieck duality for schemes was introduced in the book "Residues and Duality" [15] by R. Hartshorne. This duality theory has applications in various areas of algebraic geometry, including moduli spaces, resolution of singularities, arithmetic geometry, enumerative geometry and more.

In the 40 years since the publication of [15] a number of related papers appeared in the literature. Some of these papers provided elaborations on, or more explicit versions of Grothendieck duality (e.g. [6, 9, 11, 12, 16, 21, 22]). Other papers contained alternative approaches (e.g. [15, Appendix], [19] and [14]). The recent book [2] is a complement to [15] that fills gaps in the proofs, and also contains the first proof of the Base Change Theorem. A noncommutative version of Grothendieck duality was developed in [20], which has applications in algebra (e.g. [3]) and even in mathematical physics (e.g. [8]). Other papers sought to extend the scope of Grothendieck duality to formal schemes (e.g. [1] and [13]) or to differential graded algebras (see [4]).

One of the fascinating features of Grothendieck duality is the complicated interplay between its local and global components. Another feature of this theory (in some sense parallel to the first) is the gap between formal categorical statements and their concrete realizations. Much of the effort in studying Grothendieck duality was aimed at clarifying the local-global interplay, and at bridging the above-mentioned gap.

In this paper we present a new approach to Grothendieck duality for commutative rings (i.e. affine schemes). The sequel (Yekutieli, in preparation) will treat schemes in general (including duality for proper morphisms). The key idea in our approach is the use of *rigid dualizing complexes*. This notion was introduced by Van den Bergh [18] in the context of noncommutative algebraic geometry, and was developed further, in the noncommutative direction, in our papers [24–27]. In the paper [28] we worked out the fundamental properties of rigid complexes over commutative rings relative to an arbitrary commutative base ring (as opposed to a base field). Attaching a rigid structure to a dualizing complex eliminates all nontrivial automorphisms. Moreover, rigid dualizing complexes admit several useful operations, such as localization and traces.

The general concept of "rigidity" is familiar in other areas of algebraic geometry (e.g. level structures on elliptic curves, or marked points on higher genus curves). Actually, in Grothendieck's original treatment [15] duality (local and global) itself was used as a sort of "rigid structure" on dualizing complexes, but this was very cumbersome (amounting to big commutative diagrams) and hard to employ. On the other hand, Van den Bergh's rigidity is very neat, and enjoys remarkable functorial properties.

The background material we need in this paper is standard commutative algebra, the theory of derived categories, and the theory of rigid complexes over commutative rings from [28]. All of that is reviewed is Section 1 of the paper, for the convenience of the reader. We also need a few isolated results on dualizing complexes from [15].

Let us explain what are rigid dualizing complexes and how they are used in our paper. Fix for the rest of the introduction a finite dimensional, regular, noetherian, commutative base ring $\mathbb{K}$ (e.g. a field, or the ring of integers). Recall that an essentially finite type commutative $\mathbb{K}$-algebra A is by definition a localization (with

respect to a multiplicatively closed subset) of some finitely generated $\mathbb{K}$-algebra. Note that the rings $\Gamma(U, \mathcal{O}_X)$ and $\mathcal{O}_{X,x}$, where X is a finite type $\mathbb{K}$-scheme, $U \subset X$ is an affine open set and $x \in X$ is a point, are both essentially finite type $\mathbb{K}$-algebras. We denote by EFTAlg $/\mathbb{K}$ the category of essentially finite type commutative $\mathbb{K}$-algebras, and by default we will stay within this category.

We shall use notation such as $f^* : A \to B$ for a homomorphism of algebras, corresponding to a morphism of schemes $f : \text{Spec } B \to \text{Spec } A$. This convention, although perhaps awkward at first sight, fits better with the usual notation for associated functors. Thus there are functors $f_* : \text{Mod } B \to \text{Mod } A$ (restriction of scalars, which is push-forward geometrically) and $f^* : \text{Mod } A \to \text{Mod } B$ (extension of scalars, i.e. $B \otimes_A -$, which is pull-back geometrically). For composable homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ we sometimes write $(f \circ g)^*$ instead of $g^* \circ f^*$.

For a $\mathbb{K}$-algebra A the derived category of complexes of A-modules is denoted by $\text{D}(\text{Mod } A)$, with the usual modifiers (e.g. $\text{D}^{\text{b}}_{\text{f}}(\text{Mod } A)$ is the full subcategory of bounded complexes with finitely generated cohomologies).

Given a complex $M \in \text{D}(\text{Mod } A)$ we define its *square* $\text{Sq}_{A/\mathbb{K}} M \in \text{D}(\text{Mod } A)$. The functor $\text{Sq}_{A/\mathbb{K}}$ from the category $\text{D}(\text{Mod } A)$ to itself is quadratic, in the sense that given a morphism $\phi : M \to N$ in $\text{D}(\text{Mod } A)$ and an element $a \in A$, one has $\text{Sq}_{A/\mathbb{K}}(a\phi) = a^2 \text{Sq}_{A/\mathbb{K}}(\phi)$.

A *rigidifying isomorphism* for M is an isomorphism $\rho : M \xrightarrow{\simeq} \text{Sq}_{A/\mathbb{K}} M$ in $\text{D}(\text{Mod } A)$. If $M \in \text{D}^{\text{b}}_{\text{f}}(\text{Mod } A)$ then the pair (M, ρ) is called a *rigid complex over A relative to* $\mathbb{K}$. Suppose (M, ρ_M) and (N, ρ_N) are two rigid complexes. A *rigid morphism* $\phi : (M, \rho_M) \to (N, \rho_N)$ is a morphism $\phi : M \to N$ in $\text{D}(\text{Mod } A)$ such that $\rho_N \circ \phi = \text{Sq}_{A/\mathbb{K}}(\phi) \circ \rho_M$. Observe that if (M, ρ_M) is a rigid complex such that $\text{RHom}_A(M, M) = A$, and $\phi : (M, \rho_M) \to (M, \rho_M)$ is a rigid isomorphism, then ϕ is multiplication by some invertible element $a \in A$ satisfying $a = a^2$; and therefore $a = 1$. We conclude that *the identity is the only rigid automorphism of* (M, ρ_M).

Let B be another $\mathbb{K}$-algebra, and let $f^* : A \to B$ be a finite $\mathbb{K}$-algebra homomorphism. Define

$$f^{\flat} M := \text{RHom}_A(B, M) \in \text{D}^+_{\text{f}}(\text{Mod } B).$$

If $f^{\flat} M$ has bounded cohomology then there is an induced rigidifying isomorphism $f^{\flat}(\rho_M) : f^{\flat} M \xrightarrow{\simeq} \text{Sq}_{B/\mathbb{K}} f^{\flat} M$. We write $f^{\flat}(M, \rho_M) := (f^{\flat} M, f^{\flat}(\rho_M))$.

Next we consider *essentially smooth* homomorphisms. By definition $f^* : A \to B$ is essentially smooth if it is essentially finite type and formally smooth. Then B is flat over A, and the module of differentials $\Omega^1_{B/A}$ is a finitely generated projective B-module. The rank of $\Omega^1_{B/A}$ might vary on $\text{Spec } B$; but if it has constant rank n then we say f^* is essentially smooth of relative dimension n. An essentially smooth homomorphism of relative dimension 0 is called an *essentially étale* homomorphism.

Let $f^* : A \to B$ be an essentially smooth homomorphism, let $B = \prod B_i$ be the decomposition of $\text{Spec } B$ into connected components, and for any i let $n_i := \text{rank}_{B_i} \Omega^1_{B_i/A}$. We then define

$$f^{\sharp} M := \bigoplus_i \Omega^{n_i}_{B_i/A}[n_i] \otimes_A M \in \text{D}^{\text{b}}_{\text{f}}(\text{Mod } B).$$

There is an induced rigidifying isomorphism $f^\sharp(\rho_M) : f^\sharp M \xrightarrow{\simeq} \mathrm{Sq}_{B/\mathbb{K}} f^\sharp M$, and thus a new rigid complex $f^\sharp(M, \rho_M) := (f^\sharp M, f^\sharp(\rho_M))$.

Now let's consider dualizing complexes. Recall that a complex $R \in \mathsf{D}_\mathrm{f}^\mathrm{b}(\mathsf{Mod}\, A)$ is dualizing if it has finite injective dimension over A, and if the canonical morphism $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism. A *rigid dualizing complex over A relative to* $\mathbb{K}$ is a rigid complex (R, ρ) such that R is dualizing.

Here is the main result of our paper.

Theorem 1.1 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra.*

(1) *The algebra A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*
(2) *Given a finite homomorphism $f^* : A \to B$, there is a unique rigid isomorphism $f^\flat(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.*
(3) *Given an essentially smooth homomorphism $f^* : A \to B$, there is a unique rigid isomorphism $f^\sharp(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.*

This theorem is repeated as Theorem 3.6 in the body of the paper.

The next result is one we find quite surprising. Its significance is not yet understood.

Theorem 1.2 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra. Assume A is regular, and $\mathrm{Spec}\, A$ is connected and nonempty. Then, up to isomorphism, the only nonzero rigid complex over A relative to $\mathbb{K}$ is the rigid dualizing complex (R_A, ρ_A).*

This theorem is repeated as Theorem 3.10.

Let A be some $\mathbb{K}$-algebra. Using the rigid dualizing complex R_A we define the *auto-duality functor* $\mathrm{D}_A := \mathrm{RHom}_A(-, R_A)$ of $\mathsf{D}_\mathrm{f}(\mathsf{Mod}\, A)$. Due to Theorem 1.1(1) this functor is independent of the rigid dualizing complex R_A chosen. Given a $\mathbb{K}$-algebra homomorphism $f^* : A \to B$ we define the *twisted inverse image functor*

$$f^! : \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A) \to \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, B)$$

as follows. If $A = B$ and $f^* = \mathbf{1}_A$ (the identity automorphism) then $f^! := \mathbf{1}_{\mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A)}$ (the identity functor). Otherwise we define $f^! := \mathrm{D}_B \, \mathrm{L}f^* \mathrm{D}_A$.

As explained in Corollary 4.8, the base ring $\mathbb{K}$ can sometimes be "factored out" of the construction of the twisted inverse image functor $f^!$.

Suppose we are given two homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ in $\mathsf{EFTAlg}\,/\mathbb{K}$. Then there is an obvious isomorphism

$$\phi_{f,g} : (f \circ g)^! \xrightarrow{\simeq} g^! f^!$$

of functors $\mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A) \to \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, C)$, coming from the adjunction isomorphism $\mathbf{1} \xrightarrow{\simeq} \mathrm{D}_B \mathrm{D}_B$ on $\mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, B)$. For three homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} D$ in $\mathsf{EFTAlg}\,/\mathbb{K}$ the isomorphisms $\phi_{-,-}$ satisfy the compatibility condition

$$\phi_{g,h} \circ \phi_{f,g \circ h} = \phi_{f,g} \circ \phi_{f \circ g, h} : (f \circ g \circ h)^! \xrightarrow{\simeq} h^! g^! f^!$$

(see Proposition 4.4). This means that the assignment $f^* \mapsto f^!$ is the 1-component of a two-functor $\mathsf{EFTAlg}\,/\mathbb{K} \to \mathsf{Cat}$, whose 0-component is $A \mapsto \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$. Here Cat denotes the two-category of all categories (the notion of two-functor is recalled in Section 4).

The next theorem (which is a consequence of Theorem 1.1) describes the variance properties of the twisted inverse image two-functor $f^* \mapsto f^!$.

Theorem 1.3 *Let* $f^* : A \to B$ *be a homomorphism in* $\mathsf{EFTAlg}\,/\mathbb{K}$.

(1) *If* f^* *is finite, then there is an isomorphism*

$$\psi_f^{\flat} : f^{\flat} \xrightarrow{\simeq} f^!$$

of functors $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,B)$. *These isomorphisms are two-functorial for finite homomorphisms.*

(2) *If* f^* *is essentially smooth, then there is an isomorphism*

$$\psi_f^{\sharp} : f^{\sharp} \xrightarrow{\simeq} f^!$$

of functors $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,B)$. *These isomorphisms are two-functorial for essentially smooth homomorphisms.*

For more detailed statements and proofs see Theorems 4.5 and 4.6.

In the situation of a finite homomorphism $f^* : A \to B$, part (1) of the theorem gives rise to the *functorial trace map*

$$\mathrm{Tr}_f : f_* f^! \to \mathbf{1}.$$

This is a nondegenerate morphism of functors from $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$ to itself. See Proposition 5.2 for details.

If $f^* : A \to B$ is essentially étale, then from part (2) of the theorem we get the *functorial localization map*

$$\mathsf{q}_f : \mathbf{1} \to f_* f^!,$$

which is a nondegenerate morphism of functors from $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$ to itself. See Proposition 5.4 for details. The relation between functorial localization maps and functorial trace maps is explained in Proposition 5.5.

Here is an application to differential forms, which is a corollary to Theorem 1.3.

Corollary 1.4 *Suppose* $A \to B \to C$ *are homomorphisms in* $\mathsf{EFTAlg}\,/\mathbb{K}$, *with* $A \to B$ *and* $A \to C$ *essentially smooth of relative dimension n, and* $B \to C$ *finite. Then there is a nondegenerate trace map*

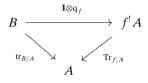$$\mathrm{Tr}_{C/B/A} : \Omega_{C/A}^n \to \Omega_{B/A}^n.$$

The trace maps $\mathrm{Tr}_{-/-/A}$ *are functorial for such finite homomorphisms* $B \to C$, *and commute with the localization maps for a localization homomorphism* $B \to B'$.

This is restated (in more detail) as Theorem 6.2 and Proposition 6.8. In some cases we can compute $\mathrm{Tr}_{C/B/A}$; see Propositions 6.9 and 6.10.

For a finite flat homomorphism $A \to B$ let $\mathrm{tr}_{B/A} : B \to A$ be the usual trace map, i.e. $\mathrm{tr}_{B/A}(b) \in A$ is the trace of the operator b acting of the locally free A-module B.

Here is another result relating traces and localization. It is not a consequence of Theorem 1.3, but instead relies on the fact that for an étale homomorphism $A \to B$ one has a canonical ring isomorphism $B \otimes_A B \cong B \times B'$, where B' is the kernel of the multiplication map $B \otimes_A B \to B$. This is interpreted in terms of rigidity.

Theorem 1.5 *Suppose* $f^* : A \to B$ *is a finite étale homomorphism in* $\mathsf{EFTAlg}\,/\mathbb{K}$, *so the localization map* $\mathsf{q}_f : A \to f^! A$ *induces an isomorphism* $\mathbf{1} \otimes \mathsf{q}_f : B \overset{\simeq}{\to} f^! A$. *Then the diagram*

$$
\begin{array}{ccc}
B & \xrightarrow{\;\;\mathbf{1}\otimes\mathsf{q}_f\;\;} & f^! A \\
& {\scriptstyle \mathrm{tr}_{B/A}} \searrow \quad \swarrow {\scriptstyle \mathrm{Tr}_{f;A}} & \\
& A &
\end{array}
$$

is commutative.

This result is repeated (in slightly more general form) as Theorem 5.6.

To conclude the introduction let us mention how our twisted inverse image two-functor compares with the original constructions in [15]. We shall restrict attention to the category $\mathsf{FTAlg}\,/\mathbb{K}$ of finite type $\mathbb{K}$-algebras. (This is of course the opposite of the category of finite type affine $\mathbb{K}$-schemes.) Given a homomorphism $f^* : A \to B$ in $\mathsf{FTAlg}\,/\mathbb{K}$ let us denote by

$$
f^{!(\mathrm{G})} : \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,B)
$$

the twisted inverse image from [15]. In particular, for an algebra A, with structural homomorphism $\pi_A^* : \mathbb{K} \to A$, we obtain the complex $R_A^{(\mathrm{G})} := \pi_A^{!(\mathrm{G})} \mathbb{K} \in \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$, which is known to be dualizing. In Theorem 4.10 we show that for any $A \in \mathsf{FTAlg}\,/\mathbb{K}$ there is an isomorphism $R_A^{(\mathrm{G})} \cong R_A$ in $\mathsf{D}(\mathsf{Mod}\,A)$. This implies that there is an isomorphism $f^! \cong f^{!(\mathrm{G})}$ of two-functors $\mathsf{FTAlg}\,/\mathbb{K} \to \mathsf{Cat}$. In general we do not know an easy way to make the isomorphisms $R_A^{(\mathrm{G})} \cong R_A$ canonical; but see Remark 4.11.

2 Review of Rigid Complexes

In this section we recall definitions and results from the paper [28].

Throughout the paper all rings and algebras are assumed to be commutative by default. Given two rings A and B we denote a homomorphism between them by an expression such as $f^* : A \to B$. This of course signifies that the corresponding morphism of schemes is $f : \mathrm{Spec}\,B \to \mathrm{Spec}\,A$. The benefit of this notation is that it is compatible with the customary notation for various related functors, such as $f_* : \mathsf{Mod}\,B \to \mathsf{Mod}\,A$ (restriction of scalars, which is direct image for schemes), and $f^* : \mathsf{Mod}\,A \to \mathsf{Mod}\,B$ (base change, i.e. $f^* M = B \otimes_A M$, which is inverse image for schemes).

Given a ring A we denote by $\mathsf{D}(\mathsf{Mod}\, A)$ the derived category of complexes of A-modules. If A is noetherian then we denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the full subcategory consisting of bounded complexes with finitely generated cohomology modules.

In [28, Section 2] we introduced the *squaring operation*. Let A be a ring, B an A-algebra and $M \in \mathsf{D}(\mathsf{Mod}\, B)$. The square of M over B relative to A is a complex $\mathrm{Sq}_{B/A}\, M \in \mathsf{D}(\mathsf{Mod}\, B)$. In case B is flat over A one has the simple formula

$$\mathrm{Sq}_{B/A}\, M = \mathrm{RHom}_{B \otimes_A B}\left(B,\, M \otimes^{\mathrm{L}}_A M\right).$$

But in general it is necessary to replace $B \otimes_A B$ with $\tilde{B} \otimes_A \tilde{B}$ in the formula defining $\mathrm{Sq}_{B/A}\, M$, where $\tilde{B}$ is a suitable differential graded A-algebra, quasi-isomorphic to B.

Suppose C is another A-algebra, $f^* : B \to C$ is an A-algebra homomorphism, $N \in \mathsf{D}(\mathsf{Mod}\, C)$, and $\phi : N \to M$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, B)$. (Strictly speaking, ϕ is a morphism $f_* N \to M$). Then there is an induced morphism

$$\mathrm{Sq}_{f^*/A}(\phi) : \mathrm{Sq}_{C/A}\, N \to \mathrm{Sq}_{B/A}\, M$$

in $\mathsf{D}(\mathsf{Mod}\, B)$. The formation of $\mathrm{Sq}_{f^*/A}(\phi)$ is functorial in f^* and in ϕ.

Specializing to the case $C = B$ and $f^* = \mathbf{1}_B$ (the identity homomorphism), we obtain a functor

$$\mathrm{Sq}_{B/A} := \mathrm{Sq}_{\mathbf{1}_B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, B).$$

This functor is quadratic, in the sense that for any $\phi \in \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(M, N)$ and $b \in B$ one has

$$\mathrm{Sq}_{B/A}(b\phi) = b^2\, \mathrm{Sq}_{B/A}(\phi).$$

The next definition is a variant of the original definition of Van den Bergh [18].

Definition 2.1 Let A be a ring and B a noetherian A-algebra. A *rigid complex over B relative to A* is a pair (M, ρ), where:

(1) M is a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ which has finite flat dimension over A.
(2) ρ is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}\, M$$

in $\mathsf{D}(\mathsf{Mod}\, B)$, called a *rigidifying isomorphism*.

Definition 2.2 Let A be a ring, let B and C be noetherian A-algebras, let $f^* : B \to C$ be an A-algebra homomorphism, let (M, ρ) be a rigid complex over B relative to A, and let (N, σ) be a rigid complex over C relative to A. A *rigid trace morphism relative to A*

$$\phi : (N, \sigma) \to (M, \rho)$$

is a morphism $\phi : N \to M$ in $\mathsf{D}(\mathsf{Mod}\, B)$, such that the diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\ \sigma\ } & \mathrm{Sq}_{C/A}\, N \\
{\scriptstyle \phi}\Big\downarrow & & \Big\downarrow {\scriptstyle \mathrm{Sq}_{f^*/A}(\phi)} \\
M & \xrightarrow{\ \rho\ } & \mathrm{Sq}_{B/A}\, M
\end{array}
$$

is commutative.

Clearly the composition of two rigid trace morphisms is again a rigid trace morphism.

Specializing Definition 2.2 to the case $C = B$ and $f^* = \mathbf{1}_B$, we call such a morphism ϕ a *rigid morphism over B relative to A*. Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ be the category whose objects are the rigid complexes over B relative to A, and whose morphisms are the rigid morphisms.

Recall that a ring homomorphism $f^* : A \to B$ is called *finite* if B is a finitely generated A-module. Given a finite homomorphism $f^* : A \to B$ we define a functor

$$
f^\flat : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)
$$

by

$$
f^\flat M := \mathrm{RHom}_A(B, M).
$$

For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ there is a morphism

$$
\mathrm{Tr}^\flat_{f;M} : f_* f^\flat M \to M \tag{2.3}
$$

called the *trace map*, which is induced from the homomorphism $\phi \mapsto \phi(1)$ for $\phi \in \mathrm{Hom}_A(B, M)$. In this way we get a morphism $\mathrm{Tr}^\flat_f : f_* f^\flat \to \mathbf{1}$ of functors from $\mathsf{D}(\mathsf{Mod}\, A)$ to itself. Note that for any $N \in \mathsf{D}(\mathsf{Mod}\, B)$ the homomorphism of B-modules

$$
\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(N, f^\flat M) \to \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(N, M), \ \psi \mapsto \mathrm{Tr}^\flat_{f;M} \circ \psi
$$

is bijective (this is the adjunction isomorphism).

Definition 2.4 Let $f^* : A \to B$ be a ring homomorphism, $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $N \in \mathsf{D}(\mathsf{Mod}\, B)$. A morphism $\phi : N \to M$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called a *nondegenerate trace morphism* if the corresponding morphism $N \to f^\flat M$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is an isomorphism.

Here's the first result about rigid complexes, which explains their name.

Theorem 2.5 *Let A be a ring, let $f^* : B \to C$ be a homomorphism between noetherian A-algebras, let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ and let $(N, \sigma) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/A}$. Assume the canonical ring homomorphism $C \to \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\, C)}(N)$ is bijective. Then there is at most one nondegenerate rigid trace morphism $(N, \sigma) \to (M, \rho)$ over B relative to A.*

Proof This is a slight generalization of [28, Theorem 1.2]. Suppose we are given two nondegenerate rigid trace morphisms $\phi, \phi' : (N, \sigma) \to (M, \rho)$. Since ϕ is nondegenerate it follows that $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,B)}(N, M)$ is a free C-module with basis ϕ. So $\phi' = c\phi$ for some unique element $c \in C$. Because ϕ' is nondegenerate too we see that c is an invertible element. Next, since both ϕ and ϕ' are rigid, [28, Corollary 2.7] says that $c^2 = c$. Thus $c = 1$ and $\phi' = \phi$. □

Corollary 2.6 [28, Theorem 1.2] *Taking $B = C$, $f^* = \mathbf{1}_B$ and $(N, \sigma) = (M, \rho)$ in Theorem 2.5, we see that the only automorphism of (M, ρ) in $\mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\,B)_{\mathrm{rig}/A}$ is the identity $\mathbf{1}_M$.*

Here are a few results about pullbacks of rigid complexes.

Theorem 2.7 [28, Theorem 5.3] *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be a finite A-algebra homomorphism, and let $(M, \rho) \in \mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\,B)_{\mathrm{rig}/A}$. Assume $f^\flat M$ has finite flat dimension over A.*

(1) *The complex $f^\flat M$ has an induced rigidifying isomorphism*

$$f^\flat(\rho) : f^\flat M \xrightarrow{\cong} \mathrm{Sq}_{C/A}\, f^\flat M.$$

(2) *The rigid complex*

$$f^\flat(M, \rho) := \left(f^\flat M,\, f^\flat(\rho) \right) \in \mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\,C)_{\mathrm{rig}/A}$$

depends functorially on (M, ρ) and on f^*.

(3) *Assume moreover that $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,C)}(f^\flat M, f^\flat M) = C$. Then $\mathrm{Tr}^\flat_{f;M}$ is the unique nondegenerate rigid trace morphism $f^\flat(M, \rho_M) \to (M, \rho_M)$ over B relative to A.*

Let A be a noetherian ring. Recall that an A-algebra B is called formally smooth (resp. formally étale) if it has the lifting property (resp. the unique lifting property) for infinitesimal extensions. The A-algebra B is called smooth (resp. étale) if it is finitely generated and formally smooth (resp. formally étale).

In [28, Section 4] we introduced a slightly more general kind of ring homomorphism than a smooth homomorphism. Again A is noetherian. Recall that an A-algebra B is called essentially finite type if it is a localization of some finitely generated A-algebra. We say that B is *essentially smooth* (resp. *essentially étale*) over A if it is essentially finite type and formally smooth (resp. formally étale). The composition of two essentially smooth homomorphisms is essentially smooth. If $A \to B$ is essentially smooth then B is flat over A, and $\Omega^1_{B/A}$ is a finitely generated projective B-module. One can show that the homomorphism $A \to B$ is essentially étale, if and only if it is essentially smooth and $\Omega^1_{B/A} = 0$.

Let A be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Let $\mathrm{Spec}\,B = \coprod_i \mathrm{Spec}\,B_i$ be the decomposition into connected components, and for every i let n_i be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^\sharp : \mathsf{D}(\mathsf{Mod}\,A) \to \mathsf{D}(\mathsf{Mod}\,B)$$

by

$$f^\sharp M := \bigoplus_i \Omega^{n_i}_{B_i/A}[n_i] \otimes_A M.$$

Theorem 2.8 [28, Theorem 6.3] *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be an essentially smooth A-algebra homomorphism, and let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$.*

(1) *The complex $f^\sharp M$ has an induced rigidifying isomorphism*

$$f^\sharp(\rho) : f^\sharp M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\sharp M.$$

(2) *The rigid complex*

$$f^\sharp(M, \rho) := \left(f^\sharp M,\, f^\sharp(\rho) \right) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/A}$$

depends functorially on (M, ρ) and on f^*.

Definition 2.9 Suppose $f^* : A \to B$ is essentially étale, so that $f^\sharp M = B \otimes_A M$ for any $M \in \mathsf{D}(\mathsf{Mod}\, A)$. Let $\mathrm{q}^\sharp_{f;M} : M \to f^\sharp M$ be the morphism $m \mapsto 1 \otimes m$. On the level of functors this gives a morphism $\mathrm{q}^\sharp_f : \mathbf{1} \to f_* f^\sharp$ of functors from $\mathsf{D}(\mathsf{Mod}\, A)$ to itself.

In the situation of the definition, given $N \in \mathsf{D}(\mathsf{Mod}\, B)$, there is a canonical bijection

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(f^\sharp M, N), \ \phi \mapsto \mathbf{1} \otimes \phi.$$

In particular, for $N := f^\sharp M$, the morphism $\mathrm{q}^\sharp_{f;M}$ corresponds to the identity $\mathbf{1}_N$.

Definition 2.10 Let A be a noetherian ring, let $A \to B$ be an essentially étale ring homomorphism, let $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $N \in \mathsf{D}(\mathsf{Mod}\, B)$. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called a *nondegenerate localization morphism* if the corresponding morphism $\mathbf{1} \otimes \phi : f^\sharp M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is an isomorphism.

Definition 2.11 Let A be a noetherian ring, let B and C be essentially finite type A-algebras, let $f^* : B \to C$ be an essentially étale A-algebra homomorphism, let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ and let $(N, \sigma) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/A}$. A *rigid localization morphism* is a morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\, B)$, such that the corresponding morphism $\mathbf{1} \otimes \phi : f^\sharp(M, \rho) \to (N, \sigma)$ is a rigid morphism over C relative to A.

Proposition 2.12 [28, Proposition 6.8] *Let A be a noetherian ring, let B and C be essentially finite type A-algebras, let $f^* : B \to C$ be an essentially étale A-algebra homomorphism, and let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$. Assume that $\mathrm{RHom}_B(M, M) = B$. Then the morphism $\mathrm{q}^\sharp_{f;M}$ is the unique nondegenerate rigid localization morphism $(M, \rho) \to f^\sharp(M, \rho)$.*

The next result is about tensor products of rigid complexes.

Theorem 2.13 [28, Theorem 0.4] *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be an A-algebra homomorphism, and let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ and $(N, \sigma) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/B}$. Assume that the canonical homomorphism $B \to \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(M, M)$ is bijective. Then the complex $M \otimes^{\mathrm{L}}_B N$ is in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)$, it has finite flat dimension over A, and it has an induced rigidifying isomorphism*

$$\rho \otimes \sigma : M \otimes^{\mathrm{L}}_B N \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\left(M \otimes^{\mathrm{L}}_B N\right).$$
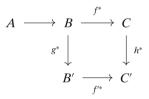
The rigid complex

$$(M, \rho) \otimes^{\mathrm{L}}_B (N, \sigma) := \left(M \otimes^{\mathrm{L}}_B N, \rho \otimes \sigma\right) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/A}$$

depends functorially of (M, ρ) and (N, σ).

Finally, a base change result.

Theorem 2.14 [28, Theorem 6.9] *Consider a commutative diagram of ring homomorphisms*

$$
\begin{array}{ccccc}
A & \longrightarrow & B & \xrightarrow{f^*} & C \\
& & \downarrow{\scriptstyle g^*} & & \downarrow{\scriptstyle h^*} \\
& & B' & \xrightarrow{f'^*} & C'
\end{array}
$$

where A is a noetherian ring, and B, B', C and C' are essentially finite type A-algebras. Assume moreover that $g^* : B \to B'$ is a localization, and the square is cartesian (namely $C' \cong B' \otimes_B C$). Let $(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$, let $(N, \sigma) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, C)_{\mathrm{rig}/A}$, and let

$$\phi : (N, \sigma) \to (M, \rho)$$

be a rigid trace morphism over B relative to A. Define $M' := g^\sharp M$ and $N' := h^\sharp N$. There is a morphism $\phi' : N' \to M'$ in $\mathsf{D}(\mathsf{Mod}\, B')$ obtained by composing the canonical isomorphism $N' = C' \otimes_C N \cong B' \otimes_B N = g^\sharp N$ with $g^\sharp(\phi) : g^\sharp N \to g^\sharp M = M'$. So the diagram

$$
\begin{array}{ccc}
M & \xleftarrow{\phi} & N \\
{\scriptstyle \mathrm{q}^\sharp_{g;M}} \downarrow & & \downarrow {\scriptstyle \mathrm{q}^\sharp_{h;N}} \\
M' & \xleftarrow{\phi'} & N'
\end{array}
$$

is commutative. Then

$$\phi' : \left(N', h^\sharp(\sigma)\right) \to \left(M', g^\sharp(\rho)\right)$$

is a rigid trace morphism over B' relative to A.

Observe that there is no particular assumption on f^*.

Remark 2.15 The construction of the functor $\mathrm{Sq}_{B/A}$, and the proof of the theorems above in [28], required heavy use of DG algebras. For the convenience of the reader we eliminated all reference to DG algebras in the definitions and statements in the present paper. However, we could not avoid using DG algebras in some of the proofs.

3 Rigid Dualizing Complexes

In this section $\mathbb{K}$ is a fixed regular noetherian ring of finite Krull dimension. All algebras are by default essentially finite type $\mathbb{K}$-algebras, and all algebra homomorphisms are over $\mathbb{K}$. We denote by $\mathsf{EFTAlg}\,/\mathbb{K}$ the category of essentially finite type $\mathbb{K}$-algebras.

Let us recall the definition of *dualizing complex* over a $\mathbb{K}$-algebra A from [15]. A complex $R \in \mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\,A)$ is called a dualizing complex if it has finite injective dimension, and the canonical morphism $A \to \mathrm{RHom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}\,A)$ is an isomorphism. It follows that the functor $\mathrm{RHom}_A(-, R)$ is an auto-duality (i.e. a contravariant equivalence) of $\mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\,A)$. Note that since the ground ring $\mathbb{K}$ has finite global dimension, the complex R has finite flat dimension over it.

Following Van den Bergh [18] we make the following definition.

Definition 3.1 Let A be an essentially finite type $\mathbb{K}$-algebra and let R be a dualizing complex over A. Suppose R has a rigidifying isomorphism $\rho : R \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}} R$. Then the pair (R, ρ) is called a *rigid dualizing complex over A relative to* $\mathbb{K}$.

By default all rigid dualizing complexes are relative to the ground ring $\mathbb{K}$.

Example 3.2 Take the $\mathbb{K}$-algebra $A := \mathbb{K}$. The complex $R := \mathbb{K}$ is a dualizing complex over $\mathbb{K}$, since this ring is regular and finite dimensional. Let

$$\rho^{\mathrm{tau}} : \mathbb{K} \xrightarrow{\simeq} \mathrm{RHom}_{\mathbb{K} \otimes_{\mathbb{K}} \mathbb{K}}(\mathbb{K}, \mathbb{K} \otimes_{\mathbb{K}} \mathbb{K}) = \mathrm{Sq}_{\mathbb{K}/\mathbb{K}} \mathbb{K}$$

be the *tautological rigidifying isomorphism*. Then $(\mathbb{K}, \rho^{\mathrm{tau}})$ is a rigid dualizing complex over $\mathbb{K}$ relative to $\mathbb{K}$.

In [18] it was proved that when $\mathbb{K}$ is a field, a rigid dualizing complex (R, ρ) is unique up to isomorphism. And in [24] we proved that (R, ρ) is in fact unique up to a unique rigid isomorphism (again, only when $\mathbb{K}$ is a field). These results are true in our setup too:

Theorem 3.3 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, let A be an essentially finite type $\mathbb{K}$-algebra, and let (R, ρ) be a rigid dualizing complex over A relative to $\mathbb{K}$. Then (R, ρ) is unique up to a unique rigid isomorphism.*

Proof In view of [28, Lemma 6.1] and [28, Theorem 1.6] we may assume that Spec A is connected. Suppose (R', ρ') is another rigid dualizing complex over A. Then there is an isomorphism $R' \cong R \otimes_A L[n]$ for some invertible A-module L and some integer n. Indeed $L[n] \cong \mathrm{RHom}_A(R, R')$; see [15, Section V.3].

Choose a K-flat DG algebra resolution $\mathbb{K} \to \tilde{A} \to A$ of $\mathbb{K} \to A$. (If $\mathbb{K}$ is a field just take $\tilde{A} := A$). So
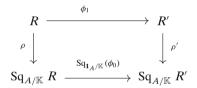
$$
\begin{aligned}
\mathrm{Sq}_{A/\mathbb{K}} \, R' &\cong \mathrm{Sq}_{A/\mathbb{K}} (R \otimes_A L[n]) \\
&= \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}} \big(A, (R \otimes_A L[n]) \otimes_{\mathbb{K}}^{\mathrm{L}} (R \otimes_A L[n]) \big) \\
&\cong^{\dagger} \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}} \big(A, R \otimes_{\mathbb{K}}^{\mathrm{L}} R \big) \otimes_A^{\mathrm{L}} L[n] \otimes_A^{\mathrm{L}} L[n] \\
&= (\mathrm{Sq}_{A/\mathbb{K}} \, R) \otimes_A^{\mathrm{L}} L[n] \otimes_A^{\mathrm{L}} L[n] \cong^{\diamond} R \otimes_A L[n] \otimes_A L[n].
\end{aligned}
$$

The isomorphism marked † exists by [28, Proposition 1.12] (with its condition (iii.b)), and the isomorphism marked $\diamond$ comes from $\rho^{-1} : \mathrm{Sq}_{A/\mathbb{K}} \, R \xrightarrow{\simeq} R$. On the other hand we have $\rho' : R' \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}} \, R'$, which gives an isomorphism

$$
R \otimes_A L[n] \cong R \otimes_A L[n] \otimes_A L[n].
$$

Applying $\mathrm{RHom}_A(R, -)$ to this isomorphism we get $L[n] \cong L[n] \otimes_A L[n]$, and hence $L \cong A$ and $n = 0$. Thus we get an isomorphism $\phi_0 : R \xrightarrow{\simeq} R'$.

The isomorphism ϕ_0 might not be rigid, but there is some isomorphism ϕ_1 making the diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\phi_1} & R' \\
\rho \downarrow & & \downarrow \rho' \\
\mathrm{Sq}_{A/\mathbb{K}} \, R & \xrightarrow{\mathrm{Sq}_{\mathbf{1}_{A/\mathbb{K}}}(\phi_0)} & \mathrm{Sq}_{A/\mathbb{K}} \, R'
\end{array}
$$

commutative. Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(R, R')$ is a free A-module with basis ϕ_0, it follows that $\phi_1 = a\phi_0$ for some $a \in A^{\times}$. Then the isomorphism $\phi := a^{-1}\phi_0$ is the unique rigid isomorphism $R \xrightarrow{\simeq} R'$. □

In view of this result we are allowed to talk about *the* rigid dualizing complex over A (if it exists). It is denoted by (R_A, ρ_A).

Suppose (M, ρ) is a rigid complex over A relative to $\mathbb{K}$, and $f^* : A \to B$ is a finite homomorphism of $\mathbb{K}$-algebras. Assume $f^{\flat} M$ has bounded cohomology. Then $f^{\flat} M$ has finite flat dimension over $\mathbb{K}$, and according to Theorem 2.7(1) we get an induced rigid complex $f^{\flat}(M, \rho)$ over B relative to $\mathbb{K}$.

Proposition 3.4 *Let $f^* : A \to B$ be a finite homomorphism of $\mathbb{K}$-algebras. Assume the rigid dualizing complex (R_A, ρ_A) over A exists. Define $R_B := f^{\flat} R_A \in \mathsf{D}(\mathsf{Mod}\,B)$ and $\rho_B := f^{\flat}(\rho_A)$. Then (R_B, ρ_B) is a rigid dualizing complex over B.*

Proof The fact that R_B is a dualizing complex over B is proved in [15, Proposition V.2.4]. In particular R_B has bounded cohomology. □

Suppose (M, ρ) is a rigid complex over A relative to $\mathbb{K}$, and $f^* : A \to B$ is an essentially smooth homomorphism of $\mathbb{K}$-algebras. Then by Theorem 2.8(1) we get an induced rigid complex $f^{\sharp}(M, \rho)$ over B relative to $\mathbb{K}$.

Proposition 3.5 *Let A be a $\mathbb{K}$-algebra, and assume A has a rigid dualizing complex (R_A, ρ_A). Let $f^* : A \to B$ be an essentially smooth homomorphism. Define $R_B := f^\sharp R_A$ and $\rho_B := f^\sharp(\rho_A)$. Then (R_B, ρ_B) is a rigid dualizing complex over B.*

Proof The complex R_B is bounded. Hence to check it is dualizing is a local calculation on Spec B. By [28, Proposition 3.2(1)] we can assume that $A \to B$ is smooth. Now [15, Theorem V.8.3] implies R_B is dualizing. □

Theorem 3.6 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra.*

(1) *The algebra A has a rigid dualizing complex (R_A, ρ_A) relative to $\mathbb{K}$, which is unique up to a unique rigid isomorphism.*

(2) *Given a finite homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$\phi_f^{\flat,\text{rig}} : f^\flat(R_A, \rho_A) \xrightarrow{\cong} (R_B, \rho_B).$$

(3) *Given an essentially smooth homomorphism $f^* : A \to B$, there is a unique rigid isomorphism*

$$\phi_f^{\sharp,\text{rig}} : f^\sharp(R_A, \rho_A) \xrightarrow{\cong} (R_B, \rho_B).$$

Proof

(1) We can find algebras and homomorphisms $\mathbb{K} \xrightarrow{f^*} C \xrightarrow{g^*} B \xrightarrow{h^*} A$, where $C = \mathbb{K}[t_1, \ldots, t_n]$ is a polynomial algebra, g^* is surjective and h^* is a localization. By Example 3.2, $(\mathbb{K}, \rho^{\text{tau}})$ is a rigid dualizing complex over $\mathbb{K}$. By Propositions 3.4 and 3.5 the complex

$$R_A := h^\sharp g^\flat f^\sharp \mathbb{K} = A \otimes_B \text{RHom}_C \left(B, \Omega_{C/\mathbb{K}}^n[n] \right)$$

is a rigid dualizing complex over A, with rigidifying isomorphism $\rho_A := h^\sharp g^\flat f^\sharp(\rho^{\text{tau}})$. Uniqueness was proved in Theorem 3.3.

(2/3) Use Propositions 3.4 and 3.5 and Theorem 3.3. □

Corollary 3.7 *Let $f^* : A \to B$ be a finite homomorphism in $\textsf{EFTAlg}/\mathbb{K}$. There exists a unique nondegenerate rigid trace morphism*

$$\text{Tr}_f = \text{Tr}_{B/A} : (R_B, \rho_B) \to (R_A, \rho_A).$$

Proof Since $f^\flat R_A$ is a dualizing complex over B we know that $\text{Hom}_{\text{D}(\text{Mod } B)}(f^\flat R_A, f^\flat R_A) = B$. So by [28, Corollary 5.11], $\text{Tr}_{f;R_A}^\flat : f^\flat R_A \to R_A$ is the unique nondegenerate rigid trace morphism between these two objects.

Composing $\text{Tr}_{f;R_A}^\flat$ with the unique rigid isomorphism $R_B \cong f^\flat R_A$ guaranteed by Theorem 3.3, we get the unique rigid trace $\text{Tr}_f : R_B \to R_A$. □

Corollary 3.8 *Let A and A' be in $\textsf{EFTAlg}/\mathbb{K}$, with rigid dualizing complexes (R_A, ρ_A) and $(R_{A'}, \rho_{A'})$ respectively. Suppose $f^* : A \to A'$ is an essentially étale homomorphism. Then there is exactly one nondegenerate rigid localization morphism*

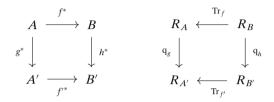$$\text{q}_f = \text{q}_{A'/A} : (R_A, \rho_A) \to (R_{A'}, \rho_{A'}).$$

Proof By Proposition 3.5 we have a rigid complex $f^\sharp R_A$ over A', and by Proposition 2.12 there is a unique rigid localization morphism $q_{f;R_A}^\sharp : R_A \to f^\sharp R_A$. According to Theorem 3.3 there is a unique rigid isomorphism $f^\sharp R_A \cong R_{A'}$. By composing them we get the unique rigid localization map q_f. □

Corollary 3.9 *Let A, B and A' be in* EFTAlg $/\mathbb{K}$, *let $f^* : A \to B$ be a finite homomorphism, and let $g^* : A \to A'$ be a localization. Define $B' := A' \otimes_A B$, and let $f'^* : A' \to B'$ and $h^* : B \to B'$ be the induced homomorphisms. Then*

$$q_g \circ \mathrm{Tr}_f = \mathrm{Tr}_{f'} \circ q_h$$

in $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(R_B, R_{A'})$.

$$
\begin{array}{ccc}
A \xrightarrow{\;f^*\;} B & \qquad & R_A \xleftarrow{\;\mathrm{Tr}_f\;} R_B \\
g^* \downarrow \qquad \downarrow h^* & & q_g \downarrow \qquad \downarrow q_h \\
A' \xrightarrow{\;f'^*\;} B' & & R_{A'} \xleftarrow{\;\mathrm{Tr}_{f'}\;} R_{B'}
\end{array}
$$

Proof The B'-module $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(R_B, R_{A'})$ is free of rank 1, and both $q_g \circ \mathrm{Tr}_f$ and $\mathrm{Tr}_{f'} \circ q_h$ are generators. So there is a unique invertible element $b' \in B'$ such that $\mathrm{Tr}_{f'} \circ q_h = b'\, q_g \circ \mathrm{Tr}_f$. Now by Theorem 2.13 the morphism

$$g^\sharp(\mathrm{Tr}_f) : h^\sharp(R_B, \rho_B) \to g^\sharp(R_A, \rho_A)$$

is a rigid trace morphism relative to $\mathbb{K}$. And so are

$$\mathbf{1} \otimes q_g : g^\sharp(R_A, \rho_A) \to (R_{A'}, \rho_{A'}),$$

$$\mathbf{1} \otimes q_h : h^\sharp(R_B, \rho_B) \to (R_{B'}, \rho_{B'})$$

and

$$\mathrm{Tr}_{f'} : (R_{B'}, \rho_{B'}) \to (R_{A'}, \rho_{A'}).$$

We conclude that both $(\mathbf{1} \otimes q_g) \circ g^\sharp(\mathrm{Tr}_f)$ and $\mathrm{Tr}_{f'} \circ (\mathbf{1} \otimes q_h)$ are nondegenerate rigid trace morphisms $h^\sharp(R_B, \rho_B) \to (R_{A'}, \rho_{A'})$ over A' relative to $\mathbb{K}$, and therefore they must be equal (cf. Theorem 2.5). But $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A')}(h^\sharp R_B, R_{A'})$ is also a free B'-module of rank 1, So $b' = 1$. □

Next comes a surprising result that basically says "all rigid complexes are dualizing". The significance of this result is yet unknown.

Theorem 3.10 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra. Assume A is regular and* Spec A *is connected. Let (M, ρ) be a nonzero rigid complex over A relative to $\mathbb{K}$. Then M is a dualizing complex over A. Hence there exists a unique rigid isomorphism $(M, \rho) \cong (R_A, \rho_A)$.*

The proof is after these two lemmas.

Lemma 3.11 *Suppose* $L \in \mathsf{D}_{\mathsf{f}}^{\mathsf{b}}(\mathsf{Mod}\,A)$ *satisfies* $\mathrm{Ext}_A^i(L, A/\mathfrak{m}) = 0$ *for all* $i \neq 0$ *and all maximal ideals* $\mathfrak{m} \subset A$. *Then* L *is isomorphic to a finitely generated projective module concentrated in degree* 0.

Proof This can be checked locally. Over a local ring $A_\mathfrak{m}$, a minimal free resolution of $L_\mathfrak{m}$ must have a single nonzero term, in degree 0. □

Lemma 3.12 *If* A *is a field then the theorem is true, and moreover* $M \cong A[d]$ *for some integer* d.

Proof Since any dualizing complex over the field A is isomorphic to a shift of A, we see that the rigid dualizing complex satisfies $R_A \cong A[d]$ for some d. Therefore $\mathrm{Sq}_{A/\mathbb{K}}(A[d]) \cong A[d]$.

Consider the rigid complex (M, ρ). We can decompose $M \cong \bigoplus_{i=1}^r A[d_i]$ in $\mathsf{D}(\mathsf{Mod}\,A)$. Then in the setup of Theorem [28, Theorem 2.2] we have

$$\mathrm{Sq}_{A/\mathbb{K}}\,M = \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}\left(A, M \otimes_{\mathbb{K}}^{\mathsf{L}} M\right)$$

$$\cong \bigoplus_{i,j} \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}\left(A, A[d_i] \otimes_{\mathbb{K}}^{\mathsf{L}} A[d_j]\right)$$

$$\cong \bigoplus_{i,j} \left(\mathrm{Sq}_{A/\mathbb{K}}(A[d])\right)[d_i + d_j - 2d].$$

But $M \cong \mathrm{Sq}_{A/\mathbb{K}}\,M$, from which we obtain

$$\bigoplus_{i=1}^r A[d_i] \cong \bigoplus_{i,j=1}^r A[d_i + d_j - d].$$

Thus $r = 1$ and $d_1 = d$. □

Proof of Theorem 3.10 Consider the object $L := \mathrm{RHom}_A(M, R_A) \in \mathsf{D}_{\mathsf{f}}^{\mathsf{b}}(\mathsf{Mod}\,A)$. Take an arbitrary maximal ideal $\mathfrak{m} \subset A$, and define $B := A/\mathfrak{m}$. We get a finite homomorphism $f^* : A \to B$. By Theorem 2.7(1) the complex $\mathrm{RHom}_A(B, M) = f^\flat M$ is a rigid complex over B relative to $\mathbb{K}$. So either $f^\flat M$ is zero, or, by Lemma 3.12, $f^\flat M \cong R_B \cong B[d]$ for some integer d. On the other hand by Theorem 3.6(2) we have an isomorphism $f^\flat R_A \cong R_B$. Thus there are isomorphisms

$$\mathrm{RHom}_A(L, B) \cong \mathrm{RHom}_A(L, B[d])[-d]$$

$$\cong \mathrm{RHom}_A\left(\mathrm{RHom}_A(M, R_A), \mathrm{RHom}_A(B, R_A)\right)[-d]$$

$$\cong \mathrm{RHom}_A(B, M)[-d] = f^\flat M[-d] \cong B^r \qquad (3.13)$$

where $r = 0, 1$. In particular $\mathrm{Ext}_A^i(L, B) = 0$ for $i \neq 0$. Since $\mathfrak{m}$ was arbitrary, Lemma 3.11 tells us that we can assume L is a finitely generated locally free A-module, concentrated in degree 0. Moreover, from the isomorphisms (3.13) we see that the rank of L at any point of $\mathrm{Spec}\,A$ is at most 1. Since L is nonzero and $\mathrm{Spec}\,A$ is connected it follows that L has constant rank 1.

At this stage we have $M \cong R_A \otimes_A L^\vee$, where $L^\vee := \mathrm{Hom}_A(L, A)$. We conclude that M is a dualizing complex over A. By Theorem 3.6(1) there is a unique rigid isomorphism $R_A \cong M$. □

We end this section with some remarks and examples.

Remark 3.14 The standing assumption that the base ring $\mathbb{K}$ has finite global dimension seems superfluous. It is needed for technical reasons (bounded complexes have finite flat dimension), yet we don't know how to remove it. However, it seems necessary for $\mathbb{K}$ to be Gorenstein—see Example 3.16. Also the finiteness of A over $\mathbb{K}$ is important, as Example 3.17 shows. A similar reservation applies to the assumption that A is regular in Theorem 3.10 (Note the mistake in [28, Theorem 0.6]: there too A has to be regular).

Remark 3.15 A result of Van den Bergh (valid even in the noncommutative setup) says the following: if A is a Gorenstein ring, then there is a canonical isomorphism

$$R_A \cong \mathrm{RHom}_A(\mathrm{Sq}_{A/\mathbb{K}} A, A)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$.

Example 3.16 Consider a field $\Bbbk$, and let $\mathbb{K} = A := \Bbbk[t_1, t_2]/(t_1^2, t_2^2, t_1 t_2)$. Then A does not have a rigid dualizing complex relative to $\mathbb{K}$. The reason is that any dualizing complex over the artinian local ring A must be $R \cong A^*[n]$ for some integer n, where $A^* := \mathrm{Hom}_{\Bbbk}(A, \Bbbk)$. Now $\mathrm{Sq}_{A/\mathbb{K}} R \cong R \otimes_{\mathbb{K}}^{\mathrm{L}} R$, which has infinitely many nonzero cohomology modules. So there can be no isomorphism $R \cong \mathrm{Sq}_{A/\mathbb{K}} R$.

Example 3.17 Take any field $\mathbb{K}$, and let $A := \mathbb{K}(t_1, t_2, \ldots)$, the field of rational functions in countably many variables. So A is a noetherian $\mathbb{K}$ algebra, but it is not of essentially finite type. Clearly A has a dualizing complex (e.g. $R := A$), but as shown in [24, Example 3.13], there does not exist a rigid dualizing complex over A relative to $\mathbb{K}$.

Remark 3.18 The paper [17] by de Salas uses an idea similar to Van den Bergh's rigidity to define residues on local rings. However the results there are pretty limited. Lipman (unpublished notes) has an approach to duality using the fundamental class of the diagonal, which is close in spirit to the idea of rigidity.

4 The Twisted Inverse Image Two-Functor

In this section we translate properties of rigid dualizing complexes that were established in Section 3 into properties of certain functors. As before we assume that $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension. All algebras are by default essentially finite type $\mathbb{K}$-algebras, and all algebra homomorphisms are over $\mathbb{K}$.

Here is a review of the notion of two-functor, following [5, Section 5.15]. Let Cat be the two-category of all categories. The objects of Cat are the categories; the 1-morphisms are the functors between categories; and the two-morphisms are the natural transformations between functors. Suppose A is some category. A two-functor (or pseudofunctor) $F : \mathsf{A} \to \mathsf{Cat}$ is a triple $F = (F_0, F_1, F_2)$ consisting of functions of the types explained below. The function F_0 is from the class of objects of A to the class of objects of Cat; i.e. $F_0(A)$ is a category for any $A \in \mathsf{A}$. The function F_1 assigns to any morphism $\alpha_0 \in \mathrm{Hom}_{\mathsf{A}}(A_0, A_1)$ a functor $F_1(\alpha_0) : F_0(A_0) \to F_0(A_1)$.

The function F_2 assigns to a composable pair of morphisms $\alpha_0 \in \mathrm{Hom}_{\mathsf{A}}(A_0, A_1)$ and $\alpha_1 \in \mathrm{Hom}_{\mathsf{A}}(A_1, A_2)$ a natural isomorphism

$$F_2(\alpha_0, \alpha_1) : F_1(\alpha_1 \circ \alpha_0) \xrightarrow{\simeq} F_1(\alpha_1) \circ F_1(\alpha_0)$$

between functors $F_0(A_0) \to F_0(A_2)$. The data (F_0, F_1, F_2) have to satisfy the compatibility condition

$$F_2(\alpha_0, \alpha_1) \circ F_2(\alpha_1 \circ \alpha_0, \alpha_2) = F_2(\alpha_1, \alpha_2) \circ F_2(\alpha_0, \alpha_2 \circ \alpha_1)$$

for any composable triple $A_0 \xrightarrow{\alpha_0} A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3$ of morphisms in A. Moreover for any object $A \in \mathsf{A}$, with identity morphism $\mathbf{1}_A$, it is required that $F_1(\mathbf{1}_A) = \mathbf{1}_{F_0(A)}$, the identity functor of the category $F_0(A)$; and also $F_2(\mathbf{1}_A, \mathbf{1}_A)$ has to be the identity automorphism of the functor of $\mathbf{1}_{F_0(A)}$.

We are going to construct a two-functor $F : \mathsf{EFTAlg}/\mathbb{K} \to \mathsf{Cat}$. Its 0-component F_0 will assign the category $F_0(A) := \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A)$ to any algebra A. The 1-component F_1 will assign a functor $F_1(f^*) = f^! : \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, B)$ to every algebra homomorphism $f^* : A \to B$. And for every composable pair of homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ there will be a natural isomorphism $F_2(f^*, g^*) = \phi_{f,g}$. Furthermore there will be isomorphisms of two-functors $f^! \cong f^{\flat}$ and $f^! \cong f^{\sharp}$ on suitable subcategories. These constructions will require a lot of notation, and we tried to make this notation sensible. Isomorphisms between functors of the same family (i.e. belonging to the same two-functor) will be labelled by "ϕ" with modifiers (e.g. $\phi_{f,g} : (f \circ g)^! \xrightarrow{\simeq} g^! f^!$). Isomorphisms between functors belonging to different families will be labelled by "ψ" with modifiers (e.g. $\psi_f^{\sharp} : f^{\sharp} \xrightarrow{\simeq} f^!$). When applying an isomorphism such as $\psi_f^{\sharp}$ to a particular object, say M, the notation will be $\psi_{f;M}^{\sharp} : f^{\sharp} M \xrightarrow{\simeq} f^! M$.

By Theorem 3.6 any $\mathbb{K}$-algebra A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism. For the sake of legibility we will often keep the rigidifying isomorphism ρ_A implicit, and refer to the rigid dualizing complex R_A.

Definition 4.1 Given a $\mathbb{K}$-algebra A, with rigid dualizing complex R_A, define the *auto-duality functor* of $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$ relative to $\mathbb{K}$ to be $\mathsf{D}_A := \mathrm{RHom}_A(-, R_A)$.

Note that the functor D_A exchanges the subcategories $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A)$ and $\mathsf{D}_{\mathrm{f}}^-(\mathsf{Mod}\, A)$. Given a homomorphism of algebras $f^* : A \to B$ the functor $\mathrm{L}f^* = B \otimes_A^{\mathrm{L}} -$ sends $\mathsf{D}_{\mathrm{f}}^-(\mathsf{Mod}\, A)$ into $\mathsf{D}_{\mathrm{f}}^-(\mathsf{Mod}\, B)$. This permits the next definition.

Definition 4.2 Let $f^* : A \to B$ be a homomorphism in $\mathsf{EFTAlg}/\mathbb{K}$. We define the *twisted inverse image* functor

$$f^! : \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, B)$$

relative to $\mathbb{K}$ as follows.

(1) If $A = B$ and $f^* = \mathbf{1}_A$ (the identity automorphism) then we let $f^! := \mathbf{1}_{\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A)}$ (the identity functor).
(2) Otherwise we define $f^! := \mathsf{D}_B \mathrm{L}f^* \mathsf{D}_A$.

Recall that for composable homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ we sometimes write $(f \circ g)^*$ instead of $g^* \circ f^*$.

Definition 4.3 Given two homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ in $\mathsf{EFTAlg}/\mathbb{K}$ we define an isomorphism

$$\phi_{f,g} : (f \circ g)^! \xrightarrow{\cong} g^! f^!$$

of functors $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,C)$ as follows.

(1) If either $A = B$ and $f^* = \mathbf{1}_A$, or $B = C$ and $g^* = \mathbf{1}_B$, then $\phi_{f,g}$ is just the identity automorphism of $(f \circ g)^! = g^! f^!$.

(2) Otherwise we use the adjunction isomorphism $\mathbf{1}_{\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,B)} \xrightarrow{\cong} \mathrm{D}_B \mathrm{D}_B$, together with the obvious isomorphism $\mathrm{L}(f \circ g)^* = \mathrm{L}(g^* \circ f^*) \cong \mathrm{L}g^* \mathrm{L}f^*$, to obtain an isomorphism

$$(f \circ g)^! = \mathrm{D}_C \mathrm{L}(f \circ g)^* \mathrm{D}_A \cong \mathrm{D}_C \mathrm{L}g^* \mathrm{L}f^* \mathrm{D}_A$$

$$\cong \mathrm{D}_C \mathrm{L}g^* \mathrm{D}_B \mathrm{D}_B \mathrm{L}f^* \mathrm{D}_A = g^! f^!.$$

Proposition 4.4 *For three homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} D$ in $\mathsf{EFTAlg}/\mathbb{K}$ the isomorphisms $\phi_{-,-}$ satisfy the compatibility condition*

$$\phi_{g,h} \circ \phi_{f,g\circ h} = \phi_{f,g} \circ \phi_{f\circ g,h} : (f \circ g \circ h)^! \xrightarrow{\cong} h^! g^! f^!.$$

Thus the assignment $f^* \mapsto f^!$ is the 1-component of a two-functor $\mathsf{EFTAlg}/\mathbb{K} \to \mathsf{Cat}$, whose 0-component is $A \mapsto \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$.

In stating the proposition we were a bit sloppy with notation; for instance we wrote $\phi_{f,g}$, whereas the correct expression is $h^!(\phi_{f,g})$. This was done for the sake of legibility, and we presume the reader can fill in the omissions (also in what follows).

Proof By definition

$$(f \circ g \circ h)^! M = \mathrm{D}_D \mathrm{L}(f \circ g \circ h)^* \mathrm{D}_A M$$

and

$$h^! g^! f^! M = \mathrm{D}_D \mathrm{L}h^* \mathrm{D}_C \mathrm{D}_C \mathrm{L}g^* \mathrm{D}_B \mathrm{D}_B \mathrm{L}f^* \mathrm{D}_A M.$$

The two isomorphism $\phi_{g,h} \circ \phi_{f,g\circ h}$ and $\phi_{f,g} \circ \phi_{f\circ g,h}$ differ only by the order in which the adjunction isomorphisms $\mathbf{1}_{\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,B)} \cong \mathrm{D}_B \mathrm{D}_B$ and $\mathbf{1}_{\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,C)} \cong \mathrm{D}_C \mathrm{D}_C$ are applied, and correspondingly an isomorphism $C \cong C \otimes_B^{\mathrm{L}} B$ is replaced by $D \cong D \otimes_B^{\mathrm{L}} B$. Due to standard identities the net effect is that $\phi_{g,h} \circ \phi_{f,g\circ h} = \phi_{f,g} \circ \phi_{f\circ g,h}$. $\qquad\square$

Suppose $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ are finite homomorphisms in $\mathsf{EFTAlg}/\mathbb{K}$. Adjunction gives rise to an isomorphism

$$g^\flat f^\flat M = \mathrm{RHom}_B(C, \mathrm{RHom}_A(B, M)) \cong \mathrm{RHom}_A(C, M) = (f \circ g)^\flat M$$

for any $M \in \mathsf{D}(\mathsf{Mod}\,A)$, and thus there is an isomorphism of functors

$$\phi_{f,g}^\flat : (f \circ g)^\flat \xrightarrow{\cong} g^\flat f^\flat.$$

So $f^* \mapsto f^\flat$ is a two-functor on the subcategory of $\mathsf{EFTAlg}/\mathbb{K}$ consisting of all algebras, but only finite homomorphisms.

Theorem 4.5

(1) *Let $f^* : A \to B$ be a finite homomorphism in $\mathsf{EFTAlg}/\mathbb{K}$. The isomorphism $\phi_f^{\flat,\mathrm{rig}} : f^\flat R_A \xrightarrow{\simeq} R_B$ of Theorem 3.6(2) induces an isomorphism*

$$\psi_f^\flat : f^\flat \xrightarrow{\simeq} f^!$$

of functors $\mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\,A) \to \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\,B)$.

(2) *Given two finite homomorphism $A \xrightarrow{f^*} B \xrightarrow{g^*} C$, there is equality*

$$\psi_g^\flat \circ \psi_f^\flat \circ \phi_{f,g}^\flat = \phi_{f,g} \circ \psi_{f \circ g}^\flat$$

as isomorphisms of functors $(f \circ g)^\flat \xrightarrow{\simeq} g^! f^!$. Thus the isomorphisms $\psi_f^\flat$ are two-functorial.
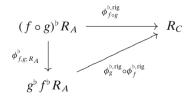
Proof

(1) Take $M \in \mathsf{D}_\mathrm{f}^\mathrm{b}(\mathsf{Mod}\,A)$. We then have a series of isomorphisms

$$\begin{aligned}
f^! M &= \mathrm{RHom}_B\big(B \otimes_A^\mathrm{L} \mathrm{RHom}_A(M, R_A), R_B\big) \\
&\cong^\dagger \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_A), R_B\big) \\
&\cong^\diamond \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_A), \mathrm{RHom}_A(B, R_A)\big) \\
&\cong^\nabla \mathrm{RHom}_A(B, M) = f^\flat M,
\end{aligned}$$

where the isomorphism marked † is by the Hom-tensor adjunction; the isomorphism marked ◇ is induced by $\phi_f^{\flat,\mathrm{rig}}$; the isomorphism ▽ is because R_A is a dualizing complex.

(2) Given a second finite homomorphism $g^* : B \to C$ we obtain a commutative diagram of rigid isomorphisms

$$\begin{array}{ccc}
(f \circ g)^\flat R_A & \xrightarrow{\;\;\phi_{f \circ g}^{\flat,\mathrm{rig}}\;\;} & R_C \\
{\scriptstyle \phi_{f,g;R_A}^\flat} \downarrow & \nearrow {\scriptstyle \phi_g^{\flat,\mathrm{rig}} \circ \phi_f^{\flat,\mathrm{rig}}} & \\
g^\flat f^\flat R_A & &
\end{array}$$

This is due to the fact that $\phi_{f,g;R_A}^\flat$ is a rigid isomorphism [28, Theorem 5.3(2)], together with the uniqueness in Theorem 3.6(2). From this, using standard identities, we deduce the desired equality. □

Next let $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ be essentially smooth homomorphisms. Then according to [28, Proposition 3.4] there is an isomorphism of functors $\phi_{f,g}^\sharp : (f \circ g)^\sharp \xrightarrow{\simeq} g^\sharp f^\sharp$. The

isomorphisms $\phi^{\sharp}_{-,-}$ are two-functorial on the subcategory of $\mathsf{EFTAlg}\,/\mathbb{K}$ consisting of essentially smooth homomorphisms.

Theorem 4.6

(1) *Let $f^* : A \to B$ be an essentially smooth homomorphism in $\mathsf{EFTAlg}\,/\mathbb{K}$. The isomorphism $\phi^{\sharp,\mathrm{rig}}_f : f^{\sharp} R_A \xrightarrow{\simeq} R_B$ of Theorem 3.6(3) induces an isomorphism*

$$\psi^{\sharp}_f : f^{\sharp} \xrightarrow{\simeq} f^!$$

of functors $\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\,A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\,B)$.

(2) *Given two essentially smooth homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$, there is equality*

$$\psi^{\sharp}_g \circ \psi^{\sharp}_f \circ \phi^{\sharp}_{f,g} = \phi_{f,g} \circ \psi^{\sharp}_{f \circ g}$$

as isomorphisms of functors $(f \circ g)^{\sharp} \xrightarrow{\simeq} g^! f^!$. Thus the isomorphisms $\psi^{\sharp}_f$ are two-functorial.

Proof

(1) We can assume that f^* has relative dimension n (see [28, Lemma 6.1]). Take any $M \in \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\,A)$. Then

$$\begin{aligned}
f^! M &= \mathrm{RHom}_B\big(B \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(M, R_A), R_B\big) \\
&\cong^{\dagger} \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_A), R_B\big) \\
&\cong^{\diamond} \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_A), \Omega^n_{B/A}[n] \otimes_A R_A\big) \\
&\cong^{\triangledown} \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_A), R_A\big) \otimes_A \Omega^n_{B/A}[n] \\
&\cong^{\ddagger} M \otimes_A \Omega^n_{B/A}[n] = f^{\sharp} M,
\end{aligned}$$

where the isomorphism marked $\dagger$ is by the Hom-tensor adjunction; the isomorphism marked $\diamond$ is induced by $\phi^{\sharp,\mathrm{rig}}_f$; the isomorphism $\triangledown$ is due to [28, Proposition 1.12]; and the isomorphism $\ddagger$ is by the adjunction isomorphism $M \cong \mathrm{D}_A \mathrm{D}_A M$. We let $\psi^{\sharp}_{f;M} : f^{\sharp} M \xrightarrow{\simeq} f^! M$ be the composed isomorphism.

(2) Given a second essentially smooth homomorphism $g^* : B \to C$ we obtain a commutative diagram of rigid isomorphisms

$$\begin{array}{ccc}
(f \circ g)^{\sharp} R_A & \xrightarrow{\phi^{\sharp,\mathrm{rig}}_{f \circ g}} & R_C \\
\phi^{\sharp}_{f,g;R_A} \Big\downarrow & \nearrow{\scriptstyle \phi^{\sharp,\mathrm{rig}}_g \circ \phi^{\sharp,\mathrm{rig}}_f} & \\
g^{\sharp} f^{\sharp} R_A & &
\end{array}$$

This is due to the fact that $\phi^{\sharp}_{f,g;R_A}$ is a rigid isomorphism [28, Theorem 6.3(3)], together with the uniqueness in Theorem 3.6(3). From this, using standard identities, we deduce the desired equality. $\qquad\square$

The next result explains the dependence of the twisted inverse image two-functor $f \mapsto f^!$ on the base ring $\mathbb{K}$. Assume $\mathbb{L}$ is an essentially finite type $\mathbb{K}$-algebra which is regular (but maybe not essentially smooth over $\mathbb{K}$). Just like for $\mathbb{K}$, any essentially finite type $\mathbb{L}$-algebra A has a rigid dualizing complex relative to $\mathbb{L}$, which we denote by $(R_{A/\mathbb{L}}, \rho_{A/\mathbb{L}})$. For any homomorphism $f^* : A \to B$ of $\mathbb{L}$-algebras there is a corresponding twisted inverse image functor $f^{!/\mathbb{L}} : \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, B)$, constructed using $R_{A/\mathbb{L}}$ and $R_{B/\mathbb{L}}$. Let $(R_{\mathbb{L}}, \rho_{\mathbb{L}})$ be the rigid dualizing complex of $\mathbb{L}$ relative to $\mathbb{K}$.

Proposition 4.7 *Let A be an essentially finite type $\mathbb{L}$-algebra. Then $R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}}$ is a dualizing complex over A, and it has an induced rigidifying isomorphism relative to $\mathbb{K}$. Hence there is a unique isomorphism $R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}} \cong R_A$ in $\mathsf{D}_{\mathrm{f}}^{\mathrm{b}}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.*

Proof We might as well assume $\mathrm{Spec}\,\mathbb{L}$ to be connected (cf. [28, Lemma 6.1]). Since $\mathbb{L}$ is regular, one has $R_{\mathbb{L}} \cong P[n]$ for some invertible $\mathbb{L}$-module P and some integer n. Therefore $R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}}$ is a dualizing complex over A. According to Theorem 2.12 the complex $R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}}$ has an induced rigidifying isomorphism $\rho_{\mathbb{L}} \otimes \rho_{A/\mathbb{L}}$. Now use Theorem 3.3. $\qquad\square$

Corollary 4.8 *There is a canonical isomorphism*

$$(f^* \mapsto f^!) \cong (f^* \mapsto f^{!/\mathbb{L}})$$

of two-functors $\mathsf{EFTAlg}\,/\mathbb{L} \to \mathsf{Cat}$.

Proof The twist $R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} -$ gets canceled out in

$$f^! M \cong \mathrm{RHom}_A\big(\mathrm{RHom}_A(M, R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}}), R_{\mathbb{L}} \otimes_{\mathbb{L}}^{\mathrm{L}} R_{A/\mathbb{L}}\big). \qquad\square$$

Example 4.9 Take $\mathbb{K} := \mathbb{Z}$ and $\mathbb{L} := \mathbb{F}_p = \mathbb{Z}/(p)$ for some prime number p. Then $R_{\mathbb{L}} = \mathbb{L}[-1]$, and for any $A \in \mathsf{EFTAlg}\,/\mathbb{L}$ we have $R_{A/\mathbb{L}} \cong R_A[1]$.

The final result of this section connects our constructions to those of [15]. We shall restrict attention to the category $\mathsf{FTAlg}\,/\mathbb{K}$ of finite type $\mathbb{K}$-algebras. Given a homomorphism $f^* : A \to B$ let us denote by

$$f^{!(\mathrm{G})} : \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, A) \to \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\, B)$$

the twisted inverse image from [15].

Theorem 4.10 *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring.*

(1) *Given $A \in \mathsf{FTAlg}\,/\mathbb{K}$ let $\pi_A^* : \mathbb{K} \to A$ be the structural homomorphism, let $R_A^{(\mathrm{G})} := \pi_A^{!(\mathrm{G})}\mathbb{K}$, and let R_A be the rigid dualizing complex of A. Then there is an isomorphism $R_A^{(\mathrm{G})} \cong R_A$ in $\mathsf{D}(\mathsf{Mod}\, A)$.*

(2) *There is an isomorphism $(f^* \mapsto f^!) \cong (f^* \mapsto f^{!(\mathrm{G})})$ of two-functors $\mathsf{FTAlg}\,/\mathbb{K} \to \mathsf{Cat}$.*

Proof

(1) Take homomorphisms $\mathbb{K} \xrightarrow{f^*} C \xrightarrow{g^*} B \xrightarrow{h^*} A$ as in the proof of Theorem 3.6(1). Then $R_A \cong h^\sharp g^\flat f^\sharp \mathbb{K}$, and also $R_A^{(G)} \cong h^\sharp g^\flat f^\sharp \mathbb{K}$.

(2) For any A fix an isomorphism $\psi_A^{(G)} : R_A^{(G)} \xrightarrow{\cong} R_A$. Let $\mathrm{D}_A^{(G)} :=$ $\mathrm{RHom}_A(-, R_A^{(G)})$, so we get an induced isomorphism $\psi_A^{(G)} : \mathrm{D}_A^{(G)} \xrightarrow{\cong} \mathrm{D}_A$ between the associated auto-duality functors. It is known that the two-functor $(f^* \mapsto f^{!(G)})$ also satisfies $f^{!(G)} \cong \mathrm{D}_B^{(G)} \, \mathrm{L} f^* \, \mathrm{D}_A^{(G)}$. In this way we obtain an isomorphism of two-functors $\psi^{(G)} : f^{!(G)} \xrightarrow{\cong} f^!$. □

Remark 4.11 If A is a flat $\mathbb{K}$-algebra, then flat base change for the theory in [15] endows $R_A^{(G)}$ with a rigidifying isomorphism, thus making the isomorphism $R_A^{(G)} \cong R_A$ canonical. (see Yekutieli, in preparation). So in case $\mathbb{K}$ is a field one has a canonical isomorphism $f^! \cong f^{!(G)}$ of two-functors, and one may try to ask more precise questions, such as compatibility with the transformations in Theorems 4.5 and 4.6.

5 Functorial Traces and Localizations

In this section again we work over a fixed base ring $\mathbb{K}$, which is assumed to be regular, noetherian and of finite Krull dimension. Recall that to each homomorphism $f^* : A \to B$ in $\mathsf{EFTAlg}/\mathbb{K}$ we constructed a twisted inverse image functor $f^! : \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A) \to \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, B)$.

Definition 5.1 Let $f^* : A \to B$ be a finite homomorphism in $\mathsf{EFTAlg}/\mathbb{K}$. Take any $M \in \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A)$. Then by Theorem 4.5(1) there is an isomorphism $\psi_{f;M}^\flat : f^\flat M \xrightarrow{\cong} f^! M$, and by formula (2.3) there is a morphism $\mathrm{Tr}_{f;M}^\flat : f^\flat M \to M$. Define

$$\mathrm{Tr}_{f;M} := \mathrm{Tr}_{f;M}^\flat \circ (\psi_{f;M}^\flat)^{-1} : f^! M \to M.$$

On the level of functors this becomes a morphism

$$\mathrm{Tr}_f : f_* f^! \to \mathbf{1}$$

of functors from $\mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A)$ to itself, called the *functorial trace map*.

Proposition 5.2

(1) *The trace maps* Tr_f *are nondegenerate. Namely, given a finite homomorphism* $f^* : A \to B$ *and an object* $M \in \mathsf{D}_\mathrm{f}^+(\mathsf{Mod}\, A)$, *the morphism* $\mathrm{Tr}_{f;M} : f^! M \to M$ *is a nondegenerate trace morphism, in the sense of Definition* 2.3.

(2) *The trace maps* Tr_f *are two-functorial for finite homomorphisms. I.e. in the setup of Definition* 4.3, *with both* f^*, g^* *finite, one has*

$$\mathrm{Tr}_{f \circ g} = \mathrm{Tr}_f \circ \mathrm{Tr}_g \circ \phi_{f,g}.$$

Proof

(1) This is true because $\mathrm{Tr}_f : R_B \to R_A$ is a nondegenerate trace morphism. See Corollary 3.7.
(2) This follows from Theorem 4.5(2). □

Definition 5.3 Let $f^* : A \to B$ be an essentially étale homomorphism in $\mathsf{EFTAlg}\,/\mathbb{K}$. Composing the localization map $\mathsf{q}_f^\sharp : \mathbf{1} \to f_* f^\sharp$ of Definition 2.8 with the isomorphism $\psi_f^\sharp : f^\sharp \xrightarrow{\simeq} f^!$ of Theorem 4.6(1), we define the *functorial localization map*

$$\mathsf{q}_f : \mathbf{1} \to f_* f^!,$$

which is a morphism of functors from $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$ to itself.

Proposition 5.4

(1) *The localization maps* q_f *are nondegenerate. Namely, given an essentially étale homomorphism* $f^* : A \to B$ *and an object* $M \in \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$, *the morphism* $\mathsf{q}_{f;M} : M \to f^! M$ *is a nondegenerate localization morphism, in the sense of Definition* 2.9.
(2) *The localization maps* q_f *are two-functorial for essentially étale homomorphisms. I.e. in the setup of Definition 4.3, with both* f^*, g^* *essentially étale, one has*

$$\phi_{f,g} \circ \mathsf{q}_{f \circ g} = \mathsf{q}_g \circ \mathsf{q}_f .$$

Proof Assertion (1) is true because $\mathsf{q}_{f;M}^\sharp : M \to f^\sharp M$ is nondegenerate; see Proposition 2.11. Assertion (2) is an immediate consequence of Theorem 4.6(2). □
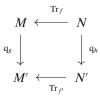
Proposition 5.5 *In the setup of Corollary* 3.9 *there is equality*

$$\mathsf{q}_g \circ \mathrm{Tr}_f = \mathrm{Tr}_{f'} \circ \phi_{g,f'} \circ \phi_{f,h}^{-1} \circ \mathsf{q}_h$$

of morphisms of functors $f_* f^! \to g_* g^!$ *from* $\mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$ *to itself.*

Proof The functorial trace maps Tr_f are induced from the corresponding trace maps between the rigid dualizing complexes, via double dualization (cf. Definition 5.1). Likewise for the functorial localization maps q_f. Thus the corollary is a consequence of Corollary 3.9. □
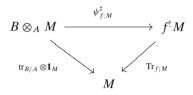
Here is an illustration of Proposition 5.5. Take $M \in \mathsf{D}_{\mathrm{f}}^+(\mathsf{Mod}\,A)$, and define $N := f^! M$, $M' := g^! M$ and $N' := h^! N$. Using the isomorphism $\phi_{g,f'} \circ \phi_{f,h}^{-1}$ we identify $N' = f'^! M'$. Then the diagram

$$
\begin{array}{ccc}
M & \xleftarrow{\ \mathrm{Tr}_f\ } & N \\
{\scriptstyle \mathsf{q}_g}\downarrow & & \downarrow{\scriptstyle \mathsf{q}_h} \\
M' & \xleftarrow{\ \mathrm{Tr}_{f'}\ } & N'
\end{array}
$$

is commutative.

For a finite flat homomorphism $f^* : A \to B$ we denote by $\mathrm{tr}_{B/A} : B \to A$ the usual trace map. Thus $\mathrm{tr}_{B/A}(b)$ is the trace of the operator b acting by multiplication on the locally free A-module B.

Theorem 5.6 *Suppose* $f^* : A \to B$ *is a finite étale homomorphism in* $\mathsf{EFTAlg}\,/\mathbb{K}$. *Then for any* $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ *the diagram*

$$
\begin{array}{ccc}
B \otimes_A M & \xrightarrow{\ \ \psi^{\sharp}_{f;M}\ \ } & f^! M \\
& \searrow & \swarrow \\
\mathrm{tr}_{B/A} \otimes \mathbf{1}_M & & \mathrm{Tr}_{f;M} \\
& M &
\end{array}
$$

is commutative.

Note that

$$
\psi^{\sharp}_{f;M}(b \otimes m) = b \cdot \mathsf{q}_{f;M}(m)
$$

for any $b \in B$ and $m \in M$.

We need a few lemmas for the proof of the theorem.

Lemma 5.7 *Suppose* $M, N, R, S \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$, *with* R, S *having finite injective dimension. Then there is an isomorphism*

$$
M \otimes^{\mathrm{L}}_A \mathrm{RHom}_A\big(\mathrm{RHom}_A(N, R), S\big) \to \mathrm{RHom}_A\big(\mathrm{RHom}_A(M \otimes^{\mathrm{L}}_A N, R), S\big),
$$

which is functorial in these four complexes, and, when $R = S$, *it commutes with the adjunction morphisms*

$$
M \otimes^{\mathrm{L}}_A N \to M \otimes^{\mathrm{L}}_A \mathrm{RHom}_A\big(\mathrm{RHom}_A(N, R), R\big)
$$

and

$$
M \otimes^{\mathrm{L}}_A N \to \mathrm{RHom}_A\big(\mathrm{RHom}_A(M \otimes^{\mathrm{L}}_A N, R), R\big).
$$

Proof First consider A-modules P, K, L. If P is a finitely generated projective module, then the obvious homomorphisms of A-modules

$$
P \otimes_A \mathrm{Hom}_A(K, L) \to \mathrm{Hom}_A\big(\mathrm{Hom}_A(P, A) \otimes_A K, L\big) \tag{5.1}
$$

and

$$
\mathrm{Hom}_A(P, A) \otimes_A \mathrm{Hom}_A(K, L) \to \mathrm{Hom}_A(P \otimes_A K, L) \tag{5.2}
$$

are bijective.

Now choose a resolution $P \to M$ by a bounded above complex of finitely generated projective A-modules. Also choose resolutions $R \to I$ and $S \to J$ by bounded complexes of injective modules. So

$$
M \otimes^{\mathrm{L}}_A \mathrm{RHom}_A\big(\mathrm{RHom}_A(N, R), S\big) = P \otimes_A \mathrm{Hom}_A\big(\mathrm{Hom}_A(N, I), J\big)
$$

and

$$\text{RHom}_A\big(\text{RHom}_A(M \otimes_A^{\text{L}} N, R), S\big) = \text{Hom}_A\big(\text{Hom}_A(P \otimes_A N, I), J\big).$$

Because each P^i is a finitely generated projective A-module, we have a bijection

$$P^i \otimes_A \text{Hom}_A\big(\text{Hom}_A(N^j, I^k), J^l\big)$$
$$\cong \text{Hom}_A\big(\text{Hom}_A(P^i, A) \otimes_A \text{Hom}_A(N^j, I^k), J^l\big)$$

for any i, j, k, l. Indeed, this is exactly Eq. 5.1 with $K := \text{Hom}_A(N^j, I^k)$ and $L := J^l$. Since the three complexes N, I, J are bounded it follows that we have a canonical isomorphism of DG A-modules

$$P \otimes_A \text{Hom}_A\big(\text{Hom}_A(N, I), J\big)$$
$$\cong \text{Hom}_A\big(\text{Hom}_A(P, A) \otimes_A \text{Hom}_A(N, I), J\big). \tag{5.3}$$

Similarly, using Eq. 5.2, we have a canonical isomorphisms of DG A-modules

$$\text{Hom}_A\big(\text{Hom}_A(P, A) \otimes_A \text{Hom}_A(N, I), J\big)$$
$$\cong \text{Hom}_A\big(\text{Hom}_A(P \otimes_A N, I), J\big). \tag{5.4}$$

Finally, is $S = R$ then we can take $J = I$, and then it is easy to track where $P \otimes_A N$ gets mapped in Eqs. 5.3 and 5.4. □

Given $M \in \mathsf{D}_{\text{f}}^{\text{b}}(\mathsf{Mod}\, A)$ let's write

$$\chi_M := \text{Tr}_{f;M} \circ \psi_{f;M}^{\sharp} : f^{\sharp} M \to M$$

(this is a temporary definition). Since $f^{\sharp} A = B$ and $f^{\sharp} M = B \otimes_A M$, there is a functorial isomorphism $M \otimes_A^{\text{L}} f^{\sharp} A \xrightarrow{\cong} f^{\sharp} M$.

Lemma 5.8 *The morphism* $\chi_M : f^{\sharp} M \to M$ *is induced from* $\chi_A : f^{\sharp} A \to A$ *via the tensor operation* $M \otimes_A^{\text{L}} -$. *Namely the diagram*

$$
\begin{array}{ccc}
M \otimes_A^{\text{L}} f^{\sharp} A & \xrightarrow{\ \cong\ } & f^{\sharp} M \\
{\scriptstyle \mathbf{1}_M \otimes \chi_A} \downarrow & & \downarrow {\scriptstyle \chi_M} \\
M \otimes_A^{\text{L}} A & \xrightarrow{\ \cong\ } & M
\end{array}
$$

is commutative.

Proof Take any perfect complex $N \in \mathsf{D}_f^b(\mathsf{Mod}\, A)$. Consider the sequence of morphisms

$$
\begin{aligned}
M \otimes_A^L f^\sharp N &= M \otimes_A^L (B \otimes_A N) \\
&\stackrel{\simeq}{\to}^\dagger M \otimes_A^L \left(B \otimes_A \mathrm{RHom}_A(\mathrm{RHom}_A(N, R_A), R_A) \right) \\
&\stackrel{\simeq}{\to}^\diamond M \otimes_A^L \mathrm{RHom}_A(\mathrm{RHom}_A(N, R_A), R_B) \\
&\to^\nabla M \otimes_A^L \mathrm{RHom}_A(\mathrm{RHom}_A(N, R_A), R_A) \\
&\stackrel{\simeq}{\to}^\dagger M \otimes_A^L N
\end{aligned}
\tag{5.5}
$$

in which † come from the adjunction isomorphism

$$
N \cong \mathrm{RHom}_A(\mathrm{RHom}_A(N, R_A), R_A);
$$

$\diamond$ comes from the localization map $\mathsf{q}_f : R_A \to R_B$; and ∇ comes from the trace map $\mathrm{Tr}_f : R_B \to R_A$. By comparing these morphisms to the definition of $\psi_{f;M}^\sharp$ in Theorem 4.6(1), and the definition of $\mathrm{Tr}_{f;M}$ in Definition 5.1, we see that the composition of all the morphisms in Eq. 5.5 is precisely

$$
\mathbf{1}_M \otimes \chi_N : M \otimes_A^L f^\sharp N \to M \otimes_A^L N.
$$

Now $M \otimes_A^L N$ is also in $\mathsf{D}_f^b(\mathsf{Mod}\, A)$, because N is perfect. In parallel to the sequence of morphisms (5.5) there is another sequence

$$
\begin{aligned}
f^\sharp \left(M \otimes_A^L N \right) &= B \otimes_A M \otimes_A^L N \\
&\stackrel{\simeq}{\to}^\dagger B \otimes_A \mathrm{RHom}_A(\mathrm{RHom}_A(M \otimes_A^L N, R_A), R_A) \\
&\stackrel{\simeq}{\to}^\diamond \mathrm{RHom}_A(\mathrm{RHom}_A(M \otimes_A^L N, R_A), R_B) \\
&\to^\nabla \mathrm{RHom}_A(\mathrm{RHom}_A(M \otimes_A^L N, R_A), R_A) \\
&\stackrel{\simeq}{\to}^\dagger M \otimes_A^L N.
\end{aligned}
\tag{5.6}
$$

The composition of all these morphisms is $\chi_{M \otimes_A^L N}$.

Since $A \to B$ is flat it follows that R_B has finite injective dimension over A. According to Lemma 5.7 at each step there is a canonical isomorphism from the object in Eq. 5.5 to the corresponding object in Eq. 5.6, and together these form a big commutative ladder. Therefore we get a commutative diagram

$$
\begin{array}{ccc}
M \otimes_A^L f^\sharp N & \xrightarrow{\;\cong\;} & f^\sharp(M \otimes_A^L N) \\
{\scriptstyle \mathbf{1}_M \otimes \chi_N} \downarrow & & \downarrow {\scriptstyle \chi_{M \otimes_A^L N}} \\
M \otimes_A^L N & \xrightarrow{\;=\;} & M \otimes_A^L N
\end{array}
$$

functorial in M and N. Taking $N := A$ we get the desired assertion. $\qquad\square$

Lemma 5.9 *The morphism*

$$\chi_A : f^\sharp \left(A, \rho_A^{\text{tau}} \right) \to \left(A, \rho_A^{\text{tau}} \right)$$

is a nondegenerate rigid trace morphism relative to A.

Proof Since $f^\sharp \cong f^\flat$ and $\text{Tr}_{f;A}^\flat : f^\flat A \to A$ is a nondegenerate rigid trace morphism relative to A, it follows that $b\,\chi_A$ is a nondegenerate rigid trace morphism relative to A for some unique $b \in B^\times$. We are going to prove that $b = 1$.

Take the rigid dualizing complex $(R_A, \rho_A) \in \mathsf{D}_{\text{f}}^{\text{b}}(\mathsf{Mod}\,A)_{\text{rig}/\mathbb{K}}$. Recall the tensor operation for rigid complexes (Theorem 2.12). Trivially the isomorphism

$$(R_A, \rho_A) \otimes_A^{\text{L}} (A, \rho^{\text{tau}}) \cong (R_A, \rho_A)$$

is rigid over A relative to $\mathbb{K}$. According to [28, Theorem 6.3(2)] the isomorphism

$$(R_A, \rho_A) \otimes_A^{\text{L}} f^\sharp(A, \rho^{\text{tau}}) \cong f^\sharp(R_A, \rho_A)$$

is rigid over B relative to $\mathbb{K}$. Since the tensor operation of rigid complexes is functorial, it follows that the morphism

$$\mathbf{1}_{R_A} \otimes b\,\chi_A : f^\sharp(R_A, \rho_A) \to (R_A, \rho_A)$$

is a nondegenerate rigid trace morphism. Next, from Lemma 5.8 we know that

$$\mathbf{1}_{R_A} \otimes \chi_A = \chi_{R_A} : f^\sharp R_A \to R_A.$$

We conclude that

$$b\,\chi_{R_A} : f^\sharp(R_A, \rho_A) \to (R_A, \rho_A)$$

nondegenerate rigid trace morphism over A relative to $\mathbb{K}$.

On the other hand

$$\chi_{R_A} = \text{Tr}_f^\flat \circ \left(\psi_f^{\flat,\text{rig}} \right)^{-1} \circ \psi_f^{\sharp,\text{rig}} : f^\sharp(R_A, \rho_A) \to (R_A, \rho_A),$$

which itself is a rigid trace morphism over A relative to $\mathbb{K}$. Since there is only one nondegenerate rigid trace morphism $f^\sharp(R_A, \rho_A) \to (R_A, \rho_A)$, it follows that $b\,\chi_{R_A} = \chi_{R_A}$, and hence $b = 1$. $\qquad\square$

Proof of Theorem 5.6 By definition $\text{tr}_{B/A} \otimes \mathbf{1}_M$ is induced from $\text{tr}_{B/A}$. And according to Lemma 5.8 the morphism $\chi_M = \text{Tr}_{f;M} \circ \psi_{f;M}^\sharp$ is also induced from χ_A. Therefore it is enough to look at $M = A$. We must show that $\text{Tr}_{f;A} \circ \psi_{f;A}^\sharp = \text{tr}_{B/A}$.

Let $B^{\text{e}} := B \otimes_A B$. According to [28, Proposition 3.15] we know that there is a canonical ring isomorphism $B^{\text{e}} \cong B \times B'$, where the factor B' is the kernel of the multiplication map $B^{\text{e}} \to B$. Thus the surjective B^{e}-module homomorphism $B^{\text{e}} \to B$ has a canonical splitting $\nu : B \to B^{\text{e}}$. From the proofs of [28, Theorems 6.3(2) and 3.14(3)] we see that the rigidifying isomorphism $\rho := f^\sharp(\rho^{\text{tau}})$ of $B = f^\sharp A$ is precisely

$$\nu : B \xrightarrow{\simeq} \text{Hom}_{B^{\text{e}}}(B, B^{\text{e}}) = \text{Sq}_{B/A}\,B.$$

Now Lemma 5.9 says that the morphism

$$\chi_A = \text{Tr}_{f;A} \circ \psi_{f;A}^\sharp : (B, \rho) \to (A, \rho^{\text{tau}})$$

is a nondegenerate rigid trace morphism relative to A. Since $\mathrm{tr}_{B/A}$ is also nondegenerate, it suffices to prove that $\mathrm{tr}_{B/A}$ is a rigid trace morphism relative to A.

We have finite flat ring homomorphisms $A \to B \xrightarrow{g^*} B^{\mathrm{e}}$, where g^* is $b \mapsto b \otimes 1$. Because $B^{\mathrm{e}} \cong B \times B'$ we have

$$\mathrm{tr}_{B^{\mathrm{e}}/B}\big(\nu(b)\big) = \mathrm{tr}_{(B \times B')/B}(b, 0) = b$$

for any $b \in B$. But $\mathrm{tr}_{B^{\mathrm{e}}/A} = \mathrm{tr}_{B/A} \otimes \mathrm{tr}_{B/A}$, and we know that $\rho = \nu$. Also the traces are transitive. So

$$\mathrm{tr}_{B/A}(b) = (\mathrm{tr}_{B/A} \circ \mathrm{tr}_{B^{\mathrm{e}}/B})\big(\nu(b)\big) = \mathrm{tr}_{B^{\mathrm{e}}/A}\big(\nu(b)\big) = (\mathrm{tr}_{B/A} \otimes \mathrm{tr}_{B/A})\big(\rho(b)\big).$$

This means that indeed $\mathrm{tr}_{B/A}$ is a rigid trace morphism relative to A. □

6 Traces of Differential Forms

A useful feature of Grothendieck duality theory is that it gives rise to traces of differential forms. Such traces are quite hard to construct directly (cf. [7, 11] and [10]). The aim of this section is to construct trace maps and to study some of their properties. The connection of our constructions to [15] is via Theorem 4.10.

As before $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension.

Definition 6.1 Suppose $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ are homomorphisms in $\mathsf{EFTAlg}/\mathbb{K}$, with $f^* : A \to B$ and $(f \circ g)^* : A \to C$ essentially smooth of relative dimension n, and $g^* : B \to C$ finite. According to Theorem 4.6(1) there are isomorphisms $\psi^{\sharp}_{f;A} : \Omega^n_{B/A}[n] \xrightarrow{\cong} f^! A$ and $\psi^{\sharp}_{f \circ g;A} : \Omega^n_{C/A}[n] \xrightarrow{\cong} (f \circ g)^! A$. From Definition 5.1 there is a trace map $\mathrm{Tr}_g : g_* g^! \xrightarrow{\cong} \mathbf{1}$, and from Definition 4.3 there is an isomorphism $\phi_{f,g} : (f \circ g)^! \xrightarrow{\cong} g^! f^!$. Define

$$\mathrm{Tr}_{C/B/A} = \mathrm{Tr}_{f/A} := (\psi^{\sharp}_f)^{-1} \circ \mathrm{Tr}_g \circ \phi_{f,g} \circ \psi^{\sharp}_{f \circ g}[-n].$$

Thus

$$\mathrm{Tr}_{C/B/A} : \Omega^n_{C/A} \to \Omega^n_{B/A}$$

is a B-linear homomorphism called the *trace map*.

Theorem 6.2 *Let $A \to B \to C$ be homomorphisms in $\mathsf{EFTAlg}/\mathbb{K}$ as in Definition 6.1.*

(1) *The trace map $\mathrm{Tr}_{C/B/A} : \Omega^n_{C/A} \to \Omega^n_{B/A}$ is nondegenerate, i.e. it induces a bijection $\Omega^n_{C/A} \cong \mathrm{Hom}_B(C, \Omega^n_{B/A})$.*

(2) *Suppose that $C \to D$ is a finite homomorphism, such that the composed homomorphism $A \to D$ is also essentially smooth of relative dimension n. Then there is equality*

$$\mathrm{Tr}_{D/B/A} = \mathrm{Tr}_{C/B/A} \circ \mathrm{Tr}_{D/C/A}.$$

Proof These assertions follow directly from Proposition 5.2(1,2) respectively. □

Remark 6.3 In the setup of Definition 6.1, the homomorphism $g^* : B \to C$ is actually flat. The proof will be published elsewhere; but here is the idea. It suffices to check the flatness of g^* after passing to the induced homomorphisms $\widehat{A}_{\mathfrak{p}} \to \widehat{B}_{\mathfrak{q}} \to \widehat{C}_{\mathfrak{r}}$ between the complete local rings, where $\mathfrak{r} \in \operatorname{Spec} C$ is arbitrary, $\mathfrak{q} := g(\mathfrak{r})$ and $\mathfrak{p} := f(\mathfrak{q})$. We then prove that these homomorphisms can be lifted to homomorphisms $\tilde{A} \to \tilde{B} \to \tilde{C}$, such that all three rings are regular complete local rings, and $\tilde{B} \to \tilde{C}$ is finite and injective. It is now a classical result that $\tilde{B} \to \tilde{C}$ is flat.

Remark 6.4 R. Hübl has communicated to us that results of Kunz [10] imply that the homomorphism $g^* : B \to C$ above is not only flat, but in fact a locally complete intersection. Hence according to [10, Section 16] there is a trace map $\sigma_{C/B} : \Omega_{C/A} \to \Omega_{B/A}$, which is a homomorphism of DG modules. In degree n it is a nondegenerate A-linear map $\sigma_{C/B}^n : \Omega_{C/A}^n \to \Omega_{B/A}^n$. Presumably Kunz's trace map $\sigma_{C/B}^n$ coincides with our trace map $\operatorname{Tr}_{C/B/A}$, although this is quite hard to verify (cf. Propositions 6.9 and 6.10 below).

Remark 6.5 One can show that

$$\operatorname{Tr}_{C/B/A}[n] : (f \circ g)^{\sharp}(A, \rho^{\mathrm{tau}}) \to f^{\sharp}(A, \rho^{\mathrm{tau}})$$

is the unique nondegenerate rigid trace morphism over B relative to A between these two rigid complexes. This means that the trace map $\operatorname{Tr}_{C/B/A}$ is actually independent of the base ring $\mathbb{K}$. cf. a similar phenomenon for the cohomological residue map in Yekutieli (in preparation).

Definition 6.6 Let $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ be homomorphisms in $\mathsf{EFTAlg}/\mathbb{K}$, with $f^* : A \to B$ essentially smooth of relative dimension n, and $g^* : B \to C$ essentially étale. Define a B-linear homomorphism

$$\mathsf{q}_{C/B/A} : \Omega_{B/A}^n \to \Omega_{C/A}^n$$

by the formula

$$\mathsf{q}_{C/B/A}(b_0 \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_n) = g^*(b_0)\mathrm{d}(g^*(b_1)) \wedge \cdots \wedge \mathrm{d}(g^*(b_n))$$

for any $b_0, \ldots, b_n \in B$. Here d is the de Rham differential.

It is trivial that $\mathsf{q}_{C/B/A}$ is a nondegenerate localization homomorphism, namely

$$\mathbf{1} \otimes \mathsf{q}_{C/B/A} : C \otimes_B \Omega_{B/A}^n \to \Omega_{C/A}^n$$

is bijective.

Lemma 6.7 *Let $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ be as in Definition* 6.6. *Then*

$$\mathsf{q}_{C/B/A}[n] : f^{\sharp}(A, \rho^{\mathrm{tau}}) \to (f \circ g)^{\sharp}(A, \rho^{\mathrm{tau}})$$

is the unique nondegenerate rigid localization morphism over B relative to A between these two rigid complexes.

Proof By [28, Theorem 6.3(3)] the obvious isomorphism

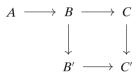$$\nu : g^\sharp f^\sharp (A, \rho^{\mathrm{tau}}) \xrightarrow{\simeq} (f \circ g)^\sharp (A, \rho^{\mathrm{tau}})$$

is rigid. And according to [28, Proposition 6.8] the localization map

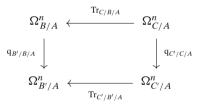$$q_g^\sharp : f^\sharp (A, \rho^{\mathrm{tau}}) \to g^\sharp f^\sharp (A, \rho^{\mathrm{tau}})$$

is the unique nondegenerate rigid localization map. But $q_{C/B/A}[n] = \nu \circ q_g^\sharp$. □

The next result says that the trace maps commute with localizations.

Proposition 6.8 *Consider a commutative diagram*

$$
\begin{array}{ccc}
A \longrightarrow B \longrightarrow C \\
\downarrow \qquad \downarrow \\
B' \longrightarrow C'
\end{array}
$$

in EFTAlg $/\mathbb{K}$, *in which the homomorphisms $A \to B$ and $A \to C$ are essentially smooth of relative dimension n; $B \to C$ is finite; $B \to B'$ is a localization; and the square is cartesian (i.e. $C' \cong B' \otimes_B C$). Then the diagram*

$$
\begin{array}{ccc}
\Omega^n_{B/A} & \xleftarrow{\mathrm{Tr}_{C/B/A}} & \Omega^n_{C/A} \\
q_{B'/B/A} \downarrow & & \downarrow q_{C'/C/A} \\
\Omega^n_{B'/A} & \xleftarrow{\mathrm{Tr}_{C'/B'/A}} & \Omega^n_{C'/A}
\end{array}
$$

is commutative.

Proof Let's write $e^* : A \to B$ and $g^* : B \to B'$. By Lemma 6.7 the localization map $q_{B'/B/A}[n] : \Omega^n_{B/A}[n] \to \Omega^n_{B'/A}[n]$ gets sent to the localization map $q_g : e^! A \to (e \circ g)^! A$ under the isomorphisms $\psi^\sharp_{e;A} : \Omega^n_{B/A}[n] \xrightarrow{\simeq} e^! A$ and $\psi^\sharp_{e \circ g; A} : \Omega^n_{B/A}[n] \xrightarrow{\simeq} (e \circ g)^! A$ of Theorem 4.6. Likewise for $q_{C'/C/A}[n]$. Now we can use Proposition 5.5. □

To finish the paper here are two nice properties of the trace map.

Proposition 6.9 *In the setup of Definition* 6.1, *assume that the ring A is reduced. Then for any $\beta \in \Omega^n_{B/A}$ and $c \in C$ one has*

$$\mathrm{Tr}_{C/B/A}(c\beta) = \mathrm{tr}_{C/B}(c) \cdot \beta \in \Omega^n_{B/A}.$$

Proof Denote by A' the total ring of fractions of A, and let $B' := A' \otimes_A B$. Since $A \to A'$ is injective, so is $\Omega^n_{B/A} \to \Omega^n_{B'/A}$. Now A' is a finite product of fields, and hence B' is a finite product of integral domains (cf. [28, Proposition 3.2]). Let B'' be the total ring of fractions of B'. Then B'' is a finite product of fields, and $\Omega^n_{B'/A} \to \Omega^n_{B''/A}$ is injective. Note that $C' := A' \otimes_A C$ is also a finite product of

integral domains, so $C'' := B'' \otimes_B C'$ is a finite product of fields. Due to Proposition 6.8 we may replace $A \to B \to C$ with $A \to B'' \to C''$. We can also localize at one of the factors of B''. Thus we might as well assume that B is a field and $C = \prod C_i$ is a finite product of fields.

Since each homomorphism $C \to C_i$ is both finite and a localization, Theorem 6.2(2) and Proposition 6.8 imply that for any $\gamma \in \Omega^n_{C_i/A}$ the i-th component of $\mathrm{Tr}_{C_i/C/A}(\gamma) \in \Omega^n_{C/A} = \bigoplus_j \Omega^n_{C_j/A}$ is γ, and all other components are zero. We conclude that we can replace C with C_i; i.e. C can be assumed to be a field.

Now there are two cases to look at. If the finite field extension $B \to C$ is separable then $\mathrm{Tr}_{C/B/A}(c\beta) = \mathrm{tr}_{C/B}(c) \cdot \beta$ by Theorem 5.6. On the other hand, if $B \to C$ is inseparable then $\mathrm{tr}_{C/B}(c) = 0$ and also $\mathrm{q}_{C/B/A}(\beta) = 0$. $\qed$

Proposition 6.10 *Let $A \in \mathsf{EFTAlg}\,/\mathbb{K}$ be an integral domain whose field of fractions has characteristic 0. Let $B := A[s]$ and $C := A[t]$ be polynomial algebras in one variable each. Define an A-algebra homomorphism $f^* : B \to C$ by $f^*(s) := t^n$ for some positive integer n. Then*

$$\mathrm{Tr}_{C/B/A}\big(t^{n-1}\mathrm{d}t\big) = \mathrm{d}s,$$

and

$$\mathrm{Tr}_{C/B/A}\big(t^i\mathrm{d}t\big) = 0 \ \text{for} \ 0 \le i \le n-2.$$

Proof Let A' be the fraction field of A. And let's write $\mathrm{Tr}_{C/B/A}\big(t^i\mathrm{d}t\big) = p_i(s)\mathrm{d}s$ with $p_i(s) \in B = A[s]$. So we need to prove that $p_{n-1}(s) = 1$ and $p_i(s) = 0$ for $0 \le i \le n-2$. Due to Proposition 6.8 we can localize to $B' := A'[s]$ and $C' := A'[t]$. Denote by $f'^* : B' \to C'$ the corresponding homomorphism. Take any nonzero $\lambda \in A'$. There are A'-algebra automorphisms $g_\lambda^* : B' \to B'$ and $h_\lambda^* : C' \to C'$, defined by $g_\lambda^*(s) := \lambda^n s$ and $h_\lambda^*(t) := \lambda t$, and these satisfy $h_\lambda^* \circ f'^* = f'^* \circ g_\lambda^*$. Since the trace is functorial (Theorem 6.2(2)) we have $\mathrm{Tr}_{f'/A} \circ \mathrm{Tr}_{h_\lambda/A} = \mathrm{Tr}_{g_\lambda/A} \circ \mathrm{Tr}_{f'/A}$. But by Proposition 6.9, $\mathrm{Tr}_{g_\lambda/A}\big(g_\lambda^*(\beta)\big) = \beta$ for any $\beta \in \Omega^n_{B'/A}$; so that $\mathrm{Tr}_{g_\lambda/A}\big(p(s)\mathrm{d}s\big) = p(\lambda^{-n}s)\mathrm{d}(\lambda^{-n}s)$ for any polynomial $p(s) \in B = A[s]$. Likewise $\mathrm{Tr}_{h_\lambda/A}\big(t^i\mathrm{d}t\big) = \lambda^{-(i+1)}t^i\mathrm{d}t$. We conclude that

$$\lambda^{-(i+1)}p_i(s)\mathrm{d}s = (\mathrm{Tr}_{f'/A} \circ \mathrm{Tr}_{h_\lambda/A})(t^i\mathrm{d}t)$$
$$= (\mathrm{Tr}_{g_\lambda/A} \circ \mathrm{Tr}_{f'/A})(t^i\mathrm{d}t)$$
$$= p_i(\lambda^{-n}s)\mathrm{d}(\lambda^{-n}s).$$

Therefore $p_i(\lambda^{-n}s) = \lambda^{n-(i+1)}p_i(s)$. Since this is true for infinitely many λ we must have $p_i(s) = 0$ for $0 \le i \le n-2$, and $p_{n-1}(s)$ is a constant.

In order to compute the value of $p_{n-1}(s) \in A$ we note that $f'^*(\mathrm{d}s) = nt^{n-1}\mathrm{d}t$. Since we are in characteristic 0 we can divide by n, and by Proposition 6.9 we get

$$\mathrm{Tr}_{C'/B'/A}\big(t^{n-1}\mathrm{d}t\big) = \mathrm{Tr}_{C'/B'/A}\big(f'^*\big(n^{-1}\mathrm{d}s\big)\big)$$
$$= \mathrm{tr}_{C'/B'/A}(1_{C'}) \cdot n^{-1}\mathrm{d}s$$
$$= \mathrm{d}s. \qquad\qed$$

Remark 6.11 The extra assumptions on the algebra A in the last two propositions are not really necessary. In Proposition 6.10 we can actually let A be an arbitrary algebra in EFTAlg /$\mathbb{K}$. For the proof we would then use "rigid base change" [which is developed in Yekutieli (in preparation) to prove results on the residue map]. Base change allows us to replace both $\mathbb{K}$ and A with the ring of integers $\mathbb{Z}$.

Similarly, in Proposition 6.9 we can let A be an arbitrary algebra in EFTAlg /$\mathbb{K}$. However the proof here requires methods that aren't available yet, namely rigid complexes over adically complete rings. The idea is to use rigid base change, and the setup explained in Remark 6.3, to replace $A \to B \to C$ with $\tilde{A} \to \tilde{B} \to \tilde{C}$. It is possible to choose the complete regular local ring $\tilde{A}$ such that it has field of fractions of characteristic 0.

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