Rigid dualizing complexes via differential graded algebras (survey)

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ABSTRACT. In this article we survey recent results on rigid dualizing complexes over commutative algebras. We begin by recalling what are dualizing complexes. Next we define rigid complexes, and explain their functorial properties. Due to the possible presence of torsion, we must use differential graded algebras in the constructions. We then discuss rigid dualizing complexes. Finally we show how rigid complexes can be used to understand Cohen-Macaulay homomorphisms and relative dualizing sheaves.

0. Introduction

This short article is based on a lecture I gave at the "Workshop on Triangulated Categories", Leeds, August 2006. It is a survey of recent results on rigid dualizing complexes over commutative rings. Most of these results are joint work of mine with James Zhang. The idea of rigid dualizing complex is due to Michel Van den Bergh.

By default all rings considered in this article are *commutative*. We begin by recalling the notion of *dualizing complex* over a noetherian ring A. Next let B be a noetherian A-algebra. We define what is a *rigid complex* of B-modules relative to A. In making this definition we must use differential graded algebras (when B is not flat over A). The functorial properties of rigid complexes are explained. We then discuss *rigid dualizing complexes*, which by definition are complexes that are both rigid and dualizing. Finally we show how rigid complexes can be used to understand Cohen-Macaulay homomorphisms and relative dualizing sheaves.

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1. Dualizing Complexes: Overview

Let A be a noetherian ring. Denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ the derived category of bounded complexes of A-modules with finitely generated cohomology modules.

Definition 1.1. (Grothendieck [RD]) A *dualizing complex* over A is a complex $R \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ satisfying the two conditions:

(i) R has finite injective dimension.
(ii) The canonical morphism $A \to \mathrm{RHom}_A(R, R)$ is an isomorphism.

Condition (i) means that there is an integer d such that $\mathrm{Ext}^i_A(M, R) = 0$ for all $i > d$ and all modules M.

Recall that a noetherian ring $\mathbb{K}$ is called *regular* if all its local rings $\mathbb{K}_{\mathfrak{p}}$, $\mathfrak{p} \in \mathrm{Spec}\, \mathbb{K}$, are regular local rings.

Example 1.2. If $\mathbb{K}$ is a regular noetherian ring of finite Krull dimension (say a field, or the ring of integers $\mathbb{Z}$) then

$$R := \mathbb{K} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, \mathbb{K})$$

is a dualizing complex over $\mathbb{K}$.

Dualizing complexes over commutative rings are part of Grothendieck's duality theory in algebraic geometry, which was developed in [RD]. This duality theory deals with dualizing complexes on schemes and relations between them. See Remark 4.4.

In Section 4 we explain a new approach to dualizing complexes over commutative rings, due to James Zhang and the author (see [YZ4] and [YZ5]). Specifically, we discuss existence and uniqueness of *rigid dualizing complexes*.

A dualizing complex R has many automorphisms; indeed, its group of automorphisms in $\mathsf{D}(\mathsf{Mod}\, A)$ is the group $A^\times$ of invertible elements. The purpose of rigidity is to eliminate automorphisms, and to make dualizing complexes functorial. See Theorem 4.2.

In a sequel paper [Ye2] we use the technique of *perverse coherent sheaves* to construct rigid dualizing complexes on schemes, and we reproduce almost all of the geometric Grothendieck duality theory.

Related work in noncommutative algebraic geometry (where rigid dualizing complexes were first introduced) can be found in [VdB, YZ1, YZ2, YZ3].

2. Rigid Complexes and DG Algebras

Let me start with a discussion of rigidity for algebras over a field. Suppose $\mathbb{K}$ is a field, B is a $\mathbb{K}$-algebra, and $M \in \mathsf{D}(\mathsf{Mod}\, B)$.

According to Van den Bergh [VdB] a *rigidifying isomorphism* for M is an isomorphism

$$\rho : M \xrightarrow{\simeq} \mathrm{RHom}_{B \otimes_{\mathbb{K}} B}(B, M \otimes_{\mathbb{K}} M) \tag{2.1}$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

Now suppose A is any ring. Trying to write A instead of $\mathbb{K}$ in formula (2.1) does not make sense: instead of $M \otimes_A M$ we must take the derived tensor product $M \otimes^{\mathrm{L}}_A M$; but then there is no obvious way to make $M \otimes^{\mathrm{L}}_A M$ into a complex of $B \otimes_A B$ -modules.

The problem is torsion: B might fail to be a flat A-algebra. This is where *differential graded algebras* (DG algebras) enter the picture.

A DG algebra is a graded ring $\tilde{A} = \bigoplus_{i \in \mathbb{Z}} \tilde{A}^i$, together with a graded derivation $\mathrm{d} : \tilde{A} \to \tilde{A}$ of degree 1, satisfying $\mathrm{d} \circ \mathrm{d} = 0$.

A DG algebra quasi-isomorphism is a homomorphism $f : \tilde{A} \to \tilde{B}$ respecting degrees, multiplications and differentials, and such that $\mathrm{H}(f) : \mathrm{H}\tilde{A} \to \mathrm{H}\tilde{B}$ is an isomorphism (of graded algebras).

We shall only consider *super-commutative non-positive* DG algebras. Super-commutative means that $ab = (-1)^{ij} ba$ and $c^2 = 0$ for all $a \in \tilde{A}^i$, $b \in \tilde{A}^j$ and $c \in \tilde{A}^{2i+1}$. Non-positive means that $\tilde{A} = \bigoplus_{i \leq 0} \tilde{A}^i$.

We view a ring A as a DG algebra concentrated in degree 0. Given a DG algebra homomorphism $A \to \tilde{A}$ we say that $\tilde{A}$ is a DG A-algebra.

Let A be a ring. A *semi-free* DG A-algebra is a DG A-algebra $\tilde{A}$, such that after forgetting the differential $\tilde{A}$ is isomorphic, as graded A-algebra, to a super-polynomial algebra on some graded set of variables.

Definition 2.2. Let A be a ring and B an A-algebra. A *semi-free DG algebra resolution of B relative to A* is a quasi-isomorphism $\tilde{B} \to B$ of DG A-algebras, where $\tilde{B}$ is a semi-free DG A-algebra.

Such resolutions always exist, and they are unique up to quasi-isomorphism.

Example 2.3. Take $A = \mathbb{Z}$ and $B = \mathbb{Z}/(6)$. Define $\tilde{B}$ to be the super-polynomial algebra $A[\xi]$ on the variable ξ of degree -1. So $\tilde{B} = A \oplus A\xi$ as free graded A-module, and $\xi^2 = 0$. Let $\mathrm{d}(\xi) := 6$. Then $\tilde{B} \to B$ is a semi-free DG algebra resolution of B relative to A.

For a DG algebra $\tilde{A}$ one has the category $\mathsf{DGMod}\, \tilde{A}$ of DG $\tilde{A}$-modules. It is analogous to the category of complexes of modules over a ring, and by a similar process of inverting quasi-isomorphisms we obtain the derived category $\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$; see [Ke], [Hi].

For a ring A (i.e. a DG algebra concentrated in degree 0) we have

$$\tilde{\mathsf{D}}(\mathsf{DGMod}\,A) = \mathsf{D}(\mathsf{Mod}\,A),$$

the usual derived category.

It is possible to derive functors of DG modules, again in analogy to $\mathsf{D}(\mathsf{Mod}\,A)$. An added feature is that for a quasi-isomorphism $\tilde{A} \to \tilde{B}$, the restriction of scalars functor

$$\tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{B}) \to \tilde{\mathsf{D}}(\mathsf{DGMod}\,\tilde{A})$$

is an equivalence.

Getting back to our original problem, suppose A is a ring and B is an A-algebra. Choose a semi-free DG algebra resolution $\tilde{B} \to B$ relative to A. For $M \in \mathsf{D}(\mathsf{Mod}\,B)$ define

$$\mathrm{Sq}_{B/A}\, M := \mathrm{RHom}_{\tilde{B}\otimes_A \tilde{B}}(B, M \otimes^{\mathrm{L}}_A M)$$

in $\mathsf{D}(\mathsf{Mod}\,B)$.

Theorem 2.4. ([YZ4]) *The functor*

$$\mathrm{Sq}_{B/A} : \mathsf{D}(\mathsf{Mod}\,B) \to \mathsf{D}(\mathsf{Mod}\,B)$$

is independent of the resolution $\tilde{B} \to B$.

The functor $\mathrm{Sq}_{B/A}$, called the *squaring operation*, is nonlinear. In fact, given a morphism $\phi : M \to M$ in $\mathsf{D}(\mathsf{Mod}\,B)$ and an element $b \in B$ one has

$$\mathrm{Sq}_{B/A}(b\phi) = b^2\, \mathrm{Sq}_{B/A}(\phi) \tag{2.5}$$

in

$$\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,B)}(\mathrm{Sq}_{B/A}\, M, \mathrm{Sq}_{B/A}\, M).$$

Definition 2.6. Let B be a noetherian A-algebra, and let M be a complex in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,B)$ that has finite flat dimension over A. Assume

$$\rho : M \xrightarrow{\simeq} \mathrm{Sq}_{B/A}\, M$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,B)$. Then the pair (M, ρ) is called a *rigid complex over B relative to A*.

Definition 2.7. Say (M, ρ) and (N, σ) are rigid complexes over B relative to A. A morphism $\phi : M \to N$ in $\mathsf{D}(\mathsf{Mod}\,B)$ is called a *rigid morphism relative* *to* A if the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \mathrm{Sq}_{B/A}\, M \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \mathrm{Sq}_{B/A}(\phi)} \\ N & \xrightarrow{\sigma} & \mathrm{Sq}_{B/A}\, N \end{array}$$

is commutative.

We denote by $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ the category of rigid complexes over B relative to A.

Example 2.8. Take $M = B = A$. Then

$$\mathrm{Sq}_{A/A}\, A = \mathrm{RHom}_{A \otimes_A A}(A, A \otimes_A A) = A,$$

and we interpret this as a rigidifying isomorphism

$$\rho^{\mathrm{tau}} : A \xrightarrow{\simeq} \mathrm{Sq}_{A/A}\, A.$$

The *tautological rigid complex* is

$$(A, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)_{\mathrm{rig}/A}.$$

3. Properties of Rigid Complexes

The first property of rigid complexes explains their name.

Theorem 3.1. ([YZ4]) *Let A be a ring, B a noetherian A-algebra, and*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

Assume the canonical ring homomorphism

$$B \to \mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\, B)}(M, M)$$

is bijective. Then the only automorphism of (M, ρ) in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}$ is the identity $\mathbf{1}_M$.

The proof is very easy: an automorphism ϕ of M has to be of the form $\phi = b\, \mathbf{1}_M$ for some invertible element $b \in B$. If ϕ is rigid then $b = b^2$ (cf. formula (2.5)), and hence $b = 1$.

We find it convenient to denote ring homomorphisms by f^* etc. Thus a ring homomorphism $f^* : A \to B$ corresponds to the morphism of schemes

$$f : \mathrm{Spec}\, B \to \mathrm{Spec}\, A.$$

Let A be a noetherian ring. Recall that an A-algebra B is called *essentially finite type* if it is a localization of some finitely generated A-algebra. We say that B is *essentially smooth* (resp. *essentially étale*) over A if it is essentially finite type and formally smooth (resp. formally étale).

Example 3.2. If A' is a localization of A then $A \to A'$ is essentially étale. If $B = A[t_1, \dots, t_n]$ is a polynomial algebra then $A \to B$ is smooth, and hence also essentially smooth.

Let A be a noetherian ring and $f^* : A \to B$ an essentially smooth homomorphism. Then $\Omega^1_{B/A}$ is a finitely generated projective B-module. Let

$$\operatorname{Spec} B = \coprod_i \operatorname{Spec} B_i$$

be the decomposition into connected components, and for every i let n_i be the rank of $\Omega^1_{B_i/A}$. We define a functor

$$f^\sharp : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\sharp M := \bigoplus_i \Omega^{n_i}_{B_i/A}[n_i] \otimes_A M.$$

Recall that a ring homomorphism $f^* : A \to B$ is called *finite* if B is a finitely generated A-module. Given such a finite homomorphism we define a functor

$$f^\flat : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B)$$

by

$$f^\flat M := \mathrm{RHom}_A(B, M).$$

Theorem 3.3. ([YZ4]) *Let A be a noetherian ring, let B, C be essentially finite type A-algebras, let $f^* : B \to C$ be an A-algebra homomorphism, and let*

$$(M, \rho) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)_{\mathrm{rig}/A}.$$

(1) *If f^* is finite and $f^\flat M$ has finite flat dimension over A, then $f^\flat M$ has an induced rigidifying isomorphism*

$$f^\flat(\rho) : f^\flat M \xrightarrow{\simeq} \mathrm{Sq}_{C/A}\, f^\flat M.$$

The assignment

$$(M, \rho) \mapsto f^\flat(M, \rho) := \big(f^\flat(\rho), f^\flat M\big)$$

is functorial.

(2) *If f^* is essentially smooth then $f^\sharp M$ has an induced rigidifying isomorphism*

$$f^\sharp(\rho) : f^\sharp M \xrightarrow{\simeq} \mathrm{Sq}_{C/A} f^\sharp M.$$

The assignment

$$(M, \rho) \mapsto f^\sharp(M, \rho) := \big(f^\sharp(\rho), f^\sharp M\big)$$

is functorial.

4. Rigid Dualizing Complexes

Let $\mathbb{K}$ be a regular noetherian ring of finite Krull dimension. We denote by $\mathsf{EFTAlg}\,/\mathbb{K}$ the category of essentially finite type $\mathbb{K}$-algebras.

Definition 4.1. A *rigid dualizing complex* over A relative to $\mathbb{K}$ is a rigid complex (R_A, ρ_A), such that R_A is a dualizing complex.

Theorem 4.2. ([YZ5]) *Let $\mathbb{K}$ be a regular finite dimensional noetherian ring, and let A be an essentially finite type $\mathbb{K}$-algebra.*

(1) *The algebra A has a rigid dualizing complex (R_A, ρ_A), which is unique up to a unique rigid isomorphism.*
(2) *Given a finite homomorphism $f^* : A \to B$, there is a unique rigid isomorphism $f^\flat(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.*
(3) *Given an essentially smooth homomorphism $f^* : A \to B$, there is a unique rigid isomorphism $f^\sharp(R_A, \rho_A) \xrightarrow{\simeq} (R_B, \rho_B)$.*

Here is how the rigid dualizing complex (R_A, ρ_A) is obtained. We begin with the tautological rigid complex

$$(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,\mathbb{K})_{\mathrm{rig}/\mathbb{K}},$$

which is dualizing (cf. Examples 1.2 and 2.8). Now the structural homomorphism $\mathbb{K} \to A$ can be factored into

$$\mathbb{K} \xrightarrow{f^*} B \xrightarrow{g^*} C \xrightarrow{h^*} A,$$

where f^* is essentially smooth (B is a polynomial algebra over $\mathbb{K}$); g^* is finite (a surjection); and h^* is also essentially smooth (a localization).

It is not hard to check (see [RD, Chapter V]) that each of the complexes $f^\sharp\mathbb{K}$, $g^\flat f^\sharp\mathbb{K}$ and $h^\sharp g^\flat f^\sharp\mathbb{K}$ is dualizing over the respective ring. In particular, $g^\flat f^\sharp\mathbb{K}$ has bounded cohomology, and hence it has finite flat dimension over $\mathbb{K}$.

According to Theorem 3.3 we then have a rigid complex

$$(R_A, \rho_A) := h^{\sharp}\, g^{\flat}\, f^{\sharp}(\mathbb{K}, \rho^{\mathrm{tau}}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)_{\mathrm{rig}/\mathbb{K}}.$$

Definition 4.3. Given a homomorphism $f^* : A \to B$ in $\mathsf{EFTAlg}\,/\mathbb{K}$, define the *twisted inverse image functor*

$$f^! : \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{+}_{\mathrm{f}}(\mathsf{Mod}\, B)$$

by the formula

$$f^! M := \mathrm{RHom}_B\big(B \otimes^{\mathrm{L}}_A \mathrm{RHom}_A(M, R_A), R_B\big).$$

It is easy to show that the assignment $f^* \mapsto f^!$ is a pseudofunctor from the category $\mathsf{EFTAlg}\,/\mathbb{K}$ to the 2-category Cat of all categories. Moreover, using Theorem 4.2 one can show that this operation has very good properties. For instance, when f^* is finite, then there is a functorial nondegenerate trace morphism

$$\mathrm{Tr}_f : f^! M \to M.$$

Remark 4.4. According to Grothendieck's duality theory in [RD], if $f : X \to Y$ is a finite type morphism between noetherian schemes, and if Y has a dualizing complex, then there is a functor

$$f^{!(\mathrm{G})} : \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_Y) \to \mathsf{D}^{+}_{\mathrm{c}}(\mathsf{Mod}\, \mathcal{O}_X),$$

with many good properties.

Let $\mathsf{FTAlg}\,/\mathbb{K}$ be the category of finite type $\mathbb{K}$-algebras. By restricting attention to affine schemes, the results of [RD] give rise to a pseudofunctor $f^* \mapsto f^{!(\mathrm{G})}$ from $\mathsf{FTAlg}\,/\mathbb{K}$ to Cat. It is not hard to show that the pseudofunctor $f^* \mapsto f^{!(\mathrm{G})}$ is isomorphic to our 2-functor $f^* \mapsto f^!$; see [YZ5, Theorem 4.10].

It should be noted that our construction works in the slightly bigger category $\mathsf{EFTAlg}\,/\mathbb{K}$. It also has the advantage of being local; whereas in [RD] some of the results require that morphisms between affine schemes be compactified.

5. Rigid Complexes and CM Homomorphisms

In this final section we discuss the relation between rigid complexes and Cohen-Macaulay homomorphisms.

Definition 5.1. A ring A is called *tractable* if there is an essentially finite type homomorphism $\mathbb{K} \to A$, for some regular noetherian ring of finite Krull dimension $\mathbb{K}$.

Such a homomorphism $\mathbb{K} \to A$ is called a *traction* for A. It is not part of the structure – the ring A does not come with any preferred traction. "Most commutative noetherian rings we know" are tractable.

Given a traction $\mathbb{K} \to A$ we denote by $R_{A/\mathbb{K}}$ the rigid dualizing complex of A relative to $\mathbb{K}$; cf. Theorem 4.2. (The rigidifying isomorphism $\rho_{A/\mathbb{K}}$ is implicit.)

Recall that a noetherian ring A is called *Cohen-Macaulay* (resp. *Gorenstein*) if all its local rings $A_{\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{Spec} A$, are Cohen-Macaulay (resp. Gorenstein) local rings. The implications are regular $\Rightarrow$ Gorenstein $\Rightarrow$ Cohen-Macaulay.

Let $f^* : A \to B$ be a ring homomorphism. For $\mathfrak{p} \in \operatorname{Spec} A$ let $\boldsymbol{k}(\mathfrak{p}) := (A/\mathfrak{p})_{\mathfrak{p}}$, the residue field. The fiber of f^* above $\mathfrak{p}$ is the $\boldsymbol{k}(\mathfrak{p})$-algebra $B \otimes_A \boldsymbol{k}(\mathfrak{p})$. Now assume f^* is an essentially finite type flat homomorphism. If all the fibers of f^* are Cohen-Macaulay (resp. Gorenstein) rings, then we call f^* an *essentially Cohen-Macaulay* (resp. *essentially Gorenstein*) homomorphism.

Theorem 5.2. ([Ye2]) *Let A be a tractable ring, and let $f^* : A \to B$ be homomorphism which is of essentially finite type and of finite flat dimension. Then there exists a rigid complex $R_{B/A}$ over B relative to A, unique up to a unique rigid isomorphism, with the following property:*

(*) *Let $\mathbb{K} \to A$ be some traction. Then*

$$R_{A/\mathbb{K}} \otimes^{\mathrm{L}}_A R_{B/A} \cong R_{B/\mathbb{K}}$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

Condition (*) implies that the support of the complex $R_{B/A}$ is $\operatorname{Spec} B$. One can prove that

$$f^! M \cong R_{B/A} \otimes^{\mathrm{L}}_A M$$

for $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

If the ring A is Gorenstein, then $R_{A/\mathbb{K}}$ is a shift of an invertible A-module. Hence:

Corollary 5.3. *Assume that in Theorem* 5.2 *the ring A is Gorenstein. Then $R_{B/A}$ is a dualizing complex over B*

The rigid complex $R_{B/A}$ allows us to characterize Cohen-Macaulay homomorphisms, as follows.

Theorem 5.4. ([Ye2]) *Let A be a tractable ring, and let $f^* : A \to B$ be an essentially finite type flat homomorphism. Then the following conditions are equivalent:*

(i) *f^* is an essentially Cohen-Macaulay homomorphism.*
(ii) *Let*

$$\operatorname{Spec} B = \coprod_i \operatorname{Spec} B_i$$

be the decomposition into connected components. Then for any i there is a finitely generated B_i-module $\boldsymbol{\omega}_{B_i/A}$, which is flat over A, and an integer n_i, such that

$$R_{B/A} \cong \bigoplus_i \boldsymbol{\omega}_{B_i/A}[n_i]$$

in $\mathsf{D}(\mathsf{Mod}\, B)$.

The module

$$\boldsymbol{\omega}_{B/A} := \bigoplus_i \boldsymbol{\omega}_{B_i/A}$$

is called the *relative dualizing module* of $f^* : A \to B$. Note that the complex $\bigoplus_i \boldsymbol{\omega}_{B_i/A}[n_i]$ is rigid, but in general it is not a dualizing complex over B. Still the fibers of $\bigoplus_i \boldsymbol{\omega}_{B_i/A}[n_i]$ are dualizing complexes – this can be seen by taking $A' = \boldsymbol{k}(\mathfrak{p})$ in the next result, and using Corollary 5.3

Here is a "rigid" version of Conrad's base change theorem [Co].

Theorem 5.5. ([Ye2]) *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array}$$

be a cartesian diagram of rings, i.e.

$$B' \cong A' \otimes_A B,$$

with A and A' tractable rings. Assume $A \to B$ is an essentially Cohen-Macaulay homomorphism. (There isn't any restriction on the homomorphism $A \to A'$.) Then:

(1) *$A' \to B'$ is an essentially Cohen-Macaulay homomorphism.*
(2) *There is a unique isomorphism of B'-modules*

$$\boldsymbol{\omega}_{B'/A'} \cong A' \otimes_A \boldsymbol{\omega}_{B/A}$$

which respects rigidity.

From this we can easily deduce the next result.

Corollary 5.6. *Let A be a tractable ring, and let $f^* : A \to B$ be an essentially Cohen-Macaulay homomorphism. Then the following conditions are equivalent:*

(i) *f^* is an essentially Gorenstein homomorphism.*
(ii) *$\boldsymbol{\omega}_{B/A}$ is an invertible B-module.*

Remark 5.7. The recent paper [AI] contains results similar to Theorem 5.4 and Corollary 5.6, obtained by different methods, and without the requirement that A is tractable.

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