RIGID DUALIZING COMPLEXES ON SCHEMES

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ABSTRACT. In this paper we present a new approach to Grothendieck duality on schemes. Our approach is based on the idea of rigid dualizing complexes, which was introduced by Van den Bergh in the context of noncommutative algebraic geometry. We obtain most of the important features of Grothendieck duality, yet manage to avoid lengthy and difficult compatibility verifications. Our results apply to finite type schemes over a regular noetherian finite dimensional base ring, and hence are suitable for arithmetic geometry.

0. INTRODUCTION

Grothendieck duality for schemes was introduced in the book "Residues and Duality" [RD] by R. Hartshorne. This duality theory has applications in various areas of algebraic geometry, including moduli spaces, resolution of singularities, arithmetic geometry, enumerative geometry and more.

In the forty years since the publication of [RD] a number of related papers appeared in the literature. Some of these papers provided elaborations on, or more explicit versions of Grothendieck duality (e.g. [Kl], [Li], [HK], [Ye2], [Ye3], [Sa]). Other papers contained alternative approaches (e.g. [RD, Appendix], [Ve] and [Ne]). The recent book [Co] is a complement to [RD] that fills gaps in the proofs, and also contains the first proof of the Base Change Theorem. A noncommutative version of Grothendieck duality was developed in [Ye1], which has applications in algebra (e.g. [EG]) and even in mathematical physics (e.g. [KKO]). Other papers sought to extend the scope of Grothendieck duality to formal schemes (e.g. [AJL] and [LNS]) or to differential graded algebras (see [FIJ]).

In this paper we present a new approach to Grothendieck duality on schemes, including Conrad's results on base change. The key idea in our approach is the use of *rigid dualizing complexes*. This notion was introduced by Van den Bergh [VdB] in the context of noncommutative algebraic geometry, and was developed further in our papers [YZ1, YZ2, YZ3].

The background material we need is standard algebraic geometry (from [EGA]), the theory of derived categories (from [RD] or [KS]), and its generalization to differential graded algebras (which is discussed in Section 1). We also need a few isolated results on dualizing complexes from [RD]. Apart from that our treatment is self-contained.

Let us explain what are rigid dualizing complexes and how they are used in our paper. Fix for the rest of the introduction a finite dimensional, regular, noetherian, commutative base ring $\mathbb{K}$. Let A be an essentially finite type commutative $\mathbb{K}$-algebra. The bounded derived category of A-modules is denoted by $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. Given a complex

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$M \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ we define its *square* $\operatorname{Sq}_{A/\mathbb{K}} M \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. If A is flat over $\mathbb{K}$ the squaring operation is very easy to define:

$$\operatorname{Sq}_{A/\mathbb{K}} M := \operatorname{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M).$$

But in general the definition is more complicated, and requires differential graded algebras (see Section 2). Given a morphism $\phi : M \to N$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ there is an induced morphism $\operatorname{Sq}_{A/\mathbb{K}}(\phi) : \operatorname{Sq}_{A/\mathbb{K}} M \to \operatorname{Sq}_{A/\mathbb{K}} N$. For any $a \in A$ one has $\operatorname{Sq}_{A/\mathbb{K}}(a\phi) = a^2 \operatorname{Sq}_{A/\mathbb{K}}(\phi)$; hence the name "squaring".

A *rigidifying isomorphism* for M is an isomorphism $\rho : M \xrightarrow{\simeq} \operatorname{Sq}_{A/\mathbb{K}} M$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. The pair (M, ρ) is called a *rigid complex over* A *relative to* $\mathbb{K}$. Suppose (M, ρ_M) and (N, ρ_N) are two rigid complexes. A *rigid morphism* $\phi : (M, \rho_M) \to (N, \rho_N)$ is a morphism $\phi : M \to N$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ such that $\rho_N \circ \phi = \operatorname{Sq}_{A/\mathbb{K}}(\phi) \circ \rho_M$. Observe that if (M, ρ_M) is a rigid complex such that $\operatorname{RHom}_A(M, M) = A$, and $\phi : (M, \rho_M) \to (M, \rho_M)$ is a rigid isomorphism, then ϕ is multiplication by some invertible element $a \in A$ satisfying $a = a^2$; and therefore $a = 1$. We conclude that *the identity is the only rigid automorphism of* (M, ρ_M).

Let B be another essentially finite type commutative $\mathbb{K}$-algebra, and let $f^* : A \to B$ be a homomorphism. First assume f^* is finite, and let $f^\flat M := \operatorname{RHom}_A(B, M) \in \mathsf{D}^{+}(\mathsf{Mod}\, B)$. If $f^\flat M$ has bounded cohomology then we there is an induced rigidifying isomorphism $f^\flat(\rho_M) : f^\flat M \xrightarrow{\simeq} \operatorname{Sq}_{B/\mathbb{K}} f^\flat M$ (see Theorem 3.14). We write $f^\flat(M, \rho_M) := (f^\flat M, f^\flat(\rho_M))$. Next assume f^* is either smooth of relative dimension n or a localization, and let $f^\sharp M := \Omega^n_{B/A}[n] \otimes_A M \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$. Then there is an induced rigidifying isomorphism $f^\sharp(\rho_M) : f^\sharp M \xrightarrow{\simeq} \operatorname{Sq}_{B/\mathbb{K}} f^\sharp M$ (see Theorem 3.22), and thus a new rigid complex $f^\sharp(M, \rho_M) := (f^\sharp M, f^\sharp(\rho_M))$.

Now let's consider dualizing complexes. Recall that a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is dualizing if it has finite injective dimension, and if the canonical morphism $A \to \operatorname{RHom}_A(R, R)$ is an isomorphism. A *rigid dualizing complex over* A *relative to* $\mathbb{K}$ is a rigid complex (R, ρ) such that R is dualizing.

Here is the first main result of our paper.

Theorem 0.1. *Let* $\mathbb{K}$ *be a regular finite dimensional noetherian ring, and let* A *be an essentially finite type* $\mathbb{K}$*-algebra.*

(1) *The algebra* A *has a rigid dualizing complex* (R_A, ρ_A)*, which is unique up to a unique rigid isomorphism.*
(2) *Given a finite homomorphism* $f^* : A \to B$*, there is a unique rigid isomorphism* $f^\flat(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.
(3) *Given a homomorphism* $f^* : A \to B$ *which is either smooth or a localization, there is a unique rigid isomorphism* $f^\sharp(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.

This theorem is a combination of Theorems 4.3, 4.6 and 4.13 in the body of the paper. Theorem 0.1 pretty much covers Grothendieck duality for affine schemes. For instance, it it gives rise to a trace morphism $\operatorname{Tr}_f : R_B \to R_A$ for a finite homomorphism $f^* : A \to B$, which is functorial and nondegenerate (see Proposition 4.8).

Let $f^* : A \to B$ be a smooth homomorphism of $\mathbb{K}$-algebras, and let $i^* : B \to \bar{B}$ be a finite homomorphism. Assume $g^* := i^* \circ f^*$ is finite and flat. Since $\operatorname{Sq}_{A/A} A = A$ we get the tautological rigid complex (A, ρ_{tau}). As explained above, there are two rigid complexes $g^\flat(A, \rho_{\mathrm{tau}})$ and $i^\flat f^\sharp(A, \rho_{\mathrm{tau}})$ over $\bar{B}$ relative to A. By Theorem 0.1 there exist isomorphisms $R_B \cong f^{\sharp} A$, $R_{\bar{B}} \cong i^{\flat} R_B$ and $R_{\bar{B}} \cong g^{\flat} A$. Composing them we obtain an isomorphism $\zeta : g^{\flat} A \xrightarrow{\simeq} i^{\flat} f^{\sharp} A$ called the *residue isomorphism*.

Theorem 0.2. *The residue isomorphism ζ is the unique rigid isomorphism $g^{\flat}(A, \rho_{\mathrm{tau}}) \xrightarrow{\simeq} i^{\flat} f^{\sharp}(A, \rho_{\mathrm{tau}})$ relative to A.*

This theorem is restated as Theorem 5.2. It implies, among other things, that the residue isomorphism of independent of the base ring $\mathbb{K}$.

The passage to schemes requires gluing dualizing complexes. We achieve this using the concept of *stack of subcategories of* $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, \mathcal{O}_X)$; see Definition 6.4. On a finite type $\mathbb{K}$-scheme X there is a dimension function that is intimately related to rigid dualizing complexes; we denote it by $\dim_{\mathbb{K}}$ (see Definition 6.10). Following [RD] we say that a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, \mathcal{O}_X)$ is a *Cohen-Macaulay complex* if the local cohomologies $\mathrm{H}^i_x \mathcal{M}$ vanish whenever $i \neq -\dim_{\mathbb{K}}(x)$. Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ the subcategory of Cohen-Macaulay complexes with quasi-coherent cohomologies.

Theorem 0.3. *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let X be a finite type $\mathbb{K}$-scheme. The assignment $U \mapsto \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_U)_{\mathrm{CM}}$, for open sets $U \subset X$, is a stack of subcategories of $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.*

This means that Cohen-Macaulay complexes can be glued. The theorem is repeated as Theorem 6.5 in the body of the paper. In an earlier version of our paper, which was entitled "Rigid Dualizing Complexes and Perverse Sheaves on Schemes", a similar result was proved using the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$. The perverse sheaf approach is indispensable for noncommutative algebraic geometry (cf. [YZ4]). However, we later realized that for commutative schemes it is possible, and easier, to prove the required result using Cousin complexes.

A *rigid structure* on a complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$ is a collection $\boldsymbol{\rho} = \{\rho_U\}$, where for every affine open set $U \subset X$, ρ_U is a rigidifying isomorphism for the complex $M_U := \mathrm{R}\Gamma(U, \mathcal{M})$ over the algebra $A_U := \Gamma(U, \mathcal{O}_X)$ relative to $\mathbb{K}$. The condition is that for an inclusion $f : V \to U$ of affine open sets, the localization isomorphism $f^{\sharp}(M_U, \rho_U) \to (M_V, \rho_V)$ should be rigid. A *rigid dualizing complex on X* is a pair $(\mathcal{R}_X, \boldsymbol{\rho}_X)$, where $\mathcal{R}_X$ is a dualizing complex and $\boldsymbol{\rho}_X$ is a rigid structure on it.

Suppose $f : X \to Y$ is a morphism between finite type $\mathbb{K}$-schemes. If f is finite then there is a functor $f^{\flat} : \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ defined by

$$f^{\flat} \mathcal{N} := \mathcal{O}_X \otimes_{f^{-1} f_* \mathcal{O}_X} f^{-1} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{N}).$$

On the other hand if f is smooth we have a functor $f^{\sharp} : \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ defined as follows. Let $X_1, \ldots, X_r$ be the connected components of X, with inclusions $g_i : X_i \to X$. Let n_i be the rank of $\Omega^1_{X_i/Y}$. Then

$$f^{\sharp} \mathcal{N} := \big(\bigoplus\nolimits_i g_{i *} \Omega^{n_i}_{X_i/Y}[n_i] \big) \otimes_{\mathcal{O}_X} f^* \mathcal{N}.$$

The combination of Theorems 0.1 and 0.3 implies, without much effort, the next result (which is repeated as Theorems 6.13 and 6.16).

Theorem 0.4. *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring.*

(1) *Let X be a finite type $\mathbb{K}$-scheme. The scheme X has a rigid dualizing complex $(\mathcal{R}_X, \boldsymbol{\rho}_X)$, which is unique up to a unique rigid isomorphism.*
(2) *Given a finite morphism $f : X \to Y$, the complex $f^{\flat} \mathcal{R}_Y$ is a dualizing complex on X, and it has an induced rigid structure $f^{\flat}(\boldsymbol{\rho}_Y)$. Hence there is a unique rigid isomorphism $\mathcal{R}_X \cong f^{\flat} \mathcal{R}_Y$ in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.*

(3) *Given a smooth morphism $f : X \to Y$, the complex $f^{\sharp}\mathcal{R}_Y$ is a dualizing complex on X, and it has an induced rigid structure $f^{\sharp}(\boldsymbol{\rho}_Y)$. Hence there is a unique rigid isomorphism $\mathcal{R}_X \cong f^{\sharp}\mathcal{R}_Y$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$.*

We can now define the *rigid auto-duality functor* $\mathrm{D}_X := \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X)$. For a morphism $f : X \to Y$ we define a functor

$$f^{!} : \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y) \to \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$$

as follows. If $X = Y$ and $f = \mathbf{1}_X$ (the identity automorphism) then $f^{!} := \mathbf{1}_{\mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)}$ (the identity functor). Otherwise we define $f^{!} := \mathrm{D}_X\, \mathrm{L}f^{*}\, \mathrm{D}_Y$. Let $\mathsf{FTSch}\,/\mathbb{K}$ be the category of finite type schemes over $\mathbb{K}$, and let Cat denote the category of all categories.

Corollary 0.5. *The assignment $f \mapsto f^{!}$ is the 1-component of a contravariant 2-functor $\mathsf{FTSch}\,/\mathbb{K} \to \mathsf{Cat}$, whose 0-component is $X \mapsto \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$.*

For details on 2-functors see [Ha, Section I.1.5]. Some authors use the term "pseudofunctor".

A *rigid residue complex* on X is a rigid dualizing complex $(\mathcal{K}_X, \boldsymbol{\rho}_X)$, such that for every p there is an isomorphism of sheaves $\mathcal{K}^{p}_X \cong \bigoplus_{\dim_{\mathbb{K}}(x)=-p} \mathcal{J}(x)$. Here $\mathcal{J}(x)$ denotes an injective hull of the residue field $\boldsymbol{k}(x)$, considered as a quasi-coherent sheaf, constant on $\overline{\{x\}}$. It is quite easy to prove that a rigid residue complex exists: apply the Cousin functor E to the rigid dualizing complex $\mathcal{R}_X$ (see Section 6). The complex $\mathcal{K}_X := \mathrm{E}\mathcal{R}_X$ is isomorphic to $\mathcal{R}_X$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$, and hence it inherits a rigid structure $\boldsymbol{\rho}_X$. This rigid residue complex is unique up to a unique isomorphism of complexes; see Proposition 7.2. Notice that the rigid auto-duality functor becomes $\mathrm{D}_X = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{K}_X)$.

For a point x with $\dim_{\mathbb{K}}(x) = -p$ let $\mathcal{K}_X(x) := \mathrm{H}^{p}_{x}\mathcal{K}_X$. Due to the structure of the complex $\mathcal{K}_X$ we see that $\mathcal{K}_X(x) \cong \mathcal{J}(x)$ and $\mathcal{K}^{p}_X = \bigoplus_{\dim_{\mathbb{K}}(x)=-p} \mathcal{K}_X(x)$. Now $\mathcal{K}_X(x)$ only depends on the local ring $\mathcal{O}_{X,x}$. This fact, plus the traces for finite algebra homomorphisms, allow us to define a trace map $\mathrm{Tr}_f : f_{*}\mathcal{K}_X \to \mathcal{K}_Y$ for any morphism of schemes $f : X \to Y$. This trace is only a map of graded $\mathcal{O}_Y$-modules, but it is functorial, i.e. $\mathrm{Tr}_{g \circ f} = \mathrm{Tr}_g \circ \mathrm{Tr}_f$ for composable morphisms (see Definition 7.6 and Proposition 7.7).

Theorem 0.6. *Let $f : X \to Y$ be a proper morphism between finite type $\mathbb{K}$-schemes. Then $\mathrm{Tr}_f : f_{*}\mathcal{K}_X \to \mathcal{K}_Y$ is a homomorphism of complexes.*

The theorem is restated as Theorem 7.14 in the body of the paper. The proof goes like this: as in [RD], we reduce to the case $Y = \operatorname{Spec} K$ with K a field, and $X = \mathbf{P}^{1}_{K}$. We then use explicit calculations (involving the residue isomorphism and using Theorem 0.2) to do this case.

Due to Theorem 0.6 we get a trace map $\mathrm{Tr}_f : \mathrm{R}f_{*}f^{!} \to \mathbf{1}$, which is a transformation of functors from $\mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$ to itself. It is not hard to deduce that this trace is nondegenerate (this is Theorem 7.17 in the body of the paper):

Theorem 0.7. *Let $f : X \to Y$ be a proper morphism of finite type $\mathbb{K}$-schemes, let $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ and let $\mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$. Then the morphism*

$$\mathrm{R}f_{*}\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, f^{!}\mathcal{N}) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_{*}\mathcal{M}, \mathcal{N})$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_Y)$ induced by $\mathrm{Tr}_f : \mathrm{R}f_{*}f^{!}\mathcal{N} \to \mathcal{N}$ is an isomorphism.

Our last results deal with the *relative dualizing sheaf*. Suppose $f : X \to Y$ is flat of relative dimension n (i.e. the fibers of f are equidimensional of dimension n). We then define $\boldsymbol{\omega}_{X/Y} := \mathrm{H}^{-n}f^{!}\mathcal{O}_Y$. This is a coherent sheaf on X with nice properties. For instance, if $U \subset X$ is an open set such that $f|_U$ is smooth, then, due to Theorem 0.4(3), we have $\boldsymbol{\omega}_{X/Y}|_U = \Omega^n_{U/Y}$. In case f is a Cohen-Macaulay morphism of relative dimension n (i.e. flat with Cohen-Macaulay fibers) then $f^! \mathcal{O}_Y = \boldsymbol{\omega}_{X/Y}[n]$ (see Proposition 9.5). We can sometimes characterize the relative dualizing sheaf explicitly: if f is generically smooth and both X and Y are integral schemes, then $\boldsymbol{\omega}_{X/Y}$ is a subsheaf of the constant quasi-coherent sheaf $\Omega^n_{\boldsymbol{k}(X)/\boldsymbol{k}(Y)}$. Moreover, under some separability assumptions (e.g. $\operatorname{char} \boldsymbol{k}(Y) = 0$) we can describe the subsheaf $\boldsymbol{\omega}_{X/Y} \subset \Omega^n_{\boldsymbol{k}(X)/\boldsymbol{k}(Y)}$ explicitly in terms of traces (see Theorem 8.7).

Finally we have this main result, which is our version of Conrad's work [Co].

Theorem 0.8. *Suppose*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a cartesian diagram in $\mathsf{FTSch}\,/\mathbb{K}$*, with* f *a Cohen-Macaulay morphism of relative dimension* n*, and* g *any morphism.*

(1) *There is a homomorphism* $\mathcal{O}_X$*-modules*

$$\theta_{f,g} : \boldsymbol{\omega}_{X/Y} \to h_* \boldsymbol{\omega}_{X'/Y'},$$

such that the induced $\mathcal{O}_{X'}$*-linear homomorphism* $h^*(\theta_{f,g}) : h^* \boldsymbol{\omega}_{X/Y} \to \boldsymbol{\omega}_{X'/Y'}$ *is an isomorphism. The homomorphism* $\theta_{f,g}$ *has a local characterization in terms of rigidity.*

(2) *Assume the morphism* f *is proper. Then*

$$g^* \circ \operatorname{Tr}_f = g_*(\operatorname{Tr}_{f'}) \circ \mathrm{R}^n f_*(\theta_{f,g}) : \mathrm{R}^n f_* \boldsymbol{\omega}_{X/Y} \to g_* \mathcal{O}_{Y'}.$$

This theorem, with full details, appears as Theorems 9.6 and 9.12 in the body of the paper. In case f is smooth of relative dimension n, the homomorphism $\theta_{f,g}$ is the usual base change homomorphism $\Omega^n_{X/Y} \to h_* \Omega^n_{X'/Y'}$; see Corollary 9.9.

To end the introduction let us mention a potential further implementation of our methods: Grothendieck duality for algebraic stacks (in the sense of [LMB]). Let $\mathfrak{X}$ be a Deligne-Mumford stack, with étale presentation $P : X \to \mathfrak{X}$ by a finite type $\mathbb{K}$-scheme X. Since our methods are local, and rigid dualizing complexes have an extremely controlled variance with respect to étale morphisms (see Theorem 0.1(3)), it is conceivable that one could glue the rigid dualizing complex $\mathcal{R}_X$ to a dualizing complex $\mathcal{R}_{\mathfrak{X}}$ on $\mathfrak{X}$.

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CONTENTS

1. Differential Graded Algebras

This section contains some technical material about differential graded algebras and their derived categories. The results are needed for treating rigid dualizing complexes when the base ring $\mathbb{K}$ is not a field. There is some overlap here with the papers [FIJ], [Ke] and [Be]. We recommend skipping this section, as well as Section 2, when first reading the paper; the reader will just have to assume that $\mathbb{K}$ is a field, and replace $\otimes^{\mathrm{L}}_{\mathbb{K}}$ with $\otimes_{\mathbb{K}}$ everywhere.

Let $\mathbb{K}$ be a commutative ring. A graded $\mathbb{K}$-algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$ is said to be *super-commutative* if $ab = (-1)^{ij} ba$ for all $a \in A^i$ and $b \in A^j$, and if $a^2 = 0$ whenever i is odd. (Some authors call such a graded algebra strictly commutative.) A is said to be *non-positive* if $A^i = 0$ for all $i > 0$. Throughout the paper all graded algebras are assumed to be non-positive, super-commutative, associative, unital $\mathbb{K}$-algebras by default, and all algebra homomorphisms are over $\mathbb{K}$.

By *differential graded algebra* (or DG algebra) over $\mathbb{K}$ we mean a graded $\mathbb{K}$-algebra $A = \bigoplus_{i \leq 0} A^i$, together with a $\mathbb{K}$-linear derivation $\mathrm{d} : A \to A$ of degree 1 satisfying $\mathrm{d} \circ \mathrm{d} = 0$. Note that the graded Leibniz rule holds:

$$\mathrm{d}(ab) = \mathrm{d}(a)b + (-1)^i a \mathrm{d}(b)$$

for $a \in A^i$ and $b \in A^j$.

A DG algebra homomorphism $f : A \to B$ is a degree 0 homomorphism of graded $\mathbb{K}$-algebras that commutes with d. It is a quasi-isomorphism if $\mathrm{H}(f)$ is an isomorphism (of graded algebras).

A differential graded (DG) A-module is a graded (left) A-module $M = \bigoplus_{i \in \mathbb{Z}} M^i$, endowed with a degree 1 $\mathbb{K}$-linear homomorphism $\mathrm{d} : M \to M$ satisfying $\mathrm{d}(am) = \mathrm{d}(a)m + (-1)^i a \mathrm{d}(m)$ for $a \in A^i$ and $m \in M^j$. Note that we can make M into a right DG A-module by the rule $ma := (-1)^{ij} am$ for $a \in A^i$ and $m \in M^j$. The category of DG A-modules is denoted by $\mathsf{DGMod}\, A$. It is an abelian category whose morphisms are degree 0 A-linear homomorphisms commuting with the differentials.

There is a forgetful functor from DG algebras to graded algebras (it forgets the differential), and we denote it by $A \mapsto \operatorname{und} A$. Likewise for $M \in \mathsf{DGMod}\, A$ we have $\operatorname{und} M \in \mathsf{GrMod}(\operatorname{und} A)$, the category of graded $\operatorname{und} A$ -modules. A DG $\mathbb{K}$-module is just a complex of $\mathbb{K}$-modules.

Given a graded algebra A and two graded A-modules M and N let us write

$$\operatorname{Hom}_{\mathbb{K}}(M, N)^i := \prod_{j \in \mathbb{Z}} \operatorname{Hom}_{\mathbb{K}}(M^j, N^{j+i}),$$

the set of homogeneous $\mathbb{K}$-linear homomorphisms of degree i from M to N, and let

$$\operatorname{Hom}_A(M, N)^i := \\ \{\phi \in \operatorname{Hom}_{\mathbb{K}}(M, N)^i \mid \phi(am) = (-1)^{ij} a \phi(m) \text{ for all } a \in A^j \text{ and } m \in M\}.$$

Then

$$\operatorname{Hom}_A(M,N) := \bigoplus\nolimits_{i \in \mathbb{Z}} \operatorname{Hom}_A(M,N)^i \tag{1.1}$$

is a graded A-module, by the formula $(a\phi)(m) := a\phi(m) = (-1)^{ij}\phi(am)$ for $a \in A^j$ and $\phi \in \operatorname{Hom}_A(M,N)^i$. Cf. [ML, Chapter VI]. The set $\operatorname{Hom}_A(M,N)$ is related to the set of A-linear homomorphisms $M \to N$ as follows. Let's denote by ungr the functor forgetting the grading. Then the map

$$\Phi : \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_{\operatorname{ungr} A}(\operatorname{ungr} M, \operatorname{ungr} N),$$

defined by $\Phi(\phi)(m) := (-1)^{ij}\phi(m)$ for $\phi \in \operatorname{Hom}_A(M,N)^i$ and $m \in M^j$, is $\operatorname{ungr} A$ -linear, and Φ is bijective if M is a finitely generated A-module.

For a DG algebra A and two DG A-modules M, N there is a differential d on $\operatorname{Hom}_{\operatorname{und} A}(\operatorname{und} M, \operatorname{und} N)$, with formula $\mathrm{d}(\phi) := \mathrm{d} \circ \phi + (-1)^{i+1}\phi \circ \mathrm{d}$ for ϕ of degree i. The resulting DG A-module is denoted by $\operatorname{Hom}_A(M,N)$. Note that $\operatorname{Hom}_{\mathsf{DGMod}\, A}(M,N)$ coincides with the set of 0-cocycles of $\operatorname{Hom}_A(M,N)$. Two homomorphisms $\phi_0, \phi_1 \in \operatorname{Hom}_{\mathsf{DGMod}\, A}(M,N)$ are said to be *homotopic* if $\phi_0 - \phi_1 = \mathrm{d}(\psi)$ for some $\psi \in \operatorname{Hom}_A(M,N)^{-1}$. The DG modules M and N are called *homotopy equivalent* if there are homomorphisms $\phi : M \to N$ and $\psi : N \to M$ in $\mathsf{DGMod}\, A$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are homotopic to the respective identity homomorphisms.

Suppose A and B are two DG $\mathbb{K}$-algebras. Then $A \otimes_{\mathbb{K}} B$ is also a DG $\mathbb{K}$-algebra; the sign rule says that $(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{ij} a_1 a_2 \otimes b_1 b_2$ for $b_1 \in B^j$ and $a_2 \in A^i$. The differential is of course $\mathrm{d}(a \otimes b) := \mathrm{d}(a) \otimes b + (-1)^i a \otimes \mathrm{d}(b)$ for $a \in A^i$. If $M \in \mathsf{DGMod}\, A$ and $N \in \mathsf{DGMod}\, B$ then $M \otimes_{\mathbb{K}} N \in \mathsf{DGMod}\, A \otimes_{\mathbb{K}} B$. If $N \in \mathsf{DGMod}\, A$ then $M \otimes_A N$, which is a quotient of $M \otimes_{\mathbb{K}} N$, is a DG A-module.

Let A be a DG algebra. Since A is non-negative one has $\mathrm{d}(A^0) = 0$; and therefore the differential $\mathrm{d} : M^i \to M^{i+1}$ of any DG A-module M is A^0-linear. This easily implies that the truncated objects

$$\begin{gathered} \tau^{\geq i} M := \big(\cdots 0 \to \operatorname{Coker}(M^{i-1} \to M^i) \to M^{i+1} \to \cdots \big) \\ \text{and} \\ \tau^{\leq i} M := \big(\cdots \to M^{i-1} \to \operatorname{Ker}(M^i \to M^{i+1}) \to 0 \to \cdots \big) \end{gathered} \tag{1.2}$$

are DG A-modules.

There is a derived category obtained from $\mathsf{DGMod}\, A$ by inverting the quasi-isomorphisms, which we denote by $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$. See [Ke] for details. Note that in case A is a usual algebra (i.e. it is concentrated in degree 0) then $\mathsf{DGMod}\, A = \mathsf{C}(\mathsf{Mod}\, A)$, the abelian category of complexes of A-modules, and $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A) = \mathsf{D}(\mathsf{Mod}\, A)$, the usual derived category of A-modules.

In order to derive functors one has several useful devices. A DG A-module P is called *K-projective* if for any acyclic DG A-module N the DG module $\operatorname{Hom}_A(P,N)$ is acyclic. (This name is due to Spaltenstein [Sp]. Keller [Ke] uses the term "property (P)" to indicate K-projective DG modules, and in [AFH] the authors use "homotopically projective". See also [Hi].) Similarly one defines *K-injective* and *K-flat* DG modules: I is K-injective, and F is K-flat, if $\operatorname{Hom}_A(N,I)$ and $F \otimes_A N$ are acyclic for all acyclic N. It is easy to see that any K-projective DG module is also K-flat. Every two objects $M, N \in \mathsf{DGMod}\, A$ admit quasi-isomorphisms $P \to M$, $N \to I$ and $F \to M$, with P K-projective, I K-injective and F K-flat. Then one defines

$$\operatorname{RHom}_A(M,N) := \operatorname{Hom}_A(P,N) \cong \operatorname{Hom}_A(M,I) \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$$

and

$$M \otimes^{\mathrm{L}}_{A} N := F \otimes_A N \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, A).$$

When A is a usual algebra, any bounded above complex of projective (resp. flat) modules is K-projective (resp. K-flat). And any bounded below complex of injective A-modules is K-injective. A single A-module M is projective (resp. injective, resp. flat) iff it is K-projective (resp. K-injective, resp. K-flat) as DG A-module.

The following useful result is partly contained in [Hi], [Ke] and [KM].

Proposition 1.3. *Let $A \to B$ be a quasi-isomorphism of DG algebras.*

(1) *Given $M \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$ and $N \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, B)$, the canonical morphisms $M \to B \otimes^{\mathrm{L}}_{A} M$ and $B \otimes^{\mathrm{L}}_{A} N \to N$ are both isomorphisms. Hence the "restriction of scalars" functor $\tilde{\mathsf{D}}(\mathsf{DGMod}\, B) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$ is an equivalence.*

(2) *Let $M, N \in \tilde{\mathsf{D}}(\mathsf{DGMod}\, B)$. Then there are functorial isomorphisms $M \otimes^{\mathrm{L}}_{B} N \cong M \otimes^{\mathrm{L}}_{A} N$ and $\mathrm{RHom}_B(M, N) \cong \mathrm{RHom}_A(M, N)$ in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$.*

Proof. (1) Choose K-projective resolutions $P \to M$ and $Q \to N$ over A. Then $M \to B \otimes^{\mathrm{L}}_{A} M$ becomes $P \cong A \otimes_A P \to B \otimes_A P$, which is evidently a quasi-isomorphism. On the other hand $B \otimes^{\mathrm{L}}_{A} N \to N$ becomes $B \otimes_A Q \to Q$; which is a quasi-isomorphism because so is $A \otimes_A Q \to B \otimes_A Q$.

(2) Choose K-projective resolutions $P \to M$ and $Q \to N$ over A. We note that $B \otimes_A P$ and $B \otimes_A Q$ are K-projective over B, and $B \otimes_A P \to M$, $B \otimes_A Q \to N$ are quasi-isomorphisms. Therefore we get isomorphisms in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$:

$$M \otimes^{\mathrm{L}}_{B} N = (B \otimes_A P) \otimes_B (B \otimes_A Q) \cong (B \otimes_A P) \otimes_A Q \cong P \otimes_A Q = M \otimes^{\mathrm{L}}_{A} N.$$

The same resolutions give

$$\mathrm{RHom}_B(M, N) = \mathrm{Hom}_B(B \otimes_A P, N) \cong \mathrm{Hom}_A(P, N) = \mathrm{RHom}_A(M, N).$$

□

There is a structural characterization of K-projective DG modules, which we shall review (since we shall elaborate on it later). This characterization works in steps. First one defines *semi-free* DG A-modules. A DG A-module Q is called semi-free if there is a subset $X \subset Q$ consisting of (nonzero) homogeneous elements, and an exhaustive nonnegative increasing filtration $\{F_i X\}_{i \in \mathbb{Z}}$ of X by subsets (i.e. $F_{-1} X = \emptyset$ and $X = \bigcup F_i X$), such that $\mathrm{und}\, Q$ is a free graded $\mathrm{und}\, A$ -module with basis X, and for every i one has $\mathrm{d}(F_i X) \subset \sum_{x \in F_{i-1} X} Ax$. The set X is called a *semi-basis* of Q. Note that X is partitioned into $X = \coprod_{i \in \mathbb{Z}} X_i$, where $X_i := X \cap Q^i$. We call such a set a *graded set*. Now a DG A-module P is K-projective iff it is homotopy equivalent to a direct summand (in $\mathsf{DGMod}\, A$) of some semi-free DG module Q. See [AFH] or [Ke] for more details and for proofs.

A *free* (*super-commutative*, *non-positive*) *graded* $\mathbb{K}$*-algebra* is a graded algebra of the following form. One starts with a graded set of variables $X = \coprod_{i \leq 0} X_i$; the elements of X_i are the variables of degree i. Let $X_{\mathrm{ev}} := \coprod_{i \text{ even}} X_i$ and $X_{\mathrm{odd}} := \coprod_{i \text{ odd}} X_i$. Consider the free associative $\mathbb{K}$-algebra $\mathbb{K}\langle X \rangle$ on this set of variables. Let I be the two-sided ideal of $\mathbb{K}\langle X \rangle$ generated by all elements of the form $xy - (-1)^{ij} yx$ or z^2, where $x \in X_i$, $y \in X_j$, $z \in X_k$, and k is odd. The free super-commutative graded $\mathbb{K}$-algebra on X is the quotient $\mathbb{K}[X] := \mathbb{K}\langle X \rangle / I$. It is useful to note that

$$\mathbb{K}[X] \cong \mathbb{K}[X_{\mathrm{ev}}] \otimes_{\mathbb{K}} \mathbb{K}[X_{\mathrm{odd}}],$$

and that $\mathbb{K}[X_{\mathrm{ev}}]$ is a commutative polynomial algebra, whereas $\mathbb{K}[X_{\mathrm{odd}}]$ is an exterior algebra.

Definition 1.4. Suppose $A \to B$ is a homomorphism of DG $\mathbb{K}$-algebras. B is called a *semi-free* (super-commutative, non-positive) *DG algebra relative to A* if there is a graded set $X = \coprod_{i \leq 0} X_i$, and an isomorphism of graded $\mathrm{und}\, A$ -algebras

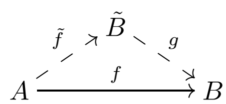
$$(\mathrm{und}\, A) \otimes_{\mathbb{K}} \mathbb{K}[X] \cong \mathrm{und}\, B.$$

Observe that the DG algebra B in the definition above, when regarded as a DG A-module, is semi-free with semi-basis consisting of the monomials in elements of X. Hence B is also K-projective and K-flat as DG A-module.

Definition 1.5. Suppose A and B are DG $\mathbb{K}$-algebras and $f : A \to B$ is a homomorphism of DG algebras. A *semi-free* (*resp. K-projective, resp. K-flat*) *DG algebra resolution of* B *relative to A* is the data $A \xrightarrow{\tilde{f}} \tilde{B} \xrightarrow{g} B$, where $\tilde{B}$ is a DG $\mathbb{K}$-algebra, $\tilde{f}$ and g are DG algebra homomorphisms, and the following conditions are satisfied:

(i) $g \circ \tilde{f} = f$.
(ii) g is a quasi-isomorphism.
(iii) $\tilde{f}$ makes $\tilde{B}$ into a semi-free DG algebra relative to A (resp. a K-projective DG A-module, resp. a K-flat DG A-module).

We also say that $A \xrightarrow{\tilde{f}} \tilde{B} \xrightarrow{g} B$ is a *semi-free* (*resp. K-projective, resp. K-flat*) *DG algebra resolution* of $A \xrightarrow{f} B$.



Proposition 1.6. *Let A and B be DG $\mathbb{K}$-algebras, and let $f : A \to B$ be a DG algebra homomorphism.*

(1) *There exists a semi-free DG algebra resolution $A \xrightarrow{\tilde{f}} \tilde{B} \xrightarrow{g} B$ of $A \xrightarrow{f} B$.*
(2) *Moreover, if $\mathrm{H}A$ is a noetherian algebra and $\mathrm{H}B$ is a finitely generated $\mathrm{H}A$ -algebra, then we can choose the semi-free DG algebra $\tilde{B}$ in part* (1) *such that $\mathrm{und}\, \tilde{B} \cong (\mathrm{und}\, A) \otimes_{\mathbb{K}} \mathbb{K}[X]$, where the graded set $X = \coprod_{i \leq 0} X_i$ has finite graded components X_i.*
(3) *If $\mathrm{H}A$ is a noetherian algebra, B is a usual algebra, and $B = \mathrm{H}^0 B$ is a finitely generated $\mathrm{H}^0 A$ -module, then there exists a K-projective DG algebra resolution $A \to \tilde{B} \to B$ of $A \to B$, such that $\mathrm{und}\, \tilde{B} \cong \bigoplus_{i=-\infty}^{0} \mathrm{und}\, A[-i]^{\mu_i}$ as graded $\mathrm{und}\, A$ -modules, and the multiplicities μ_i are finite.*

Proof. (1) We shall construct $\tilde{B}$ as the union of an increasing sequence of DG algebras $F_0 \tilde{B} \subset F_1 \tilde{B} \subset \cdots$, which will be defined recursively. At the same time we shall construct an increasing sequence of DG algebra homomorphisms $A \to F_i \tilde{B} \xrightarrow{g_i} B$, and an increasing sequence of graded sets $F_i X \subset F_i \tilde{B}$. The homomorphism g will be the union of the g_i, and the graded set $X = \coprod_{j \leq 0} X_j$ will be the union of the sets $F_i X$. For every i the following conditions will hold:

(i) $\mathrm{H}(g_i) : \mathrm{H}(F_i \tilde{B}) \to \mathrm{H}B$ is surjective in degrees $\geq -i$.
(ii) $\mathrm{H}(g_i) : \mathrm{H}(F_i \tilde{B}) \to \mathrm{H}B$ is bijective in degrees $\geq -i + 1$.
(iii) $F_i \tilde{B} = A[F_i X]$, $\mathrm{d}(F_i X) \subset F_{i-1} \tilde{B}$ and $\mathrm{und}\, F_i \tilde{B} \cong (\mathrm{und}\, A) \otimes_{\mathbb{K}} \mathbb{K}[F_i X]$.

We start by choosing a set of elements of B^0 that generate $\mathrm{H}^0 B$ as $\mathrm{H}^0 A$ -algebra. This gives us a set X_0 of elements of degree 0 with a function $g_0 : X_0 \to B^0$. Consider the DG algebra $\mathbb{K}[X_0]$ with zero differential; and define $F_0\tilde{B} := A \otimes_{\mathbb{K}} \mathbb{K}[X_0]$. Also define $F_0X := X_0$. We get a DG algebra homomorphism $g_0 : F_0\tilde{B} \to B$, and conditions (i)-(iii) hold for $i = 0$.

Now assume $i \geq 0$, and that for every $j \leq i$ we have DG algebra homomorphisms $g_j : F_j\tilde{B} \to B$ and graded sets F_jX satisfying conditions (i)-(iii). We will construct $F_{i+1}\tilde{B}$ etc.

Choose a set Y'_{i+1} of elements (of degree $-i-1$) and a function $g_{i+1} : Y'_{i+1} \to B^{-i-1}$ such that $\{g_{i+1}(y) \mid y \in Y'_{i+1}\}$ is a set of cocycles that generates $\mathrm{H}^{-i-1} B$ as $\mathrm{H}^0 A$-module. For $y \in Y'_{i+1}$ define $\mathrm{d}(y) := 0$.

Next let
$$J_{i+1} := \{b \in (F_i\tilde{B})^{-i} \mid \mathrm{d}(b) = 0 \text{ and } \mathrm{H}^{-i}(g_i)(b) = 0\}.$$
Choose a set Y''_{i+1} of elements (of degree $-i-1$) and a function $\mathrm{d} : Y''_{i+1} \to J_{i+1}$ such that $\{\mathrm{d}(y) \mid y \in Y''_{i+1}\}$ is a set of elements whose images in $\mathrm{H}^{-i}F_i\tilde{B}$ generate $\operatorname{Ker}\big(\mathrm{H}^{-i}(g_i) : \mathrm{H}^{-i}F_i\tilde{B} \to \mathrm{H}^{-i}B\big)$ as $\mathrm{H}^0 A$-module. Let $y \in Y''_{i+1}$. By definition $g_i(\mathrm{d}(y)) = \mathrm{d}(b)$ for some $b \in B^{-i}$; and we define $g_{i+1}(y) := b$.

Let $Y_{i+1} := Y'_{i+1} \sqcup Y''_{i+1}$ and $F_{i+1}X := F_iX \sqcup Y_{i+1}$. Define the DG algebra $F_{i+1}\tilde{B}$ to be
$$F_{i+1}\tilde{B} := F_i\tilde{B} \otimes_{\mathbb{K}} \mathbb{K}[Y_{i+1}]$$
with differential d extending the differential of $F_i\tilde{B}$ and the function $\mathrm{d} : Y_{i+1} \to F_i\tilde{B}$ defined above.

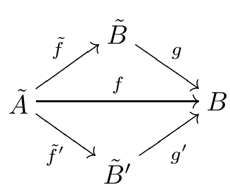
(2) This is because at each step in (1) the sets Y_i can be chosen to be finite.

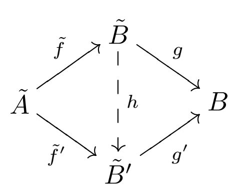
(3) Choose elements $b_1, \ldots, b_m \in B$ that generate it as A^0-algebra. Since each b_i is integral over A^0, there is some monic polynomial $p_i(y) \in A^0[y]$ such that $p_i(b_i) = 0$. Let $y_1, \ldots, y_m$ be distinct variables of degree 0. Define $Y_0 := \{y_1, \ldots, y_m\}$ and $B^\dagger := A^0[Y_0]/\big(p_1(y_1), \ldots, p_m(y_m)\big)$. This is an A^0-algebra, which is a free module of finite rank. Let $g_0 : B^\dagger \to B$ be the surjective A^0-algebra homomorphism $y_i \mapsto b_i$. Define $F_0\tilde{B} := A \otimes_{A^0} B^\dagger$ and $F_0X := \emptyset$. Then conditions (i)-(ii) hold for $i = 0$, as well as condition (iii') below.

(iii') $F_i\tilde{B} = A[Y_0 \cup F_iX]$, $\mathrm{d}(F_iX) \subset F_{i-1}\tilde{B}$ and
$$\operatorname{und} F_i\tilde{B} \cong (\operatorname{und} A) \otimes_{A^0} A^0[F_iX] \otimes_{A^0} B^\dagger.$$

For $i \geq 1$ the proof proceeds as in part (i), but always using condition (iii') instead of (iii). □

Proposition 1.7. *Suppose we are given three DG $\mathbb{K}$-algebras $\tilde{A}, \tilde{B}, \tilde{B}'$; a $\mathbb{K}$-algebra B; and five DG algebra homomorphisms $f, \tilde{f}, \tilde{f}', g, g'$ such that the first diagram below is commutative. Assume that g' is a quasi-isomorphism, and $\tilde{B}$ is semi-free DG algebra relative to $\tilde{A}$. Then there exists a DG algebra homomorphism $h : \tilde{B} \to \tilde{B}'$ such that the second diagram below is commutative.*





Proof. By definition there is a graded set $X = \coprod_{i \leq 0} X_i$ such that $\operatorname{und} \tilde{B} \cong (\operatorname{und} \tilde{A}) \otimes_{\mathbb{K}} \mathbb{K}[X]$. Let's define $F_i X := \bigcup_{j \geq -i} X_j$ and $F_i \tilde{B} := \tilde{A}[F_i X] \subset \tilde{B}$. We shall define a compatible sequence of DG algebra homomorphisms $h_i : F_i \tilde{B} \to \tilde{B}'$, whose union will be called h.

For $i = 0$ we note that $g' : \tilde{B}'^0 \to B$ is surjective. Hence there is a function $h_0 : X_0 \to \tilde{B}'^0$ such that $g'(h_0(x)) = g(x)$ for every $x \in X_0$. Since $F_0 \tilde{B} \cong \tilde{A} \otimes_{\mathbb{K}} \mathbb{K}[X_0]$ and $\mathrm{d}(h_0(X_0)) = 0$ we can extend the function h_0 uniquely to a DG algebra homomorphism $h_0 : F_0 \tilde{B} \to \tilde{B}$ such that $h_0 \circ \tilde{f} = \tilde{f}'$.

Now assume that $i \geq 0$ and $h_i : F_i \tilde{B} \to \tilde{B}'$ has been defined. Let $Y_{i+1} := F_{i+1} X - F_i X$. This is a set of degree $-i-1$ elements. Take any $y \in Y_{i+1}$. Then $\mathrm{d}(y) \in (F_i \tilde{B})^{-i}$, and we let $b := h_i(\mathrm{d}(y)) \in \tilde{B}'^{-i}$. Because $\mathrm{H} \tilde{B} \cong \mathrm{H} \tilde{B}' = B$ there exists an element $c \in \tilde{B}'^{-i-1}$ such that $\mathrm{d}(c) = b$. We now define $h_{i+1}(y) := c$. The function $h_{i+1} : Y_{i+1} \to \tilde{B}'^{-i-1}$ extends to a unique DG algebra homomorphism $h_{i+1} : F_{i+1} \tilde{B} \to \tilde{B}'$ such that $h_{i+1}|_{F_i \tilde{B}} = h_i$. □

From here to the end of this section we assume $\mathbb{K}$ is noetherian.

A homomorphism $A \to A'$ between two $\mathbb{K}$-algebras is called a *localization* if it induces an isomorphism $S^{-1} A \xrightarrow{\simeq} A'$ for some multiplicatively closed subset $S \subset A$. We then say that A' is a localization of A. A $\mathbb{K}$-algebra A is called *essentially of finite type* if A is a localization of some finitely generated $\mathbb{K}$-algebra. Such an algebra A is noetherian. If B is an essentially finite type A-algebra then it is an essentially finite type $\mathbb{K}$-algebra.

Proposition 1.8. *Let A be an essentially finite type $\mathbb{K}$-algebra. Then there is a DG algebra quasi-isomorphism $\tilde{A} \to A$ such that $\tilde{A}^0$ is an essentially finite type $\mathbb{K}$-algebra, and each $\tilde{A}^i$ is a finitely generated $\tilde{A}^0$-module and a flat $\mathbb{K}$-module. In particular $\tilde{A}$ is a K-flat DG $\mathbb{K}$-module.*

Proof. Pick a finitely generated $\mathbb{K}$-algebra A_{f} such that $S^{-1} A_{\mathrm{f}} \cong A$ for some multiplicatively closed subset $S \subset A_{\mathrm{f}}$. According to Proposition 1.6(2) we can find a semi-free DG algebra resolution $\tilde{A}_{\mathrm{f}} \to A_{\mathrm{f}}$ where $\tilde{A}_{\mathrm{f}}$ has finitely many algebra generators in each degree. Let $\tilde{S} \subset \tilde{A}_{\mathrm{f}}^0$ be the pre-image of S under the surjection $\tilde{A}_{\mathrm{f}}^0 \to A_{\mathrm{f}}$. Now define $\tilde{A} := (\tilde{S}^{-1} \tilde{A}_{\mathrm{f}}^0) \otimes_{\tilde{A}_{\mathrm{f}}^0} \tilde{A}_{\mathrm{f}}$. □

Corollary 1.9. *Let A be an essentially finite type $\mathbb{K}$-algebra, and let $\tilde{A} \to A$ be any K-flat DG algebra resolution relative to $\mathbb{K}$. Then $\mathrm{H}^0(\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$ is an essentially finite type $\mathbb{K}$-algebra, and each $\mathrm{H}^i(\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$ is a finitely generated $\mathrm{H}^0(\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$-module.*

Proof. Using Proposition 1.7, and passing via a semi-free DG algebra resolution, we can replace the given resolution $\tilde{A} \to A$ by another one satisfying the finiteness conditions in Proposition 1.8. Now the assertion is clear. □

Let M be a graded module. The *amplitude* $\operatorname{amp} M$ is defined as follows. Given $d \in \mathbb{N}$ we say that $\operatorname{amp} M \leq d$ if there exists some $i_0 \in \mathbb{Z}$ such that $\{i \mid M^i \neq 0\} \subset \{i_0, \ldots, i_0 + d\}$. Then we let $\operatorname{amp} M := \inf\{d \in \mathbb{N} \mid \operatorname{amp} M \leq d\} \in \mathbb{N} \cup \{\infty\}$. Thus M is bounded if and only if $\operatorname{amp} M < \infty$. Now let A be a DG algebra with $\mathrm{H}A$ bounded, and let M be a DG A-module. For any $d \in \mathbb{N}$ we say that $\operatorname{flat.dim}_A M \leq d$ if given any $N \in \mathsf{DGMod}\, A$ the inequality $\operatorname{amp} \mathrm{H}(M \otimes^{\mathrm{L}}_A N) \leq \operatorname{amp} \mathrm{H}N + d$ holds. The *flat dimension* of M is defined to be $\operatorname{flat.dim}_A M := \inf\{d \in \mathbb{N} \mid \operatorname{flat.dim}_A M \leq d\}$. Observe that M has finite flat dimension if and only if the functor $M \otimes^{\mathrm{L}}_A -$ is way out on both sides, in the sense of [RD, Section I.7]. Similarly one can define the projective dimension $\operatorname{proj.dim}_A M$ of a

DG A-module M, by considering the amplitude of $\mathrm{H}\,\mathrm{RHom}_A(M,N)$. For a usual algebra A and a single module M the dimensions defined above coincide with the usual ones.

Proposition 1.10. *Let A and B be DG $\mathbb{K}$-algebras, $L \in \mathsf{DGMod}\, A$, $M \in \mathsf{DGMod}\, A \otimes_{\mathbb{K}} B$ and $N \in \mathsf{DGMod}\, B$. There exists a functorial morphism*

$$\psi : \mathrm{RHom}_A(L,M) \otimes^{\mathrm{L}}_B N \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_B N)$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A \otimes_{\mathbb{K}} B)$. If conditions (i), (ii)*, and* (iii) *below hold, then the morphism ψ is an isomorphism.*

(i) $\mathrm{H}^0 A$ *is noetherian,* $\mathrm{H}L$ *is bounded above, and each of the* $\mathrm{H}^0 A$ *-modules* $\mathrm{H}^i A$ *and* $\mathrm{H}^i L$ *are finitely generated.*
(ii) $\mathrm{H}M$ *and* $\mathrm{H}N$ *are bounded.*
(iii) *Either* (a), (b) *or* (c) *is satisfied:*
  (a) $\mathrm{H}^i A = 0$ *for all $i \neq 0$, and L has finite projective dimension over A.*
  (b) $\mathrm{H}^i B = 0$ *for all $i \neq 0$, and N has finite flat dimension over B.*
  (c) $\mathrm{H}^i B = 0$ *for all $i \neq 0$,* $\mathrm{H}^0 B$ *is noetherian,* $\mathrm{H}N$ *is bounded, each* $\mathrm{H}^i N$ *is a finitely generated module over* $\mathrm{H}^0 B$*, the canonical morphism* $B \to \mathrm{RHom}_B(N,N)$ *is an isomorphism, both M and* $\mathrm{RHom}_A(L,M)$ *have finite flat dimension over B, and* $\mathrm{H}\,\mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_B N)$ *is bounded.*

Proof. The proof is in five steps.

Step 1. To define ψ we may choose a K-projective resolution $P \to L$ over A, and a K-flat resolution $Q \to N$ over B. There an obvious homomorphism of DG $A \otimes_{\mathbb{K}} B$ -modules

$$\psi_{P,Q} : \mathrm{Hom}_A(P,M) \otimes_B Q \to \mathrm{Hom}_A(P, M \otimes_B Q).$$

In the derived category this represents ψ.

Step 2. To prove that ψ is an isomorphism (or equivalently that $\psi_{P,Q}$ is a quasi-isomorphism) we may forget the $A \otimes_{\mathbb{K}} B$ -module structures, and consider ψ as a morphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$. Now by Proposition 1.3(2) we can replace A and B by quasi-isomorphic DG $\mathbb{K}$-algebras. Thus we may assume both A and B are semi-free as DG $\mathbb{K}$-modules.

Step 3. Let's suppose that condition (iii.a) holds. So $A \to \mathrm{H}^0 A$ is a quasi-isomorphism. Since B is K-flat over $\mathbb{K}$ it follows that $A \otimes_{\mathbb{K}} B \to \mathrm{H}^0 A \otimes_{\mathbb{K}} B$ is also a quasi-isomorphism. By Proposition 1.3 we can assume that $L \in \mathsf{DGMod}\, \mathrm{H}^0 A$ and $M \in \mathsf{DGMod}(\mathrm{H}^0 A \otimes_{\mathbb{K}} B)$, and that L has finite projective dimension over $\mathrm{H}^0 A$. So we may replace A with $\mathrm{H}^0 A$, and thus assume that A is a noetherian algebra.

Now choose a resolution $P \to L$, where P is a bounded complex of finitely generated projective A-modules. Take any K-flat resolution $Q \to N$ over B. Then the homomorphism $\psi_{P,Q}$ is actually bijective.

Step 4. Let's assume condition (iii.b) holds. As in step 3 we can suppose that $B = B^0$. Choose a bounded resolution $Q \to N$ by flat B-modules. By replacing M with the truncation $\tau^{\geq j_0} \tau^{\leq j_1} M$ for some $j_0 \ll 0$ and $j_1 \gg 0$ we may assume M is bounded. According to [AFH, Theorem 9.2.7] we can find a semi-free resolution $P \to L$ over A such that $\mathrm{und}\, P \cong \bigoplus_{i=-\infty}^{i_1} \mathrm{und}\, A[-i]^{\mu_i}$ with all the multiplicities μ_i finite. Because the μ_i are finite and both M and Q are bounded the homomorphism $\psi_{P,Q}$ is bijective.

Step 5. Finally we consider condition (iii.c). We can assume that $B = B^0$ is noetherian. Since $N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ and $\mathrm{RHom}_B(N,N) \cong B$ we see that the support of N is $\operatorname{Spec} B$. By Lemma 1.11 below we conclude that N generates $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$. Let

$$\psi' : \mathrm{RHom}_B\big(N, \mathrm{RHom}_A(L,M) \otimes^{\mathrm{L}}_B N\big) \to \mathrm{RHom}_B\big(N, \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_B N)\big)$$

be the morphism obtained from ψ by applying the functor $\mathrm{RHom}_B(N, -)$. Since ψ is a morphism in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$, in order to prove it is an isomorphism it suffices to prove that ψ' is an isomorphism.

The complex $\mathrm{RHom}_A(L, M)$ has finite flat dimension over B, so using the proposition with condition (iii.b), which we already proved, we have

$$\begin{aligned} \mathrm{RHom}_B\big(N, \mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_B N\big) &\cong \mathrm{RHom}_B(N, N) \otimes^{\mathrm{L}}_B \mathrm{RHom}_A(L, M) \\ &\cong \mathrm{RHom}_A(L, M). \end{aligned}$$

On the other hand the complex M has finite flat dimension over B, so using the proposition with condition (iii.b) once more (for the isomorphism marked $\diamond$), we have

$$\begin{aligned} \mathrm{RHom}_B\big(N, \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_B N)\big) &\cong \mathrm{RHom}_A\big(L, \mathrm{RHom}_B(N, M \otimes^{\mathrm{L}}_B N)\big) \\ \cong^{\diamond} \mathrm{RHom}_A\big(L, M \otimes^{\mathrm{L}}_B \mathrm{RHom}_B(N, N)\big) &\cong \mathrm{RHom}_A(L, M). \end{aligned}$$

Tracking the effect of these isomorphism on ψ' we see that it gets transformed into the identity automorphism of $\mathrm{RHom}_A(L, M)$. $\square$

Let B be a noetherian ring. Recall that given a complex $N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ its support is defined to be $\bigcup_i \operatorname{Supp} \mathrm{H}^i N \subset \operatorname{Spec} B$. The complex N is said to generate $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$ if for any nonzero object $M \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$ one has $\mathrm{RHom}_B(N, M) \neq 0$.

Lemma 1.11. *Suppose B is a noetherian ring and $N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$ is a complex whose support is $\operatorname{Spec} B$. Then N generates $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$.*

Proof. Suppose M is a nonzero object in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$. We have to prove that $\mathrm{RHom}_B(N, M) \neq 0$. Let $i_0 := \min\{i \in \mathbb{Z} \mid \mathrm{H}^i M \neq 0\}$, and choose a nonzero finitely generated submodule $M' \subset \mathrm{H}^{i_0} M$. Let $\mathfrak{p}$ be a minimal prime ideal in the support of M'; so that $M'_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_B M'$ is a nonzero finite length module over the local ring $B_{\mathfrak{p}}$. Now $N_{\mathfrak{p}}$ is a nonzero object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B_{\mathfrak{p}})$. Let $j_1 := \max\{j \in \mathbb{Z} \mid \mathrm{H}^j N_{\mathfrak{p}} \neq 0\}$. Since $\mathrm{H}^{j_1} N_{\mathfrak{p}}$ is a nonzero finitely generated $A_{\mathfrak{p}}$-module, there exists a nonzero homomorphism $\phi : \mathrm{H}^{j_1} N_{\mathfrak{p}} \to M'_{\mathfrak{p}}$. This ϕ can be interpreted as a nonzero element of $\mathrm{Ext}^{i_0 - j_1}_{B_{\mathfrak{p}}}(N_{\mathfrak{p}}, M_{\mathfrak{p}})$, which, by Proposition 1.10 with its condition (iii.b), is isomorphic to $B_{\mathfrak{p}} \otimes_B \mathrm{Ext}^{i_0 - j_1}_B(N, M)$. $\square$

Remark 1.12. Proposition 1.10 can be extended by replacing conditions (iii.a) and (iii.b) respectively with: (iii.a') $\mathrm{H}A$ is a bounded essentially finite type $\mathbb{K}$-algebra, and L has finite projective dimension over A; and (iii.b') $\mathrm{H}B$ is a bounded essentially finite type $\mathbb{K}$-algebra, $\mathrm{H}N$ is a finitely generated $\mathrm{H}B$ -module, and N has finite flat dimension over A. The trick for (iii.a') is to localize on $\operatorname{Spec} \mathrm{H}^0 A$ and to look at minimal semi-free resolutions of L. This trick also shows that $\operatorname{flat.dim}_A L = \operatorname{proj.dim}_A L$. Details will appear elsewhere.

2. The Squaring Operation

In this section we introduce a key technical notion used in the definition of rigidity, namely the squaring operation. This operation is easy to define when the base ring $\mathbb{K}$ is a field (see Corollary 2.7), but when $\mathbb{K}$ is just a commutative ring (as we assume in this section) there are complications. We solve the problem using DG algebras.

Recall that for a DG algebra A the derived category of DG modules is denoted by $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A)$. If A is a usual algebra then $\tilde{\mathsf{D}}(\mathsf{DGMod}\, A) = \mathsf{D}(\mathsf{Mod}\, A)$.

Let $M \in \mathsf{D}(\mathsf{Mod}\,\mathbb{K})$. As explained earlier the derived tensor product $M \otimes^{\mathrm{L}}_{\mathbb{K}} M \in \mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ is defined to be $M \otimes^{\mathrm{L}}_{\mathbb{K}} M := \tilde{M} \otimes_{\mathbb{K}} \tilde{M}$, where $\tilde{M} \to M$ is any K-flat resolution of M. If $M \in \mathsf{DGMod}\,A$ for some DG $\mathbb{K}$-algebra A, then we would like to be able to make $M \otimes^{\mathrm{L}}_{\mathbb{K}} M$ into an object of $\tilde{\mathsf{D}}(\mathsf{DGMod}\,A \otimes_{\mathbb{K}} A)$. But this is not always possible, at least not in any obvious way, due to torsion. (For instance take $\mathbb{K} := \mathbb{Z}$ and $M = A := \mathbb{Z}/(2)$). Fortunately there is a way to get around this problem.

Lemma 2.1. *Let $\tilde{A} \to A$ be a quasi-isomorphism of DG $\mathbb{K}$-algebras, and assume $\tilde{A}$ is K-flat as DG $\mathbb{K}$-module. Then the* (*non-additive*) *functor $\tilde{\mathsf{D}}(\mathsf{DGMod}\,A) \to \mathsf{D}(\mathsf{Mod}\,\mathbb{K})$, $M \mapsto M \otimes^{\mathrm{L}}_{\mathbb{K}} M$, factors canonically through $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$.*

Proof. Choose any quasi-isomorphism $\tilde{M} \to M$ in $\mathsf{DGMod}\,\tilde{A}$ with $\tilde{M}$ K-flat over $\mathbb{K}$. This is possible since any K-flat DG $\tilde{A}$-module is K-flat over $\mathbb{K}$. We get $M \otimes^{\mathrm{L}}_{\mathbb{K}} M = \tilde{M} \otimes_{\mathbb{K}} \tilde{M} \in \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$. □

Theorem 2.2. *Let A be a $\mathbb{K}$-algebra and let M be a DG A-module. Choose a K-flat DG algebra resolution $\mathbb{K} \to \tilde{A} \to A$ of $\mathbb{K} \to A$. Then the object*

$$\mathrm{Sq}_{A/\mathbb{K}}\, M := \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \in \mathsf{D}(\mathsf{Mod}\,A),$$

where the A-module structure is via the action on the first argument of RHom, *is independent of this choice.*

Proof. The idea for the proof was communicated to us by Bernhard Keller. Choose some semi-free DG algebra resolution $\mathbb{K} \to \tilde{A}' \to A$ of $\mathbb{K} \to A$. We will show that there is a canonical isomorphism

$$\mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \xrightarrow{\simeq} \mathrm{RHom}_{\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)$$

in $\mathsf{D}(\mathsf{Mod}\,A)$.

Let us choose a K-projective resolution $\tilde{M} \to M$ over $\tilde{A}$, and a K-injective resolution $\tilde{M} \otimes_{\mathbb{K}} \tilde{M} \to \tilde{I}$ over $\tilde{A} \otimes_{\mathbb{K}} \tilde{A}$. So

$$\mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) = \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{I}).$$

Likewise let's choose resolutions $\tilde{M}' \to M$ and $\tilde{M}' \otimes_{\mathbb{K}} \tilde{M}' \to \tilde{I}'$ over $\tilde{A}'$ and $\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'$ respectively.

By Proposition 1.7 there is a DG algebra quasi-isomorphism $f_0 : \tilde{A}' \to \tilde{A}$ that's compatible with the quasi-isomorphisms to A. By the categorical properties of K-projective resolutions there is an $\tilde{A}'$-linear quasi-isomorphism $\phi_0 : \tilde{M}' \to \tilde{M}$, that's compatible up to homotopy with the quasi-isomorphisms to M. We obtain an $\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'$ -linear quasi-isomorphism $\phi_0 \otimes \phi_0 : \tilde{M}' \otimes_{\mathbb{K}} \tilde{M}' \to \tilde{M} \otimes_{\mathbb{K}} \tilde{M}$. Next by the categorical properties of K-injective resolutions there is an $\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'$ -linear quasi-isomorphism $\psi_0 : \tilde{I} \to \tilde{I}'$ that's compatible up to homotopy with the quasi-isomorphisms from $\tilde{M}' \otimes_{\mathbb{K}} \tilde{M}'$. We thus get an A-linear homomorphism

$$\chi_0 : \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{I}) \to \mathrm{Hom}_{\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'}(A, \tilde{I}').$$

Proposition 1.3 shows that χ_0 is in fact an isomorphism in $\mathsf{D}(\mathsf{Mod}\,A)$.

Now suppose $f_1 : \tilde{A}' \to \tilde{A}$, $\phi_1 : \tilde{M}' \to \tilde{M}$ and $\psi_1 : \tilde{I} \to \tilde{I}'$ are other choices of quasi-isomorphisms of the same respective types as f_0, ϕ_0 and ψ_0. Then we get an induced isomorphism

$$\chi_1 : \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{I}) \to \mathrm{Hom}_{\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'}(A, \tilde{I}')$$

in $\mathsf{D}(\mathsf{Mod}\,A)$. We shall prove that $\chi_1 = \chi_0$.

Here we have to introduce an auxiliary DG $\mathbb{K}$-module $\mathrm{C}(\tilde{M})$, the cylinder module. As graded module one has $\mathrm{C}(\tilde{M}) := \left[\begin{smallmatrix} \tilde{M} & \tilde{M}[-1] \\ 0 & \tilde{M} \end{smallmatrix}\right]$, a triangular matrix module, and the differential is $\mathrm{d}(\left[\begin{smallmatrix} m_0 & n \\ 0 & m_1 \end{smallmatrix}\right]) := \left[\begin{smallmatrix} \mathrm{d}(m_0) & m_0 - m_1 - \mathrm{d}(n) \\ 0 & \mathrm{d}(m_1) \end{smallmatrix}\right]$ for $m_0, m_1, n \in \tilde{M}$. There are DG module quasi-isomorphisms $\epsilon : \tilde{M} \to \mathrm{C}(\tilde{M})$ and $\eta_0, \eta_1 : \mathrm{C}(\tilde{M}) \to \tilde{M}$, with formulas $\epsilon(m) := \left[\begin{smallmatrix} m & 0 \\ 0 & m \end{smallmatrix}\right]$ and $\eta_i(\left[\begin{smallmatrix} m_0 & n \\ 0 & m_1 \end{smallmatrix}\right]) := m_i$. The cylinder module $\mathrm{C}(\tilde{M})$ is a DG module over $\tilde{A}$ by the formula $a \cdot \left[\begin{smallmatrix} m_0 & n \\ 0 & m_1 \end{smallmatrix}\right] := \left[\begin{smallmatrix} am_0 & an \\ 0 & am_1 \end{smallmatrix}\right]$.

There is a quasi-isomorphism of DG $\tilde{A}$-modules $\mathrm{C}(\tilde{M}) \to \left[\begin{smallmatrix} \tilde{M} & M[-1] \\ 0 & \tilde{M} \end{smallmatrix}\right]$ which is the identity on the diagonal elements, and the given quasi-isomorphism $\tilde{M} \to M$ in the corner. The two $\tilde{A}'$-linear quasi-isomorphisms ϕ_0 and ϕ_1 fit into an $\tilde{A}'$-linear quasi-morphism $\tilde{M}' \xrightarrow{\left[\begin{smallmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{smallmatrix}\right]} \left[\begin{smallmatrix} \tilde{M} & M[-1] \\ 0 & \tilde{M} \end{smallmatrix}\right]$. Since $\tilde{M}'$ is K-projective over $\tilde{A}'$ we can lift $\left[\begin{smallmatrix} \phi_0 & 0 \\ 0 & \phi_1 \end{smallmatrix}\right]$ to a quasi-isomorphism $\phi : \tilde{M}' \to \mathrm{C}(\tilde{M})$ such that $\eta_i \circ \phi = \phi_i$ up to homotopy.

Let's choose a K-injective resolution $\mathrm{C}(\tilde{M}) \otimes_{\mathbb{K}} \mathrm{C}(\tilde{M}) \to \tilde{K}$ over $\tilde{A} \otimes_{\mathbb{K}} \tilde{A}$. Then for $i = 0, 1$ we have a diagram

$$\begin{array}{ccccc} \tilde{I}' & \xleftarrow{\psi} & \tilde{K} & \xleftarrow{\beta_i} & \tilde{I} \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{M}' \otimes_{\mathbb{K}} \tilde{M}' & \xrightarrow{\phi \otimes \phi} & \mathrm{C}(\tilde{M}) \otimes_{\mathbb{K}} \mathrm{C}(\tilde{M}) & \xrightarrow{\eta_i \otimes \eta_i} & \tilde{M} \otimes_{\mathbb{K}} \tilde{M} \end{array}$$

that's commutative up to homotopy. Here ψ and β_i are some DG module homomorphisms, which exist due to the K-injectivity of $\tilde{I}'$ and $\tilde{K}$ respectively. Because $\phi_i \otimes \phi_i = (\eta_i \otimes \eta_i) \circ (\phi \otimes \phi)$ up to homotopy, and $\tilde{I}'$ is K-injective, it follows that the $\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'$ -linear DG module quasi-isomorphisms $\psi \circ \beta_i$ and ψ_i are homotopic. Therefore in order to prove that $\chi_0 = \chi_1$ it suffices to prove that the two isomorphisms in $\mathsf{D}(\mathsf{Mod}\, A)$

$$\theta_0, \theta_1 : \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{I}) \to \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{K}),$$

that are induced by β_0, β_1 respectively, are equal.

For $i = 0, 1$ consider the diagram

$$\begin{array}{ccccc} \tilde{I} & \xleftarrow{\gamma} & \tilde{K} & \xleftarrow{\beta_i} & \tilde{I} \\ \uparrow & & \uparrow & & \uparrow \\ \tilde{M} \otimes_{\mathbb{K}} \tilde{M} & \xrightarrow{\epsilon \otimes \epsilon} & \mathrm{C}(\tilde{M}) \otimes_{\mathbb{K}} \mathrm{C}(\tilde{M}) & \xrightarrow{\eta_i \otimes \eta_i} & \tilde{M} \otimes_{\mathbb{K}} \tilde{M} \end{array}$$

where γ is some $\tilde{A} \otimes_{\mathbb{K}} \tilde{A}$ -linear DG module homomorphism, chosen so as to make the left square commute up to homotopy. As before, since $(\eta_i \otimes \eta_i) \circ (\epsilon \otimes \epsilon) = \mathbf{1}_{\tilde{M} \otimes_{\mathbb{K}} \tilde{M}}$ it follows that $\gamma \circ \beta_i$ and $\mathbf{1}_{\tilde{I}}$ are homotopic. Hence both θ_0 and θ_1 are inverses of the isomorphism

$$\mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{K}) \xrightarrow{\simeq} \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, \tilde{I})$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ induced by γ, so $\theta_0 = \theta_1$. □

Theorem 2.3. *Let A and B be $\mathbb{K}$-algebras, and let $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $N \in \mathsf{D}(\mathsf{Mod}\, B)$. Suppose $f : A \to B$ is an algebra homomorphism and $\phi : N \to M$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)$. Then there is an induced morphism*

$$\mathrm{Sq}_{f/\mathbb{K}}(\phi) : \mathrm{Sq}_{B/\mathbb{K}}\, N \to \mathrm{Sq}_{A/\mathbb{K}}\, M$$

in $\mathsf{D}(\mathsf{Mod}\, B)$*. This construction is functorial; namely if* C *is another* $\mathbb{K}$*-algebra,* $P \in \mathsf{D}(\mathsf{Mod}\, C)$*,* $g : B \to C$ *is an algebra homomorphism and* $\psi : P \to N$ *is a morphism in* $\mathsf{D}(\mathsf{Mod}\, B)$*, then*

$$\mathrm{Sq}_{g\circ f/\mathbb{K}}(\phi \circ \psi) = \mathrm{Sq}_{f/\mathbb{K}}(\phi) \circ \mathrm{Sq}_{g/\mathbb{K}}(\psi).$$

Also for the identity morphisms $\mathrm{Sq}_{\mathbf{1}_A/\mathbb{K}}(\mathbf{1}_M) = \mathbf{1}_{\mathrm{Sq}_{A/\mathbb{K}} M}$.

Proof. Let's choose a semi-free DG algebra resolution $\mathbb{K} \to \tilde{A} \to A$ of $\mathbb{K} \to A$, and a semi-free DG algebra resolution $\tilde{A} \xrightarrow{\tilde{f}} \tilde{B} \to B$ of $\tilde{A} \to B$. Note that $\tilde{B}$ is also semi-free relative to $\mathbb{K}$, so it may be used to calculate $\mathrm{Sq}_{B/\mathbb{K}} N$. Next let's choose DG module resolutions $\tilde{M} \to M$, $\tilde{N} \to N$, $\tilde{M} \otimes_{\mathbb{K}} \tilde{M} \to \tilde{I}$ and $\tilde{N} \otimes_{\mathbb{K}} \tilde{N} \to \tilde{J}$ by K-projective or K-injective DG modules over the appropriate DG algebras, as was done in the proof of Theorem 2.2. Since $\tilde{N}$ is a K-projective DG $\tilde{A}$-module we get an actual DG module homomorphism $\tilde{\phi} : \tilde{N} \to \tilde{M}$ representing ϕ. Therefore there is an $\tilde{A} \otimes_{\mathbb{K}} \tilde{A}$ -linear DG module homomorphism $\tilde{\phi} \otimes \tilde{\phi} : \tilde{N} \otimes_{\mathbb{K}} \tilde{N} \to \tilde{M} \otimes_{\mathbb{K}} \tilde{M}$. Because $\tilde{I}$ is K-injective we obtain a DG module homomorphism $\psi : \tilde{J} \to \tilde{I}$ lifting $\tilde{N} \otimes_{\mathbb{K}} \tilde{N} \xrightarrow{\tilde{\phi}\otimes\tilde{\phi}} \tilde{M} \otimes_{\mathbb{K}} \tilde{M} \to \tilde{I}$. Applying $\mathrm{Hom}(A, -)$ we then have a homomorphism

$$\mathrm{Sq}_{f/\mathbb{K}}(\phi) : \mathrm{Hom}_{\tilde{B}\otimes_{\mathbb{K}}\tilde{B}}(B, \tilde{J}) \to \mathrm{Hom}_{\tilde{A}\otimes_{\mathbb{K}}\tilde{A}}(A, \tilde{I})$$

in $\mathsf{DGMod}\, A$.

Given $g : B \to C$ and $\psi : P \to N$ it is now clear how to define $\mathrm{Sq}_{g/\mathbb{K}}(\psi)$ such that $\mathrm{Sq}_{g\circ f/\mathbb{K}}(\phi \circ \psi) = \mathrm{Sq}_{f/\mathbb{K}}(\phi) \circ \mathrm{Sq}_{g/\mathbb{K}}(\psi)$, for these particular choices.

It remains to prove that after passing to $\mathsf{D}(\mathsf{Mod}\, A)$ the morphism $\mathrm{Sq}_{f/\mathbb{K}}(\phi)$ becomes independent of choices. The independence on choices of K-projective and K-injective resolutions, and on the DG module homomorphisms $\tilde{\phi}$ and ψ, is standard. Now suppose we choose another semi-free DG algebra resolution $\mathbb{K} \to \tilde{A}' \to A$ of $\mathbb{K} \to A$, and a semi-free DG algebra resolution $\tilde{A}' \xrightarrow{\tilde{f}'} \tilde{B}' \to B$ of $\tilde{A}' \to B$. After choosing DG module resolutions $\tilde{M}' \to M$, $\tilde{N}' \to N$, $\tilde{M}' \otimes_{\mathbb{K}} \tilde{M}' \to \tilde{I}'$ and $\tilde{N}' \otimes_{\mathbb{K}} \tilde{N}' \to \tilde{J}'$ by K-projective or K-injective DG modules over the appropriate DG algebras, we obtain a homomorphism

$$\mathrm{Sq}'_{f/\mathbb{K}}(\phi) : \mathrm{Hom}_{\tilde{B}'\otimes_{\mathbb{K}}\tilde{B}'}(B, \tilde{J}') \to \mathrm{Hom}_{\tilde{A}'\otimes_{\mathbb{K}}\tilde{A}'}(A, \tilde{I}')$$

in $\mathsf{DGMod}\, A$.

Applying Proposition 1.7 twice we can find DG algebra homomorphisms g_0 and h_0 such that the diagram of DG algebra homomorphisms

(2.4)
$$\begin{array}{ccccc} \tilde{A}' & \xrightarrow{g_0} & \tilde{A} & \longrightarrow & A \\ {\scriptstyle \tilde{f}'}\downarrow & & {\scriptstyle \tilde{f}}\downarrow & & {\scriptstyle f}\downarrow \\ \tilde{B}' & \xrightarrow{h_0} & \tilde{B} & \longrightarrow & B. \end{array}$$

is commutative. As in the proof of Theorem 2.2 we pick quasi-isomorphisms $\psi_{M,0} : \tilde{I} \to \tilde{I}'$ and $\psi_{N,0} : \tilde{J} \to \tilde{J}'$ over $\tilde{A}' \otimes_{\mathbb{K}} \tilde{A}'$ and $\tilde{B}' \otimes_{\mathbb{K}} \tilde{B}'$ respectively. Then we get a commutative up to homotopy diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\tilde{A}\otimes_{\mathbb{K}}\tilde{A}}(A, \tilde{I}) & \xrightarrow{\chi_{M,0}} & \mathrm{Hom}_{\tilde{A}'\otimes_{\mathbb{K}}\tilde{A}'}(A, \tilde{I}') \\ {\scriptstyle \mathrm{Sq}_{f/\mathbb{K}}(\phi)}\uparrow & & \uparrow{\scriptstyle \mathrm{Sq}'_{f/\mathbb{K}}(\phi)} \\ \mathrm{Hom}_{\tilde{B}\otimes_{\mathbb{K}}\tilde{B}}(B, \tilde{J}) & \xrightarrow{\chi_{N,0}} & \mathrm{Hom}_{\tilde{B}'\otimes_{\mathbb{K}}\tilde{B}'}(B, \tilde{J}') \end{array}$$

in $\mathsf{DGMod}\, A$, where the horizontal arrows are quasi-isomorphisms. If we were to choose another pair of DG algebra quasi-isomorphisms $g_1 : \tilde{A}' \to \tilde{A}$ and $h_1 : \tilde{B}' \to \tilde{B}$ so as to make diagram 2.4 commutative, then according to Theorem 2.2 there would be equalities $\chi_{M,0} = \chi_{M,1}$ and $\chi_{N,0} = \chi_{N,1}$ of isomorphisms in $\mathsf{D}(\mathsf{Mod}\, A)$. Therefore $\mathrm{Sq}'_{f/\mathbb{K}}(\phi) = \mathrm{Sq}_{f/\mathbb{K}}(\phi)$ as morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$. □

For the identity homomorphism $\mathbf{1}_A : A \to A$ we write $\mathrm{Sq}_{A/\mathbb{K}}(\phi) := \mathrm{Sq}_{\mathbf{1}_A/\mathbb{K}}(\phi)$.

Definition 2.5. The (nonlinear) functor $\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$ from Theorems 2.2 and 2.3 is called the *squaring operation* over A relative to $\mathbb{K}$.

The next result explains the name "squaring".

Corollary 2.6. *In the situation of Theorem* 2.3 *let* $b \in B$. *Then*

$$\mathrm{Sq}_{f/\mathbb{K}}(b\phi) = b^2\, \mathrm{Sq}_{f/\mathbb{K}}(\phi).$$

Proof. It suffices to consider $f = \mathbf{1}_B : B \to B$ and $\phi = \mathbf{1}_N : N \to N$. Choose any lifting of b to $\tilde{b} \in \tilde{B}^0$. Then multiplication by $\tilde{b} \otimes \tilde{b}$ on $\tilde{J}$ has the same effect on $\mathrm{Hom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}(B, \tilde{J})$, up to homotopy, as multiplication by b^2 on B. □

Corollary 2.7. *Suppose A is a flat $\mathbb{K}$-algebra, and M is a bounded above complex of A-modules that are flat as $\mathbb{K}$-modules. Then there is a functorial isomorphism*

$$\mathrm{Sq}_{A/\mathbb{K}}\, M \cong \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes_{\mathbb{K}} M).$$

Proof. This is because A and M are K-flat DG $\mathbb{K}$-modules. □

Remark 2.8. One might be tempted to use the notation $\mathrm{RHom}_{A \otimes^{\mathrm{L}}_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)$ instead of $\mathrm{Sq}_{A/\mathbb{K}}\, M$. Indeed, we think it is possible to make sense of the "DG algebra" $A \otimes^{\mathrm{L}}_{\mathbb{K}} A$, as an object of a suitable Quillen localization of the category of DG $\mathbb{K}$-algebras. Cf. [Hi], and also [Qu], where an analogous construction was made using simplicial algebras rather than DG algebras. Then one should show that the triangulated category "$\tilde{\mathsf{D}}(\mathsf{DGMod}\, A \otimes^{\mathrm{L}}_{\mathbb{K}} A)$" is well-defined, etc. See also [Dr, Appendix V].

3. Rigid Complexes

In this section all rings are are by default commutative and noetherian. We shall use notation such as $f^* : A \to B$ for a ring homomorphism; so that $f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A$ is the corresponding morphism of schemes. This will make our notation for various functors more uniform. For instance restriction of scalars becomes $f_* : \mathsf{Mod}\, B \to \mathsf{Mod}\, A$, and extension of scalars (i.e. $M \mapsto B \otimes_A M$) becomes $f^* : \mathsf{Mod}\, A \to \mathsf{Mod}\, B$. See also Definitions 3.13 and 3.21. Given another algebra homomorphism $g^* : B \to C$ we shall sometimes write $(f \circ g)^* := g^* \circ f^*$.

Let us begin with a bit of commutative algebra. Recall that an A-algebra B is called formally smooth (resp. formally étale) if it has the lifting property (resp. the unique lifting property) for infinitesimal extensions. The A-algebra B is called smooth (resp. étale) if it is finitely generated and formally smooth (resp. formally étale). If B is smooth over A then it is flat, and $\Omega^1_{B/A}$ is a finitely generated projective B-module. See [EGA, Section 0_{IV}.19.3] and [EGA, Section IV.17.3] for details.

Definition 3.1. Let A and B be noetherian rings. A ring homomorphism $f^* : A \to B$ is called *essentially smooth* (resp. *essentially étale*) if it is of essentially finite type and formally smooth (resp. formally étale). In this case B is called an essentially smooth (resp. essentially étale) A-algebra.

Observe that smooth homomorphisms and localizations are essentially smooth.

Proposition 3.2. *Let $f^* : A \to B$ be an essentially smooth homomorphism.*

1. *There is an open covering $\operatorname{Spec} B = \bigcup_i \operatorname{Spec} B_i$ such that for every i the homomorphism $A \to B_i$ is the composition of a smooth homomorphism $A \to B_i^{\mathrm{sm}}$ and a localization $B_i^{\mathrm{sm}} \to B_i$.*
2. *f^* is flat, and $\Omega^1_{B/A}$ is a finitely generated projective B-module.*
3. *f^* is essentially étale if and only if $\Omega^1_{B/A} = 0$.*
4. *Let $g^* : B \to C$ be another essentially smooth homomorphism. Then $g^* \circ f^* : A \to C$ is also essentially smooth.*

Proof. (1) Choose a finitely generated A-subalgebra $B^{\mathrm{f}} \subset B$ such that B is a localization of B^{f}. We can identify $U := \operatorname{Spec} B$ with a subset of $U^{\mathrm{f}} := \operatorname{Spec} B^{\mathrm{f}}$. Take a point $x \in U$, and let $y := f(x) \in \operatorname{Spec} A$. Then the local ring $\mathcal{O}_{U^{\mathrm{f}},x} = \mathcal{O}_{U,x} = B_x$ is a formally smooth A_y-algebra. According to [EGA, Chapitre IV Théorème 17.5.1] there is an open neighborhood W of x in U^{f} which is smooth over $\operatorname{Spec} A$. Choose an element $b \in B^{\mathrm{f}}$ such that the localization $B^{\mathrm{f}}_b = B^{\mathrm{f}}[b^{-1}]$ satisfies $x \in \operatorname{Spec} B^{\mathrm{f}}_b \subset W$. Then B^{f}_b is a smooth A-algebra, B_b is a localization of B^{f}_b, $\operatorname{Spec} B_b$ is open in $\operatorname{Spec} B$, and $x \in \operatorname{Spec} B_b$. Finally let i be an index corresponding to the point x, and define $B_i^{\mathrm{sm}} := B^{\mathrm{f}}_b$ and $B_i := B_b$.

(2) follows from (1).

(3) See [EGA, Chapitre 0_{IV} Proposition 20.7.4].

(4) Both conditions in Definition 3.1 are transitive. □

Definition 3.3. Let $f^* : A \to B$ be an essentially smooth homomorphism. If $\operatorname{rank}_B \Omega^1_{B/A} = n$ then f^* is called an *essentially smooth homomorphism of relative dimension n*, and B is called an *essentially smooth A-algebra of relative dimension n*.

By Proposition 3.2(3), an essentially étale homomorphism is the same as an essentially smooth homomorphism of relative dimension 0.

Proposition 3.4. *Suppose $f^* : A \to B$ and $g^* : B \to C$ are essentially smooth homomorphism of relative dimensions m and n respectively. Then $g^* \circ f^* : A \to C$ is an essentially smooth homomorphism of relative dimension $m+n$, and there is a canonical isomorphism of C-modules $\Omega^{m+n}_{C/A} \cong \Omega^m_{B/A} \otimes_B \Omega^n_{C/B}$.*

Proof. By [EGA, Chapitre 0_{IV} Théorème 20.5.7] the sequence

$$0 \to C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

is split-exact. □

Proposition 3.5. *Let $f^* : A \to B$ be an essentially smooth homomorphism of relative dimension m.*

1. *The $B \otimes_A B$ -module B has finite projective dimension.*
2. *There is a canonical isomorphism*

$$\operatorname{Ext}^i_{B\otimes_A B}(B, \Omega^{2m}_{(B\otimes_A B)/A}) \cong \begin{cases} \Omega^m_{B/A} & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

(3) *Suppose* $g^* : B \to C$ *is an essentially smooth homomorphism of relative dimension* n*. Let's write* $\mathrm{E}(A,B) := \mathrm{Ext}^m_{B\otimes_A B}(B, \Omega^{2m}_{(B\otimes_A B)/A})$ *etc. Then the diagram*

$$\begin{array}{ccc} \Omega^m_{B/A} \otimes_B \Omega^n_{C/B} & \xrightarrow{\cong} & \Omega^{m+n}_{C/A} \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{E}(A,B) \otimes_B \mathrm{E}(B,C) & \xrightarrow{\cong} & \mathrm{E}(A,C) \end{array}$$

in which the vertical arrows are from part (2)*, and the horizontal arrows come from Proposition* 3.4*, is commutative.*

Proof. First assume that $B \otimes_A B \to B$ is a complete intersection, i.e. the ideal $\mathrm{Ker}(B \otimes_A B \to B)$ is generated by a regular sequence $\boldsymbol{b} = (b_1, \ldots, b_m)$. This implies that B has projective dimension m over $B \otimes_A B$, and that the Ext's in part (2) vanish for $i \neq m$. Define $\mathrm{d}\boldsymbol{b} := \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_m \in \Omega^m_{(B\otimes_A B)/A}$. Then the map $\Omega^m_{B/A} \to \mathrm{E}(A,B)$, $\beta \mapsto \left[\begin{smallmatrix} \mathrm{d}\boldsymbol{b}\wedge\beta \\ \boldsymbol{b} \end{smallmatrix}\right]$, is bijective. Here $\left[\begin{smallmatrix} \mathrm{d}\boldsymbol{b}\wedge\beta \\ \boldsymbol{b} \end{smallmatrix}\right]$ is the generalized fraction, cf. Definition 5.7. According to [RD, Proposition III.7.2] this bijection is independent of the regular sequence $\boldsymbol{b}$.

Now suppose $f^* : A \to B$ is an essentially smooth homomorphism of relative dimension m. Combining Proposition 3.2(1) and [EGA, Chapitre IV Proposition 17.12.4] we see that there is an open covering $\operatorname{Spec} B = \bigcup_i \operatorname{Spec} B_i$, such that for every i the homomorphism $B_i \otimes_A B_i \to B_i$ is a complete intersection. Using the previous paragraph we deduce parts (1) and (2). For part (3) we utilize a similar open covering of $\operatorname{Spec} C$. □

From now on in this section $\mathbb{K}$ is a fixed noetherian base ring. As references for the results on derived categories needed here we recommend [RD] or [KS].

Let A be a $\mathbb{K}$-algebra. In Section 2 we constructed a functor $\mathrm{Sq}_{A/\mathbb{K}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$, the squaring operation. When $\mathbb{K}$ is a field one has the easy formula

$$\mathrm{Sq}_{A/\mathbb{K}}\, M = \mathrm{RHom}_{A\otimes_{\mathbb{K}} A}(A, M \otimes_{\mathbb{K}} M)$$

(see Corollary 2.7). The squaring is functorial for algebra homomorphisms too. Given a homomorphism of algebras $f^* : A \to B$, complexes $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $N \in \mathsf{D}(\mathsf{Mod}\, B)$, and a morphism $\phi : N \to M$ in $\mathsf{D}(\mathsf{Mod}\, A)$, there is an induced morphism $\mathrm{Sq}_{f^*/\mathbb{K}}(\phi) : \mathrm{Sq}_{B/\mathbb{K}}\, N \to \mathrm{Sq}_{A/\mathbb{K}}\, M$ in $\mathsf{D}(\mathsf{Mod}\, A)$. Again when $\mathbb{K}$ is a field the formula for $\mathrm{Sq}_{f^*/\mathbb{K}}$ is obvious; complications arise only when the base ring $\mathbb{K}$ is not a field.

Definition 3.6. Let A be a $\mathbb{K}$-algebra and let $M \in \mathsf{D}(\mathsf{Mod}\, A)$. Assume M has finite flat dimension over $\mathbb{K}$. A *rigidifying isomorphism* for M relative to $\mathbb{K}$ is an isomorphism

$$\rho : M \to \mathrm{Sq}_{A/\mathbb{K}}\, M$$

in $\mathsf{D}(\mathsf{Mod}\, A)$. The pair (M, ρ) is called a *rigid complex over* A *relative to* $\mathbb{K}$.

Example 3.7. Take $A = M := \mathbb{K}$. Since $\mathrm{Sq}_{\mathbb{K}/\mathbb{K}}\, \mathbb{K} = \mathbb{K}$ it follows that $\mathbb{K}$ has a tautological rigidifying isomorphism $\rho_{\mathrm{tau}} : \mathbb{K} \xrightarrow{\simeq} \mathrm{Sq}_{\mathbb{K}/\mathbb{K}}\, \mathbb{K}$. We call $(\mathbb{K}, \rho_{\mathrm{tau}})$ the *tautological rigid complex over* $\mathbb{K}$ *relative to* $\mathbb{K}$.

Definition 3.8. Let $f^* : A \to B$ be a homomorphism between $\mathbb{K}$-algebras, and let (M, ρ_M) and (N, ρ_N) be rigid complexes over A and B respectively, both relative to $\mathbb{K}$. A morphism $\phi : N \to M$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called a *rigid trace-like morphism relative to* $\mathbb{K}$

if the diagram

$$\begin{array}{ccc} N & \xrightarrow{\rho_N} & \operatorname{Sq}_{B/\mathbb{K}} N \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \operatorname{Sq}_{f^*/\mathbb{K}}(\phi)} \\ M & \xrightarrow{\rho_M} & \operatorname{Sq}_{A/\mathbb{K}} M \end{array}$$

of morphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ is commutative. If $A = B$ (and f^* is the identity) then we say $\phi : N \to M$ is a *rigid morphism over A relative to $\mathbb{K}$*.

It is easy to see that the composition of two rigid trace-like morphisms relative to $\mathbb{K}$ is a rigid trace-like morphism relative to $\mathbb{K}$. In particular, for a fixed $\mathbb{K}$-algebra A the rigid complexes over A relative to $\mathbb{K}$ form a category, which we denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.

Theorem 3.9. *Let $\mathbb{K}$ be a noetherian ring, let A and B be essentially finite type $\mathbb{K}$-algebras, and let $A \to B$ be a $\mathbb{K}$-algebra homomorphism. Let $(L, \rho_L) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ and $(M, \rho_M) \in \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$. Assume either of the conditions* (i), (ii) *or* (iii) *holds.*

(i) *$A \to B$ is essentially smooth.*
(ii) *L has finite flat dimension over A.*
(iii) *The A-modules $\mathrm{H}^i L$ are finitely generated, the canonical morphism $A \to \mathrm{RHom}_A(L, L)$ is an isomorphism, and $\mathrm{H}\operatorname{Sq}_{B/\mathbb{K}}(L \otimes^{\mathrm{L}}_A M)$ is bounded.*

Then:

(1) *The complex $L \otimes^{\mathrm{L}}_A M \in \mathsf{D}(\mathsf{Mod}\, B)$ has finite flat dimension over $\mathbb{K}$, and an induced rigidifying isomorphism*
$$\rho_L \otimes \rho_M : L \otimes^{\mathrm{L}}_A M \xrightarrow{\simeq} \operatorname{Sq}_{B/\mathbb{K}}(L \otimes^{\mathrm{L}}_A M).$$
(2) *Let $\phi : (L, \rho_L) \to (L', \rho_{L'})$ be a morphism in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$, and let $\psi : (M, \rho_M) \to (M', \rho_{M'})$ be a morphism in $\mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$. Under conditions* (ii) *or* (iii) *assume L' and M' also have the corresponding properties. Then the morphism*
$$\phi \otimes \psi : L \otimes^{\mathrm{L}}_A M \to L' \otimes^{\mathrm{L}}_A M'$$
in $\mathsf{D}(\mathsf{Mod}\, B)$ is rigid relative to $\mathbb{K}$.

Proof. (1) Since L has finite flat dimension over $\mathbb{K}$ and M has finite flat dimension over A (cf. Definition 3.6), it follows that $L \otimes^{\mathrm{L}}_A M$ has finite flat dimension over $\mathbb{K}$.

Choose K-flat DG algebra resolutions $\mathbb{K} \to \tilde{A} \to A$ and $\tilde{A} \to \tilde{B} \to B$ of $\mathbb{K} \to A$ and $\tilde{A} \to B$ respectively. (If $\mathbb{K}$ is a field and $A \to B$ is flat one may just take $\tilde{A} := A$ and $\tilde{B} := B$.) There is a sequence of isomorphisms in $\mathsf{D}(\mathsf{Mod}\, B)$:

$$\begin{aligned} &\operatorname{Sq}_{B/\mathbb{K}}(L \otimes^{\mathrm{L}}_A M) = \mathrm{RHom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}\big(B, (L \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_{\mathbb{K}} (L \otimes^{\mathrm{L}}_A M)\big) \\ &\quad \cong \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}\Big(B, \mathrm{RHom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}\big(\tilde{B} \otimes_{\tilde{A}} \tilde{B}, (L \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_{\mathbb{K}} (L \otimes^{\mathrm{L}}_A M)\big)\Big) \\ &\quad \cong \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}\Big(B, \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}\big(\tilde{A}, (L \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_{\mathbb{K}} (L \otimes^{\mathrm{L}}_A M)\big)\Big). \end{aligned}$$

These isomorphisms come from the Hom-tensor adjunction for the DG algebra homomorphisms $\tilde{B} \otimes_{\mathbb{K}} \tilde{B} \to \tilde{B} \otimes_{\tilde{A}} \tilde{B} \to B$, plus the fact that

$$\tilde{A} \otimes_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}} (\tilde{B} \otimes_{\mathbb{K}} \tilde{B}) \cong \tilde{B} \otimes_{\tilde{A}} \tilde{B}.$$

Now using tensor product identities we get an isomorphism

$$(L \otimes^{\mathrm{L}}_A M) \otimes^{\mathrm{L}}_{\mathbb{K}} (L \otimes^{\mathrm{L}}_A M) \cong M \otimes^{\mathrm{L}}_{\tilde{A}} (L \otimes^{\mathrm{L}}_{\mathbb{K}} L) \otimes^{\mathrm{L}}_{\tilde{A}} M$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{B} \otimes_{\mathbb{K}} \tilde{B})$. There are functorial morphisms

$$
\begin{aligned}
& M \otimes^{\mathrm{L}}_{\tilde{A}} \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, L \otimes^{\mathrm{L}}_{\mathbb{K}} L) \otimes^{\mathrm{L}}_{\tilde{A}} M \\
& \quad \to \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}\big(A, M \otimes^{\mathrm{L}}_{\tilde{A}} (L \otimes^{\mathrm{L}}_{\mathbb{K}} L)\big) \otimes^{\mathrm{L}}_{\tilde{A}} M \\
& \quad \to \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}\big(A, M \otimes^{\mathrm{L}}_{\tilde{A}} (L \otimes^{\mathrm{L}}_{\mathbb{K}} L) \otimes^{\mathrm{L}}_{\tilde{A}} M\big)
\end{aligned} \tag{3.10}
$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\, \tilde{B} \otimes_{\mathbb{K}} \tilde{B})$, which we claim are isomorphisms. To prove this we can forget the $\tilde{B} \otimes_{\mathbb{K}} \tilde{B}$ -module structure, and consider (3.10) as morphisms in $\mathsf{D}(\mathsf{Mod}\, \mathbb{K})$. According to Corollary 1.9 the algebra $\mathrm{H}^0(\tilde{A} \otimes_{\mathbb{K}} \tilde{A}) \cong A \otimes_{\mathbb{K}} A$ is noetherian, and each $\mathrm{H}^i(\tilde{A} \otimes_{\mathbb{K}} \tilde{A})$ is a finitely generated module over it. Since M has finite flat dimension over $\tilde{A}$, and both $\mathrm{H}(L \otimes^{\mathrm{L}}_{\mathbb{K}} L)$ and $\mathrm{H}(M \otimes^{\mathrm{L}}_{\tilde{A}} L \otimes^{\mathrm{L}}_{\mathbb{K}} L)$ are bounded, we can use Proposition 1.10, with its condition (iii.b).

At this point we have a functorial isomorphism

$$
\mathrm{Sq}_{B/\mathbb{K}}(L \otimes^{\mathrm{L}}_{A} M) \cong \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}\big(B, (M \otimes^{\mathrm{L}}_{A} M) \otimes^{\mathrm{L}}_{A} \mathrm{Sq}_{A/\mathbb{K}} L\big)
$$

in $\mathsf{D}(\mathsf{Mod}\, B)$. The DG module $M \otimes^{\mathrm{L}}_{A} M$ has bounded cohomology, and so does $\mathrm{Sq}_{A/\mathbb{K}} L$, since the latter is isomorphic to L. If $A \to B$ is essentially smooth then $\tilde{B} \otimes_{\tilde{A}} \tilde{B} \to B \otimes_A B$ is a quasi-isomorphism, and moreover B has finite projective dimension over $B \otimes_A B$. Thus under either condition (i), (ii) or (iii) of the theorem we may apply Proposition 1.10, with its conditions (iii.a), (iii.b) or (iii.c) respectively, to get an isomorphism

$$
\begin{aligned}
& \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}\big(B, (M \otimes^{\mathrm{L}}_{A} M) \otimes^{\mathrm{L}}_{A} \mathrm{Sq}_{A/\mathbb{K}} L\big) \\
& \quad \cong \mathrm{RHom}_{\tilde{B} \otimes_{\tilde{A}} \tilde{B}}(B, M \otimes^{\mathrm{L}}_{A} M) \otimes^{\mathrm{L}}_{A} \mathrm{Sq}_{A/\mathbb{K}} L.
\end{aligned}
$$

Thus we have obtained an isomorphism

$$
\mathrm{Sq}_{B/\mathbb{K}}(L \otimes^{\mathrm{L}}_{A} M) \cong (\mathrm{Sq}_{A/\mathbb{K}} L) \otimes^{\mathrm{L}}_{A} (\mathrm{Sq}_{B/A} M) \tag{3.11}
$$

in $\mathsf{D}(\mathsf{Mod}\, B)$. The rigidifying isomorphism we want is $\rho_L \otimes \rho_M$.

(2) This is because the isomorphism (3.11) is functorial in L and M. □

Henceforth in the situation of the theorem we shall write

$$
(L, \rho_L) \otimes^{\mathrm{L}}_{A} (M, \rho_M) := (L \otimes^{\mathrm{L}}_{A} M, \rho_L \otimes \rho_M) \in \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{rig}/\mathbb{K}}. \tag{3.12}
$$

Definition 3.13. Let $f^* : A \to B$ be a finite homomorphism between two essentially finite type $\mathbb{K}$-algebras. Define a functor $f^\flat : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$ by $f^\flat M := \mathrm{RHom}_A(B, M)$. Let $\mathrm{Tr}^\flat_{f;M} : f^\flat M \to M$ be the morphism "evaluation at 1". This becomes a morphism of functors $\mathrm{Tr}^\flat_f : f_* \, f^\flat \to \mathbf{1}_{\mathsf{D}(\mathsf{Mod}\, A)}$.

Theorem 3.14. *Let $\mathbb{K}$ be a noetherian ring, let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be a finite algebra homomorphism. Suppose we are given a rigid complex $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$, such that $f^\flat M$ has finite flat dimension over $\mathbb{K}$.*

(1) *The complex $f^\flat M \in \mathsf{D}(\mathsf{Mod}\, B)$ has an induced rigidifying isomorphism*

$$
f^\flat(\rho) : f^\flat M \xrightarrow{\simeq} \mathrm{Sq}_{B/\mathbb{K}} f^\flat M.
$$

The rigid complex $f^\flat(M, \rho) := \big(f^\flat M, f^\flat(\rho)\big)$ depends functorially on (M, ρ).

(2) *The morphism $\mathrm{Tr}^\flat_{f;M} : f^\flat M \to M$ is a rigid trace-like morphism relative to $\mathbb{K}$.*

(3) *Suppose $g^* : B \to C$ is another finite homomorphism. Assume that $(f \circ g)^\flat M$ has finite flat dimension over $\mathbb{K}$. Then under the standard isomorphism $g^\flat f^\flat M \cong (f \circ g)^\flat M$ one has $g^\flat f^\flat(\rho) = (f \circ g)^\flat(\rho)$.*

(4) *Let (A, ρ_{tau}) be the tautological rigid complex. Assume that B has finite projective dimension over A. Then under the standard isomorphism $f^\flat M \cong M \otimes^{\mathrm{L}}_A f^\flat A$ one has $f^\flat(\rho) = \rho \otimes f^\flat(\rho_{\mathrm{tau}})$.*

For the proof we will need a lemma. The catch in this lemma is that the complex P of flat $\mathbb{K}$-module is bounded *below*, not above.

Lemma 3.15. *Let P and N be bounded below complexes of $\mathbb{K}$-modules. Assume that each P^i is a flat $\mathbb{K}$-module, and that N has finite flat dimension over $\mathbb{K}$. Then the canonical morphism $P \otimes^{\mathrm{L}}_{\mathbb{K}} N \to P \otimes_{\mathbb{K}} N$ in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ is an isomorphism.*

Proof. Choose a bounded flat resolution $Q \to N$ over $\mathbb{K}$. We have to show that $P \otimes_{\mathbb{K}} Q \to P \otimes_{\mathbb{K}} N$ is a quasi-isomorphism. Let L be the cone of $Q \to N$. It is enough to show that the complex $P \otimes_{\mathbb{K}} L$ is acyclic. We note that L is a bounded below acyclic complex and P is a bounded below complex of flat modules. To prove that $\mathrm{H}^i(P \otimes_{\mathbb{K}} L) = 0$ for any given i we might as well replace P with a truncation $P' := (\cdots \to P^{j_1 - 1} \to P^{j_1} \to 0 \to \cdots)$ for $j_1 \gg i$. Now P' is K-flat, so $P' \otimes_{\mathbb{K}} L$ is acyclic. □

Proof of the theorem. (1) Let's pick a semi-free DG algebra resolution $\mathbb{K} \to \tilde{A} \to A$ of $\mathbb{K} \to A$. Next let's pick a K-projective DG algebra resolution $\tilde{A} \to \tilde{B} \to B$ of $\tilde{A} \to B$, such that $\mathrm{und}\,\tilde{B} \cong \bigoplus_{i=-\infty}^{0} \mathrm{und}\,\tilde{A}[-i]^{\mu_i}$ with finite multiplicities μ_i; see Proposition 1.6(3). Choose a bounded above semi-free resolution $P' \to M$ over $\tilde{A}$. Since M has finite flat dimension over $\mathbb{K}$ it follows that for $i \ll 0$ the truncated DG $\tilde{A}$-module $P := \tau^{\geq i} P'$ is a bounded complex of flat $\mathbb{K}$-modules, and also $P \cong M$ in $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$.

We have an isomorphism $\mathrm{Hom}_{\tilde{A}}(\tilde{B}, P) \cong \mathrm{RHom}_A(B, M)$ in $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B})$, and an isomorphism

$$\mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B}, P \otimes_{\mathbb{K}} P) \cong \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B}, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B} \otimes_{\mathbb{K}} \tilde{B})$. Because the multiplicities μ_i are finite and P is bounded, the obvious DG module homomorphism

$$\mathrm{Hom}_{\tilde{A}}(\tilde{B}, P) \otimes_{\mathbb{K}} \mathrm{Hom}_{\tilde{A}}(\tilde{B}, P) \to \mathrm{Hom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B}, P \otimes_{\mathbb{K}} P)$$

is bijective. Now $\mathrm{Hom}_{\tilde{A}}(\tilde{B}, P)$ is a bounded below complex of flat $\mathbb{K}$-modules, which also has finite flat dimension over $\mathbb{K}$. Therefore by Lemma 3.15 we obtain

$$\mathrm{Hom}_{\tilde{A}}(\tilde{B}, P) \otimes_{\mathbb{K}} \mathrm{Hom}_{\tilde{A}}(\tilde{B}, P) \cong \mathrm{RHom}_A(B, M) \otimes^{\mathrm{L}}_{\mathbb{K}} \mathrm{RHom}_A(B, M)$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B} \otimes_{\mathbb{K}} \tilde{B})$. We conclude that there is a functorial isomorphism

$$\mathrm{RHom}_A(B, M) \otimes^{\mathrm{L}}_{\mathbb{K}} \mathrm{RHom}_A(B, M) \cong \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B}, M \otimes^{\mathrm{L}}_{\mathbb{K}} M) \tag{3.16}$$

in $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B} \otimes_{\mathbb{K}} \tilde{B})$. (If $\mathbb{K}$ is a field we may disregard the previous sentences, and just take $\tilde{A} := A$ and $\tilde{B} := B$.) We thus have a sequence of isomorphisms in $\mathsf{D}(\mathsf{Mod}\,B)$:

$$\begin{aligned} \mathrm{Sq}_{B/\mathbb{K}} f^\flat M &= \mathrm{RHom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}\big(B, \mathrm{RHom}_A(B, M) \otimes^{\mathrm{L}}_{\mathbb{K}} \mathrm{RHom}_A(B, M)\big) \\ &\cong^{\diamondsuit} \mathrm{RHom}_{\tilde{B} \otimes_{\mathbb{K}} \tilde{B}}\big(B, \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(\tilde{B} \otimes_{\mathbb{K}} \tilde{B}, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)\big) \\ &\cong^{\ddagger} \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(B, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)) \\ &\cong^{\ddagger} \mathrm{RHom}_A\big(B, \mathrm{RHom}_{\tilde{A} \otimes_{\mathbb{K}} \tilde{A}}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)\big) = f^\flat \mathrm{Sq}_{A/\mathbb{K}} M, \end{aligned} \tag{3.17}$$

where the isomorphism marked $\diamondsuit$ is by (3.16), and the isomorphisms $\ddagger$ come from the Hom-tensor adjunction formula. The rigidifying isomorphism we want is $f^\flat M \xrightarrow{f^\flat(\rho)} f^\flat \mathrm{Sq}_{B/\mathbb{K}} M \cong \mathrm{Sq}_{A/\mathbb{K}} f^\flat M$.

(2) Going over the sequence of isomorphisms (3.17) we see that the diagram

$$\begin{array}{ccc} f^\flat M & \xrightarrow{f^\flat(\rho)} & \mathrm{Sq}_{B/\mathbb{K}}\, f^\flat M \\ {\scriptstyle \mathrm{Tr}^\flat_{f;M}}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}_{f^*/\mathbb{K}}(\mathrm{Tr}^\flat_{f;M})} \\ M & \xrightarrow{\rho} & \mathrm{Sq}_{A/\mathbb{K}}\, M \end{array} \tag{3.18}$$

is commutative. This says that $\mathrm{Tr}^\flat_{f;M}$ is a rigid morphism.

(3) This is because the rigidifying isomorphism $f^\flat(\rho)$ in part (1) depends only on standard identities and on the given rigidifying isomorphism ρ.

(4) According to Proposition 1.10, under its condition (iii.a), we have a canonical isomorphism $M \otimes^{\mathrm{L}}_A f^\flat A \cong f^\flat M$. Combine this with the isomorphisms (3.17). □

Suppose $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $N \in \mathsf{D}(\mathsf{Mod}\, B)$. A morphism $\tau : N \to M$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called *nondegenerate* if the induced morphism $N \to \mathrm{RHom}_A(B, M)$ in $\mathsf{D}(\mathsf{Mod}\, B)$ is an isomorphism.

Corollary 3.19. *Let $f^* : A \to B$ be a finite homomorphism between two essentially finite type $\mathbb{K}$-algebras, and let $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$. Assume that $\mathrm{H}M$ is a finitely generated A-module, $f^\flat M$ has finite flat dimension over $\mathbb{K}$, and $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(f^\flat M, M)$ is a free B-module with basis $\mathrm{Tr}^\flat_{f;M}$. Then $\mathrm{Tr}^\flat_{f;M}$ is the unique nondegenerate rigid trace-like morphism $f^\flat M \to M$ relative to $\mathbb{K}$.*

Proof. By Theorem 3.14, $\mathrm{Tr}^\flat_{f;M} : f^\flat M \to M$ is a rigid morphism. Suppose $\tau : f^\flat M \to M$ is some other nondegenerate rigid trace-like morphism. Then $\tau = u\, \mathrm{Tr}^\flat_{f;M}$ for some $u \in B^\times$, so we get isomorphisms

$$u\, \mathrm{Tr}^\flat_{f;M} = \tau = \mathrm{Sq}_{f^*/\mathbb{K}}(\tau) = \mathrm{Sq}_{f^*/\mathbb{K}}(u\, \mathrm{Tr}^\flat_{f;M}) = u^2\, \mathrm{Sq}_{f^*/\mathbb{K}}(\mathrm{Tr}^\flat_{f;M}) = u^2\, \mathrm{Tr}^\flat_{f;M}\, .$$

Therefore $u = 1$. □

We shall need the following easy fact.

Lemma 3.20. *Suppose $B = \prod_{i=1}^m B_i$, i.e. $\mathrm{Spec}\, B = \coprod_{i=1}^m \mathrm{Spec}\, B_i$. Then the functor $N \mapsto \prod_i (B_i \otimes_B N)$ is an equivalence $\mathsf{D}(\mathsf{Mod}\, B) \to \prod_i \mathsf{D}(\mathsf{Mod}\, B_i)$.*

Definition 3.21. Suppose $f^* : A \to B$ is an essentially smooth homomorphism of $\mathbb{K}$-algebras. Let $\mathrm{Spec}\, B = \coprod_i \mathrm{Spec}\, B_i$ be the (finite) decomposition of $\mathrm{Spec}\, B$ into connected components. For each i the B_i-module $\Omega^1_{B_i/A}$ is projective of constant rank, say n_i. Given $M \in \mathsf{D}(\mathsf{Mod}\, A)$ define

$$f^\sharp M := \prod\nolimits_i (\Omega^{n_i}_{B_i/A}[n_i] \otimes_A M).$$

This is a functor $f^\sharp : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$.

Note that if $f^* : A \to B$ is essentially étale then one simply has $f^\sharp M = B \otimes_A M$.

Theorem 3.22. *Let $\mathbb{K}$ be a noetherian ring, let A and B be essentially finite type $\mathbb{K}$-algebras, and let $f^* : A \to B$ be an essentially smooth algebra homomorphism. Let $(L, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.*

(1) *The complex $f^\sharp L$ has an induced rigidifying isomorphism*

$$f^\sharp(\rho) : f^\sharp L \xrightarrow{\simeq} \mathrm{Sq}_{B/\mathbb{K}}\, f^\sharp L.$$

We get a functor $f^\sharp : \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}} \to \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{rig}/\mathbb{K}}$.

(2) *Let* (A, ρ_{tau}) *be the tautological rigid complex. Then under the standard isomorphism* $f^{\sharp} L \cong L \otimes^{\mathrm{L}}_{A} f^{\sharp} A$ *one has* $f^{\sharp}(\rho) = \rho \otimes f^{\sharp}(\rho_{\mathrm{tau}})$.
(3) *Let* $g^* : B \to C$ *be either a smooth homomorphism or a localization homomorphism. Then under the isomorphism* $(f \circ g)^{\sharp} L \cong g^{\sharp} f^{\sharp} L$ *of Proposition* 3.4 *one has* $(f \circ g)^{\sharp}(\rho) = g^{\sharp} f^{\sharp} L(\rho)$.

Proof. In view of Lemma 3.20 we might as well assume $\Omega^1_{B/A}$ has constant rank m. Using the canonical isomorphism $\Omega^{2m}_{(B \otimes_A B)/A} \cong \Omega^m_{B/A} \otimes_A \Omega^m_{B/A}$ we can interpret Proposition 3.5(2) as a canonical rigidifying isomorphism for the complex $\Omega^m_{B/A}[m]$ relative to A, which we denote by ρ_{Ω}. Thus we obtain an object

$$(\Omega^m_{B/A}[m], \rho_{\Omega}) \in \mathsf{D}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

Now using Theorem 3.9 we can define the rigidifying isomorphism $f^{\sharp}(\rho) := \rho \otimes \rho_{\Omega}$. The assertion in part (2) is clear.

For part (3) one may assume $\operatorname{rank}_C \Omega^1_{C/B} = n$. Then the claim follows from Proposition 3.5(3). □

Definition 3.23. Let $f^* : A \to A'$ be an essentially étale homomorphism between essentially finite type $\mathbb{K}$-algebras. For $M \in \mathsf{D}(\mathsf{Mod}\, A)$ let $\mathrm{q}^{\sharp}_{f;M} : M \to f^{\sharp} M$ be the morphism $m \mapsto 1 \otimes m$. This is a functorial morphism $\mathrm{q}^{\sharp}_{f} : \mathbf{1}_{\mathsf{D}(\mathsf{Mod}\, A)} \to f_* \, f^{\sharp}$.

In the situation of the definition above, given $M' \in \mathsf{D}(\mathsf{Mod}\, A')$, there is a canonical bijection

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(M, M') \cong \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A')}(f^{\sharp} M, M')$$

coming from Hom-tensor adjunction. In particular, for $M' := f^{\sharp} M$, the morphism $\mathrm{q}^{\sharp}_{f;M}$ corresponds to the identity $\mathbf{1}_{M'}$.

Definition 3.24. Let $f^* : A \to A'$ be an essentially étale homomorphism between essentially finite type $\mathbb{K}$-algebras, let $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ and let $(M', \rho') \in \mathsf{D}(\mathsf{Mod}\, A')_{\mathrm{rig}/\mathbb{K}}$. A *rigid localization morphism* is a morphism $\phi : M \to M'$ in $\mathsf{D}(\mathsf{Mod}\, A)$, such that the corresponding morphism $\phi' : f^{\sharp} M \to M'$ in $\mathsf{D}(\mathsf{Mod}\, A)'$ is a rigid isomorphism relative to $\mathbb{K}$.

Proposition 3.25. *Let* $f^* : A \to A'$ *be an essentially étale homomorphism, and let* $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$. *Define* $(M', \rho') := f^{\sharp}(M, \rho)$. *Then:*

(1) *The morphism* $\mathrm{q}^{\sharp}_{f;M} : M \to M'$ *is a rigid localization morphism.*
(2) *Moreover, if* $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ *and* $\mathrm{RHom}_A(M, M) = A$, *then* $\mathrm{q}^{\sharp}_{f;M}$ *is the unique rigid localization morphism* $M \to M'$.

Proof. (1) Since the corresponding morphism $M' \to M'$ is the identity automorphism of M', it is certainly rigid.

(2) Here we have $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A')}(M', M') = A'$. The uniqueness of $\mathrm{q}^{\sharp}_{f;M}$ is proved like in Corollary 3.19. □

Theorem 3.26. *Let* $\mathbb{K}$ *be a noetherian ring, let* A *be an essentially finite type* $\mathbb{K}$*-algebra, let* $g^* : A \to A'$ *be an essentially smooth homomorphism, and let* $f^* : A \to B$ *be a finite homomorphism. Define* $B' := A' \otimes_A B$*; so we get a cartesian diagram of algebra* *homomorphisms*

$$\begin{array}{ccc} A & \xrightarrow{f^*} & B \\ {\scriptstyle g^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ A' & \xrightarrow{f'^*} & B' \end{array}$$

in which f'^* *is finite and* h^* *is essentially smooth. Let* $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$*. Assume* $f^\flat M$ *has finite flat dimension over* $\mathbb{K}$*, and* $\mathrm{H}(A \otimes^{\mathrm{L}}_{\mathbb{K}} A)$ *is bounded. Then there is a functorial isomorphism*

$$h^\sharp f^\flat (M, \rho) \cong f'^\flat g^\sharp (M, \rho)$$

in $\mathsf{D}(\mathsf{Mod}\, B')_{\mathrm{rig}/\mathbb{K}}$.

Proof. Suppose $\operatorname{rank}_B \Omega^1_{A'/A} = n$. Using the base change isomorphism for differential forms, namely $\Omega^n_{B'/B} \cong B \otimes_A \Omega^n_{A'/A}$, we obtain isomorphisms

$$\begin{aligned} h^\sharp f^\flat M &= \Omega^n_{B'/B}[n] \otimes_B \mathrm{RHom}_A(B, M) \\ &\cong \Omega^n_{A'/A}[n] \otimes_A \mathrm{RHom}_A(B, M) \cong \mathrm{RHom}_A(B, \Omega^n_{A'/A}[n] \otimes_A M) \\ &\cong \mathrm{RHom}_{A'}(B', \Omega^n_{A'/A}[n] \otimes_A M) = f'^\flat g^\sharp M. \end{aligned}$$

Now regarding the rigidifying isomorphisms, use Proposition 1.10, with condition (iii.b), to insert $\Omega^n_{A'/A}[n]$ into the sequence of isomorphisms (3.17) at various positions. □

Corollary 3.27. *In the situation of Theorem* 3.26 *assume* g^* *is essentially étale. Define* $N := f^\flat M$*,* $M' := g^\sharp M$ *and* $N' := h^\sharp f^\flat M \cong f'^\flat g^\sharp M$*, with their induced rigidifying isomorphisms. Then*

$$\mathrm{q}^\sharp_{g;M} \circ \mathrm{Tr}^\flat_{f;M} = \mathrm{Tr}^\flat_{f';M'} \circ \mathrm{q}^\sharp_{h;N} \in \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(N, M').$$

Proof. This is because $\mathrm{Tr}^\flat_{f';M'} : N' \to M'$ is gotten from $\mathrm{Tr}^\flat_{f;M} : N \to M$ by applying $A' \otimes_A -$. □

Suppose $f^* : \mathbb{K} \to A$ is a flat ring homomorphism, and $g^* : \mathbb{K} \to \mathbb{K}'$ is another ring homomorphism. We do not impose any finiteness conditions on f^* or g^*. Define $A' := \mathbb{K}' \otimes_{\mathbb{K}} A$. Let $M \in \mathsf{D}(\mathsf{Mod}\, A)$ and $M' \in \mathsf{D}(\mathsf{Mod}\, A')$. Then $\mathrm{Sq}_{A/\mathbb{K}}\, M = \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M \otimes^{\mathrm{L}}_{\mathbb{K}} M)$, and

$$\mathrm{Sq}_{A'/\mathbb{K}'}\, M' = \mathrm{RHom}_{A' \otimes_{\mathbb{K}'} A'}(A', M' \otimes^{\mathrm{L}}_{\mathbb{K}'} M') \cong \mathrm{RHom}_{A \otimes_{\mathbb{K}} A}(A, M' \otimes^{\mathrm{L}}_{\mathbb{K}'} M').$$

If $\phi : M \to M'$ is a morphism in $\mathsf{D}(\mathsf{Mod}\, A)$, we obtain an induced morphism $\mathrm{Sq}_{A/\mathbb{K}}\, M \to \mathrm{Sq}_{A'/\mathbb{K}'}\, M'$ in $\mathsf{D}(\mathsf{Mod}\, A)$, which we denote by $\mathrm{Sq}_{f,g}(\phi)$.

Definition 3.28. With $A, \mathbb{K}'$ and A' as above, let $(M, \rho) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$ and $(M', \rho') \in \mathsf{D}(\mathsf{Mod}\, A')_{\mathrm{rig}/\mathbb{K}'}$. A morphism $\phi : M \to M'$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is called a *rigid base change morphism relative to* $\mathbb{K}$ if

$$\rho' \circ \phi = \mathrm{Sq}_{f,g}(\phi) \circ \rho.$$

Proposition 3.29. *In the situation of Definition* 3.28*, assume that the canonical morphism* $A' \to \mathrm{RHom}_{A'}(M', M')$ *is an isomorphism, and also* $M' \cong A' \otimes^{\mathrm{L}}_A M$*. Then there is a unique rigid base change morphism* $\phi : (M, \rho) \to (M', \rho')$.

Proof. Take any morphism $\tilde{\phi} : M \to M'$ which induces an isomorphism $A' \otimes^{\mathrm{L}}_A M \to M'$. Then $\rho' \circ \tilde{\phi} = u\, \mathrm{Sq}_{f,g}(\tilde{\phi}) \circ \rho$ for a unique invertible element $u \in A'$. Define $\phi := u^{-1} \tilde{\phi}$. □

4. Rigid Dualizing Complexes over $\mathbb{K}$-Algebras

In this section we assume that $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension. All algebras are by default essentially finite type $\mathbb{K}$-algebras, and all algebra homomorphisms are over $\mathbb{K}$.

Let us recall the definition of dualizing complex over a $\mathbb{K}$-algebra A from [RD]. The derived category of bounded complexes with finitely generated cohomology modules is denoted by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. A complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is called a *dualizing complex* if it has finite injective dimension, and the canonical morphism $A \to \mathrm{RHom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is an isomorphism. It follows that the functor $\mathrm{RHom}_A(-, R)$ is an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. Note since the ground ring $\mathbb{K}$ has finite global dimension, the complex R has finite flat dimension over it.

Following Van den Bergh [VdB] we make the following definition.

Definition 4.1. Let A be a $\mathbb{K}$-algebra and let R be a dualizing complex over A. Suppose R has a rigidifying isomorphism $\rho : R \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}\, R$. Then the pair (R, ρ) is called a *rigid dualizing complex over A relative to* $\mathbb{K}$.

By default all rigid dualizing complexes are relative to the ground ring $\mathbb{K}$.

Example 4.2. Take the $\mathbb{K}$-algebra $A := \mathbb{K}$. The complex $R := \mathbb{K}$ is a dualizing complex over $\mathbb{K}$, since this ring is regular. Let $\rho_{\mathrm{tau}} : \mathbb{K} \xrightarrow{\simeq} \mathrm{Sq}_{\mathbb{K}/\mathbb{K}}\, \mathbb{K}$ be the tautological rigidifying isomorphism. Then $(\mathbb{K}, \rho_{\mathrm{tau}})$ is a rigid dualizing complex over $\mathbb{K}$ relative to $\mathbb{K}$.

In [VdB] it was proved that when $\mathbb{K}$ is a field, a rigid dualizing complex R is unique up to isomorphism. And in [YZ1] we proved that the pair (R, ρ) is in fact unique up to a unique rigid isomorphism (again, only when $\mathbb{K}$ is a field). These results are true in our setup too:

Theorem 4.3. *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, let A be an essentially finite type $\mathbb{K}$-algebra, and let (R, ρ) be a rigid dualizing complex over A relative to $\mathbb{K}$. Then (R, ρ) is unique up to a unique rigid isomorphism.*

Proof. In view of Lemma 3.20 and Theorem 3.22 we may assume that $\operatorname{Spec} A$ is connected. Suppose (R', ρ') is another rigid dualizing complex over A. Then there is an isomorphism $R' \cong R \otimes_A L[n]$ for some invertible A-module L and some integer n. Indeed $L[n] \cong \mathrm{RHom}_A(R, R')$. See [RD, Section V.3] or [VdB].

Choose a K-flat DG algebra resolution $\mathbb{K} \to \tilde{A} \to A$ of $\mathbb{K} \to A$. (If $\mathbb{K}$ is a field just take $\tilde{A} := A$.) So

$$\begin{aligned}
\mathrm{Sq}_{A/\mathbb{K}}\, R' &\cong \mathrm{Sq}_{A/\mathbb{K}}(R_A \otimes_A L[n]) \\
&= \mathrm{RHom}_{\tilde{A}\otimes_{\mathbb{K}}\tilde{A}}\big(A, (R_A \otimes_A L[n]) \otimes^{\mathrm{L}}_{\mathbb{K}} (R_A \otimes_A L[n])\big) \\
&\cong^{\dagger} \mathrm{RHom}_{\tilde{A}\otimes_{\mathbb{K}}\tilde{A}}(A, R_A \otimes^{\mathrm{L}}_{\mathbb{K}} R_A) \otimes^{\mathrm{L}}_{A} L[n] \otimes^{\mathrm{L}}_{A} L[n] \\
&= (\mathrm{Sq}_{A/\mathbb{K}}\, R_A) \otimes^{\mathrm{L}}_{A} L[n] \otimes^{\mathrm{L}}_{A} L[n] \cong^{\diamondsuit} R_A \otimes_A L[n] \otimes_A L[n].
\end{aligned}$$

The isomorphism marked † exists by Proposition 1.10 (with its condition (iii.b)), and the isomorphism marked $\diamondsuit$ comes from $\rho : \mathrm{Sq}_{A/\mathbb{K}}\, R_A \xrightarrow{\simeq} R_A$. On the other and we have $\rho' : R' \xrightarrow{\simeq} \mathrm{Sq}_{A/\mathbb{K}}\, R'$, which gives an isomorphism

$$R_A \otimes_A L[n] \cong R_A \otimes_A L[n] \otimes_A L[n].$$

Since R_A is a dualizing complex it follows that $L \cong A$ and $n = 0$. Thus we get an isomorphism $\phi_0 : R_A \xrightarrow{\simeq} R'$.

The isomorphism ϕ_0 might not be rigid, but there is some isomorphism ϕ_1 making the diagram

$$\begin{array}{ccc} R_A & \xrightarrow{\phi_1} & R' \\ {\scriptstyle \rho_A}\downarrow & & \downarrow{\scriptstyle \rho'} \\ \operatorname{Sq}_{A/\mathbb{K}} R_A & \xrightarrow{\operatorname{Sq}_{\mathbf{1}_{A/\mathbb{K}}}(\phi_0)} & \operatorname{Sq}_{A/\mathbb{K}} R' \end{array}$$

commutative. Since $\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\, A)}(R_A, R')$ is a free A-module with basis ϕ_0, it follows that $\phi_1 = u\phi_0$ for some $u \in A^{\times}$. Then the isomorphism $\phi := u^{-1}\phi_0$ is the unique rigid isomorphism $R_A \xrightarrow{\simeq} R'$. □

In view of this result we are allowed to talk about *the* rigid dualizing complex over A (if it exists).

The functors $f^{\flat}$ and $f^{\sharp}$ associated to an algebra homomorphism $f^* : A \to B$ were introduced in Definitions 3.13 and 3.21 respectively.

Proposition 4.4. *Let $f^* : A \to B$ be a finite homomorphism of $\mathbb{K}$-algebras. Assume a rigid dualizing complex (R_A, ρ_A) over A exists. Define $R_B := f^{\flat} R_A \in \mathsf{D}(\mathsf{Mod}\, B)$ and $\rho_B := f^{\flat}(\rho_A)$. Then (R_B, ρ_B) is a rigid dualizing complex over B.*

Proof. The fact that R_B is a dualizing complex over B is an easy calculation; see [RD, Proposition V.2.4]. Since R_B has bounded cohomology and $\mathbb{K}$ has finite global dimension it follows that R_B has finite flat dimension over $\mathbb{K}$. So Theorem 3.14(1) can be applied. □

Proposition 4.5. *Let A be a $\mathbb{K}$-algebra, and assume A has a rigid dualizing complex (R_A, ρ_A). Let $f^* : A \to B$ be an essentially smooth homomorphism. Define $R_B := f^{\sharp} R_A$ and $\rho_B := f^{\sharp}(\rho_A)$. Then (R_B, ρ_B) is a rigid dualizing complex over B.*

Proof. A calculation, using Proposition 3.2(1), shows that R_B is a dualizing complex over B. The only tricky part is to show that R_B has finite injective dimension; see [RD, Theorem V.8.3]. Theorem 3.22(1) tells us (R_B, ρ_B) is a rigid complex over B relative to $\mathbb{K}$. □

Theorem 4.6. *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra. Then A has a rigid dualizing complex (R_A, ρ_A) relative to $\mathbb{K}$.*

Proof. We can find algebras and homomorphisms $\mathbb{K} \xrightarrow{f^*} C \xrightarrow{g^*} B \xrightarrow{h^*} A$, where $C = \mathbb{K}[t_1, \ldots, t_n]$ is a polynomial algebra, g^* is surjective and h^* is a localization. By Example 4.2, $(\mathbb{K}, \rho_{\mathrm{tau}})$ is a rigid dualizing complex over $\mathbb{K}$. By Propositions 4.5 and 4.4 the complex $h^{\sharp} g^{\flat} f^{\sharp} \mathbb{K} = A \otimes_B \operatorname{RHom}_C(B, \Omega^n_{C/\mathbb{K}}[n])$ is a rigid dualizing complex over A, with rigidifying isomorphism $h^{\sharp} g^{\flat} f^{\sharp}(\rho_{\mathrm{tau}})$. □

Definition 4.7. Let A and B be $\mathbb{K}$-algebras, with rigid dualizing complexes (R_A, ρ_A) and (R_B, ρ_B) respectively. Let $f^* : A \to B$ be a finite homomorphism and let $\phi : R_B \to R_A$ be a morphism in $\mathsf{D}(\mathsf{Mod}\, A)$. We say ϕ is a *rigid trace* if it satisfies the following two conditions:

(i) ϕ is nondegenerate, i.e. the morphism $R_B \to \operatorname{RHom}_A(B, R_A)$ in $\mathsf{D}(\mathsf{Mod}\, B)$ induced by ϕ is an isomorphism.
(ii) ϕ is a rigid trace-like morphism, in the sense of Definition 3.8.

Proposition 4.8. *Let $f^* : A \to B$ be a finite homomorphism between $\mathbb{K}$-algebras. There is a unique rigid trace $\operatorname{Tr}_f = \operatorname{Tr}_{B/A} : R_B \to R_A$.*

Proof. By Corollary 3.19 the morphism $\mathrm{Tr}^{\flat}_{f;R_A} : f^{\flat} R_A \to R_A$, namely "evaluation at 1", is the unique nondegenerate rigid trace-like morphism between these two objects. And by Proposition 4.4 and Theorem 4.3 there exist a unique rigid isomorphism $R_B \cong f^{\flat} R_A$. □

Here is an immediate consequence of the uniqueness:

Corollary 4.9 (Transitivity). *Let $A \to B \to C$ be finite homomorphisms of $\mathbb{K}$-algebras. Then $\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \circ \mathrm{Tr}_{C/B}$.*

The notion of rigid localization morphism was introduced in Definition 3.24.

Proposition 4.10. *Let A and A' be $\mathbb{K}$-algebras, with rigid dualizing complexes (R_A, ρ_A) and $(R_{A'}, \rho_{A'})$ respectively. Suppose $f^* : A \to A'$ is an essentially étale homomorphism. Then there is exactly one rigid localization morphism $\mathrm{q}_f = \mathrm{q}_{A'/A} : R_A \to R_{A'}$.*

Proof. By Proposition 4.5 we have a rigid dualizing complex $f^{\sharp} R_A$ over A', and by Proposition 3.24 there is a unique rigid localization morphism $\mathrm{q}^{\sharp}_{f;R_A} : R_A \to f^{\sharp} R_A$. According to Theorem 4.3 there is a unique rigid isomorphism $f^{\sharp} R_A \cong R_{A'}$. □

Definition 4.11. Given a $\mathbb{K}$-algebra A, with rigid dualizing complex R_A, define the *rigid auto-duality functor* to be $\mathrm{D}_A := \mathrm{RHom}_A(-, R_A)$.

Note that D_A is a duality of $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$, and it exchanges the subcategories $\mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{-}_{\mathrm{f}}(\mathsf{Mod}\, A)$. Given a homomorphism $f^* : A \to B$ the functor $\mathrm{L}f^* = B \otimes^{\mathrm{L}}_A -$ sends $\mathsf{D}^{-}_{\mathrm{f}}(\mathsf{Mod}\, A)$ into $\mathsf{D}^{-}_{\mathrm{f}}(\mathsf{Mod}\, B)$. This permits the next definition.

Definition 4.12. Let $f^* : A \to B$ be a homomorphism between two $\mathbb{K}$-algebras. We define the *twisted inverse image* functor $f^! : \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, B)$ as follows. If $A = B$ and $f = \mathbf{1}_A$ (the identity automorphism) then we let $f^! := \mathbf{1}_{\mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A)}$ (the identity functor). Otherwise we define $f^! := \mathrm{D}_B\, \mathrm{L}f^*\, \mathrm{D}_A$.

Let $\psi^{\mathrm{tau}}_f : f^! R_A = \mathrm{D}_B\, \mathrm{L}f^*\, \mathrm{D}_A\, R_A \xrightarrow{\simeq} R_B$ be the isomorphism in $\mathsf{D}(\mathsf{Mod}\, B)$ determined by the standard isomorphisms $A \cong \mathrm{D}_A R_A$, $B \cong B \otimes^{\mathrm{L}}_A A$ and $R_B \cong \mathrm{D}_B B$.

Theorem 4.13. *Let $\mathbb{K}$ be a finite dimensional regular noetherian ring.*

(1) *Given two homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C$ between essentially finite type $\mathbb{K}$-algebras, there is an isomorphism*
$$\phi_{f,g} : (f \circ g)^! \xrightarrow{\simeq} g^!\, f^!$$
of functors $\mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, C)$.

(2) *For three homomorphisms $A \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} D$, the isomorphisms $\phi_{-,-}$ satisfy the compatibility condition*
$$\phi_{g,h} \circ \phi_{f,g\circ h} = \phi_{f,g} \circ \phi_{f\circ g,h} : (f \circ g \circ h)^! \xrightarrow{\simeq} h^! g^! f^!.$$

(3) *For a finite homomorphism $f^* : A \to B$ there is an isomorphism $\psi^{\flat}_f : f^{\flat} \xrightarrow{\simeq} f^!$ of functors $\mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, B)$.*

(4) *For an essentially smooth homomorphism $f^* : A \to B$ there is an isomorphism $\psi^{\sharp}_f : f^{\sharp} \xrightarrow{\simeq} f^!$ of functors $\mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, B)$.*

(5) *In the situation of (1) there is equality*
$$\psi^{\mathrm{tau}}_{f\circ g} = \psi^{\mathrm{tau}}_g \circ \psi^{\mathrm{tau}}_f \circ \phi_{f,g;R_A} : (f \circ g)^! R_A \to R_C.$$
In the situations of (3) and (4) the isomorphisms $\psi^{\mathrm{tau}}_f \circ \psi^{\flat}_{f;R_A} : f^{\flat} R_A \xrightarrow{\simeq} R_B$ and $\psi^{\mathrm{tau}}_f \circ \psi^{\sharp}_{f;R_A} : f^{\sharp} R_A \xrightarrow{\simeq} R_B$ respectively are rigid relative to $\mathbb{K}$.

In stating the theorem we were a bit sloppy with notation; for instance in part (5) we wrote "$\psi_g^{\mathrm{tau}} \circ \psi_f^{\mathrm{tau}}$", whereas the correct expression is "$\psi_g^{\mathrm{tau}} \circ g^!(\psi_f^{\mathrm{tau}})$". This was done for the sake of legibility, and we presume the reader can fill in the omissions.

Proof. (1) The adjunction isomorphism $\mathbf{1}_{\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)} \xrightarrow{\simeq} \mathrm{D}_B \mathrm{D}_B$, together with the obvious isomorphism $C \xrightarrow{\simeq} C \otimes^{\mathrm{L}}_B B$, give rise to functorial isomorphisms

$$\begin{aligned}(f \circ g)^! M = \mathrm{D}_C(C \otimes^{\mathrm{L}}_A \mathrm{D}_A M) &\cong \mathrm{D}_C(C \otimes^{\mathrm{L}}_B B \otimes^{\mathrm{L}}_A \mathrm{D}_A M) \\ &\cong \mathrm{D}_C\big(C \otimes^{\mathrm{L}}_B \mathrm{D}_B \mathrm{D}_B (B \otimes^{\mathrm{L}}_A \mathrm{D}_A M)\big) = g^! f^! M\end{aligned}$$

for $M \in \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A)$. The composed isomorphism $(f \circ g)^! M \xrightarrow{\simeq} g^! f^! M$ is called $\phi_{f,g;M}$.

(2) By definition

$$(f \circ g \circ h)^! M = \mathrm{D}_D(D \otimes^{\mathrm{L}}_A \mathrm{D}_A M)$$

and

$$h^! g^! f^! M = \mathrm{D}_D\Big(D \otimes^{\mathrm{L}}_C \mathrm{D}_C \mathrm{D}_C\big(C \otimes^{\mathrm{L}}_B \mathrm{D}_B \mathrm{D}_B (B \otimes^{\mathrm{L}}_A \mathrm{D}_A M)\big)\Big).$$

The two isomorphism $\phi_{g,h} \circ \phi_{f,g\circ h}$ and $\phi_{f,g} \circ \phi_{f\circ g,h}$ differ only by the order in which the adjunction isomorphisms $\mathbf{1}_{\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)} \cong \mathrm{D}_B \mathrm{D}_B$ and $\mathbf{1}_{\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, C)} \cong \mathrm{D}_C \mathrm{D}_C$ are applied, and correspondingly an isomorphism $C \cong C \otimes^{\mathrm{L}}_B B$ is replaced by $D \cong D \otimes^{\mathrm{L}}_B B$. Due to standard identities the net effect is that $\phi_{g,h} \circ \phi_{f,g\circ h} = \phi_{f,g} \circ \phi_{f\circ g,h}$.

(3) Let $\chi : f^\flat R_A \xrightarrow{\simeq} R_B$ be the unique rigid isomorphism (see Proposition 4.4 and Theorem 4.3). Since $f^\flat R_A = \mathrm{RHom}_A(B, R_A) = \mathrm{D}_A B$, we may view χ as an isomorphism $\chi : \mathrm{D}_A B \xrightarrow{\simeq} R_B$ in $\mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)$. Applying D_A to it we obtain $\mathrm{D}_A(\chi) : \mathrm{D}_A R_B \xrightarrow{\simeq} \mathrm{D}_A \mathrm{D}_A B \cong B$. Now for any $M \in \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A)$ we have

$$f^! M = \mathrm{D}_B(B \otimes^{\mathrm{L}}_A \mathrm{D}_A M) = \mathrm{RHom}_B(B \otimes^{\mathrm{L}}_A \mathrm{D}_A M, R_B) \cong \mathrm{RHom}_A(\mathrm{D}_A M, R_B).$$

Next, using $\mathrm{D}_A(\chi)$ and $\mathrm{D}_A \mathrm{D}_A M \cong M$, we arrive at isomorphisms

$$\mathrm{RHom}_A(\mathrm{D}_A M, R_B) \cong \mathrm{RHom}_A(\mathrm{D}_A R_B, \mathrm{D}_A \mathrm{D}_A M) \cong \mathrm{RHom}_A(B, M) = f^\flat M.$$

The composed isomorphism $f^\flat M \xrightarrow{\simeq} f^! M$ is $\psi^\flat_{f;M}$.

(4) By Proposition 4.5 and Theorem 4.3 there is a unique rigid isomorphism $\chi : f^\sharp R_A \xrightarrow{\simeq} R_B$. We may assume that $\Omega^1_{B/A}$ has constant rank n, so that $f^\sharp M = \Omega^n_{B/A}[n] \otimes_A M$ for any $M \in \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A)$. In particular we have an isomorphism $\chi : \Omega^n_{B/A}[n] \otimes_A R_A \xrightarrow{\simeq} R_B$. Using χ, Proposition 1.10 and the adjunction isomorphism $M \cong \mathrm{D}_A \mathrm{D}_A M$, we obtain

$$f^! M \cong \mathrm{RHom}_A(\mathrm{D}_A M, R_B) \cong \Omega^n_{B/A}[n] \otimes_A \mathrm{RHom}_A(\mathrm{D}_A M, R_A) \cong f^\sharp M.$$

The composed isomorphism $f^\sharp M \xrightarrow{\simeq} f^! M$ is called $\psi^\sharp_{f;M}$.

(5) These assertions are immediate consequences of the construction of $\phi_{f,g}$, $\psi^\flat_f$ and $\psi^\sharp_g$. □

The notion of 2-functor between categories is explained in [Ha, Section 5.15]. Let $\mathsf{EFTAlg}\,/\mathbb{K}$ be the category of essentially finite type $\mathbb{K}$-algebras, and let Cat be the category of all categories. Due to part (2) of the theorem we have:

Corollary 4.14. *The isomorphisms $\phi_{-,-}$ in part* (1) *of the theorem make $f^* \mapsto f^!$ the* 1*-component of a* 2*-functor* $\mathsf{EFTAlg}\,/\mathbb{K} \to \mathsf{Cat}$, *whose* 0*-component is* $A \mapsto \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A)$.

The last result in this section explains the dependence of the twisted inverse image 2-functor $f \mapsto f^!$ on the base ring $\mathbb{K}$. Assume $\mathbb{K}'$ is an essentially finite type $\mathbb{K}$-algebra that's regular (but maybe not smooth over $\mathbb{K}$). Just like for $\mathbb{K}$, any essentially finite type $\mathbb{K}'$-algebra A has a rigid dualizing complex relative to $\mathbb{K}'$, which we denote by (R'_A, ρ'_A). For any homomorphism $f^* : A \to B$ there is a corresponding functor $f^{!'} : \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^+_{\mathrm{f}}(\mathsf{Mod}\, B)$, constructed using R'_A and R'_B. Let $(R_{\mathbb{K}'}, \rho_{\mathbb{K}'})$ be the rigid dualizing complex of $\mathbb{K}'$ relative to $\mathbb{K}$.

Proposition 4.15. *Let A be an essentially finite type $\mathbb{K}'$-algebra. Then $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} R'_A$ is a dualizing complex over A, and it has an induced is a rigidifying isomorphism relative to $\mathbb{K}$. Hence there is a unique isomorphism $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} R'_A \cong R_A$ in $\mathsf{D}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}$.*

Proof. We might as well assume $\operatorname{Spec} \mathbb{K}'$ is connected. Since $\mathbb{K}'$ is regular, one has $R_{\mathbb{K}'} \cong L[n]$ for some invertible $\mathbb{K}'$-module L and some integer n. Therefore $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} R'_A$ is a dualizing complex over A. According to Theorem 3.9 the complex $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} R'_A$ has an induced rigidifying isomorphism $\rho_{\mathbb{K}'} \otimes \rho'_A$. Now use Theorem 4.3. □

Example 4.16. Take $\mathbb{K} := \mathbb{Z}$ and $\mathbb{K}' := \mathbb{F}_p = \mathbb{Z}/(p)$ for some prime number p. Then $R_{\mathbb{K}'} = \mathbb{K}'[-1]$, and for any $A \in \mathsf{EFTAlg}\,/\mathbb{K}'$ we have $R'_A \cong R_A[1]$.

Remark 4.17. The assumption that the base ring $\mathbb{K}$ has finite global dimension seems superfluous. It is needed for technical reasons (bounded complexes have finite flat dimension), yet we don't know how to remove it. However, it seems necessary for $\mathbb{K}$ to be Gorenstein – see next example. Also finiteness is important, as Example 4.19 shows.

Example 4.18. Consider a field $\Bbbk$, and let $\mathbb{K} = A := \Bbbk[t_1, t_2]/(t_1^2, t_2^2, t_1 t_2)$. Then A does not have a rigid dualizing complex relative to $\mathbb{K}$. The reason is that any dualizing complex over the artinian local ring A must be $R \cong A^*[n]$ for some integer n, where $A^* := \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$. Now $\operatorname{Sq}_{A/\mathbb{K}} R \cong R \otimes^{\mathrm{L}}_{\mathbb{K}} R$, which has infinitely many nonzero cohomology modules. So there can be no isomorphism $R \cong \operatorname{Sq}_{A/\mathbb{K}} R$.

Example 4.19. Take any field $\mathbb{K}$, and let $A := \mathbb{K}(t_1, t_2, \ldots)$, the field of rational functions in countably many variables. So A is a noetherian $\mathbb{K}$ algebra, but it is not of essentially finite type. Clearly A has a dualizing complex (e.g. $R := A$), but as shown in [YZ1, Example 3.13], there does not exist a rigid dualizing complex over A relative to $\mathbb{K}$.

Remark 4.20. The paper [SdS] by de Salas uses an idea similar to Van den Bergh's rigidity to define residues on local rings. However the results there are pretty limited. Lipman (unpublished notes) has an approach to duality using the fundamental class of the diagonal, which is close in spirit to the idea of rigidity; cf. Remark 6.20.

5. The Residue Symbol

In this section we apply our methods to the residue symbol of [RD, Section III.9]. Throughout $\mathbb{K}$ is a finite dimensional regular noetherian ring. All rings are commutative essentially finite type $\mathbb{K}$-algebras, and all homomorphisms are over $\mathbb{K}$.

Definition 5.1. Suppose $f^* : A \to B$ is an essentially smooth homomorphism of relative dimension n, $i^* : B \to \bar{B}$ is a finite homomorphism, and the composition $g^* := i^* \circ f^* : A \to \bar{B}$ is finite and flat. Let $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. According to Theorem 4.13 there are isomorphisms $\psi^\flat_i \circ \psi^\sharp_f : i^\flat f^\sharp M \xrightarrow{\simeq} i^! f^! M$, $\psi^\flat_g : g^\flat M \xrightarrow{\simeq} g^! M$ and $\phi_{f,i} : g^! M \xrightarrow{\simeq} i^! f^! M$ in $\mathsf{D}(\mathsf{Mod}\, \bar{B})$. The isomorphism

$$\zeta_M := (\psi^\flat_i \circ \psi^\sharp_f)^{-1} \circ \phi_{f,i} \circ \psi^\flat_g : g^\flat M \xrightarrow{\simeq} i^\flat f^\sharp M$$

in $\mathsf{D}(\mathsf{Mod}\,\bar{B})$ is called the *residue isomorphism*.

If M is a single A-module then we have $g^\flat M \cong \mathrm{H}^0 g^\flat M$, and there are $\bar{B}$-linear isomorphisms

$$\mathrm{H}^0(\zeta_M) : \mathrm{Hom}_A(\bar{B}, M) = \mathrm{H}^0 g^\flat M \xrightarrow{\simeq} \mathrm{H}^0 i^\flat f^\sharp M = \mathrm{Ext}^n_B(\bar{B}, \Omega^n_{B/A} \otimes_A M).$$

Recall that A has the tautological rigidifying isomorphism ρ_{tau}, so we have an object $(A, \rho_{\mathrm{tau}}) \in \mathsf{D}(\mathsf{Mod}\,A)_{\mathrm{rig}/A}$. By Theorems 3.14 and 3.22 we get rigid complexes $g^\flat(A, \rho_{\mathrm{tau}})$ and $i^\flat f^\sharp(A, \rho_{\mathrm{tau}})$ in $\mathsf{D}(\mathsf{Mod}\,\bar{B})_{\mathrm{rig}/A}$.

Theorem 5.2. *In the situation of Definition* 5.1*, the residue isomorphism ζ_A is the unique rigid isomorphism $g^\flat(A, \rho_{\mathrm{tau}}) \xrightarrow{\simeq} i^\flat f^\sharp(A, \rho_{\mathrm{tau}})$ over $\bar{B}$ relative to A.*

The proof is after this lemma.

Lemma 5.3. *In the setup of the theorem, for any $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ the diagram*

$$\begin{array}{ccc} g^\flat M & \xrightarrow{\cong} & M \otimes^{\mathrm{L}}_A g^\flat A \\ {\scriptstyle \zeta_M}\downarrow & & \downarrow{\scriptstyle \mathbf{1}_M \otimes \zeta_A} \\ i^\flat f^\sharp M & \xrightarrow{\cong} & M \otimes^{\mathrm{L}}_A i^\flat f^\sharp A \end{array} \tag{5.4}$$

with horizontal arrows coming from Theorems 3.14(4) *and* 3.22(2)*, is commutative.*

Proof. Going over the proof of Theorem 4.13 we see that there are similar commutative diagrams with pairs of vertical arrows $(\psi^\flat_{g;M}, \mathbf{1}_M \otimes \psi^\flat_{g;A})$, $(\psi^\sharp_{f;M}, \mathbf{1}_M \otimes \psi^\sharp_{f;A})$, $(\psi^\flat_{i;f^\sharp M}, \mathbf{1}_{f^\sharp M} \otimes \psi^\flat_{i;f^\sharp A})$ and $(\phi_{f,i;M}, \mathbf{1}_M \otimes \phi_{f,i;A})$. □

Proof of Theorem 5.2. Since $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,\bar{B})}(g^\flat A, i^\flat f^\sharp A)$ is a free $\bar{B}$-module with basis ζ_A, it follows that $u\zeta_A : g^\flat A \to i^\flat f^\sharp A$ is a rigid isomorphism for a unique $u \in \bar{B}^\times$. We will show that $u = 1$.

Since $g^\flat \cong \mathrm{Hom}_A(\bar{B}, -)$ there are isomorphisms $R_A \otimes^{\mathrm{L}}_A g^\flat A \cong g^\flat R_A \cong R_{\bar{B}}$. We also know that $i^\flat f^\sharp A \cong i^! f^! A \cong g^! A \cong g^\flat A$, implying that $R_A \otimes^{\mathrm{L}}_A i^\flat f^\sharp A \cong R_{\bar{B}}$. Because $R_{\bar{B}} \cong \mathrm{Sq}_{\bar{B}/\mathbb{K}} R_{\bar{B}}$ we see that Theorem 3.9(2) applies, with its condition (iii). Thus we obtain a rigid isomorphism

$$\mathbf{1}_{R_A} \otimes u\zeta_A : R_A \otimes^{\mathrm{L}}_A g^\flat A \xrightarrow{\simeq} R_A \otimes^{\mathrm{L}}_A i^\flat f^\sharp A$$

over $\bar{B}$ relative to $\mathbb{K}$. Now the commutativity of the diagram (5.4), with $M := R_A$, says that $\zeta_{R_A} = \mathbf{1}_{R_A} \otimes \zeta_A$. Therefore $u\zeta_{R_A} = \mathbf{1}_{R_A} \otimes u\zeta_A$, implying that $u\zeta_{R_A} : g^\flat R_A \xrightarrow{\simeq} i^\flat f^\sharp R_A$ is a rigid isomorphism relative to $\mathbb{K}$. However, by Theorem 4.13, the isomorphism ζ_{R_A} is itself rigid relative to $\mathbb{K}$. The uniqueness in Theorem 4.3 implies that $u = 1$. □

Definition 5.5. The *residue map*

$$\mathrm{Res}_{B/A} : \mathrm{Ext}^n_B(\bar{B}, \Omega^n_{B/A}) \to A$$

is defined to be $\mathrm{Res}_{B/A} := \mathrm{Tr}^\flat_{g;A} \circ \zeta_A^{-1}$, where $\zeta_A : g^\flat A \xrightarrow{\simeq} i^\flat f^\sharp A$ is the residue isomorphism, and $\mathrm{Tr}^\flat_{g;A} : g^\flat A \to A$ is "evaluation at 1".

Consider the object

$$\mathrm{Ext}^n_B(\bar{B}, \Omega^n_{B/A}) = i^\flat f^\sharp A \in \mathsf{D}(\mathsf{Mod}\,\bar{B})_{\mathrm{rig}/A}.$$

The rigidifying isomorphism is $i^\flat f^\sharp(\rho_{\mathrm{tau}})$. In this notation, Theorem 5.2 says that $\zeta_A : g^\flat A \xrightarrow{\simeq} \operatorname{Ext}^n_B(\bar{B}, \Omega^n_{B/A})$ is a rigid isomorphism relative to A. Using Corollary 3.19 we obtain:

Corollary 5.6. *The residue map* $\operatorname{Res}_{B/A}$ *is the unique nondegenerate rigid trace-like morphism* $\operatorname{Ext}^n_B(\bar{B}, \Omega^n_{B/A}) \to A$ *relative to* A*.*

The corollary shows that (as would be expected) the residue map is independent of the base ring $\mathbb{K}$ and of the twisted inverse image functor $f \mapsto f^!$ associated to it. Indeed, the only data needed to characterize $\operatorname{Res}_{B/A}$ is the two ring homomorphisms $A \to B \to \bar{B}$.

We shall now look at a special case: $f^* : A \to B$ is a smooth homomorphism of relative dimension n, and $\boldsymbol{b} = (b_1, \ldots, b_n)$ is a sequence of elements in B such that the algebra $\bar{B} := B/(\boldsymbol{b})$ is finite over A. It follows that $\boldsymbol{b}$ is a regular sequence, and $\bar{B}$ is flat over A; cf. [EGA, Chapitre IV, Section 11]. Let $i^* : B \to \bar{B}$ and $g^* : A \to \bar{B}$ be the corresponding homomorphisms.

Let $\mathbf{K}(B, \boldsymbol{b})$ be the Koszul complex associated to the sequence $\boldsymbol{b}$. Recall that for any i the Koszul complex $\mathbf{K}(B, b_i)$ is the free graded B-module $Be_i \oplus B$, with $\deg(e_i) := -1$ and differential $\mathrm{d}(e_i) := b_i$. The total Koszul complex is $\mathbf{K}(B, \boldsymbol{b}) := \mathbf{K}(B, b_1) \otimes_B \cdots \otimes_B \mathbf{K}(B, b_n)$. Since $\boldsymbol{b}$ is a regular sequence we get a quasi-isomorphism $\mathbf{K}(B, \boldsymbol{b}) \to \bar{B}$, which is a free resolution over B, and

$$\operatorname{Ext}^n_B(\bar{B}, \Omega^n_{B/A}) = \mathrm{H}^0 \operatorname{Hom}_B\big(\mathbf{K}(B, \boldsymbol{b}), \Omega^n_{B/A}[n]\big).$$

Definition 5.7. Given a differential form $\beta \in \Omega^n_{B/A}$, the *generalized fraction*

$$\begin{bmatrix} \beta \\ \boldsymbol{b} \end{bmatrix} \in \operatorname{Ext}^n_B(\bar{B}, \Omega^n_{B/A})$$

is the cohomology class of the homomorphism $\mathbf{K}(B, \boldsymbol{b})^{-n} \to \Omega^n_{B/A}$, $e_1 \wedge \cdots \wedge e_n \mapsto \beta$.

Combining the Definitions 5.7 and 5.5 we obtain the *residue symbol* $\operatorname{Res}_{B/A} \left[\begin{smallmatrix} \beta \\ \boldsymbol{b} \end{smallmatrix}\right] \in A$. In view of Proposition 6.19 (see also Remark 6.20) this definition of the residue symbol coincides (perhaps up to sign) with the one in [RD, Section III.9].

For the remainder of this section we will write ρ_A for the tautological rigidifying isomorphism ρ_{tau} of A relative to itself, and likewise for other rings.

Let's define $E := \operatorname{Ext}^n_B(\bar{B}, \Omega^n_{B/A})$. This $\bar{B}$-module has a rigidifying isomorphism $\rho_E := i^\flat f^\sharp(\rho_A) : E \xrightarrow{\simeq} \operatorname{Sq}_{\bar{B}/A} E$ relative to A. Since $A \to \bar{B}$ is flat we have

$$\operatorname{Sq}_{\bar{B}/A} E = \operatorname{Hom}_{\bar{B} \otimes_A \bar{B}}(\bar{B}, E \otimes_A E),$$

which is a $\bar{B} \otimes_A \bar{B}$ -submodule of $E \otimes_A E$. We are going to find an explicit formula for the homomorphism $\rho_E : E \to E \otimes_A E$ in a special case (see Proposition 5.12).

In Lemma 5.8 and Proposition 5.9 below we will look at the following setup. The $\mathbb{K}$-algebras $A, B, \bar{B}$ are as before; $\mathbb{K}'$ is another regular noetherian ring of finite Krull dimension; $\mathbb{K} \to \mathbb{K}'$ is a ring homomorphism (without any finiteness assumptions); A', B' and $\bar{B}'$ are essentially finite type $\mathbb{K}'$-algebras; and there is a commutative diagram of ring homomorphisms

$$\begin{array}{ccccccc} \mathbb{K} & \longrightarrow & A & \xrightarrow{f^*} & B & \xrightarrow{i^*} & \bar{B} \\ \big\downarrow & & {\scriptstyle h^*}\big\downarrow & & \big\downarrow & & \big\downarrow \\ \mathbb{K}' & \longrightarrow & A' & \xrightarrow{f'^*} & B' & \xrightarrow{i'^*} & \bar{B}'. \end{array}$$

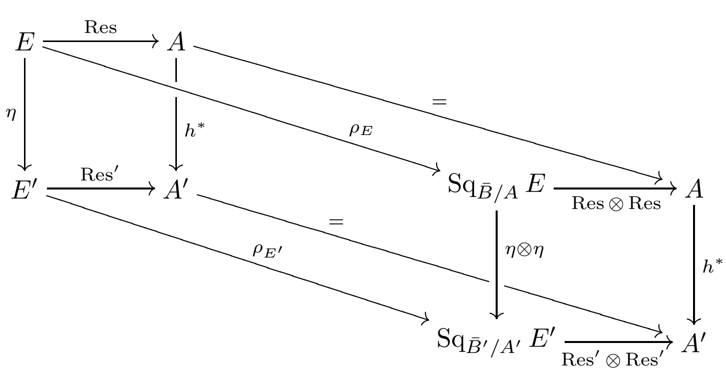
We assume that $B' \cong B \otimes_A A'$ and $\bar{B}' \cong \bar{B} \otimes_A A'$. Let $E' := \mathrm{Ext}^n_{B'}(\bar{B}', \Omega^n_{B'/A'})$. There is an induced isomorphism $E' \cong E \otimes_A A'$ (cf. Proposition 1.10 with condition (iii.a)), and we denote by $\eta : E \to E'$ the corresponding $\bar{B}$-linear homomorphism. Let $\rho_{A'}$ be the tautological rigidifying isomorphism of A' relative to itself, and let $\rho_{E'} := i'^{\flat} f'^{\sharp}(\rho_{A'})$ be the rigidifying isomorphism of E' over $\bar{B}'$ relative to A'.

Lemma 5.8.
$$(\eta \otimes \eta) \circ \rho_E = \rho_{E'} \circ \eta : E \to E' \otimes_{A'} E',$$
i.e. η is a rigid base change morphism relative to A.

Proof. From the proof of Theorem 3.22 we see that the canonical morphism
$$\eta_0 : f^{\sharp} A = \Omega^n_{B/A}[n] \to \Omega^n_{B'/A'}[n] = f'^{\sharp} A'$$
satisfies
$$(\eta_0 \otimes \eta_0) \circ f^{\sharp}(\rho_A) = f^{\sharp}(\rho_{A'}) \circ \eta_0.$$
So η_0 is rigid base change morphism relative to A. Similarly, the proof of Theorem 3.14 shows that the canonical morphism
$$\eta_1 : i^{\flat} B = \mathrm{Ext}^n_B(\bar{B}, B)[-n] \to \mathrm{Ext}^n_{B'}(\bar{B}', B')[-n] = i'^{\flat} B'$$
satisfies
$$(\eta_1 \otimes \eta_1) \circ i^{\flat}(\rho_B) = i'^{\flat}(\rho_{B'}) \circ \eta_1.$$
This says that η_1 is a rigid base change morphism relative to B. Combine this with Theorem 3.14(4). □

Proposition 5.9. *In the situation described above one has*
$$h^* \circ \mathrm{Res}_{B/A} = \mathrm{Res}_{B'/A'} \circ \eta : E \to A'.$$

Proof. Since $\mathrm{Hom}_A(E, A')$ is a free $\bar{B}'$-module of rank 1, we see that $\mathrm{Res}' \circ \eta = u h^* \circ \mathrm{Res}$ for a unique invertible element $u \in \bar{B}'$. Here $\mathrm{Res} := \mathrm{Res}_{B/A}$ and $\mathrm{Res}' := \mathrm{Res}_{B'/A'}$. So the rear square in the diagram below commutes up to a factor of u. Because Res and Res' are rigid morphisms, the two horizontal rectangles are commutative. By Lemma 5.8 the left-facing vertical rectangle is commutative, and trivially the right-facing vertical rectangle is commutative. Finally the front square commutes up to a factor of u^2. We conclude that $u = 1$.



□

Lemma 5.10. *Assume $\mathbb{K} = A = \mathbb{Z}$; $B = \mathbb{Z}[t]$, the polynomial algebra in one variable; and $b = t^{m+1}$ for some $m \geq 0$. Then*

$$\rho_E(\begin{bmatrix} \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \epsilon \sum_{j=0}^{m} (\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix} \otimes \begin{bmatrix} t^{m-j} \, \mathrm{d}t \\ t^{m+1} \end{bmatrix})$$

for some $\epsilon \in \{1, -1\}$.

Proof. The A-module E is free of rank $m+1$ with basis $\left[\begin{smallmatrix} \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right], \ldots, \left[\begin{smallmatrix} t^m \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right]$. Therefore

$$\rho_E(\begin{bmatrix} \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \sum_{j,k=0}^{m} a_{j,k} (\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix} \otimes \begin{bmatrix} t^k \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) \in E \otimes_A E \tag{5.11}$$

for some $a_{j,k} \in A$.

Define $A_{\mathbb{Q}} := \mathbb{Q}$, $B_{\mathbb{Q}} := \mathbb{Q}[t]$, $\bar{B}_{\mathbb{Q}} := \mathbb{Q}[t]/(t^{m+1})$ and $E_{\mathbb{Q}} := \mathrm{Ext}^1_{B_{\mathbb{Q}}}(\bar{B}_{\mathbb{Q}}, \Omega^1_{B_{\mathbb{Q}}/A_{\mathbb{Q}}})$. So $E_{\mathbb{Q}} \cong E \otimes_A A_{\mathbb{Q}}$, and by Lemma 5.8 the rigidifying isomorphism $\rho_{E_{\mathbb{Q}}}$ also satisfies equation (5.11). Take any $\lambda \in \mathbb{Q} - \{0, 1, -1\}$, and consider the automorphism $h^* : B_{\mathbb{Q}} \to B_{\mathbb{Q}}$, $t \mapsto \lambda t$. Let $\eta : E \to E$ be the corresponding homomorphism. Again by Lemma 5.8 we see that

$$(\rho_E \circ \eta)(\begin{bmatrix} \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \sum_{j,k=0}^{m} a_{j,k} \left(\eta(\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) \otimes \eta(\begin{bmatrix} t^k \, \mathrm{d}t \\ t^{m+1} \end{bmatrix})\right).$$

Since $\eta(\left[\begin{smallmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right]) = \lambda^{j-m} \left[\begin{smallmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right]$, we conclude that $a_{j,k} = 0$ unless $j + k = m$.

Let $\bar{t}$ denote the class of t in $\bar{B}$. So $1 \otimes \bar{t} - \bar{t} \otimes 1 \in \mathrm{Ker}(\bar{B} \otimes_A \bar{B} \to \bar{B})$, and therefore

$$(1 \otimes \bar{t} - \bar{t} \otimes 1) \cdot \rho_E(\begin{bmatrix} \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = 0.$$

Now $\bar{t}\left[\begin{smallmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right] = \left[\begin{smallmatrix} t^{j+1} \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right]$. We conclude that $a_{0,m} = a_{1,m-1} = \cdots = a_{m,0}$, which we denote by ϵ. Since

$$\sum_{j=0}^{m} (\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix} \otimes \begin{bmatrix} t^{m-j} \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) \in \mathrm{Hom}_{\bar{B} \otimes_A \bar{B}}(\bar{B}, E \otimes_A E) = \mathrm{Sq}_{\bar{B}/A} E,$$

yet $\rho_E(\left[\begin{smallmatrix} \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right])$ is part of a basis of the A-module $\mathrm{Sq}_{\bar{B}/A} E$, it follows that ϵ must be invertible. Thus $\epsilon \in \{1, -1\}$. □

Proposition 5.12. *Let A be any essentially finite type $\mathbb{K}$-algebra, $B := A[t]$ and $\bar{B} := B/(t^{m+1})$. Then*

$$\mathrm{Res}_{B/A} \begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix} = \begin{cases} \epsilon & \text{if } j = m \\ 0 & \text{otherwise.} \end{cases}$$

Here $\epsilon \in \{1, -1\}$ is some universal constant.

Proof. According to Proposition 5.9 we can assume that $\mathbb{K} = A = \mathbb{Z}$. Let ϵ be the number occurring in Lemma 5.10. Define an A-linear homomorphism $\phi : E \to A$ by $\phi(\left[\begin{smallmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{smallmatrix}\right]) := \epsilon$ if $j = m$, and 0 otherwise. We have to prove that $\phi = \mathrm{Res}_{B/A}$. In view of Corollary 5.6 it suffices to show that ϕ is a rigid morphism relative to A.

Thus we have to verify that

$$(\phi \otimes \phi) \circ \rho_E = \rho_A \circ \phi : E \to A.$$

By Lemma 5.10, for any $j \in \{0, \ldots, m\}$ we have

$$\rho_E(\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = t^j \rho_E(\begin{bmatrix} \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \epsilon \sum_{k=0}^{m} (\begin{bmatrix} t^{j+k} \, \mathrm{d}t \\ t^{m+1} \end{bmatrix} \otimes \begin{bmatrix} t^{m-k} \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}).$$

Thus

$$((\phi \otimes \phi) \circ \rho_E)(\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \epsilon \cdot \phi(\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) \cdot \phi(\begin{bmatrix} t^m \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \begin{cases} \epsilon^3 & \text{if } j = m \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand ρ_A is the identity, and

$$\phi(\begin{bmatrix} t^j \, \mathrm{d}t \\ t^{m+1} \end{bmatrix}) = \begin{cases} \epsilon & \text{if } j = m \\ 0 & \text{otherwise.} \end{cases}$$

But $\epsilon^3 = \epsilon$. □

Remark 5.13. The actual value of ϵ is not so easy to determine. Since we will not need it, we did not do the calculation.

6. Gluing Rigid Dualizing Complexes on Schemes

In the beginning of this section X is some finite dimensional noetherian scheme. A *dimension function* on X is a function $\dim : X \to \mathbb{Z}$ such that $\dim(y) = \dim(x) - 1$ whenever y is an immediate specialization of x. Thus $-\dim$ is a codimension function, in the sense of [RD, Section V.7]. Note that any closed subset $Z \subset X$ has a dimension, namely $\dim Z := \sup\{\dim(x) \mid x \in Z\}$.

Let $\dim$ be a dimension function on X. This determines a Cousin functor $\mathrm{E} : \mathsf{D}^+(\mathsf{Mod}\, \mathcal{O}_X) \to \mathsf{C}(\mathsf{Mod}\, \mathcal{O}_X)$, the latter being the category of complexes of $\mathcal{O}_X$-modules. Let us recall the construction of the Cousin functor from [RD, Chapter IV]. Given an $\mathcal{O}_X$-module $\mathcal{M}$, denote by $F_i \mathcal{M}$ the subsheaf of sections whose support has dimension $\leq i$. Now let $\mathcal{M} \in \mathsf{D}^+(\mathsf{Mod}\, \mathcal{O}_X)$, and choose a bounded below injective resolution $\mathcal{M} \to \mathcal{J}$. Let $\{E_r^{p,q}\}$ be the spectral sequence associated to the filtered complex $\{F_i \mathcal{J}\}$. The Cousin complex $\mathrm{E}\mathcal{M}$ is the row $q = 0$ in the page $r = 1$ of this spectral sequence. According to [RD, Section IV.2], for any p one has $(\mathrm{E}\mathcal{M})^p \cong \bigoplus_{\dim(x) = -p} \mathrm{H}^p_x \mathcal{M}$, where we view $\mathrm{H}^p_x \mathcal{M}$ as a constant sheaf supported on the closed set $\overline{\{x\}}$. If $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_X)$ then each $\mathrm{H}^p_x \mathcal{M}$ is quasi-coherent, so $\mathrm{E}\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\, \mathcal{O}_X)$

Definition 6.1. A complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, \mathcal{O}_X)$ is called a *Cohen-Macaulay complex* (relative to the dimension function $\dim$) if $\mathrm{H}^i_x \mathcal{M} = 0$ for all x and $i \neq \dim(x)$.

According to [RD, Proposition IV.2.6] or [YZ2, Theorem 2.11], $\mathcal{M}$ is a Cohen-Macaulay complex if and only if $\mathcal{M} \cong \mathrm{E}\mathcal{M}$ in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$. Let us denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ the full subcategory of $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_X)$ consisting of Cohen-Macaulay complexes.

Recall that a quasi-coherent $\mathcal{O}_X$-module $\mathcal{J}$ is injective as object of the category $\mathsf{QCoh}\, \mathcal{O}_X$ iff it is injective in the bigger category $\mathsf{Mod}\, \mathcal{O}_X$. Moreover, for such an injective quasi-coherent module there is an isomorphism $\mathcal{J} \cong \bigoplus_{x \in X} \mathcal{J}(x)^{(\mu_x)}$, where $\mathcal{J}(x)$ denotes an injective hull of the residue field $\boldsymbol{k}(x)$, considered as a quasi-coherent sheaf; μ_x is a cardinal number; and $\mathcal{J}(x)^{(\mu_x)}$ denotes the direct sum of μ_x copies of $\mathcal{J}(x)$. See [RD, Section II.7].

A bounded below complex $\mathcal{J}$ of injective quasi-coherent $\mathcal{O}_X$-modules is called a *minimal injective quasi-coherent complex* if for any q the module of cocycles $\mathrm{Ker}(\mathcal{J}^q \to \mathcal{J}^{q+1})$ is an essential submodule of $\mathcal{J}^q$ in the category $\mathsf{QCoh}\, \mathcal{O}_X$. Given a complex $\mathcal{N} \in$

$\mathsf{D}^{+}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_X)$, a *minimal injective quasi-coherent resolution* of $\mathcal{N}$ is a quasi-isomorphism $\mathcal{N} \to \mathcal{J}$, with $\mathcal{J}$ a minimal injective quasi-coherent complex.

Lemma 6.2. *Let $\mathcal{N} \in \mathsf{D}^{+}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_X)$.*

(1) *There exists a minimal injective quasi-coherent resolution $\mathcal{N} \to \mathcal{J}$. Moreover $\mathcal{J}$ is unique up to isomorphism.*

(2) *For any $q \in \mathbb{Z}$ and any $x \in X$ let $\mu_{x,q}$ be the multiplicity of $\mathcal{J}(x)$ in $\mathcal{J}^q$. Then*
$$\mu_{x,q} = \operatorname{rank}_{\boldsymbol{k}(x)} \operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\boldsymbol{k}(x), \mathcal{N}_x),$$
where $\mathcal{N}_x$ is the stalk at x.

(3) *If $\mathcal{N}$ is a Cohen-Macaulay complex then $\mu_{x,q} = 0$ whenever $q < -\dim(x)$.*

Proof. (1) By [RD, Corollary II.7.19] we may assume that $\mathcal{N} \in \mathsf{D}^{+}(\mathsf{QCoh}\,\mathcal{O}_X)$. Now we may apply [Ye1, Lemma 4.2].

(2) The complex $\mathcal{J}_x$ is a minimal injective resolution of $\mathcal{N}_x$ over the local ring $\mathcal{O}_{X,x}$. Now use [YZ2, Lemma 4.12(2)].

(3) Note that
$$\operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\boldsymbol{k}(x), \mathcal{N}_x) \cong \operatorname{Ext}^q_{\mathcal{O}_{X,x}}(\boldsymbol{k}(x), \mathrm{R}\Gamma_x \mathcal{N}).$$
The Cohen-Macaulay assumption says that the cohomology of $\mathrm{R}\Gamma_x\mathcal{N}$ is concentrated in degree $-\dim(x)$. □

Lemma 6.3. *Suppose $\mathcal{M}, \mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\,\mathcal{O}_X)_{\mathrm{CM}}$. Then the assignment*
$$U \mapsto \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathcal{M}|_U, \mathcal{N}|_U)$$
is a sheaf on X.

Proof. As explained above there is an isomorphism $\phi : \mathcal{M} \xrightarrow{\simeq} \mathrm{E}\mathcal{M}$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$. Choose a minimal injective quasi-coherent resolution $\psi : \mathcal{N} \to \mathcal{J}$. By Lemma 6.2(3) the multiplicities of the complex $\mathcal{J}$ satisfy $\mu_{x,q} = 0$ for all $q < -\dim(x)$.

For an open set $U \subset X$ consider the canonical homomorphism
$$\lambda_U : \operatorname{Hom}_{\mathsf{C}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathrm{E}\mathcal{M}|_U, \mathcal{J}|_U) \to \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathrm{E}\mathcal{M}|_U, \mathcal{J}|_U).$$
Since $\mathcal{J}|_U$ is a bounded below complex of injective $\mathcal{O}_U$-modules it follows that λ_U is surjective. On the other hand, any local section of $(\mathrm{E}\mathcal{M})^q$ has support in dimension $\leq -q$, but there are no nonzero local sections of $\mathcal{J}^{q-1}$ with support in dimension $\leq -q$. It follows that $\operatorname{Hom}_{\mathsf{C}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathrm{E}\mathcal{M}|_U, \mathcal{J}|_U)^{-1} = 0$, and so λ_U is also injective.

Now the isomorphisms $\phi|_U : \mathcal{M}|_U \xrightarrow{\simeq} \mathrm{E}\mathcal{M}|_U$ and $\psi|_U : \mathcal{N}|_U \xrightarrow{\simeq} \mathcal{J}|_U$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)$ give rise to a bijection
$$\operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathcal{M}|_U, \mathcal{N}|_U) \to \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathrm{E}\mathcal{M}|_U, \mathcal{J}|_U).$$
We conclude that the presheaves $U \mapsto \operatorname{Hom}_{\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathcal{M}|_U, \mathcal{N}|_U)$ and $U \mapsto \operatorname{Hom}_{\mathsf{C}(\mathsf{Mod}\,\mathcal{O}_U)}(\mathrm{E}\mathcal{M}|_U, \mathcal{J}|_U)$ are isomorphic. But the latter is a sheaf. □

A stack on X is a "sheaf of categories." The general definition (cf. [LMB]) is quite forbidding; but we shall only need the following special instance (cf. [KS, Section X.10]).

Definition 6.4. Suppose that for every open set $U \subset X$ we are given a full subcategory $\mathsf{C}(U) \subset \mathsf{D}(\mathsf{Mod}\,\mathcal{O}_U)$. The collection of categories $\mathsf{C} = \{\mathsf{C}(U)\}$ is called a *stack of subcategories of* $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ if the following axioms hold.

(a) Let $V \subset U$ be open sets in X and $\mathcal{M} \in \mathsf{C}(U)$. Then $\mathcal{M}|_V \in \mathsf{C}(V)$.

(b) Descent for objects: given an open covering $U = \bigcup V_i$, objects $\mathcal{M}_i \in \mathsf{C}(V_i)$ and isomorphisms $\phi_{i,j} : \mathcal{M}_i|_{V_i \cap V_j} \xrightarrow{\simeq} \mathcal{M}_j|_{V_i \cap V_j}$ satisfying the cocycle condition $\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}$ on triple intersections, there exists an object $\mathcal{M} \in \mathsf{C}(U)$ and isomorphisms $\phi_i : \mathcal{M}|_{V_i} \xrightarrow{\simeq} \mathcal{M}_i$ such that $\phi_{i,j} \circ \phi_i = \phi_j$.

(c) Descent for morphisms: given two objects $\mathcal{M}, \mathcal{N} \in \mathsf{C}(U)$, an open covering $U = \bigcup V_i$ and morphisms $\psi_i : \mathcal{M}|_{V_i} \to \mathcal{N}|_{V_i}$ such that $\psi_i|_{V_i \cap V_j} = \psi_j|_{V_i \cap V_j}$, there is a unique morphism $\psi : \mathcal{M} \to \mathcal{N}$ such that $\psi|_{V_i} = \psi_i$.

Theorem 6.5. *Let X be a finite dimensional noetherian scheme with dimension function $\dim$. The assignment $U \mapsto \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_U)_{\mathrm{CM}}$ is a stack of subcategories of $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.*

Proof. Axiom (a) is clear, since the Cohen-Macaulay property is local. Axiom (c) is Lemma 6.3. Let us prove axiom (b). Since X is noetherian, and in view of axiom (c), we may assume $I = \{1, \ldots, n\}$. Let us define $W_i := \bigcup_{j=1}^{i} V_j$. By induction on i we will construct a complex $\mathcal{N}_i \in \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_{W_i})_{\mathrm{CM}}$ with isomorphisms $\psi_{i,j} : \mathcal{N}_i|_{V_j} \xrightarrow{\simeq} \mathcal{M}_j$ for all $j \leq i$ that are compatible with the $\phi_{j,k}$. Then $\mathcal{M} := \mathcal{N}_n$ will be the desired global object on $V = W_n$.

So assume $i < n$ and $\mathcal{N}_i$ has already been defined. For any $j \leq i$ we have an isomorphism

$$\phi_{j,i+1} \circ \psi_{i,j} : \mathcal{N}_i|_{V_j \cap V_{i+1}} \xrightarrow{\simeq} \mathcal{M}_j|_{V_j \cap V_{i+1}} \xrightarrow{\simeq} \mathcal{M}_{i+1}|_{V_j \cap V_{i+1}},$$

and these satisfy the cocycle condition. According to axiom (c) there is an isomorphism

$$\psi_{i,i+1} : \mathcal{N}_i|_{W_i \cap V_{i+1}} \xrightarrow{\simeq} \mathcal{M}_{i+1}|_{W_i \cap V_{i+1}}$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{W_i \cap V_{i+1}})$. Denote by $f_{i+1} : W_i \to W_{i+1}$, $g_{i+1} : V_{i+1} \to W_{i+1}$ and $h_{i+1} : W_i \cap V_{i+1} \to W_{i+1}$ the inclusions. Define $\mathcal{N}_{i+1} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, \mathcal{O}_{W_{i+1}})$ to be the cone of the morphism

$$h_{(i+1)!}(\mathcal{N}_i|_{W_i \cap V_{i+1}}) \xrightarrow{(\gamma, \psi_{i,i+1})} f_{(i+1)!}\mathcal{N}_i \oplus g_{(i+1)!}\mathcal{M}_{i+1}$$

where $h_{(i+1)!}$ etc. are extension by zero, and γ is the canonical morphism. We obtain a distinguished triangle

$$h_{(i+1)!}(\mathcal{N}_i|_{W_i \cap V_{i+1}}) \to f_{(i+1)!}\mathcal{N}_i \oplus g_{(i+1)!}\mathcal{M}_{i+1} \to \mathcal{N}_{i+1} \to h_{(i+1)!}(\mathcal{N}_i|_{W_i \cap V_{i+1}})[1]$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{W_{i+1}})$. Upon restriction to W_i we get an isomorphism $\mathcal{N}_i \cong \mathcal{N}_{i+1}|_{W_i}$; and upon restriction to V_{i+1} we get an isomorphism $\mathcal{N}_{i+1}|_{V_{i+1}} \xrightarrow{\simeq} \mathcal{M}_{i+1}$ which we call $\psi_{i+1,i+1}$. From these isomorphisms it follows that $\mathcal{N}_{i+1} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_{W_{i+1}})_{\mathrm{CM}}$. □

Remark 6.6. Assume X is a finite type scheme over a field $\mathbb{K}$, and consider the dimension function $\dim_{\mathbb{K}}$ (see Definition 6.10 below). Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ be the category of Cohen-Macaulay complexes with coherent cohomology sheaves. In [YZ4] we show that $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ is the heart of the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$. Moreover, $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ is the image of $\mathsf{Coh}\, \mathcal{O}_X$ under the rigid auto-duality functor D_X. Therefore $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ is an artinian abelian category. We do not know of a similar statement for the bigger category $\mathsf{D}^{\mathrm{b}}_{\mathrm{qc}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$.

From here on in this section $\mathbb{K}$ is a finite dimensional noetherian regular ring. All rings are by default essentially finite type $\mathbb{K}$-algebras, all schemes are by default finite type $\mathbb{K}$-schemes, and all morphisms are over $\mathbb{K}$.

For a scheme X we write $\mathsf{Aff}\, X$ for the set of affine open sets in it.

Definition 6.7. Let X be a scheme and $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$. For $U \in \mathsf{Aff}\,X$ we write $A_U := \Gamma(U, \mathcal{O}_X)$ and $M_U := \mathrm{R}\Gamma(U, \mathcal{M})$. Assume that for every $U \in \mathsf{Aff}\,X$ we are given a rigidifying isomorphism $\rho_U : M_U \xrightarrow{\simeq} \mathrm{Sq}_{A_U/\mathbb{K}}\, M_U$ in $\mathsf{D}(\mathsf{Mod}\,A_U)$. Moreover, assume that for every pair of affine open sets $V \subset U$ the localization morphism $\mathrm{q}_{V/U} : M_U \to M_V$ in $\mathsf{D}(\mathsf{Mod}\,A_U)$ is a rigid localization morphism (cf. Definition 3.24). Then we call $\boldsymbol{\rho} = \{\rho_U\}_{U \in \mathsf{Aff}\,X}$ a *rigid structure on $\mathcal{M}$ relative to* $\mathbb{K}$, and the pair $(\mathcal{M}, \boldsymbol{\rho})$ is called a *rigid complex of $\mathcal{O}_X$-modules relative to* $\mathbb{K}$.

Definition 6.8. Suppose $(\mathcal{M}, \boldsymbol{\rho}_{\mathcal{M}})$ and $(\mathcal{N}, \boldsymbol{\rho}_{\mathcal{N}})$ are rigid complexes of $\mathcal{O}_X$-modules relative to $\mathbb{K}$. A *rigid morphism* from $(\mathcal{M}, \boldsymbol{\rho}_{\mathcal{M}})$ to $(\mathcal{N}, \boldsymbol{\rho}_{\mathcal{N}})$ is a morphism $\phi : \mathcal{M} \to \mathcal{N}$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$, such that for every affine open set $U \subset X$ the morphism $\mathrm{R}\Gamma(U, \phi) : M_U \to N_U$ in $\mathsf{D}(\mathsf{Mod}\,A_U)$ is a rigid morphism relative to $\mathbb{K}$, in the sense of Definition 3.8.

We denote the category of rigid complexes by $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)_{\mathrm{rig}/\mathbb{K}}$.

Definition 6.9. A *rigid dualizing complex on X relative to* $\mathbb{K}$ is a rigid complex $(\mathcal{R}, \boldsymbol{\rho})$ such that $\mathcal{R}$ is a dualizing complex.

For a field K any dualizing complex is of the form $K[n]$ for some integer n.

Definition 6.10.
(1) Suppose K is an essentially finite type $\mathbb{K}$-algebra that's a field, and let R_K be its rigid dualizing complex. Let n be the integer such that $R_K \cong K[n]$, and define $\dim_{\mathbb{K}}(K) := n$.
(2) Let X be a finite type $\mathbb{K}$-scheme, $x \in X$ a point and $\boldsymbol{k}(x)$ the residue field of x. We define $\dim_{\mathbb{K}}(x) := \dim_{\mathbb{K}}(\boldsymbol{k}(x))$.

Lemma 6.11. *The function* $\dim_{\mathbb{K}} : X \to \mathbb{Z}$ *is a dimension function.*

Proof. Choose an affine open neighborhood $U = \operatorname{Spec} A$ of x in X. Denote by $\mathcal{R}$ the sheafification of the rigid dualizing complex R_A to U, and let $\mathcal{R}_x$ be its stalk at x. According to [RD, Proposition V.3.4] there is an integer n such that

$$\mathrm{Ext}^i_{\mathcal{O}_{X,x}}(\boldsymbol{k}(x), \mathcal{R}_x) \cong \begin{cases} \boldsymbol{k}(x) & \text{if } i = -n \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathrm{RHom}_{\mathcal{O}_{X,x}}(\boldsymbol{k}(x), \mathcal{R}_x)$ is a rigid dualizing complex over $\boldsymbol{k}(x)$ we see that $n = \dim_{\mathbb{K}}(x)$. By [RD, Proposition V.7.1] we see that $\dim_{\mathbb{K}}(y) = \dim_{\mathbb{K}}(x) - 1$ for an immediate specialization. □

Example 6.12. If $\mathbb{K}$ is equidimensional of dimension d (i.e. every maximal ideal has height d) then $\dim_{\mathbb{K}}(x) = \dim \overline{\{x\}} - d$. Thus in the case of a field $\mathbb{K}$ one has $\dim_{\mathbb{K}}(x) = \dim \overline{\{x\}}$. On the other hand, for $\mathbb{K} = \mathbb{Z}$ and $X = \operatorname{Spec} \mathbb{Z}$, a closed point $x = (p)$ has $\dim_{\mathbb{K}}(x) = -1$.

Theorem 6.13. *Let $\mathbb{K}$ be a finite dimensional regular noetherian ring, and let X be a finite type $\mathbb{K}$-scheme. Then X has a rigid dualizing complex $(\mathcal{R}_X, \boldsymbol{\rho}_X)$, which is unique up to a unique rigid isomorphism.*

Proof. Let $U = \operatorname{Spec} A$ be an affine open set in X. By Theorem 4.3 the $\mathbb{K}$-algebra A has a rigid dualizing complex (R_A, ρ_A). If $U' = \operatorname{Spec} A' \subset U$ is a smaller affine open set, then by Proposition 4.10 there is a unique rigid localization morphism $\mathrm{q}_{A'/A} : (R_A, \rho_A) \to (R_{A'}, \rho_{A'})$. In this way we get an isomorphism $\phi_{A'/A} : A' \otimes_A R_A \xrightarrow{\simeq} R_{A'}$ in $\mathsf{D}(\mathsf{Mod}\,A')$. Given another affine open set $U'' = \operatorname{Spec} A'' \subset U'$ the localization morphisms satisfy $\mathrm{q}_{A''/A} = \mathrm{q}_{A''/A'} \circ \mathrm{q}_{A'/A}$, and hence $\phi_{A''/A} = \phi_{A''/A'} \circ \phi_{A'/A}$.

Now let's pass to sheaves. For an affine open set $U = \operatorname{Spec} A \subset X$ let $\mathcal{R}_U$ be the sheafification of R_A to U, which is a dualizing complex on U. Given an affine open set $U' \subset U$ there is an isomorphism $\phi_{U'/U} : \mathcal{R}_U|_{U'} \xrightarrow{\simeq} \mathcal{R}_{U'}$ in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{U'})$. For a third affine open set $U'' \subset U'$ these isomorphisms satisfy the condition $\phi_{U''/U} = \phi_{U''/U'} \circ \phi_{U'/U}$.

By [RD, Proposition V.7.3] each of the complexes $\mathcal{R}_U$ is Cohen-Macaulay with respect to $\dim_{\mathbb{K}}$. Given two affine open sets $U_1, U_2 \subset X$, and any affine open set $W \subset U_1 \cap U_2$, we get isomorphisms $\phi_{W/U_i} : \mathcal{R}_{U_i}|_W \xrightarrow{\simeq} \mathcal{R}_W$. According to Theorem 6.5 the Cohen-Macaulay complexes form a stack on X. By axiom (c) of Definition 6.4 the isomorphisms $\phi^{-1}_{W/U_2} \circ \phi_{W/U_1} : \mathcal{R}_{U_1}|_W \xrightarrow{\simeq} \mathcal{R}_{U_2}|_W$ can be patched to an isomorphism $\phi_{U_1,U_2} : \mathcal{R}_{U_1}|_{U_1 \cap U_2} \xrightarrow{\simeq} \mathcal{R}_{U_2}|_{U_1 \cap U_2}$; and these isomorphisms satisfy the cocycle condition on triple intersections. By axiom (b) there is a complex $\mathcal{R}_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)_{\mathrm{CM}}$ with isomorphisms $\mathcal{R}_X|_U \cong \mathcal{R}_U$ for any affine open set U. The complex $\mathcal{R}_X$ is dualizing, and by construction it comes equipped with a rigid structure $\boldsymbol{\rho}_X$.

Regarding uniqueness: this is immediate from the uniqueness of the rigid dualizing complexes R_A over the $\mathbb{K}$-algebras A, and by the uniqueness of the rigid localization morphisms $\mathrm{q}_{A'/A}$. □

The *rigid auto-duality functor* of X is

$$\mathrm{D}_X := \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{R}_X).$$

Let $\mathsf{FTSch}\,/\mathbb{K}$ be the category of finite type schemes over $\mathbb{K}$.

Definition 6.14. For a morphism $f : X \to Y$ in $\mathsf{FTSch}\,/\mathbb{K}$ we define a functor

$$f^! : \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$$

as follows. If $X = Y$ and $f = \mathbf{1}_X$ (the identity automorphism) then $f^! := \mathbf{1}_{\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)}$ (the identity functor). Otherwise we define $f^! := \mathrm{D}_X \,\mathrm{L}f^*\, \mathrm{D}_Y$.

Note that since $\mathrm{D}_Y \mathcal{O}_Y = \mathcal{R}_Y$, $\mathrm{L}f^* \mathcal{O}_Y = \mathcal{O}_X$ and $\mathrm{D}_X \mathcal{O}_Y = \mathcal{R}_X$, one has $f^! \mathcal{R}_Y = \mathcal{R}_X$. In Section 4 we were more pedantic, so we introduced the tautological isomorphism $\psi^{\mathrm{tau}}_f : f^! \mathcal{R}_Y \xrightarrow{\simeq} \mathcal{R}_X$, using standard identities; see Definition 4.12.

Corollary 6.15. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms in $\mathsf{FTSch}\,/\mathbb{K}$. Then there is an isomorphism of functors $\phi_{g,f} : (g \circ f)^! \xrightarrow{\simeq} f^! g^!$. Given another morphism $h : Z \to W$ in $\mathsf{FTSch}\,/\mathbb{K}$ the compatibility relation*

$$\phi_{h,g} \circ \phi_{h \circ g, f} = \phi_{g,f} \circ \phi_{h, g \circ f} : (h \circ g \circ f)^! \xrightarrow{\simeq} f^! g^! h^!$$

holds. Thus $f \mapsto f^!$ is the 1-component of a contravariant 2-functor $\mathsf{FTSch}\,/\mathbb{K} \to \mathsf{Cat}$, whose 0-component is $X \mapsto \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.

Proof. Use the adjunction isomorphism $\mathbf{1}_Y \xrightarrow{\simeq} \mathrm{D}_Y\, \mathrm{D}_Y$. Cf. Theorem 4.13(1,2). □

Recall that for a finite morphism of schemes $f : X \to Y$ there is a functor $f^\flat : \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ defined by

$$f^\flat \mathcal{N} := \mathcal{O}_X \otimes_{f^{-1} f_* \mathcal{O}_X} f^{-1} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{N}).$$

For a smooth morphism f we have a functor $f^\sharp : \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ defined as follows. Let $X = \coprod X_i$ be the decomposition of X into connected components, and for each i let n_i be the rank of the locally free $\mathcal{O}_{X_i}$-module $\Omega^1_{X_i/Y}$. Denote by $g_i : X_i \to X$ the inclusion. Then

$$f^\sharp \mathcal{N} := \big(\bigoplus\nolimits_i g_{i*}\, \Omega^{n_i}_{X_i/Y}[n_i] \big) \otimes_{\mathcal{O}_X} f^* \mathcal{N}.$$

Cf. [RD, Sections III.2 and III.6].

Theorem 6.16. *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, let $f : X \to Y$ be a finite* (*resp. smooth*) *morphism between finite type $\mathbb{K}$-schemes, and let $(\mathcal{R}_X, \boldsymbol{\rho}_X)$ and $(\mathcal{R}_Y, \boldsymbol{\rho}_Y)$ be the rigid dualizing complexes. Then the complex $f^\flat \mathcal{R}_Y$* (*resp. $f^\sharp \mathcal{R}_Y$*) *is a dualizing complex over X, and it has an induced rigid structure $f^\flat(\boldsymbol{\rho}_Y)$* (*resp. $f^\sharp(\boldsymbol{\rho}_Y)$*)*. Therefore there is a unique rigid isomorphism $f^\flat \mathcal{R}_Y \cong \mathcal{R}_X$* (*resp. $f^\sharp \mathcal{R}_Y \cong \mathcal{R}_X$*).

Proof. The fact that $f^\flat \mathcal{R}_Y$ (resp. $f^\sharp \mathcal{R}_Y$) is a dualizing complex on X is quite easy to verify; see [RD, Proposition V.2.4] (resp. [RD, Theorem V.8.3]). We need to provide it with a rigid structure $f^\flat(\boldsymbol{\rho}_Y)$ (resp. $f^\sharp(\boldsymbol{\rho}_Y)$). We will do only the case of a finite morphism. The smooth case is similar (but easier).

Let $V \subset Y$ be an affine open set. Define $A := \Gamma(V, \mathcal{O}_Y)$ and $B := \Gamma(f^{-1}(V), \mathcal{O}_X)$. So $f^* : A \to B$ is a finite homomorphism of $\mathbb{K}$-algebras. Let $R_V := \mathrm{R}\Gamma(V, \mathcal{R}_Y)$, and let ρ_V be its rigidifying isomorphism. By Theorem 3.14 the complex $f^\flat R_V$ is a dualizing complex over B, with rigidifying isomorphism $f^\flat(\rho_V)$. If $V' \subset V$ is a smaller affine open set, and we let $A' := \Gamma(V', \mathcal{O}_Y)$, $B' := \Gamma(f^{-1}(V'), \mathcal{O}_X)$ and $R_{V'} := \mathrm{R}\Gamma(V', \mathcal{R}_Y)$, then under the isomorphism $f^\flat R_{V'} \cong B' \otimes_B f^\flat R_V$ one has $f^\flat(\rho_{V'}) = \mathbf{1}_{B'} \otimes_B f^\flat(\rho_V)$. This is due to Theorem 3.26.

We want to show that for every affine open set $U \subset X$ the complex $R_U := \mathrm{R}\Gamma(U, f^\flat \mathcal{R}_Y)$ has a rigidifying isomorphism ρ_U. If $U \subset f^{-1}(V)$ for some affine open set $V \subset Y$ then this follows from the previous paragraph. Indeed, with A, B and R_V as defined above, and $B' := \Gamma(U, \mathcal{O}_X)$, we have an isomorphism $R_U \cong B' \otimes_B f^\flat R_V$; so we can use the rigidifying isomorphism $f^\flat(\rho_V)$. And this rigidifying isomorphism of R_U does not depend on the choice of V.

Now for an arbitrary affine open set $U \subset X$, let us cover it by affine open sets $U_1, \ldots, U_n$ such that each $U_i \subset f^{-1}(V_i)$ for some affine open set $V_i \subset Y$. We have to find a rigidifying isomorphism $\rho_U : R_U \xrightarrow{\simeq} \mathrm{Sq}_{B/\mathbb{K}}\, R_U$ in $\mathsf{D}(\mathsf{Mod}\, B)$. Let us denote by $\mathcal{S} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_U)$ the sheafification of $\mathrm{Sq}_{B/\mathbb{K}}\, R_U$. Since both $(f^\flat \mathcal{R}_Y)|_U$ and $\mathcal{S}$ are Cohen-Macaulay complexes on U, and on each of the open sets U_i we have an isomorphism $f^\flat(\rho_{V_i}) : (f^\flat \mathcal{R}_Y)|_{U_i} \xrightarrow{\simeq} \mathcal{S}|_{U_i}$, that agree on double intersections, we can glue them to obtain the desired rigidifying isomorphism ρ_U.

By construction the various rigidifying isomorphisms ρ_U respect localizations, so we have a rigid structure on $f^\flat \mathcal{R}_Y$, which we denote by $f^\flat(\boldsymbol{\rho}_Y)$. By the uniqueness in Theorem 6.13 we get a rigid isomorphism $(f^\flat \mathcal{R}_Y, f^\flat(\boldsymbol{\rho}_Y)) \xrightarrow{\simeq} (\mathcal{R}_X, \boldsymbol{\rho}_X)$. □

Corollary 6.17. *In the situation of Theorem* 6.16 *there is a functorial isomorphism $f^! \mathcal{N} \cong f^\flat \mathcal{N}$* (*resp. $f^! \mathcal{N} \cong f^\sharp \mathcal{N}$*) *for $\mathcal{N} \in \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$.*

Proof. Use Theorem 6.16 and standard adjunction formulas. □

For more details on the isomorphisms $f^! \cong f^\flat$ and $f^! \cong f^\sharp$ see Theorem 4.13.

In the next two results we shall consider only *embeddable* morphisms, in order to avoid complications. Most likely they are true without this assumption. Recall that a morphism $f : X \to Y$ is called embeddable if it can be factored into $f = h \circ g$, where g is finite and h is smooth.

Proposition 6.18 (Flat Base Change)**.** *Let $f : X \to Y$ be an embeddable morphism, and let $g : Y' \to Y$ be a flat morphism. Define a scheme $X' := Y' \times_Y X$, with projections $f' : X' \to Y'$ and $h : X' \to X$. Then there is an isomorphism ${f'}^! g^* \cong h^* f^!$ of functors $\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_{X'})$.*

Proof. This is immediate when f is either finite or smooth, by Corollary 6.17. Cf. proof of Theorem 3.26. □

For a morphism of schemes $f : X \to Y$ let's denote by

$$f^{!(\mathrm{G})} : \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X)$$

Grothendieck's twisted inverse image functor from [RD].

Proposition 6.19 (Comparison to [RD])**.** *If $f : X \to Y$ is an embeddable morphism then there is an isomorphism of functors $f^{!(\mathrm{G})} \cong f^!$. In particular, if the structural morphism $\pi : X \to \operatorname{Spec} \mathbb{K}$ is embeddable, then there is an isomorphism $\pi^{!(\mathrm{G})} \mathbb{K} \cong \pi^! \mathbb{K} = \mathcal{R}_X$ in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.*

Proof. Choose a factorization $f = h \circ g$, with g finite and h smooth. Then, according to [RD, Theorem III.8.7], there are isomorphisms $g^{!(\mathrm{G})} \cong g^\flat$, $h^{!(\mathrm{G})} \cong h^\sharp$ and $f^{!(\mathrm{G})} \cong g^{!(\mathrm{G})} h^{!(\mathrm{G})}$. On the other hand, by Corollaries 6.17 and 6.15 we have $g^! \cong g^\flat$, $h^! \cong h^\sharp$ and $f^! \cong g^! h^!$. □

Remark 6.20. In case X is a separated flat embeddable $\mathbb{K}$-scheme, Proposition 6.19 can be strengthened significantly. Indeed, one can prove that the dualizing complex $\mathcal{R}' := \pi^{!(\mathrm{G})} \mathbb{K}$ has a rigid structure, which determined by the variance properties of the twisted inverse image 2-functor $f \mapsto f^{!(\mathrm{G})}$, as stated in [RD, Theorem III.8.7]. Here is an outline. Letting $X^2 := X \times_{\mathbb{K}} X$, there are the diagonal embedding $\Delta : X \to X^2$, which is a finite morphism; and the two projections $\mathrm{p}_i : X^2 \to X$, which are flat. See diagram below. Using flat base change one can obtain a canonical isomorphism

$$\mathcal{R}' \cong \Delta^{-1} \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}} \big(\Delta_* \mathcal{O}_X, (\mathrm{p}_1^* \mathcal{R}') \otimes^{\mathrm{L}}_{\mathcal{O}_{X^2}} (\mathrm{p}_2^* \mathcal{R}') \big). \tag{6.21}$$

Now take any affine open set $U \subset X$, and let $A_U := \Gamma(U, \mathcal{O}_X)$ and $R'_U := \mathrm{R}\Gamma(U, \mathcal{R}')$. Applying the functor $\mathrm{R}\Gamma(U, -)$ to the isomorphism (6.21) we obtain a rigidifying isomorphism $\rho'_U : R'_U \xrightarrow{\simeq} \operatorname{Sq}_{A_U / \mathbb{K}} R'_U$ in $\mathsf{D}(\mathsf{Mod}\, A_U)$. The collection of isomorphisms $\{\rho'_U\}$ is compatible with localizations, so it's a rigid structure on $\mathcal{R}'$.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta} & X^2 & \xrightarrow{\mathrm{p}_2} & X \\ & & {\scriptstyle \mathrm{p}_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ & & X & \xrightarrow{\pi} & \operatorname{Spec} \mathbb{K} \end{array}$$

To conclude this section we address the question of dependence of the twisted inverse image 2-functor $f \mapsto f^!$ on the base ring $\mathbb{K}$. Assume $\mathbb{K}'$ is an essentially finite type $\mathbb{K}$-algebra that's regular (but maybe not smooth over $\mathbb{K}$). Consider the category $\mathsf{FTSch} / \mathbb{K}'$, with the faithful functor $\mathsf{FTSch} / \mathbb{K}' \to \mathsf{FTSch} / \mathbb{K}$. Just like for $\mathbb{K}$, any finite type $\mathbb{K}'$-scheme X has a rigid dualizing complex relative to $\mathbb{K}'$, which we denote by $\mathcal{R}'_X$. Also there is a 2-functor $\mathsf{FTSch} / \mathbb{K}' \to \mathsf{Cat}$, constructed using the complexes $\mathcal{R}'_X$; we denote it by $f \mapsto f^{!'}$. Let $R_{\mathbb{K}'}$ be the rigid dualizing complex of $\mathbb{K}'$ relative to $\mathbb{K}$. Note that since $\mathbb{K}'$ is regular, one has $R_{\mathbb{K}'} \cong L[n]$ for some invertible $\mathbb{K}'$-module L and some integer n.

Proposition 6.22. *Given $X \in \mathsf{FTSch}\,/\mathbb{K}'$, the complex $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} \mathcal{R}'_X \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ has an induced rigid structure relative to $\mathbb{K}$. Therefore there is a unique rigid isomorphism $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} \mathcal{R}'_X \cong \mathcal{R}_X$.*

Proof. This is immediate from Proposition 4.15. □

Because the twisting $R_{\mathbb{K}'} \otimes^{\mathrm{L}}_{\mathbb{K}'} -$ is an auto-equivalence of $\mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ for any scheme $X \in \mathsf{FTSch}\,/\mathbb{K}'$, we obtain:

Corollary 6.23. *There is an isomorphism of 2-functors*

$$(f \mapsto f^{!'}) \cong (f \mapsto f^{!}) : \mathsf{FTSch}\,/\mathbb{K}' \to \mathsf{Cat}\,.$$

7. The Residue Theorem and Duality

In this section $\mathbb{K}$ denotes a regular finite dimensional noetherian ring. All schemes are by default finite type $\mathbb{K}$-schemes, all algebras are by default essentially finite type $\mathbb{K}$-algebras, and all morphisms are over $\mathbb{K}$.

Let X be a scheme. The dimension function $\dim_{\mathbb{K}}$ was introduced in Definition 6.10. For a point $x \in X$ we denote by $\mathcal{J}(x)$ an injective hull of the residue field $\boldsymbol{k}(x)$, considered as an $\mathcal{O}_X$-module. So $\mathcal{J}(x)$ is a quasi-coherent $\mathcal{O}_X$-module supported on $\overline{\{x\}}$; see [RD, Section II.7].

Definition 7.1. A *rigid residue complex on X* (relative to $\mathbb{K}$) is a rigid dualizing complex $(\mathcal{K}_X, \boldsymbol{\rho}_X)$, such that for every integer p there is an isomorphism of $\mathcal{O}_X$-modules $\mathcal{K}^p_X \cong \bigoplus_{\dim_{\mathbb{K}}(x)=-p} \mathcal{J}(x)$.

Proposition 7.2. *The scheme X has a rigid residue complex $(\mathcal{K}_X, \boldsymbol{\rho}_X)$, which is unique up to a unique isomorphism in $\mathsf{C}(\mathsf{Mod}\,\mathcal{O}_X)$.*

Proof. Define $\mathcal{K}_X := \mathrm{E}\mathcal{R}_X$, the Cousin complex with respect to $\dim_{\mathbb{K}}$. According to [RD, Proposition VI.1.1], for any p there is an isomorphism $\mathcal{K}^p_X \cong \bigoplus_{\dim_{\mathbb{K}}(x)=-p} \mathcal{J}(x)$. By [RD, Proposition IV.3.1] or [YZ2, Theorem 2.11] there is an isomorphism $\mathcal{K}_X \cong \mathcal{R}_X$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$. Using this isomorphism we obtain a rigid structure $\boldsymbol{\rho}_X$ on $\mathcal{K}_X$.

Now suppose $(\mathcal{K}', \boldsymbol{\rho}')$ is another residue complex on X. According to Theorem 6.13 there is a unique rigid isomorphism $\phi : \mathcal{K}_X \xrightarrow{\simeq} \mathcal{K}'$ in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$. Like in the proof of Lemma 6.3 we see that ϕ is a uniquely determined isomorphism in $\mathsf{C}(\mathsf{Mod}\,\mathcal{O}_X)$. □

Since $\mathcal{K}_X$ is a bounded complex of injective $\mathcal{O}_X$-modules the rigid duality functor is $\mathrm{D}_X = \mathcal{H}om_{\mathcal{O}_X}(-, \mathcal{K}_X)$. Furthermore, for any complex $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,\mathcal{O}_X)$ the complex $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X)$ is a bounded complex of flasque sheaves, and hence $\mathrm{R}f_* \,\mathrm{D}_X \mathcal{M} = f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X)$.

Let $x \in X$ be a point with $\dim_{\mathbb{K}}(x) = -p$. Define $\mathcal{K}_X(x) := \mathrm{H}^p_x \mathcal{R}_X$; so $\mathcal{K}_X(x) \cong \mathcal{J}(x)$, and $\mathcal{K}^p_X = \bigoplus_{\dim_{\mathbb{K}}(x)=-p} \mathcal{K}_X(x)$.

Definition 7.3. Let A be a local essentially finite type $\mathbb{K}$-algebra, with maximal ideal $\mathfrak{m}$, residue field K and rigid dualizing complex R_A. Let $d := \dim_{\mathbb{K}}(K)$, and define $\mathcal{K}(A) := \mathrm{H}^{-d}_{\mathfrak{m}} R_A$.

In the setup of the definition above, the A-module $\mathcal{K}(A)$ is an injective hull of the field K. If A is artinian then $\mathrm{H}^i R_A = 0$ for all $i \neq -d$, and hence there is a canonical isomorphism $R_A \cong \mathcal{K}(A)[d]$ in $\mathsf{D}(\mathsf{Mod}\,A)$.

Lemma 7.4. *Let X be a $\mathbb{K}$-scheme, $x \in X$ a point and $A := \mathcal{O}_{X,x}$. Then there is a canonical isomorphism $\mathcal{K}_X(x) \cong \mathcal{K}(A)$.*

Proof. Let $U = \operatorname{Spec} C$ be an affine open neighborhood of x in X, and let $\mathfrak{p} \subset C$ be the prime ideal of x. By definition $\mathcal{K}_X(x) = \mathrm{H}^{-d}_{\mathfrak{p}} R_C$, where $d := \dim_{\mathbb{K}}(x)$. Now according to Proposition 4.10 we get a canonical isomorphism $R_A \cong A \otimes_C R_C$ in $\mathsf{D}(\mathsf{Mod}\, A)$, inducing an isomorphism $\mathrm{H}^{-d}_{\mathfrak{m}} R_A \cong \mathrm{H}^{-d}_{\mathfrak{p}} R_C$. □

Let A be as in Definition 7.3. For $i \geq 0$ let $A_i := A/\mathfrak{m}^{i+1}$. Corresponding to the finite homomorphisms $A \to A_i$ there are rigid trace morphisms $\mathrm{Tr}_{A_i/A} : R_{A_i} \to R_A$ in $\mathsf{D}(\mathsf{Mod}\, A)$; see Proposition 4.8. From these we can extract A-linear homomorphisms $\mathrm{H}^{-d}_{\mathfrak{m}}(\mathrm{Tr}_{A_i/A}) : \mathcal{K}(A_i) \to \mathcal{K}(A)$.

Lemma 7.5. *In the setup of Definition* 7.3*, the homomorphisms* $\mathrm{H}^{-d}_{\mathfrak{m}}(\mathrm{Tr}_{A_i/A})$ *give rise to a bijection* $\lim_{i\to} \mathcal{K}(A_i) \xrightarrow{\simeq} \mathcal{K}(A)$.

Proof. Because $\mathrm{Tr}_{A_i/A}$ is nondegenerate it induces an isomorphism $\mathcal{K}(A_i) \cong \operatorname{Hom}_A(A_i, \mathcal{K}(A))$. □

Definition 7.6. Let $f : X \to Y$ be a morphism of schemes. Define a homomorphism of graded $\mathcal{O}_Y$-modules $\mathrm{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$ as follows. There is a decomposition $f_*\mathcal{K}_X = \bigoplus_{x\in X} f_*\mathcal{K}_X(x)$. Consider a point $x \in X$, and let $y := f(x)$. There are two cases:

- (i) If x is closed in its fiber, define $A_i := \mathcal{O}_{Y,y}/\mathfrak{m}_y^{i+1}$ and $B_i := \mathcal{O}_{X,x}/\mathfrak{m}_x^{i+1}$. Then A_i is an essentially finite type $\mathbb{K}$-algebra, and $A_i \to B_i$ is a finite homomorphism. Using the isomorphisms from Lemmas 7.4 and 7.5 we define $\mathrm{Tr}_f|_{f_*\mathcal{K}_X(x)} := \lim_{i\to} \mathrm{Tr}_{B_i/A_i}$.
- (ii) If x is not closed in the fiber $f^{-1}(y)$ then we let $\mathrm{Tr}_f|_{f_*\mathcal{K}_X(x)} := 0$.

Proposition 7.7. *Given two morphisms of schemes* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *there is equality*

$$\mathrm{Tr}_{g\circ f} = \mathrm{Tr}_g \circ g_*(\mathrm{Tr}_f) : (g\circ f)_*\mathcal{K}_X \to \mathcal{K}_Z.$$

Proof. This is an immediate consequence of Corollary 4.9. □

The next thing we want to do is to study residues on curves, and to prove Theorem 7.13. This will lead us to the general residue theorem for proper morphisms 7.14. The strategy we shall use is taken from [RD, Chapter VII], but the considerations are much easier in our context.

Let A be an artinian, local, essentially finite type $\mathbb{K}$-algebra, and let X be a smooth irreducible curve over A. Let x_0 be the generic point of X, and let x_1 be some closed point. The corresponding local rings are denoted by $L := \mathcal{O}_{X,x_0}$ and $B := \mathcal{O}_{X,x_1}$, and we denote by $\mathfrak{n}$ the maximal ideal of B. The Cousin complex of the sheaf $\Omega^1_{X/A}$ on X gives rise to a B-linear homomorphism

$$\delta_{(x_0,x_1)} : \Omega^1_{L/A} \cong \mathrm{H}^0_{x_0}\Omega^1_{X/A} \to \mathrm{H}^1_{x_1}\Omega^1_{X/A} \cong \mathrm{H}^1_{\mathfrak{n}}\Omega^1_{B/A}.$$

For any $k \geq 0$ let $B_k := B/\mathfrak{n}^{k+1}$. Since $A \to B$ is essentially smooth of relative dimension 1, and $A \to B_k$ is finite and flat, there is a residue map $\mathrm{Res}_{B_k/A} : \mathrm{Ext}^1_B(B_k, \Omega^1_{B/A}) \to A$. See Definition 5.5. There is a canonical isomorphism

$$\mathrm{H}^1_{\mathfrak{n}}\Omega^1_{B/A} \cong \lim_{k\to} \mathrm{Ext}^1_B(B_k, \Omega^1_{B/A}),$$

and we define

$$\mathrm{Res}_{B/A} := \lim_{k\to} \mathrm{Res}_{B_k/A} : \mathrm{H}^1_{\mathfrak{n}}\Omega^1_{B/A} \to A. \tag{7.8}$$

Definition 7.9. Let A be an artinian, local, essentially finite type $\mathbb{K}$-algebra, let X be a smooth irreducible curve over A, and let x be a closed point of X. Denote by $\boldsymbol{k}(X)$ the total ring of fractions of X. The *residue map at* x is the A-linear homomorphism $\operatorname{Res}_x : \Omega^1_{\boldsymbol{k}(X)/A} \to A$ which is the composition of the homomorphisms

$$\Omega^1_{\boldsymbol{k}(X)/A} \xrightarrow{\delta_{(x_0,x_1)}} \mathrm{H}^1_x \Omega^1_{\mathcal{O}_{X,x}/A} \xrightarrow{\operatorname{Res}_{\mathcal{O}_{X,x}/A}} A$$

described above.

Note that the kernel of Res_x is $\Omega^1_{X/A,x}$. More generally we make the next definition.

Definition 7.10. With the data of the previous definition, let M be an A-module. Define

$$\operatorname{Res}_{x;M} : \Omega^1_{\boldsymbol{k}(X)/A} \otimes_A M \to M$$

by the formula

$$\operatorname{Res}_{x;M}(\alpha \otimes m) := \operatorname{Res}_x(\alpha) \cdot m$$

for $\alpha \in \Omega^1_{\boldsymbol{k}(X)/A}$ and $m \in M$.

Lemma 7.11. *In the situation of Definition* 7.9*, suppose* $X = \operatorname{Spec} B$ *is an affine curve over* A*. Let* $b \in B$ *be some element,* $\bar{B} := B/(b)$*, and* $\beta \in \Omega^1_{B/A}$*. Assume that* $A \to \bar{B}$ *is finite, and let* $\operatorname{Res}_{B/A} : \operatorname{Ext}^1_B(\bar{B}, \Omega^1_{B/A}) \to A$ *be the residue map from Definition* 5.5*. Then*

$$\operatorname{Res}_{B/A} \begin{bmatrix} \beta \\ b \end{bmatrix} = \sum_{x \in X \text{ closed}} \operatorname{Res}_x(\tfrac{\beta}{b}).$$

Proof. First we note that b is a regular element of B, so it is invertible in the fraction ring $L := \boldsymbol{k}(X)$. Another thing to note is that $\operatorname{Res}_x(\frac{\beta}{b}) = 0$ if $b(x) \neq 0$, i.e. if $x \notin \operatorname{Spec} \bar{B}$.

Let us denote the generic point of X by x_0. The homomorphism $\delta_{(x_0,x_1)} : \mathrm{H}^0_{x_0} \Omega^1_{X/A} \to \mathrm{H}^1_{x_1} \Omega^1_{X/A}$ sends the fraction $\frac{\beta}{b}$ to the generalized fraction $\begin{bmatrix} \beta \\ b \end{bmatrix}$. The artinian ring $\bar{B}$ is semi-local: $\bar{B} = \prod_{x \in \operatorname{Spec} \bar{B}} \bar{B}_x$, and the projection $B \to \bar{B}_x$ factors via B_x. Looking at the definitions we see that the residue map $\operatorname{Res}_{B/A} : \operatorname{Ext}^1_B(\bar{B}, \Omega^1_{B/A}) \to A$ factors via

$$\sum \operatorname{Res}_{B_x/A} : \bigoplus\nolimits_{x \in \operatorname{Spec} \bar{B}} \mathrm{H}^1_x \, \Omega^1_{B_x/A} \to A.$$

□

Lemma 7.12. *In the situation of Definition* 7.9*, suppose* $X = \mathbf{P}^1_A$*. Let* A' *be another artinian local* $\mathbb{K}$*-algebra, and let* $f^* : A \to A'$ *be a finite homomorphism. Define* $X' := \mathbf{P}^1_{A'}$*. We get an induced finite morphism of schemes* $g : X' \to X$*. Then for any differential form* $\alpha \in \Omega^1_{\boldsymbol{k}(X)/A}$ *one has*

$$\sum_{x \in X \text{ closed}} f^*(\operatorname{Res}_x(\alpha)) = \sum_{x' \in X' \text{ closed}} \operatorname{Res}_{x'}(g^*(\alpha)).$$

Proof. Let us write $X = U \cup \{\infty\}$ with $U := \mathbf{A}^1_A = \operatorname{Spec} B$ and $B := A[t]$. So $\alpha = \frac{\beta}{b}$ for some differential form $\beta \in \Omega^1_{B/A}$ and some regular element $b \in B$. Define $B' := A'[t]$ and $U' := \mathbf{A}^1_{A'} = \operatorname{Spec} B'$. So X' decomposes into $U' \cup \{\infty'\}$. By Lemma 7.11 and Proposition 5.9 we have

$$\sum_{x \in U \text{ closed}} f^*(\operatorname{Res}_x(\alpha)) = f^*(\operatorname{Res}_{B/A} \begin{bmatrix} \beta \\ b \end{bmatrix}) =$$

$$\operatorname{Res}_{B'/A'} \begin{bmatrix} g^*(\beta) \\ g^*(b) \end{bmatrix} = \sum_{x' \in U' \text{ closed}} \operatorname{Res}_{x'}(g^*(\alpha)).$$

At ∞ we will use the coordinate $s := \frac{1}{t}$. Then $\alpha = \frac{\gamma}{s^e}$ for some $e \geq 0$ and $\gamma \in \Omega^1_{X/A,\infty}$. Choose a sufficiently small affine open neighborhood $V = \operatorname{Spec} C$ of ∞ in X, so that $\gamma \in \Gamma(V, \Omega^1_{X/A})$, but the origin O is not in V; so that $s \in \Gamma(V, \mathcal{O}_X)$. Define $C' := A' \otimes_A C$. Again using Lemma 7.11 and Proposition 5.9 we obtain

$$f^*(\operatorname{Res}_\infty(\tfrac{\gamma}{s^e})) = f^*(\operatorname{Res}_{C/A} \left[\begin{smallmatrix}\gamma \\ s^e\end{smallmatrix}\right]) = \\ \operatorname{Res}_{C'/A'} \left[\begin{smallmatrix}g^*(\gamma) \\ g^*(s^e)\end{smallmatrix}\right] = \operatorname{Res}_{\infty'}(g^*(\tfrac{\gamma}{s^e})).$$

□

Theorem 7.13 (Residue Theorem for $\mathbf{P}^1$). *Suppose A is an artinian, local, essentially finite type $\mathbb{K}$-algebra, and $X = \mathbf{P}^1_A$. Let M be an A-module and let $\alpha \in \Omega^1_{\boldsymbol{k}(X)/A} \otimes_A M$. Then*

$$\sum_{x \in X \text{ closed}} \operatorname{Res}_{x;M}(\alpha) = 0.$$

Proof. The homomorphism

$$\sum_{x \in X \text{ closed}} \operatorname{Res}_{x;M} : \Omega^1_{\boldsymbol{k}(X)/A} \otimes_A M \to M$$

is functorial in M. Thus we can assume that M is finitely generated; and by induction on length, we can also assume M is simple. Thus we may assume $M \cong K$, where K is the residue field of A. Consider the ring homomorphism $f^* : A \to K$. Because $A \otimes_A M \cong K \otimes_K M$, and by Lemma 7.12, we can replace A with K.

So let us assume that $A = K$ is a field, and $M = K$. Let t be the coordinate on the finite part of X, i.e. $X = \mathbf{A}^1_K \cup \{\infty\}$ and $\mathbf{A}^1_K = \operatorname{Spec} K[t]$. We can write α as a fraction $\alpha = \frac{f(t)}{g(t)} \mathrm{d}t$ where $f(t), g(t)$ are polynomials. Choose some finite field extension $K \to K'$ which splits the polynomials $f(t)$ and $g(t)$. By Lemma 7.12 we can replace K with K'. So we can assume $f(t)$ and $g(t)$ are products of linear terms. Now we may apply partial fraction decomposition to the rational function $\frac{f(t)}{g(t)}$. So we can assume that either $\alpha = (t-a)^{-e}\mathrm{d}t$ for some $a \in K$ and $e \geq 1$; or that $\alpha = t^e \mathrm{d}t$ for $e \geq 0$.

Consider the case $\alpha = (t-a)^{-e}\mathrm{d}t$. Applying the linear change of coordinates $t \mapsto t-a$ (which is permitted by Lemma 7.11 and Proposition 5.9) we can assume that $a = 0$. Let $O \in X$ be the origin. At any closed point $x \in X$ except O and ∞ one has $\operatorname{Res}_x(\alpha) = 0$. According to Lemma 7.11 and Proposition 5.12 we have $\operatorname{Res}_O(\alpha) = \epsilon$ if $e = 1$, and $\operatorname{Res}_O(\alpha) = 0$ otherwise. By change of coordinates $t \mapsto s := t^{-1}$ we get $\alpha = -s^{e-2}\mathrm{d}s$, and the same calculation gives $\operatorname{Res}_\infty(\alpha) = -\epsilon$ if $e = 1$, and $\operatorname{Res}_\infty(\alpha) = 0$ otherwise.

Finally consider the case $\alpha = t^e \mathrm{d}t$. Then by Lemma 7.11 and Proposition 5.12 we get $\operatorname{Res}_x(\alpha) = 0$ at all points. □

Theorem 7.14 (Residue Theorem for Proper Morphisms). *Let $f : X \to Y$ be a proper morphism between finite type $\mathbb{K}$-schemes. Then $\operatorname{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$ is a homomorphism of complexes.*

Proof. The proof is in several steps.

Step 1. First consider the case of a finite morphism f. According to Theorem 6.16 we have an isomorphism $\mathcal{K}_X \xrightarrow{\simeq} f^\flat \mathcal{K}_Y$, which is the same as an isomorphism

$$\psi : f_* \mathcal{K}_X \xrightarrow{\simeq} f_* f^\flat \mathcal{K}_Y = \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{K}_Y).$$

By definition 7.6, the trace $\mathrm{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$ is transformed by ψ into the homomorphism $\mathrm{Tr}^{\flat}_{f;\mathcal{K}_Y} : f_* f^{\flat} \mathcal{K}_Y \to \mathcal{K}_Y$, namely evaluation at 1. And the latter is a homomorphism of complexes.

Step 2. Now let f be any proper morphism. For any point $x \in X$ let $y := f(x) \in Y$, and denote by $\mathrm{Tr}_{f,x} : \mathcal{K}_X(x) \to \mathcal{K}_Y(y)$ the corresponding component of Tr_f (see Definition 7.6). Also, for any point $x' \in X$ which is an immediate specialization of x, let us denote by $\delta_{X,(x,x')} : \mathcal{K}_X(x) \to \mathcal{K}_X(x')$ the corresponding component of the coboundary operator $\delta_X : \mathcal{K}_X \to \mathcal{K}_X$.

There are two kinds of identities we must check. The first is when $\dim_{\mathbb{K}}(x) = \dim_{\mathbb{K}}(y)$, and $y' \in Y$ is any point which is an immediate specialization of y. We must then show that given any element $\alpha \in \mathcal{K}_X(x)$ the equality

$$\sum_{x'} (\mathrm{Tr}_{f,x'} \circ \delta_{X,(x,x')})(\alpha) = (\delta_{Y,(y,y')} \circ \mathrm{Tr}_{f,x})(\alpha)$$

holds in $\mathcal{K}_Y(y')$. The sum is over all points $x' \in f^{-1}(y')$ which are immediate specializations of x. This case will be treated now; the other case will be taken care of in the subsequent steps.

It is possible to choose a nilpotent closed subscheme X_0 of X which is supported on the closed set $\overline{\{x\}}$, and such that $\alpha \in \mathcal{K}_{X_0}(x)$. The transitivity of traces (Proposition 7.7) implies that we may replace X with X_0. Now $f : X \to Y$ is proper and quasi-finite, hence finite; and we can apply step 1.

Step 3. In this step we assume that $\dim_{\mathbb{K}}(x) = \dim_{\mathbb{K}}(y) + 1$, and we must show that

$$\sum_{x'} (\mathrm{Tr}_{f,x'} \circ \delta_{X,(x,x')})(\alpha) = 0. \tag{7.15}$$

Here the sum is over the points $x' \in X$ which are immediate specializations of x; these points necessarily lie in $f^{-1}(y)$. As done in step 2, we can find nilpotent closed subschemes X_0 and Y_0, supported on $\overline{\{x\}}$ and $\overline{\{y\}}$ respectively, such that $\alpha \in \mathcal{K}_{X_0}(x)$ and $f : X_0 \to Y$ factors via Y_0. Observe that the identity (7.15) depends only on the homomorphisms $\mathcal{O}_{Y_0,y} \to \mathcal{O}_{X_0,x'} \to \mathcal{O}_{X_0,x}$, when these rings are considered as essentially finite type $\mathbb{K}$-algebras.

Define $A := \mathcal{O}_{Y_0,y}$. Let us choose a ring $\mathbb{K}'$ which is a localization of a polynomial algebra $\mathbb{K}[t_1, \ldots, t_n]$, and which admits a finite homomorphism $\mathbb{K}' \to A$. Then $\mathbb{K}'$ is also a regular noetherian ring of finite Krull dimension, and the rigid dualizing complex of $\mathbb{K}'$ relative to $\mathbb{K}$ is $\Omega^n_{\mathbb{K}'/\mathbb{K}}[n]$. In view of Proposition 4.15 and Corollary 6.23, we can replace $\mathbb{K}$ with $\mathbb{K}'$ – it amounts to twisting by the inverse of $\Omega^n_{\mathbb{K}'/\mathbb{K}}[n]$, which does not effect (7.15). We conclude that we may replace Y with $\operatorname{Spec} A$ and X with $X_0 \times_{Y_0} \operatorname{Spec} A$.

Step 4. In this step we assume that A is a local artinian finite $\mathbb{K}$-algebra, and X is a proper curve over A, with generic point x. Given $\alpha \in \mathcal{K}_X(x)$, we have to verify (7.15); and the sum is over all closed points $x' \in X$. As explained in [RD, p. 373], the morphism $f : X \to Y = \operatorname{Spec} A$ factors via a finite morphism $X \to \mathbf{P}^1_A$. Because of step 1 we can replace X with $\mathbf{P}^1_A$.

Let x' be any closed point in $X = \mathbf{P}^1_A$. It is immediate from Definitions 7.9 and 7.10 that under the isomorphisms

$$\mathcal{K}_X(x) \cong \mathrm{H}^0_x(\Omega^1_{X/A} \otimes_A \mathcal{K}(A)) \cong (\mathrm{H}^0_x \Omega^1_{X/A}) \otimes_A \mathcal{K}(A) \cong \Omega^1_{\boldsymbol{k}(X)/A} \otimes_A \mathcal{K}(A)$$

the homomorphism

$$\mathrm{Tr}_{f,x'} \circ \delta_{X,(x,x')} : \mathcal{K}_X(x) \to \mathcal{K}(A)$$

goes to the residue map

$$\mathrm{Res}_{x';\mathcal{K}(A)} : \Omega^1_{\boldsymbol{k}(X)/A} \otimes_A \mathcal{K}(A) \to \mathcal{K}(A).$$

So Theorem 7.13 applies with $M := \mathcal{K}(A)$. □

Corollary 7.16. (1) *Let $f : X \to Y$ be a proper morphism between finite type $\mathbb{K}$-schemes. Then there is a morphism of functors $\mathrm{Tr}_f : \mathrm{R}f_* f^! \to \mathbf{1}$ of functors from $\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$ to itself.*

(2) *The assignment $f \mapsto \mathrm{Tr}_f$ above is 2-functorial for proper morphisms. Namely, given another proper morphism $g : Y \to Z$, the diagram*

$$\begin{array}{ccccc}
\mathrm{R}(g\circ f)_*\,(g\circ f)^! & \xrightarrow{\phi_{g,f}} & \mathrm{R}(g\circ f)_*\,f^!\,g^! & \xrightarrow{\cong} & \\
\downarrow{\scriptstyle \mathrm{Tr}_{g\circ f}} & & & & \\
\mathbf{1}_{\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Z)} & \xleftarrow{\mathrm{Tr}_g} & \mathrm{R}g_*\,g^! & \xleftarrow{\mathrm{Tr}_f} & \mathrm{R}g_*\,\mathrm{R}f_*\,f^!\,g^!
\end{array}$$

is commutative, where the isomorphism marked "≅" is the standard isomorphism of functors $\mathrm{R}(g\circ f)_* \cong \mathrm{R}g_*\,\mathrm{R}f_*$, and $\phi_{g,f}$ is from Corollary 6.15. If $X = Y$ and $f = \mathbf{1}_X$, then Tr_f is the identity automorphism of $f^! = \mathbf{1}_{\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)}$.

In the diagram there is a little bit of sloppiness; for instance, instead of "$\phi_{g,f}$" we should have really written "$\mathrm{R}(g\circ f)_*(\phi_{g,f})$".

Proof. (1) Take any $\mathcal{N} \in \mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$. By definition of $f^!$ we have

$$f^!\mathcal{N} = \mathcal{H}om_{\mathcal{O}_X}\big(\mathrm{L}f^*\,\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M},\mathcal{K}_Y),\mathcal{K}_X\big),$$

and hence

$$\mathrm{R}f_*f^!\mathcal{N} = \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M},\mathcal{K}_Y),f_*\mathcal{K}_X\big).$$

According to Theorem 7.14 the trace map $\mathrm{Tr}_f : f_*\mathcal{K}_X \to \mathcal{K}_Y$ is a homomorphism of complexes, and so we obtain a morphism

$$\tau : \mathrm{R}f_*f^!\mathcal{N} \to \mathcal{H}om_{\mathcal{O}_Y}\big(\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M},\mathcal{K}_Y),\mathcal{K}_Y\big) = \mathrm{D}_Y\mathrm{D}_Y\mathcal{N}$$

in $\mathsf{D}^+_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$. Using the adjunction isomorphism $\phi_Y : \mathcal{N} \xrightarrow{\simeq} \mathrm{D}_Y\mathrm{D}_Y\mathcal{N}$ we define

$$\mathrm{Tr}_{f;\mathcal{N}} := \phi_Y^{-1}\circ\tau : \mathrm{R}f_*f^!\mathcal{N} \to \mathcal{N}.$$

As $\mathcal{N}$ varies this becomes a morphism of functors $\mathrm{Tr}_f : \mathrm{R}f_*f^! \to \mathbf{1}$.

(2) Recall that the isomorphism $\phi_{g,f}$ was defined solely using adjunction formulas. By Proposition 7.7 the traces are transitive: $\mathrm{Tr}_g\circ\mathrm{Tr}_f = \mathrm{Tr}_{g\circ f}$. This implies the commutativity of the diagram. Finally, for the identity automorphism $\mathbf{1}_X : X \to X$, the trace $\mathrm{Tr}_{\mathbf{1}_X} : \mathcal{K}_X \to \mathcal{K}_X$ is the identity automorphism of this complex. □

Theorem 7.17 (Duality for Proper Morphisms)**.** *Let $f : X \to Y$ be a proper morphism of finite type $\mathbb{K}$-schemes, let $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_X)$ and let $\mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\,\mathcal{O}_Y)$. Then the morphism*

$$\mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M},f^!\mathcal{N}) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_*\mathcal{M},\mathcal{N})$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_Y)$ induced by $\mathrm{Tr}_f : \mathrm{R}f_*f^!\mathcal{N} \to \mathcal{N}$ is an isomorphism.

Proof. Using the same reduction as in the proof of [RD, Theorem VII.3.3] we can assume Y is affine, $X = \mathbf{P}^n_Y$ and f is the projection.

For fixed $\mathcal{N} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y)$ the contravariant functors $\mathrm{R}f_* \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(-, f^! \mathcal{N})$ and $\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_* -, \mathcal{N})$ are way-out left, in the sense of [RD, Section I.7]. Let $\boldsymbol{\omega} := \Omega^n_{X/Y}$, and for any integer i let $\boldsymbol{\omega}(i)$ be the Serre twist. As explained in the proof of [RD, Theorem III.5.1], any coherent $\mathcal{O}_X$-module is a quotient of a finite direct sum $\bigoplus_{j=1}^m \boldsymbol{\omega}(-i_j)$ for some $i_1, \ldots, i_m > 0$. Therefore, using [RD, Proposition I.7.1], reversed so as to handle contravariant functors, we can assume that $\mathcal{M} = \boldsymbol{\omega}(-i)[n]$ with $i > 0$.

Now the coherent sheaves $\boldsymbol{\omega}(-i)$ and $\mathrm{R}^n f_* \boldsymbol{\omega}(-i)$ are locally free, and $\mathrm{R}^j f_* \boldsymbol{\omega}(-i) = 0$ for $j \neq n$. Also we know that $f^! - \cong \boldsymbol{\omega}[n] \otimes_{\mathcal{O}_X} f^* -$ (see Corollary 6.17). Therefore the functors $\mathrm{R}f_* \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}\big(\boldsymbol{\omega}(-i)[n], f^! -\big)$ and $\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], -\big)$ are way-out in both directions. Once again using [RD, Proposition I.7.1] we can reduce to the case $\mathcal{N} = \mathcal{O}_Y$.

At this stage we have to prove that the morphism

$$\theta : \mathrm{R}f_* \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}\big(\boldsymbol{\omega}(-i)[n], \boldsymbol{\omega}[n]\big) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], \mathcal{O}_Y\big)$$

is an isomorphism. By definition $\theta = \gamma \circ \beta$, where

$$\beta : \mathrm{R}f_* \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}\big(\boldsymbol{\omega}(-i)[n], \boldsymbol{\omega}[n]\big) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], \mathrm{R}f_* \boldsymbol{\omega}[n]\big)$$

is the canonical morphism, and

$$\gamma : \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], \mathrm{R}f_* \boldsymbol{\omega}[n]\big) \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], \mathcal{O}_Y\big)$$

is induced by

$$\mathrm{Tr}_f : \mathrm{R}f_* \boldsymbol{\omega}[n] \cong \mathrm{R}f_* f^! \mathcal{O}_Y \to \mathcal{O}_Y.$$

Consider the canonical isomorphism

$$\alpha : \mathrm{R}f_* \mathcal{O}_X(i) \xrightarrow{\simeq} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}\big(\mathrm{R}f_* \boldsymbol{\omega}(-i)[n], \mathrm{R}f_* \boldsymbol{\omega}[n]\big)$$

The composite $\beta \circ \alpha$ is an isomorphism, because the cup product pairing

$$f_* \mathcal{O}_X(i) \times \mathrm{R}^n f_* \boldsymbol{\omega}(-i) \to \mathrm{R}^n f_* \boldsymbol{\omega}$$

is perfect (see [RD, Theorem III.3.4]). Hence β is an isomorphism. It remains to prove that γ is also an isomorphism.

To accomplish this we will prove that the trace $\mathrm{Tr}_f : \mathrm{R}f_* f^! \mathcal{O}_Y \to \mathcal{O}_Y$ is an isomorphism. We know that $f^! \mathcal{O}_Y \cong \boldsymbol{\omega}[n]$, $\mathrm{R}^j f_* f^! \mathcal{O}_Y \cong \mathrm{R}^{j+n} f_* \boldsymbol{\omega} = 0$ for $j \neq 0$, and that $\mathrm{R}^0 f_* f^! \mathcal{O}_Y \cong \mathrm{R}^n f_* \boldsymbol{\omega} \cong \mathcal{O}_Y$. Since $\mathrm{R}^0 f_* f^! \mathcal{O}_Y$ is a free $\mathcal{O}_Y$-module of rank 1, it suffices to show that $\mathrm{H}^0(\mathrm{Tr}_f) : \mathrm{R}^0 f_* f^! \mathcal{O}_Y \to \mathcal{O}_Y$ is surjective. Choose any section $g : Y \to X$ of f. Then according to Corollary 7.16(2) we have $\mathrm{H}^0(\mathrm{Tr}_f) \circ \mathrm{H}^0(\mathrm{Tr}_g) = \mathrm{H}^0(\mathrm{Tr}_{\mathbf{1}_Y})$, implying that indeed $\mathrm{H}^0(\mathrm{Tr}_f)$ is surjective. □

8. The Relative Dualizing Sheaf

In this section $\mathbb{K}$ denotes a regular noetherian commutative ring of finite Krull dimension. All schemes are by default finite type $\mathbb{K}$-schemes, all algebras are by default essentially finite type $\mathbb{K}$-algebras, and all morphisms are over $\mathbb{K}$. The base ring $\mathbb{K}$ will have no visible role here; it will only be in the background, making rigid residue complexes and the 2-functor $f \mapsto f^!$ available.

Let us begin with a few facts about $f^!$ for flat morphisms.

Lemma 8.1. *Suppose $f : X \to Y$ is a flat morphism. Then*

$$f^! \mathcal{M} = \mathcal{H}om_{f^{-1}\mathcal{O}_Y}\big(f^{-1}\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{K}_Y), \mathcal{K}_X\big)$$

for $\mathcal{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y)$.

Proof. Due to flatness the rigid residue complex $\mathcal{K}_X$ is a bounded complex of injective $f^{-1}\mathcal{O}_Y$-modules. □

Given an algebra A we define the rigid residue complex of A to be $\mathcal{K}_A := \Gamma(U, \mathcal{K}_U)$ where $U := \operatorname{Spec} A$.

Proposition 8.2. *Let $f^* : A \to B$ be a flat homomorphism of $\mathbb{K}$-algebras.*

1. *The complex $f^! A$ has finite flat dimension over A.*
2. *There is a functorial isomorphism $f^! M \cong M \otimes^{\mathrm{L}}_A f^! A$ for $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.*

Proof. Since $A \to B$ is flat it follows that each $\mathcal{K}^p_B$ is an injective A-module. Let d_1, d_2 be the amplitudes of the complexes $\mathcal{K}_A$ and $\mathcal{K}_B$ respectively.

Given $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, let M' be the complex obtained from M by truncating above and below the degrees where the cohomology is nonzero. According to Lemma 8.1 one has $f^! M \cong \operatorname{Hom}_A\big(\operatorname{Hom}_A(M', \mathcal{K}_A), \mathcal{K}_B\big)$. Therefore

$$\operatorname{amp} \mathrm{H} f^! M \leq \operatorname{amp} \mathrm{H} M + d_1 + d_2.$$

Next choose a resolution $P \to M$ by a bounded above complex of finitely generated free A-modules. Then

$$f^! M \cong \operatorname{Hom}_A\big(\operatorname{Hom}_A(P, \mathcal{K}_A), \mathcal{K}_B\big) \cong P \otimes_A \operatorname{Hom}_A(\mathcal{K}_A, \mathcal{K}_B) \cong M \otimes^{\mathrm{L}}_A f^! A.$$

This proves part (2), and also shows that $\operatorname{flat.dim}_A f^! A \leq d_1 + d_2$. □

Proposition 8.3. *Let $f^* : A \to B$ be a flat homomorphism of $\mathbb{K}$-algebras. The canonical morphism*

$$(f^! A) \otimes^{\mathrm{L}}_A R_A \cong \operatorname{RHom}_A(R_A, R_B) \otimes^{\mathrm{L}}_A R_A \to R_B$$

is an isomorphism.

Proof. Let us denote this morphism by ψ, and let N be the cone on ψ. So ψ is an isomorphism if and only $N = 0$. By Proposition 8.2 the complex $(f^! A) \otimes^{\mathrm{L}}_A R_A$ has bounded cohomology, and hence $N \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$. According to Lemma 1.11 the complex R_A generates $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, and we conclude that $N = 0$ if and only if the morphism

$$\psi' : \operatorname{RHom}_A\big(R_A, \operatorname{RHom}_A(R_A, R_B) \otimes^{\mathrm{L}}_A R_A\big) \to \operatorname{RHom}_A(R_A, R_B)$$

induced by ψ is an isomorphism. We know that $\operatorname{RHom}_A(R_A, R_B)$ has finite flat dimension over A – again, this is by Proposition 8.2. Using Proposition 1.10, under its assumption (iii.b), we can pass from ψ' to the morphism

$$\operatorname{RHom}_A(R_A, R_B) \otimes^{\mathrm{L}}_A \operatorname{RHom}_A(R_A, R_A) \to \operatorname{RHom}_A(R_A, R_B),$$

which is evidently an isomorphism. □

A flat morphism of schemes $f : X \to Y$ is said to have *relative dimension* n if all its fibers are equidimensional of dimension n.

Definition 8.4. Let $f : X \to Y$ be a flat morphism of relative dimension n. The *relative dualizing sheaf* is

$$\boldsymbol{\omega}_{X/Y} := \mathrm{H}^{-n} f^! \mathcal{O}_Y.$$

Some authors refer to $\boldsymbol{\omega}_{X/Y}$ as the *relative canonical sheaf*. The $\mathcal{O}_X$-module $\boldsymbol{\omega}_{X/Y}$ is coherent. We now study some more of its properties.

Proposition 8.5. *Let $f : X \to Y$ be flat of relative dimension n, and let U be an open subset of X such that $f|_U : U \to Y$ is smooth. Then there is a canonical isomorphism $\boldsymbol{\omega}_{X/Y}|_U \cong \Omega^n_{X/Y}|_U$.*

Proof. Define $f' := f|_U$. By Corollary 6.17 there is an isomorphism $f'^{!}\mathcal{O}_Y \cong f'^{\sharp}\mathcal{O}_Y = \Omega^n_{U/Y}[n]$. □

Proposition 8.6. *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then $\mathrm{H}^i f^! \mathcal{O}_Y = 0$ for all $i < -n$. Consequently, truncation gives rise to a canonical morphism*

$$\gamma_f : \boldsymbol{\omega}_{X/Y}[n] \to f^! \mathcal{O}_Y$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.

Proof. In view of Lemma 8.1, it suffices to show that the complex $\mathcal{H}om_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{K}_Y, \mathcal{K}_X)$ is concentrated in degrees $\geq -n$. Suppose we are given a local section $\phi \in \mathcal{H}om_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{K}_Y, \mathcal{K}_X)^i$ which has a nonzero component going from $\mathcal{J}(y)$ to $\mathcal{J}(x)$, for some points $y \in Y$ and $x \in X$; see Section 7 for notation. Then we must have $x \in f^{-1}(y)$. But the dimension of the fiber $f^{-1}(y)$ is n, and hence $\dim_{\mathbb{K}}(x) \leq \dim_{\mathbb{K}}(y) + n$, so $i \geq -n$. □

Recall that our notation for the residue field of a point $x \in X$ is $\boldsymbol{k}(x)$. In case X is an integral scheme with generic point x, we also write $\boldsymbol{k}(X)$ for this field, which is of course the function field of X.

Suppose K is a field, and L is a finitely generated extension field of K (i.e. L is an essentially finite type K-algebra). Let M be a finite separable field extension of L, and denote by $\mathrm{tr}_{M/L} : M \to L$ the trace map. Since the homomorphism $M \otimes_L \Omega_{L/K} \to \Omega_{M/K}$ is bijective, we obtain an induced $\Omega_{L/K}$-linear homomorphism $\mathrm{tr}_{M/L} : \Omega_{M/K} \to \Omega_{L/K}$.

Theorem 8.7. *Let X and Y be integral schemes, and let $f : X \to Y$ be a flat morphism of relative dimension n. Assume f is generically smooth.*

(1) *The coherent sheaf $\boldsymbol{\omega}_{X/Y}$ is a subsheaf of the constant quasi-coherent sheaf $\Omega^n_{\boldsymbol{k}(X)/\boldsymbol{k}(Y)}$.*
(2) *Suppose $U \subset X$ is a nonempty affine open set, and there is a commutative diagram*

$$\begin{array}{ccc} U & \xrightarrow{\subset} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{h} & Y \end{array}$$

with Z an integral affine scheme; h a smooth morphism; and g a finite, dominant, separable morphism. Then

$$\Gamma(U, \boldsymbol{\omega}_{X/Y}) = \big\{ \alpha \in \Omega^n_{\boldsymbol{k}(X)/\boldsymbol{k}(Y)} \mid \mathrm{tr}_{\boldsymbol{k}(X)/\boldsymbol{k}(Z)}(a\alpha) \in \Gamma(Z, \Omega^n_{Z/Y}) \ \text{for all } a \in \Gamma(U, \mathcal{O}_X) \big\}.$$

The proof of the theorem is after this lemma.

Lemma 8.8. *Let K, L, M be fields. Assume L is a finitely generated separable extension of K, and M is a finite separable extension of L. Let $n := \mathrm{tr.deg}_K\, L$.*

(1) *The rigid dualizing complexes of* L *and* M *relative to* K *are* $\Omega^n_{L/K}[n]$ *and* $\Omega^n_{M/K}[n]$ *respectively.*
(2) *The trace map* $\mathrm{tr}_{M/L} : \Omega^n_{M/K}[n] \to \Omega^n_{L/K}[n]$ *is a rigid trace morphism relative to* K.

Proof. (1) The homomorphisms $K \to L$ and $K \to M$ are essentially smooth, so Proposition 4.5 applies.

(2) Let's denote the finite étale homomorphism $L \to M$ by f^*. The trace map $\mathrm{tr}_{M/L} : M \to L$ is nondegenerate (see Definition 4.7). So by Proposition 4.8, Proposition 4.15 and Corollary 6.23, it suffices to prove that $\mathrm{tr}_{M/L}$ is a rigid trace-like morphism relative to L (see Definition 3.28). Here L has the tautological rigidifying isomorphism ρ_{tau}, and $M \cong f^\sharp L = \Omega^0_{M/L}$ has the rigidifying isomorphism $f^\sharp(\rho_{\mathrm{tau}})$.

This is an exercise in Galois theory. Using the transitivity of both the field traces $\mathrm{tr}_{-/-}$ and the rigid traces $\mathrm{Tr}_{-/-}$, we may assume that M is a Galois extension of L. Denote the Galois group by G. Then $M \otimes_L M \cong \prod_{g \in G} M^g$, where the map $M \otimes_L M \to M^g$ is $a \otimes b \mapsto a \cdot g(b)$. The rigidifying isomorphism

$$f^\sharp(\rho_{\mathrm{tau}}) : M \xrightarrow{\simeq} \mathrm{Hom}_{M \otimes_L M}(M, M \otimes_L M)$$

identifies M with M^{g_0}, where g_0 is the identity automorphism. By the properties of the trace for a Galois extension one has $\mathrm{tr}_{M/L}(a) = \sum_{g \in G} g(a)$. A convolution-type calculation shows that under the isomorphism $M \otimes_L M \cong \prod_{g \in G} M^g$, the map $\mathrm{tr}_{M/L} \otimes \mathrm{tr}_{M/L}$ is sent to $\sum_{g \in G} \mathrm{tr}_{M^g/L}$. Hence the diagram

$$\begin{array}{ccc} M & \xrightarrow{f^\sharp(\rho_{\mathrm{tau}})} & \mathrm{Hom}_{M \otimes_L M}\big(M, M \otimes_L M\big) \\ {\scriptstyle \mathrm{tr}_{M/L}} \downarrow & & \downarrow {\scriptstyle \mathrm{tr}_{M/L} \otimes \mathrm{tr}_{M/L}} \\ L & \xrightarrow{\rho_{\mathrm{tau}}} & \mathrm{Hom}_{L \otimes_L L}\big(L, L \otimes_L L\big) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\, L)$ is commutative, and thus indeed $\mathrm{tr}_{M/L}$ is a rigid trace-like morphism. □

Proof of Theorem 8.7. (1) By Proposition 8.6 there is a canonical morphism $\gamma_f : \boldsymbol{\omega}_{X/Y}[n] \to f^! \mathcal{O}_Y$. Since f is generically smooth, there is a nonempty open set $U_0 \subset X$ such that $f : U_0 \to Y$ is smooth. According to Proposition 8.5 there is a canonical isomorphism $f^! \mathcal{O}_Y|_{U_0} \cong \Omega^n_{U_0/Y}[n]$. Combining these morphisms and passage to the generic stalk, we obtain a sheaf homomorphism $\lambda : \boldsymbol{\omega}_{X/Y} \to \Omega^n_{\boldsymbol{k}(X)/\boldsymbol{k}(Y)}$. It remains to prove that λ is injective.

Pick a point $x \in X$. Using quasi-normalization and Zariski's Main Theorem [EGA, Chapter IV, Sections 8.12.3. and 13.3.1] we know that there exists an affine open neighborhood $U = \mathrm{Spec}\, D$ of x in X, and a commutative diagram

$$\begin{array}{ccccc} W & \xleftarrow{\supset} & U & \xrightarrow{\subset} & X \\ {\scriptstyle g} \downarrow & & \downarrow & & \downarrow {\scriptstyle f} \\ Z & \xrightarrow{h} & V & \xrightarrow{\subset} & Y \end{array}$$

where $V = \mathrm{Spec}\, A$ is an affine open set in Y; $Z = \mathrm{Spec}\, B$ and $W = \mathrm{Spec}\, C$ are affine integral schemes; $U \to W$ is an open immersion; $g : W \to Z$ is finite dominant; and $h : Z \to V$ is smooth. Then

$$\boldsymbol{\omega}_{C/A} = \mathrm{H}^0 (h \circ g)^! A \cong \mathrm{H}^0 g^! h^! A \cong \mathrm{H}^0 g^\flat h^\sharp A \cong \mathrm{Hom}_B(C, \Omega^n_{B/A}).$$

This is a torsion-free C-module, i.e. it embeds in

$$M \otimes_C \operatorname{Hom}_B(C, \Omega^n_{B/A}) \cong \operatorname{Hom}_L(M, \Omega^n_{L/A}) \cong \Omega^n_{M/K},$$

where $K := \operatorname{Frac} A$, $L := \operatorname{Frac} B$ and $M := \operatorname{Frac} C$. But on the other hand

$$\Gamma(U, \boldsymbol{\omega}_{X/Y}) = \boldsymbol{\omega}_{D/A} \cong D \otimes_C \boldsymbol{\omega}_{C/A}.$$

(2) Here, in the notation of the proof of part (1), we have $U = W$, and in addition $L \to M$ is separable. The rigid trace Tr_g gives rise to an isomorphism

$$\Gamma(U, \boldsymbol{\omega}_{X/Y}) = \boldsymbol{\omega}_{D/A} \cong \operatorname{Hom}_B(D, \Omega^n_{B/A}).$$

Now $D \to M$ is essentially étale, $D \to B$ is finite, and $L \cong B \otimes_D M$. Hence by Corollary 3.27 and Lemma 8.8 the diagram

$$\begin{array}{ccc} \boldsymbol{\omega}_{D/A} & \xrightarrow{\subset} & \Omega^n_{M/K} \\ {\scriptstyle \operatorname{Tr}_g}\downarrow & & \downarrow{\scriptstyle \operatorname{tr}_{M/L}} \\ \Omega^n_{B/A} & \xrightarrow{\subset} & \Omega^n_{L/K} \end{array}$$

is commutative. □

Remark 8.9. There are notions of differential forms and traces for inseparable field extensions due to Kunz. Presumably these can be used to remove the separability assumption from part (2) of Theorem 8.7. Cf. [HK].

9. Base Change and Traces

As before we work in the category $\mathsf{FTSch}\,/\mathbb{K}$ of finite type $\mathbb{K}$-schemes, where $\mathbb{K}$ is a regular finite dimensional noetherian ring. The main results in this section are Theorems 9.6 and 9.12, which were first obtained by Conrad [Co, Theorems 3.6.1 and 3.6.5]. Indeed, Conrad proved somewhat more general results, since he only assumed his schemes are noetherian and admit dualizing complexes. On the other hand, our proofs, which rely on rigidity, are significantly easier (and shorter) than Conrad's.

Proposition 9.1. *Let $f^* : A \to B$ be a flat finite type homomorphism of $\mathbb{K}$-algebras. The complex $f^! A \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$ has a unique rigidifying isomorphism $\rho_{B/A} : f^! A \xrightarrow{\simeq} \operatorname{Sq}_{B/A} f^! A$, such that under the canonical isomorphism $(f^! A) \otimes^{\mathrm{L}}_A R_A \cong R_B$ from Proposition* 8.3 *one has $\rho_{B/A} \otimes \rho_A = \rho_B$.*

Proof. Let us begin by choosing a factorization $A \xrightarrow{g^*} A[\boldsymbol{t}] \xrightarrow{h^*} B$ of f^*, where $A[\boldsymbol{t}]$ is a polynomial algebra in m variables, and h^* is a surjection. According to Theorem 3.22(1) the complex $g^\sharp A = \Omega^m_{A[\boldsymbol{t}]/A}[m]$ has a rigidifying isomorphism $g^\sharp(\rho_{\mathrm{tau}})$ relative to A, where ρ_{tau} is the tautological rigidifying isomorphism of A. Next, since $h^\flat g^\sharp A \cong f^! A$ has finite flat dimension over A (see Proposition 8.2), Theorem 3.14(1) says that $h^\flat g^\sharp A$ has an induced rigidifying isomorphism $h^\flat(g^\sharp(\rho_{\mathrm{tau}}))$ relative to A, which we shall denote by ρ'.

There exists a unique element $u \in B^\times$ such that under the isomorphism $(f^! A) \otimes^{\mathrm{L}}_A R_A \cong R_B$, the rigidifying isomorphisms $u\rho' \otimes \rho_A$ coincides with ρ_B. Then $\rho_{B/A} := u\rho'$ is the desired rigidifying isomorphism of $f^! A$. □

Definition 9.2. A morphism of schemes $f : X \to Y$ is called a *Cohen-Macaulay* morphism of relative dimension n if it is flat, and all the fibers are equidimensional Cohen-Macaulay schemes of dimension n.

Proposition 9.3. *Let $f : X \to Y$ be a Cohen-Macaulay morphism of relative dimension n, and let $g : Y' \to Y$ be an arbitrary morphism. Define $X' := X \times_Y Y'$, and let $f' : X' \to Y'$ be the projection. Then f' is a Cohen-Macaulay morphism of relative dimension n.*

Proof. The fact that f' is flat is trivial. We have to prove that the fibers of f' are Cohen-Macaulay schemes. This reduces to the following question about rings: let B be an equidimensional n-dimensional Cohen-Macaulay algebra over the field K, let $K \to K'$ be a field extension, and let $B' := K' \otimes_K B$. We must show that B' is an equidimensional n-dimensional Cohen-Macaulay algebra.

Let us introduce the notation $f^* : K \to B$, $g^* : K \to K'$, $f'^* : K' \to B'$ and $h^* : B \to B'$. Since B is an equidimensional Cohen-Macaulay ring of dimension n, and $f^\sharp K$ is a dualizing complex over it (cf. Proposition 4.15), we see that $\mathrm{H}^i f^\sharp K = 0$ for all $i \neq -n$; i.e. $\boldsymbol{\omega}_{B/K}[n] \cong f^\sharp K$. By flat base change (Proposition 6.18) we get

$$f'^! K' \cong f'^! g^* K \cong h^* f^! K \cong B' \otimes_B \boldsymbol{\omega}_{B/K}[n].$$

But $f'^! K'$ is a dualizing complex over B', and therefore this ring is equidimensional n-dimensional and Cohen-Macaulay. □

Proposition 9.4. *Let $f^* : A \to B$ be a flat homomorphism, and let $\mathfrak{a} \subset A$ be an ideal. Define $\bar{A} := A/\mathfrak{a}$ and $\bar{B} := B/\mathfrak{a}B$. Let $\bar{f}^* : \bar{A} \to \bar{B}$ be the induced homomorphism. Then there is a functorial isomorphism $f^! \bar{M} \cong \bar{f}^! \bar{M}$ for $\bar{M} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\bar{A})$.*

Proof. Let $\mathcal{K}_A$, $\mathcal{K}_B$, $\mathcal{K}_{\bar{A}}$ and $\mathcal{K}_{\bar{B}}$ be the respective residue complexes of these algebras. Then $\mathcal{K}_{\bar{A}} \cong \mathrm{Hom}_A(\bar{A}, \mathcal{K}_A)$ and $\mathcal{K}_{\bar{B}} \cong \mathrm{Hom}_B(\bar{B}, \mathcal{K}_B) \cong \mathrm{Hom}_A(\bar{A}, \mathcal{K}_B)$. According to Lemma 8.1 we have

$$\begin{aligned} f^! M &\cong \mathrm{Hom}_A\left(\mathrm{Hom}_A(\bar{M}, \mathcal{K}_A), \mathcal{K}_B\right) \\ &\cong \mathrm{Hom}_{\bar{A}}\left(\mathrm{Hom}_{\bar{A}}(\bar{M}, \mathcal{K}_{\bar{A}}), \mathcal{K}_{\bar{B}}\right) \cong \bar{f}^! \bar{M}. \end{aligned}$$

□

Proposition 9.5. *Let $f : X \to Y$ be a flat morphism of relative dimension n. The following two conditions are equivalent:*

(i) *f is a Cohen-Macaulay morphism.*
(ii) *$\gamma_f : \boldsymbol{\omega}_{X/Y}[n] \to f^! \mathcal{O}_Y$ is an isomorphism, and the sheaf $\boldsymbol{\omega}_{X/Y}$ is flat over $\mathcal{O}_Y$.*

Proof. We might as well assume that $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$.

(i) ⇒ (ii): First we will prove that γ_f is an isomorphism. This amounts to proving that $\mathrm{H}^i f^! A = 0$ for all $i > -n$. The proof is by contradiction. Define $i_1 := \max\{i \mid \mathrm{H}^i f^! A \neq 0\}$, and assume $i_1 > -n$. Then there is a maximal ideal $\mathfrak{q} \subset B$ such that $(B/\mathfrak{q}) \otimes_B \mathrm{H}^i f^! A \neq 0$. Let $\mathfrak{p} := f(\mathfrak{q})$, which is a prime ideal of A. Define $\bar{A} := \mathrm{Frac}(A/\mathfrak{p})$ and $\bar{B} := B \otimes_A \bar{A}$, and let $\bar{f}^* : \bar{A} \to \bar{B}$ be the induced homomorphism. So $\bar{A}$ is a field, and $\bar{B}$ is a Cohen-Macaulay ring, equidimensional of dimension n. According to Propositions 8.2 and 9.4,

$$\boldsymbol{\omega}_{\bar{B}/\bar{A}}[n] \cong \bar{f}^! \bar{A} \cong f^! \bar{A} \cong (f^! A) \otimes^{\mathrm{L}}_A \bar{A}.$$

Hence

$$(\mathrm{H}^{i_1} f^! A) \otimes_B \bar{B} \cong (\mathrm{H}^{i_1} f^! A) \otimes_A \bar{A} \cong \mathrm{H}^{i_1}\left((f^! A) \otimes^{\mathrm{L}}_A \bar{A}\right) = 0.$$

But $B/\mathfrak{q}$ is a quotient of $\bar{B}$, so we have a contradiction.

Next we are going to prove that $\boldsymbol{\omega}_{B/A}$ is a flat A-module. It suffices to show that $\mathrm{H}^i(M \otimes^{\mathrm{L}}_A \boldsymbol{\omega}_{B/A}) = 0$ for all $i < 0$ and all cyclic A-modules M. Thus we can assume

$M = \bar{A} := A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Let $\bar{B} := B \otimes_A \bar{A}$ and let $\bar{f}^* : \bar{A} \to \bar{B}$ be the induced homomorphism. According to Proposition 9.3, $\bar{f}^*$ is a Cohen-Macaulay homomorphism of relative dimension n. Again using Propositions 8.2 and 9.4, we obtain

$$\bar{A} \otimes^{\mathrm{L}}_A \boldsymbol{\omega}_{B/A}[n] \cong \bar{A} \otimes^{\mathrm{L}}_A f^! A \cong f^! \bar{A} \cong \bar{f}^! \bar{A} \cong \boldsymbol{\omega}_{\bar{B}/\bar{A}}[n].$$

(ii) $\Rightarrow$ (i): Take any prime ideal $\mathfrak{p} \subset A$, and let $\bar{A} := A/\mathfrak{p}$, $\bar{B} := B \otimes_A \bar{A}$, $K := \operatorname{Frac} \bar{A}$ and $B_K := B \otimes_A K \cong \bar{B} \otimes_{\bar{A}} K$. We have to prove that B_K is a Cohen-Macaulay ring, equidimensional of dimension n. Let's use the notation $\bar{f}^* : \bar{A} \to \bar{B}$ and $f_K^* : K \to B_K$. Now

$$\bar{f}^! \bar{A} \cong f^! \bar{A} \cong (f^! A) \otimes^{\mathrm{L}}_A \bar{A} \cong \boldsymbol{\omega}_{B/A}[n] \otimes_A \bar{A},$$

due to the flatness of $\boldsymbol{\omega}_{B/A}$. And by flat base change,

$$f_K^! K \cong K \otimes_{\bar{A}} \bar{f}^! \bar{A} \cong K \otimes_A \boldsymbol{\omega}_{B/A}[n].$$

Because $f_K^! K$ is a dualizing complex over B_K we are done. □

Suppose A and B are essentially finite type $\mathbb{K}$-algebras, and $f^* : A \to B$ is a Cohen-Macaulay homomorphism of relative dimension n (see Definition 9.2). According to Proposition 9.1 the complex $\boldsymbol{\omega}_{B/A}[n] = f^! A$ comes equipped with a rigidifying isomorphism $\rho_{B/A}$ relative to A. The notion of rigid base change morphism was introduced in Definition 3.28.

Theorem 9.6. *Suppose*

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

is a cartesian diagram in $\mathsf{FTSch}/\mathbb{K}$*, with* f *a Cohen-Macaulay morphism of relative dimension* n*, and* g *any morphism. Then:*

(1) *There is a homomorphism* $\mathcal{O}_X$*-modules*

$$\theta_{f,g} : \boldsymbol{\omega}_{X/Y} \to h_* \boldsymbol{\omega}_{X'/Y'},$$

such that the induced $\mathcal{O}_{X'}$*-linear homomorphism* $h^*(\theta_{f,g}) : h^* \boldsymbol{\omega}_{X/Y} \to \boldsymbol{\omega}_{X'/Y'}$ *is an isomorphism.*

(2) *The homomorphism* $\theta_{f,g}$ *satisfies, and is determined by the following local condition. Let* $V = \operatorname{Spec} A \subset Y$*,* $U = \operatorname{Spec} B \subset f^{-1}(V)$ *and* $V' = \operatorname{Spec} A' \subset g^{-1}(V)$ *be affine open sets, and let* $U' = \operatorname{Spec} B' := h^{-1}(U) \cap {f'}^{-1}(V') \subset X'$*. Then*

$$\Gamma(U, \theta_{f,g}) : (\boldsymbol{\omega}_{B/A}[n], \rho_{B/A}) \to (\boldsymbol{\omega}_{B'/A'}[n], \rho_{B'/A'})$$

is a rigid base change morphism relative to A*.*

First a lemma.

Lemma 9.7. *Let* $f^* : A \to B$ *be a Cohen-Macaulay homomorphism of relative dimension* n*. Then, for* $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$*, the functorial morphism*

$$\begin{aligned} &\operatorname{RHom}_{B \otimes_A B}\big(B, \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n]\big) \otimes^{\mathrm{L}}_A M \\ &\qquad \to \operatorname{RHom}_{B \otimes_A B}\big(B, \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n] \otimes^{\mathrm{L}}_A M\big) \end{aligned} \tag{9.8}$$

is an isomorphism.

Observe the similarity to Proposition 1.10. However, the hypotheses of Proposition 1.10 might not be true here.

Proof. Let's introduce the notation $A \xrightarrow{d^*} B \otimes_A B \xrightarrow{e^*} B$; so $f^* = e^* \circ d^*$. The homomorphism d^* is a Cohen-Macaulay homomorphism of relative dimension $2n$, and by flat base change (Proposition 6.18) we have

$$d^! A \cong \boldsymbol{\omega}_{(B\otimes_A B)/A}[2n] \cong \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n].$$

Now there are isomorphisms

$$\mathrm{RHom}_{B\otimes_A B}\big(B, \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n]\big) \otimes^{\mathrm{L}}_A M \cong (f^! A) \otimes^{\mathrm{L}}_A M \cong f^! M$$

and

$$\mathrm{RHom}_{B\otimes_A B}\big(B, \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n] \otimes^{\mathrm{L}}_A M\big) \cong e^! d^! M \cong f^! M,$$

implying (see Proposition 8.2) that both these functors are way-out on both sides. According to [RD, Proposition I.7.1] it suffices to verify that (9.8) is an isomorphism when $M = A$; which is of course true. □

Proof of Theorem 9.6. The proof is in two steps.

Step 1. We will prove existence and uniqueness of $\theta_{f,g}$ on affine pieces, i.e. in the setup of part (2). Choose a factorization $A \to A[\boldsymbol{t}] \to A'$ of $g^* : A \to A'$, with $A[\boldsymbol{t}]$ a polynomial algebra in m variables, and $A[\boldsymbol{t}] \to A'$ surjective. By flat base change we know that $\boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]} \cong B[\boldsymbol{t}] \otimes_B \boldsymbol{\omega}_{B/A}$. This implies that $\mathrm{Hom}_B(\boldsymbol{\omega}_{B/A}, \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]})$ is a free $B[\boldsymbol{t}]$-module of rank 1, generated by some homomorphism $\theta_0 : \boldsymbol{\omega}_{B/A} \to \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}$. By Proposition 1.10, under its condition (iii.b), we know that the morphism

$$\begin{aligned} &\mathrm{RHom}_{B\otimes_A B}\big(B, \boldsymbol{\omega}_{B/A}[n] \otimes_A \boldsymbol{\omega}_{B/A}[n]\big) \otimes^{\mathrm{L}}_A A[\boldsymbol{t}] \\ &\quad \to \mathrm{RHom}_{B[\boldsymbol{t}]\otimes_{A[\boldsymbol{t}]} B[\boldsymbol{t}]}\big(B[\boldsymbol{t}], \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}[n] \otimes_{A[\boldsymbol{t}]} \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}[n]\big) \end{aligned}$$

induced by $\theta_0 \otimes \theta_0$ is an isomorphism. Hence there is an element $u_0 \in B[\boldsymbol{t}]^{\times}$ such that $u_0\theta_0$ is a rigid base change morphism relative to A.

Let's denote the ring homomorphisms by $f'^* : A' \to B'$ and $f^*_{\boldsymbol{t}} : A[\boldsymbol{t}] \to B[\boldsymbol{t}]$. These are Cohen-Macaulay homomorphisms of relative dimension n. Using Proposition 9.4, with A' viewed as a quotient of $A[\boldsymbol{t}]$, we get

$$\boldsymbol{\omega}_{B'/A'}[n] \cong f'^! A' \cong f^!_{\boldsymbol{t}} A' \cong \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}[n] \otimes_{A[\boldsymbol{t}]} A'.$$

So $\mathrm{Hom}_{B[\boldsymbol{t}]}(\boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}, \boldsymbol{\omega}_{B'/A'})$ is a free B'-module, generated by some θ_1. By Lemma 9.7, applied to the $A[\boldsymbol{t}]$-module A', the morphism

$$\begin{aligned} &\mathrm{RHom}_{B[\boldsymbol{t}]\otimes_{A[\boldsymbol{t}]} B[\boldsymbol{t}]}\big(B[\boldsymbol{t}], \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}[n] \otimes_{A[\boldsymbol{t}]} \boldsymbol{\omega}_{B[\boldsymbol{t}]/A[\boldsymbol{t}]}[n]\big) \otimes^{\mathrm{L}}_{A[\boldsymbol{t}]} A' \\ &\quad \to \mathrm{RHom}_{B'\otimes_{A'} B'}\big(B', \boldsymbol{\omega}_{B'/A'}[n] \otimes_{A'} \boldsymbol{\omega}_{B'/A'}[n]\big) \end{aligned}$$

induced by $\theta_1 \otimes \theta_1$ is an isomorphism. Therefore there is an element $u_1 \in B'^{\times}$ such that $u_1\theta_1$ is a rigid base change morphism relative to $A[\boldsymbol{t}]$. Then

$$\theta_{f,g} := u_1\theta_1 \circ u_0\theta_0 : \boldsymbol{\omega}_{B/A}[n] \to \boldsymbol{\omega}_{B'/A'}[n]$$

is the rigid base change morphism we want. By Proposition 3.29 it is unique.

Step 2. Gluing: in the setup of part (2), suppose $V_1 = \mathrm{Spec}\, A_1$ is an affine open set contained in V, $U_1 = \mathrm{Spec}\, B_1$ is an affine open set contained in $f^{-1}(V_1) \cap U$, and $V_1' = \mathrm{Spec}\, A_1'$ is an affine open set contained in $g^{-1}(V_1) \cap V'$. Let $B_1' := B_1 \otimes_{A_1} A_1'$. By step 1 we get homomorphisms $\theta_{f,g} : \boldsymbol{\omega}_{B/A} \to \boldsymbol{\omega}_{B'/A'}$ and $\theta_{f_1,g_1} : \boldsymbol{\omega}_{B_1/A_1} \to \boldsymbol{\omega}_{B'_1/A'_1}$. Consider the diagram

$$\begin{array}{ccc} \boldsymbol{\omega}_{B/A}[n] & \xrightarrow{\theta_{f,g}} & \boldsymbol{\omega}_{B'/A'}[n] \\ \downarrow & & \downarrow \\ \boldsymbol{\omega}_{B_1/A_1}[n] & \xrightarrow{\theta_{f_1,g_1}} & \boldsymbol{\omega}_{B'_1/A'_1}[n] \end{array}$$

where the vertical arrows are the rigid localization homomorphisms corresponding to the localizations $B \to B_1$ and $B' \to B'_1$ (see Proposition 3.25). Due to the uniqueness in step 1, this diagram is commutative.

We conclude that as the affine open sets $V \subset Y$, $U \subset f^{-1}(V)$ and $V' \subset g^{-1}(V)$ vary, the homomorphisms $\theta_{f,g}$ glue to a sheaf homomorphism $\theta_{f,g} : \boldsymbol{\omega}_{X/Y} \to h_* \boldsymbol{\omega}_{X'/Y'}$. □

Corollary 9.9. *In the situation of Theorem* 9.6*, assume f is smooth. Then under the isomorphism $\boldsymbol{\omega}_{X/Y} \cong \Omega^n_{X/Y}$ of Proposition 8.5, $\theta_{f,g}$ is the usual base change homomorphism for differential forms $\Omega^n_{X/Y} \to h_* \Omega^n_{X'/Y'}$.*

Proof. This is because on any affine piece the homomorphism $\Omega^n_{B/A}[n] \to \Omega^n_{B'/A'}[n]$ is a rigid base change morphism relative to A. □

Corollary 9.10. *In the situation of Theorem* 9.6*, suppose that $g' : Y'' \to Y'$ is another morphism. Define $X'' := Y'' \times_{Y'} X'$, and let $f'' : X'' \to Y''$ and $h' : X'' \to X'$ be the projections. Then $\theta_{f,g \circ g'} = g_*(\theta_{f',g'}) \circ \theta_{f,g}$.*

Proof. This is because of the uniqueness in part (2) of the theorem. □

Our final result, Theorem 9.12, is about the interaction of base change and traces. In order to state it we first need:

Lemma 9.11. *In the situation of Theorem* 9.6*, assume the morphism f is proper. Then $\mathrm{R}^i f'_* \boldsymbol{\omega}_{X'/Y'} = 0$ for all $i < n$. Therefore there are isomorphisms*

$$g_* \mathrm{R}^n f'_* \boldsymbol{\omega}_{X'/Y'} \cong \mathrm{H}^n \mathrm{R} g_* \mathrm{R} f'_* \boldsymbol{\omega}_{X'/Y'} \cong \mathrm{H}^n \mathrm{R} f_* \mathrm{R} h_* \boldsymbol{\omega}_{X'/Y'} \cong \mathrm{R}^n f_* h_* \boldsymbol{\omega}_{X'/Y'}.$$

Proof. By Theorem 7.17 we know that

$$\mathrm{R} f'_* \boldsymbol{\omega}_{X'/Y'}[n] \cong \mathrm{R} f'_* f'^! \mathcal{O}_Y \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R} f'_* \mathcal{O}_X, \mathcal{O}_Y).$$

□

Theorem 9.12. *In the situation of Theorem* 9.6*, assume the morphism f is proper. Then the diagram of $\mathcal{O}_Y$-linear homomorphisms*

(9.13)
$$\begin{array}{ccccc} & & \mathrm{R}^n f_* \boldsymbol{\omega}_{X/Y} & \xrightarrow{\mathrm{Tr}_f} & \mathcal{O}_Y \\ & \swarrow{\scriptstyle \mathrm{R}^n f_*(\theta_{f,g})} & & & \downarrow{\scriptstyle g^*} \\ \mathrm{R}^n f_* h_* \boldsymbol{\omega}_{X'/Y'} & \xrightarrow{\cong} & g_* \mathrm{R}^n f'_* \boldsymbol{\omega}_{X'/Y'} & \xrightarrow{g_*(\mathrm{Tr}_{f'})} & g_* \mathcal{O}_{Y'} \end{array}$$

in which the arrow marked "≅" is the one from Lemma 9.11*, is commutative.*

For the proof we shall need three lemmas.

Lemma 9.14. *Suppose that $Y = \operatorname{Spec} K$ and $Y' = \operatorname{Spec} K'$ with K and K' fields; and that $e : Z \to X$ is a finite morphism such that $f \circ e : Z \to Y$ is a Cohen-Macaulay* *morphism of relative dimension* m. *Define* $Z' := Z \times_X X'$, *with projections* $e' : Z' \to X'$ *and* $d : Z' \to Z$. *Then the diagram*

$$\begin{array}{ccc} e_* \boldsymbol{\omega}_{Z/Y}[m] & \xrightarrow{\theta_{f\circ e,g}} & d_* \boldsymbol{\omega}_{Z'/Y'}[m] \\ {\scriptstyle \mathrm{Tr}_e}\big\downarrow & & \big\downarrow {\scriptstyle h_*(\mathrm{Tr}_{e'})} \\ \boldsymbol{\omega}_{X/Y}[n] & \xrightarrow{\theta_{f,g}} & h_*\boldsymbol{\omega}_{X'/Y'}[n] \end{array} \tag{9.15}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$ *is commutative.*

Proof. Since Z is finite over K it has finitely many points. By restricting to one of the points of Z we can actually assume that Z and X are affine, say $Z = \operatorname{Spec} C$ and $X = \operatorname{Spec} B$. Hence we can also suppose that $Z' = \operatorname{Spec} C'$ and $X' = \operatorname{Spec} B'$. We now have to prove that

$$\theta_{f,g} \circ \mathrm{Tr}_e = \mathrm{Tr}_{e'} \circ \theta_{f\circ e,g} : \boldsymbol{\omega}_{C/K}[m] \to \boldsymbol{\omega}_{B'/K'}[n].$$

Due to rigidity and the fact that $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,B)}(\boldsymbol{\omega}_{C/K}[m], \boldsymbol{\omega}_{B'/K'}[n]) \cong C'$ it follows that these two morphisms are equal. □

Lemma 9.16. *Suppose* $Y = \operatorname{Spec} K$ *where* K *is a field. There exists a closed embedding* $e : Z \to X$ *such that* $f \circ e : Z \to Y$ *is finite* (*hence Cohen-Macaulay of relative dimension* 0)*; and*

$$f_*(\mathrm{Tr}_e) : (f \circ e)_* \boldsymbol{\omega}_{Z/Y} \to \mathrm{R}^n f_*\, \boldsymbol{\omega}_{X/Y}$$

is surjective.

Proof. According to Corollary 6.23 we can assume that $K = \mathbb{K}$. Then $\boldsymbol{\omega}_{X/Y} = \mathrm{H}^{-n}\mathcal{K}_X$, where $\mathcal{K}_X$ is the rigid residue complex of X. The K-module $\mathrm{R}^n f_*\, \boldsymbol{\omega}_{X/Y} = \mathrm{H}^0 f_* \mathcal{K}_X$ is finitely generated, and is a quotient of $f_*\mathcal{K}^0_X$. But $f_*\mathcal{K}^0_X \cong \lim_{\to} (f\circ e)_* \mathcal{K}^0_Z$ as Z runs over the finite length closed subschemes of X. And for any such Z one has $\mathcal{K}^0_Z = \boldsymbol{\omega}_{Z/Y}$. □

Lemma 9.17. *In the setup of Proposition* 9.4*, assume that* f^* *is a Cohen-Macaulay homomorphism of relative dimension* n. *By Propositions* 8.2 *and* 9.4 *there are isomorphisms*

$$\boldsymbol{\omega}_{\bar{B}/\bar{A}}[n] = \bar{f}^{!}\bar{A} \cong f^{!}\bar{A} \cong \bar{A} \otimes^{\mathrm{L}}_A f^{!} A = \bar{A} \otimes_A \boldsymbol{\omega}_{B/A}[n]$$

in $\mathsf{D}(\mathsf{Mod}\,B)$. *Let* $\theta : \boldsymbol{\omega}_{B/A} \to \boldsymbol{\omega}_{\bar{B}/\bar{A}}$ *be the resulting* B*-linear homomorphism. Then* $\theta : \boldsymbol{\omega}_{B/A}[n] \to \boldsymbol{\omega}_{\bar{B}/\bar{A}}[n]$ *is a rigid base change morphism relative to* A*, and consequently* $\theta = \theta_{f,g}$.

Proof. Consider the diagram

$$\begin{array}{ccccccc}
 & & & \overset{\rho_B}{\frown\!\!\!\longrightarrow} & & & \\
R_B & \xrightarrow[\cong]{} & \boldsymbol{\omega}_{B/A} \otimes_A R_A & \xrightarrow{\rho_{B/A}\otimes\rho_A} & (\mathrm{Sq}_{B/A}\, \boldsymbol{\omega}_{B/A}) \otimes^{\mathrm{L}}_A (\mathrm{Sq}_{A/\mathbb{K}}\, R_A) & \xleftarrow[\cong]{} & \mathrm{Sq}_{B/\mathbb{K}}\, R_B \\
\big\uparrow & & \big\uparrow {\scriptstyle \mathbf{1}\otimes \mathrm{Tr}_g} & & \big\uparrow {\scriptstyle \mathbf{1}\otimes \mathrm{Sq}_{g/\mathbb{K}}(\mathrm{Tr}_g)} & & \big\uparrow \\
{\scriptstyle \mathrm{Tr}_h} & & \boldsymbol{\omega}_{B/A} \otimes_A R_{\bar{A}} & \xrightarrow{\rho_{B/A}\otimes\rho_{\bar{A}}} & (\mathrm{Sq}_{B/A}\, \boldsymbol{\omega}_{B/A}) \otimes^{\mathrm{L}}_A (\mathrm{Sq}_{\bar{A}/\mathbb{K}}\, R_{\bar{A}}) & & {\scriptstyle \mathrm{Sq}_{h/\mathbb{K}}(\mathrm{Tr}_h)} \\
\big| & & \big\downarrow {\scriptstyle \theta\otimes\mathbf{1}} & ? & \big\downarrow {\scriptstyle \mathrm{Sq}_{f,g}(\theta)\otimes\mathbf{1}} & & \big| \\
R_{\bar{B}} & \xrightarrow{\cong} & \boldsymbol{\omega}_{\bar{B}/\bar{A}} \otimes_{\bar{A}} R_{\bar{A}} & \xrightarrow{\rho_{\bar{B}/\bar{A}}\otimes\rho_{\bar{A}}} & (\mathrm{Sq}_{\bar{B}/\bar{A}}\, \boldsymbol{\omega}_{\bar{B}/\bar{A}}) \otimes^{\mathrm{L}}_{\bar{A}} (\mathrm{Sq}_{\bar{A}/\mathbb{K}}\, R_{\bar{A}}) & \xleftarrow{\cong} & \mathrm{Sq}_{\bar{B}/\mathbb{K}}\, R_{\bar{B}} \\
 & & & \underset{\rho_{\bar{B}}}{\smile\!\!\!\longrightarrow} & & &
\end{array}$$

of morphisms in $\mathsf{D}(\mathsf{Mod}\, B)$. The subdiagrams on the top and on the bottom (involving ρ_B and $\rho_{\bar{B}}$) are commutative by definition of the rigidifying isomorphisms $\rho_{B/A}$ and $\rho_{\bar{B}/\bar{A}}$. The left rectangle is commutative by definition of θ. This implies that the right rectangle, which is basically obtained from the left rectangle by squaring, is commutative. The middle-upper square commutes because Tr_g is a rigid morphism. The conclusion is that the square marked "?" is commutative, and hence $\mathrm{Sq}_{f,g}(\theta) \circ \rho_{B/A} = \rho_{\bar{B}/\bar{A}} \circ \theta$; i.e. θ is a rigid base change morphism. Due to uniqueness of such morphisms we see that $\theta = \theta_{f,g}$. □

Proof of Theorem 9.12. We proceed in several steps.

Step 1. Suppose $g : Y' \to Y$ is a closed embedding. By Proposition 9.4 we can replace $g_* \, \mathrm{R}^n f'_* \, \boldsymbol{\omega}_{X'/Y'} = g_* \, \mathrm{R}^0 f'_* \, f'^{!} \mathcal{O}_{Y'}$ with $\mathrm{R}^0 f_* \, f^! \, g_* \, \mathcal{O}_{Y'}$, and then, by Lemma 9.17, instead of $\mathrm{R}^n f_*(\theta_{f,g})$ we have the homomorphism

$$\mathrm{R}^0 f_* \, f^!(g^*) : \mathrm{R}^0 f_* \, f^! \mathcal{O}_Y \to \mathrm{R}^0 f_* \, f^! \, g_* \, \mathcal{O}_{Y'},$$

corresponding to the sheaf homomorphism $g^* : \mathcal{O}_Y \to g_* \, \mathcal{O}_{Y'}$. Since the trace map $\mathrm{Tr}_f : \mathrm{R} f_* \, f^! \to \mathbf{1}$ is functorial, it follows that diagram (9.13) commutes in this case.

Step 2. Now suppose $Y = \mathrm{Spec}\, K$ and $Y' = \mathrm{Spec}\, K'$, where K and K' are fields and the homomorphism $K \to K'$ is finite. Also suppose that f is finite. Thus $X = \mathrm{Spec}\, B$ and $X' = \mathrm{Spec}\, B'$, where B is a finite K-algebra, and $B' \cong B \otimes_K K'$. In this situation $\boldsymbol{\omega}_{B/K} = \mathrm{Hom}_K(B, K)$, and the rigid trace $\mathrm{Tr}_{B/K} : \boldsymbol{\omega}_{B/K} \to K$ is evaluation at 1. Likewise for K' and B'. The rigid base change morphism $\theta_{f,g} : \boldsymbol{\omega}_{B/K} \to \boldsymbol{\omega}_{B'/K'}$ relative to K arises from the canonical isomorphism $\mathrm{Hom}_{K'}(B', K') \cong K' \otimes_K \mathrm{Hom}_K(B, K)$. Therefore diagram (9.13) commutes in this case.

Step 3. In this step we assume that $Y = \mathrm{Spec}\, K$ and $Y' = \mathrm{Spec}\, K'$, where K and K' are fields, and the homomorphism $K \to K'$ is finite. Choose a closed embedding $e : Z \to X$ as in Lemma 9.16, and let Z' and e' be as in Lemma 9.14. Let $B := \Gamma(Z, \mathcal{O}_Z)$ and $B' := \Gamma(Z', \mathcal{O}_{Z'})$. Consider the diagram

$$\begin{array}{ccccc}
\boldsymbol{\omega}_{B/K} & \xrightarrow{\mathrm{Tr}_e} & \mathrm{R}^n f_* \, \boldsymbol{\omega}_{X/K} & \xrightarrow{\mathrm{Tr}_f} & K \\
{\scriptstyle \theta_{f \circ e, g}} \downarrow & & \downarrow {\scriptstyle \theta_{f,g}} & & \downarrow {\scriptstyle g^*} \\
\boldsymbol{\omega}_{B'/K'} & \xrightarrow{\mathrm{Tr}_{e'}} & \mathrm{R}^n f'_* \, \boldsymbol{\omega}_{X'/K'} & \xrightarrow{\mathrm{Tr}_{f'}} & K'.
\end{array}$$

By Lemma 9.14 the left square is commutative. And by step 2 above the big rectangle is commutative. Since Tr_e is surjective it follows that the right square also commutes.

Step 4. Assume $Y' = \{y'\} = \mathrm{Spec}\, K'$ where K' is a field. Let $y := g(y') \in Y$. The point y might fail to be closed. However, since we are interested in $\mathcal{O}_Y$-linear homomorphisms, we can replace Y with $\mathrm{Spec}\, \mathcal{O}_{Y,y}$. The only difficulty that may arise is that the $\mathbb{K}$-scheme $\mathrm{Spec}\, \mathcal{O}_{Y,y}$ might not be of finite type. This can be repaired as follows: choose a $\mathbb{K}$-algebra $\tilde{\mathbb{K}}$, which is a localization of a polynomial $\mathbb{K}$-algebra, such that $\tilde{\mathbb{K}} \to \mathcal{O}_{Y,y}$ is finite. Since $\tilde{\mathbb{K}}$ is an essentially finite type $\mathbb{K}$-algebra which is also regular, we can replace $\mathbb{K}$ with $\tilde{\mathbb{K}}$, as explained in Corollary 6.23.

So we now have $y = g(y')$ a closed point of Y. The morphism $g : Y' \to Y$ factors through the finite morphism $Y' \to \mathrm{Spec}\, \boldsymbol{k}(y)$ and the closed embedding $\mathrm{Spec}\, \boldsymbol{k}(y) \to Y$. Combining steps 1 and 2 we conclude that diagram (9.13) commutes.

Step 5. This is the general case. We must show that two $\mathcal{O}_{Y'}$-linear homomorphisms $g^* \, \mathrm{R}^n f_* \, \boldsymbol{\omega}_{X/Y} \to \mathcal{O}_{Y'}$ are equal. It suffices to check that they become equal in the residue field $\boldsymbol{k}(y')$ for any closed point $y' \in Y'$. Using step 1 we can replace Y' with $\operatorname{Spec} \boldsymbol{k}(y')$. Now using step 4 we are done. $\square$

REFERENCES

[AFH] L. Avramov, H.-B. Foxby and S. Halperin, Differential graded homological algebra, in preparation.

[AJL] L. Alonso, A. Jeremías and J. Lipman, Duality and flat base change on formal schemes, in "Studies in Duality on Noetherian Formal Schemes and Non-Noetherian Ordinary Schemes," Contemp. Math. **244**, Amer. Math. Soc., 1999, 3-90.

[Be] K. Behrend, Differential Graded Schemes I, preprint.

[Co] B. Conrad, "Grothendieck Duality and Base Change," Lecture Notes in Math. **1750**, Springer, 2000.

[Dr] V. Drinfeld, DG quotients of DG categories, eprint math.KT/0210114 at http://arxiv.org.

[EG] P. Etingof and V. Ginzburg, Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism. Invent. Math. **147** (2002), no. 2, 243-348.

[EGA] A. Grothendieck and J. Dieudonné, "Éléments de Géometrie Algébrique." Chapitre 0_{IV}, Publ. Math. IHES **20** (1964); Chapitre IV, Publ. Math. IHES **32** (1967).

[Ha] M. Hakim, "Topos Annelés et Schémas Relatifs," Springer, 1972.

[Hi] V. Hinich, Homological algebra of homotopy algebras, Comm. Algebra **25** (1997), no. 10, 3291-3323.

[HK] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. reine angew. Math. **410** (1990), 84-108.

[FIJ] A. Frankild, S. Iyengar and P. Jørgensen, Dualizing Differential Graded modules and Gorenstein Differential Graded Algebras, J. London Math. Soc. (2) 68 (2003), 288-306.

[Ke] B. Keller, Deriving DG categories, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63-102.

[KKO] A. Kapustin, A. Kuznetsov and D. Orlov, Noncommutative Instantons and Twistor Transform, Comm. Math. Phys. **221** (2001), 385-432.

[Kl] S.L. Kleiman, Relative duality for quasi-coherent sheaves, Compositio Math. **41** (1980), no. 1, 39-60.

[KM] I. Kriz and J.P. May, "Operads, Algebras, Modules and Motives," Astérisque **233** (1995).

[KS] M. Kashiwara and P. Schapira, "Sheaves on Manifolds," Springer-Verlag, Berlin, 1990.

[Li] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Astérisque **117** (1984).

[LMB] G. Laumon and L. Moret-Bailly, "Champs algebriques," Springer, 2000.

[LNS] J. Lipman, S. Nayak and P. Sastry, Pseudofunctorial behavior of Cousin complexes on formal schemes, preprint; eprint math.AG/0310032 at http://arxiv.org.

[Ma] H. Matsumura, "Commutative Algebra", Benjamin/Cummings, Reading, Mass., 1980.

[ML] S. Maclane, "Homology," Springer-Verlag, 1994 (reprint).

[Ne] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. **9** (1996), no. 1, 205-236.

[Qu] D. Quillen, On the cohomology of commutative rings, Proc. Symp. Pure Math. (AMS) **17** (1970), pp. 65-87.

[RD] R. Hartshorne, "Residues and Duality," Lecture Notes in Math. **20**, Springer-Verlag, Berlin, 1966.

[Sa] P. Sastry, Residues and duality on algebraic schemes, Compositio Math. **101** (1996), 133-178.

[SdS] F. Sancho de Salas, Residue: a geometric construction, Canad. Math. Bull. **45** (2002), no. 2, 284-293.

[Sp] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. **65** (1988), no. 2, 121-154.

[VdB] M. Van den Bergh, Existence theorems for dualizing complexes over non-commutative graded and filtered ring, J. Algebra **195** (1997), no. 2, 662-679.

[Ve] J.L. Verdier, Base change for twisted inverse image of coherent sheaves, "Algebraic Geometry," Oxford Univ. Press, 1969, pp. 393 - 408.

[Ye1] A. Yekutieli, Dualizing Complexes over Noncommutative Graded Algebras, J. Algebra **153** no. 1 (1992), 41-84.

[Ye2] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[Ye3] A. Yekutieli, Residues and differential operators on schemes, Duke Math. J. **95** (1998), 305-341.

[YZ1] A. Yekutieli and J.J. Zhang, Rings with Auslander dualizing complexes, J. Algebra **213** (1999), no. 1, 1-51.

[YZ2] A. Yekutieli and J.J. Zhang, Residue complexes over noncommutative rings, J. Algebra **259** (2003) no. 2, 451-493.

[YZ3] A. Yekutieli and J.J. Zhang, Dualizing complexes and perverse modules over differential algebras, to appear in Compositio Math.; eprint math.RA/0301323 at http://arxiv.org.

[YZ4] A. Yekutieli and J.J. Zhang, Dualizing complexes and perverse sheaves on noncommutative ringed schemes, eprint math.AG/0211309 at http://arxiv.org.

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