

# DEFORMATION QUANTIZATION IN ALGEBRAIC GEOMETRY

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*Dedicated to Professor Michael Artin on the Occasion of his Seventieth Birthday*

ABSTRACT. We study deformation quantizations of the structure sheaf  $\mathcal{O}_X$  of a smooth algebraic variety  $X$  in characteristic 0. Our main result is that when  $X$  is  $\mathcal{D}$ -affine, any formal Poisson structure on  $X$  determines a deformation quantization of  $\mathcal{O}_X$  (canonically, up to gauge equivalence). This is an algebro-geometric analogue of Kontsevich's celebrated result.

## 0. INTRODUCTION

This article began with an attempt to understand the work of Kontsevich [Ko1, Ko3], Cattaneo-Felder-Tomassini [CFT] and Nest-Tsygan [NT] on deformation quantization of Poisson manifolds. Moreover we tried to see to what extent the methods applied in the case of  $C^\infty$  manifolds can be carried over to the algebro-geometric case.

If  $X$  is a  $C^\infty$  manifold with Poisson structure  $\alpha$  then there is always a deformation quantization of the algebra of functions  $C^\infty(X)$  with first order term  $\alpha$ . This was proved by Kontsevich in [Ko1]. Furthermore, Kontsevich proved that such a deformation quantization is unique in a suitable sense.

If  $X$  is either a complex analytic manifold or a smooth algebraic variety then one wants to deform the sheaf of functions  $\mathcal{O}_X$ . As might be expected there are potential obstructions, due to the lack of global (analytic or algebraic) functions and sections of bundles. The case of a complex analytic manifold with holomorphic symplectic structure was treated in [NT]. The algebraic case was studied in [Ko3], where several approaches were discussed. In the present paper we take a somewhat different direction than [Ko3].

First let us explain what we mean by deformation quantization in the context of algebraic geometry. Let  $\mathbb{K}$  be a field of characteristic 0, and let  $X$  be a smooth algebraic variety over  $\mathbb{K}$ . The tangent sheaf of  $X$  is denoted by  $\mathcal{T}_X$ . Given an element  $\alpha \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$  let  $\{-, -\}_\alpha$  be the  $\mathbb{K}$ -bilinear sheaf morphism  $\mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$  defined by  $\{f, g\}_\alpha := \langle \alpha, d(f) \wedge d(g) \rangle$  for local sections  $f, g \in \mathcal{O}_X$ . If  $\{-, -\}_\alpha$  is a Lie bracket on  $\mathcal{O}_X$  then it is called a *Poisson bracket*,  $\alpha$  is called a *Poisson structure on  $X$* , and the pair  $(X, \alpha)$  is called a *Poisson variety*. It is known that  $\alpha$  is a Poisson structure if and only if  $[\alpha, \alpha] = 0$  for the Schouten-Nijenhuis bracket.

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Let  $\hbar$  be an indeterminate (the ‘‘Planck constant’’). A *star product* on  $\mathcal{O}_X[[\hbar]]$  is a  $\mathbb{K}[[\hbar]]$ -bilinear sheaf morphism

$$\star : \mathcal{O}_X[[\hbar]] \times \mathcal{O}_X[[\hbar]] \rightarrow \mathcal{O}_X[[\hbar]]$$

which makes  $\mathcal{O}_X[[\hbar]]$  into a sheaf of associative unital  $\mathbb{K}[[\hbar]]$ -algebras. The unit element for  $\star$  has to be  $1 \in \mathcal{O}_X$ , and for any local sections  $f, g \in \mathcal{O}_X$  their product should satisfy  $f \star g \equiv fg \pmod{\hbar}$ . Furthermore there is a differential condition: there should be a sequence of bi-differential operators  $\beta_j : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$ , such that

$$f \star g = fg + \sum_{j=1}^{\infty} \beta_j(f, g) \hbar^j \in \mathcal{O}_X[[\hbar]].$$

A *deformation quantization* of  $\mathcal{O}_X$  is by definition a star product on  $\mathcal{O}_X[[\hbar]]$ .

Actually there is a more refined notion of deformation quantization, which has a local nature; see Section 1. In the body of the paper the deformation defined in the previous paragraph is referred to as a *globally trivialized deformation quantization*. However, according to Theorem 1.13, if  $H^1(X, \mathcal{D}_X) = 0$  then any deformation quantization is equivalent to a globally trivialized one. So for the purpose of the introduction (cf. Theorem 0.1 below) we might as well consider only globally trivialized deformation quantizations.

Suppose  $\star$  is some star product on  $\mathcal{O}_X[[\hbar]]$ . Given two local sections  $f, g \in \mathcal{O}_X$  define  $\{f, g\}_\star \in \mathcal{O}_X$  to be the unique local section satisfying

$$\hbar \{f, g\}_\star \equiv f \star g - g \star f \pmod{\hbar^2}.$$

This is a Poisson bracket on  $\mathcal{O}_X$ . Note that  $\{f, g\}_\star = \beta_1(f, g) - \beta_1(g, f)$ .

Let  $(X, \alpha)$  be a Poisson variety. A *deformation quantization of  $(X, \alpha)$*  is a deformation quantization  $\star$  such that  $\{f, g\}_\star = 2\{f, g\}_\alpha$ .

There is an obvious notion of *gauge equivalence* for deformation quantizations. First we need to define what is a gauge equivalence of  $\mathcal{O}_X[[\hbar]]$ . This is a  $\mathbb{K}[[\hbar]]$ -linear sheaf automorphism  $\gamma : \mathcal{O}_X[[\hbar]] \xrightarrow{\sim} \mathcal{O}_X[[\hbar]]$  of the following form: there is a sequence  $\gamma_j : \mathcal{O}_X \rightarrow \mathcal{O}_X$  of differential operators, such that

$$\gamma(f) = f + \sum_{j=1}^{\infty} \gamma_j(f) \hbar^j$$

for all  $f \in \mathcal{O}_X$ ; and also  $\gamma(1) = 1$ . Two star products  $\star$  and  $\star'$  on  $\mathcal{O}_X[[\hbar]]$  are said to be gauge equivalent if there is some gauge equivalence  $\gamma$  such that

$$f \star' g = \gamma^{-1}(\gamma(f) \star \gamma(g))$$

for all  $f, g \in \mathcal{O}_X$ .

To state the main result of our paper we need the notion of formal Poisson structure on  $X$ . This is a series  $\alpha = \sum_{k=1}^{\infty} \alpha_k \hbar^k \in \Gamma(X, \bigwedge_{\mathcal{O}_X}^2 \mathcal{T}_X)[[\hbar]]$  satisfying  $[\alpha, \alpha] = 0$ . For instance, if  $\alpha_1$  is a Poisson structure then  $\alpha := \alpha_1 \hbar$  is a formal Poisson structure. Two formal Poisson structure  $\alpha$  and  $\alpha'$  are called gauge equivalent if there is some  $\gamma = \sum_{k=1}^{\infty} \gamma_k \hbar^k \in \Gamma(X, \mathcal{T}_X)[[\hbar]]$  such that  $\alpha' = \exp(\text{ad}(\gamma))(\alpha)$ .

Recall that the variety  $X$  is said to be  *$\mathcal{D}$ -affine* if  $H^i(X, \mathcal{M}) = 0$  for all quasi-coherent left  $\mathcal{D}_X$ -modules  $\mathcal{M}$  and all  $i > 0$ . Here  $\mathcal{D}_X$  is the sheaf of differential operators on  $X$ .

**Theorem 0.1.** *Let  $X$  be a smooth algebraic variety over the field  $\mathbb{K}$ . Assume  $X$  is  $\mathcal{D}$ -affine and  $\mathbb{R} \subset \mathbb{K}$ . Then there is a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

*called the quantization map. The map  $Q$  preserves first order terms, and commutes with étale morphisms  $X' \rightarrow X$ . If  $X$  is affine then  $Q$  is bijective. There is an explicit formula for  $Q$ .*

This is an algebraic analogue of [Ko1, Theorem 1.3]. Theorem 0.1 is repeated as Corollaries 7.12 and 7.13 in the body of the paper. Full details, including the explicit formula for the quantization map  $Q$ , are in Theorem 7.7. By “preserving first order terms” we mean that given a formal Poisson structure  $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$  and associated deformation quantization  $Q(\alpha) = \star$ , then  $\{-, -\}_{\star} = 2\{-, -\}_{\alpha_1}$ . If  $f : X' \rightarrow X$  is an étale morphism then any formal Poisson structure  $\alpha$  on  $X$  can be pulled back to a formal Poisson structure  $f^*(\alpha)$  on  $X'$ ; and likewise any deformation quantization  $\star$  on  $X$  can be pulled back to a deformation quantization  $f^*(\star)$  on  $X'$ . The third assertion in Theorem 0.1 says that if  $X'$  is also  $\mathcal{D}$ -affine then  $Q(f^*(\alpha)) = f^*(Q(\alpha))$ .

There are two important classes of varieties satisfying the conditions of Theorem 0.1. The first consists of all smooth affine varieties. Note that even if  $X$  is affine, yet does not admit an étale morphism  $X \rightarrow \mathbf{A}_{\mathbb{K}}^n$ , the result is not trivial – since changes of coordinates have to be accounted for (cf. Corollary 3.24).

The second class of examples is that of the flag varieties  $X = G/P$ , where  $G$  is a connected reductive algebraic group and  $P$  is a parabolic subgroup. By the Beilinson-Bernstein Theorem the variety  $X$  is  $\mathcal{D}$ -affine. This class of varieties includes the projective spaces  $\mathbf{P}_{\mathbb{K}}^n$ .

Here is an outline of the paper (with some of the features simplified). There are two important sheaves of DG Lie algebras on  $X$ : the sheaf of poly vector fields  $\mathcal{T}_{\text{poly}, X}$ , and the sheaf of poly differential operators  $\mathcal{D}_{\text{poly}, X}$  (see Section 3). Their global sections  $\mathcal{T}_{\text{poly}}(X) := \Gamma(X, \mathcal{T}_{\text{poly}, X})$  and  $\mathcal{D}_{\text{poly}}(X) := \Gamma(X, \mathcal{D}_{\text{poly}, X})$  control Poisson structures and deformation quantizations respectively. If one could find an  $L_{\infty}$  quasi-isomorphism  $\mathcal{T}_{\text{poly}}(X) \rightarrow \mathcal{D}_{\text{poly}}(X)$  this would imply Theorem 0.1. However, unless  $X$  is affine and admits an étale morphism to  $\mathbf{A}_{\mathbb{K}}^n$ , there is no reason why such a quasi-isomorphism should exist.

Imitating Fedosov [Fe] and Kontsevich [Ko1], we use formal geometry to solve the global problem. The adaptation of this theory to algebraic geometry is done in Section 4. There is an infinite dimensional bundle  $\pi : \text{LCC } X \rightarrow X$ , which parameterizes formal coordinate systems on  $X$  modulo linear change of coordinates. (In [Ko1] the notation for this bundle is  $X^{\text{aff}}$ .) Let  $\mathcal{P}_X$  be the sheaf of principal parts on  $X$ . The complete pullbacks  $\pi^{\widehat{\cdot}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly}, X})$  and  $\pi^{\widehat{\cdot}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly}, X})$  are sheaves of DG Lie algebras on  $\text{LCC } X$  (see Section 5). The universal deformation formulas of Kontsevich give rise to an  $L_{\infty}$  quasi-isomorphism

$$\pi^{\widehat{\cdot}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly}, X}) \rightarrow \pi^{\widehat{\cdot}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly}, X}).$$

When  $X$  is a  $C^{\infty}$  manifold the bundle  $\text{LCC } X$  has contractible fibers, and thus it has global  $C^{\infty}$  sections. This fact is crucial for Kontsevich’s proof. However, in our algebraic setup there is no reason to assume that  $\pi : \text{LCC } X \rightarrow X$  has any global sections.

We discovered a way to get around the absence of global sections in the case of an algebraic variety: the idea is to use *simplicial sections*. This idea is inspired by a construction of Bott; see [Bo, HY]. A simplicial section  $\sigma$  of  $\pi : \text{LCC } X \rightarrow X$ , based on an open covering  $X = \bigcup U_{(i)}$ , consists of a family of morphisms  $\sigma_{\mathbf{i}} : \Delta_{\mathbb{K}}^q \times U_{\mathbf{i}} \rightarrow \text{LCC } X$ , where  $\mathbf{i} = (i_0, \dots, i_q)$  is a multi-index;  $\Delta_{\mathbb{K}}^q$  is the  $q$ -dimensional geometric simplex; and  $U_{\mathbf{i}} := U_{(i_0)} \cap \dots \cap U_{(i_q)}$ . The morphisms  $\sigma_{\mathbf{i}}$  are required to be compatible with  $\pi$  and to satisfy simplicial relations.

It is easy to show that sections of  $\pi$  exist locally. Because of the particular geometry of the bundle  $\text{LCC } X$ , if we take a sufficiently fine affine open covering  $X = \bigcup U_{(i)}$ , and choose a section  $\sigma_{(i)} : U_{(i)} \rightarrow \text{LCC } X$  for each  $i$ , then these sections can be extended to a simplicial section  $\sigma$ . (See Figure 2 for an illustration. The details of this construction are worked out in the companion paper [Ye4].)

In order to make use of the simplicial section we need *mixed resolutions*. The mixed resolution of a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{M}$  is a complex  $\text{Mix}_{\mathcal{U}}(\mathcal{M})$ , which combines a de Rham type differential related to  $\mathcal{P}_X$ , called the Grothendieck connection, together with a Čech-simplicial type differential related to the covering  $\mathcal{U} = \{U_{(i)}\}$ . (See Section 6 for a review of mixed resolutions.) We show that the inclusions  $\mathcal{T}_{\text{poly}, X} \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly}, X})$  and  $\mathcal{D}_{\text{poly}, X} \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly}, X})$  are quasi-isomorphisms of sheaves of DG Lie algebras. We then prove the following result (which is Theorem 7.1 in the body of the paper).

**Theorem 0.2.** *Let  $\mathbb{K}$  be a field containing  $\mathbb{R}$ , and let  $X$  be a smooth  $n$ -dimensional algebraic variety over  $\mathbb{K}$ . Suppose  $\mathcal{U} = \{U_{(i)}\}$  is an open covering of  $X$ , where each  $U_{(i)}$  is affine and admits an étale morphism to  $\mathbf{A}_{\mathbb{K}}^n$ . Let  $\sigma$  be the corresponding simplicial section of  $\pi : \text{LCC } X \rightarrow X$ . Then there is an induced  $L_{\infty}$  quasi-isomorphism*

$$\Psi_{\sigma} : \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly}, X}) \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly}, X})$$

between sheaves of DG Lie algebras.

We should point out that the construction of the  $L_{\infty}$  morphism  $\Psi_{\sigma}$  involves twisting, due to the presence of the Grothendieck connection in the mixed resolution  $\text{Mix}_{\mathcal{U}}(-)$ . This sort of twisting is discussed in detail in the companion paper [Ye2].

Passing to global sections we obtain an  $L_{\infty}$  quasi-isomorphism

$$\Gamma(X, \Psi_{\sigma}) : \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly}, X})) \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly}, X})).$$

There are DG Lie algebra homomorphisms

$$(0.3) \quad \mathcal{T}_{\text{poly}}(X) \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly}, X}))$$

and

$$(0.4) \quad \mathcal{D}_{\text{poly}}(X) \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly}, X})).$$

Each sheaf  $\mathcal{D}_{\text{poly}, X}^p$  is a quasi-coherent left  $\mathcal{D}_X$ -module. Hence if  $X$  is  $\mathcal{D}$ -affine the homomorphism (0.4) is a quasi-isomorphism. By standard results of deformation theory (that are reviewed in Section 3) this implies the existence of the quantization map  $Q$  in Theorem 0.1. In case  $X$  is affine the homomorphism (0.3) is also a quasi-isomorphism, and thus  $Q$  is bijective.

An earlier version of this paper was much longer. The current version contains only the main results; auxiliary results were moved to the companion papers [Ye2], [Ye3] and [Ye4].

Finally let us mention several recent papers and surveys dealing with deformation quantization: [BK1, BK2], [CDH], [CF], [CI], [Do1, Do2], and [Ke].

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## 1. DEFORMATION QUANTIZATIONS OF $\mathcal{O}_X$

Throughout the paper  $\mathbb{K}$  is a field of characteristic 0. By default all algebras and schemes in the paper are over  $\mathbb{K}$ , and so are all morphisms. The symbol  $\otimes$  denotes  $\otimes_{\mathbb{K}}$ . The letter  $\hbar$  denotes an indeterminate, and  $\mathbb{K}[[\hbar]]$  is the power series algebra.

Let  $X$  be a smooth separated irreducible  $n$ -dimensional scheme over  $\mathbb{K}$ .

**Definition 1.1.** Let  $U \subset X$  be an open set. A *star product* on  $\mathcal{O}_U[[\hbar]]$  is a  $\mathbb{K}[[\hbar]]$ -bilinear sheaf morphism

$$\star : \mathcal{O}_U[[\hbar]] \times \mathcal{O}_U[[\hbar]] \rightarrow \mathcal{O}_U[[\hbar]]$$

satisfying the following conditions:

- (i) The product  $\star$  makes  $\mathcal{O}_U[[\hbar]]$  into a sheaf of associative unital  $\mathbb{K}[[\hbar]]$ -algebras with unit  $1 \in \mathcal{O}_U$ .
- (ii) There is a sequence  $\beta_j : \mathcal{O}_U \times \mathcal{O}_U \rightarrow \mathcal{O}_U$  of bi-differential operators, such that for any two local sections  $f, g \in \mathcal{O}_U$  one has

$$f \star g = fg + \sum_{j=1}^{\infty} \beta_j(f, g) \hbar^j.$$

Note that  $f \star g \equiv fg \pmod{\hbar}$ , and also  $\beta_j(f, 1) = \beta_j(1, f) = 0$  for all  $f$  and  $j$ .

**Definition 1.2.** Let  $\mathcal{A}$  be a sheaf of  $\hbar$ -adically complete flat  $\mathbb{K}[[\hbar]]$ -algebras on  $X$ , and let  $\psi : \mathcal{A}/\hbar\mathcal{A} \xrightarrow{\cong} \mathcal{O}_X$  be an isomorphism of sheaves of  $\mathbb{K}$ -algebras. Let  $U \subset X$  be an open set. A *differential trivialization* of  $(\mathcal{A}, \psi)$  on  $U$  is an isomorphism

$$\tau : \mathcal{O}_U[[\hbar]] \xrightarrow{\cong} \mathcal{A}|_U$$

of sheaves of  $\mathbb{K}[[\hbar]]$ -modules satisfying the conditions below.

- (i) Let  $\star$  denote the product of  $\mathcal{A}$ . Then the  $\mathbb{K}[[\hbar]]$ -bilinear product  $\star_{\tau}$  on  $\mathcal{O}_U[[\hbar]]$ , defined by

$$f \star_{\tau} g := \tau^{-1}(\tau(f) \star \tau(g))$$

for local sections  $f, g \in \mathcal{O}_U$ , is a star product.

- (ii) For any local section  $f \in \mathcal{O}_U$  one has  $(\psi \circ \tau)(f) = f$ .

Condition (i) implies that  $\tau(1_{\mathcal{O}_X}) = 1_{\mathcal{A}}$ , where  $1_{\mathcal{O}_X}$  and  $1_{\mathcal{A}}$  are the unit elements of  $\mathcal{O}_X$  and  $\mathcal{A}$  respectively.

**Definition 1.3.** Let  $U \subset X$  be an open set. A *gauge equivalence* of  $\mathcal{O}_U[[\hbar]]$  is a  $\mathbb{K}[[\hbar]]$ -linear automorphism of sheaves

$$\gamma : \mathcal{O}_U[[\hbar]] \xrightarrow{\cong} \mathcal{O}_U[[\hbar]]$$

satisfying these conditions:

- (i) There is a sequence of differential operators  $\gamma_k : \mathcal{O}_U \rightarrow \mathcal{O}_U$ , such that for any local section  $f \in \mathcal{O}_U$

$$\gamma(f) = f + \sum_{k=1}^{\infty} \gamma_k(f) \hbar^k.$$

- (ii)  $\gamma(1) = 1$ .

Condition (ii) is equivalent to  $\gamma_k(1) = 0$  for all  $k$ . The gauge equivalences of  $\mathcal{O}_U[[\hbar]]$  form a group under composition.

**Definition 1.4.** Let  $(\mathcal{A}, \psi)$  be as in Definition 1.2. A *differential structure*  $\tau = \{\tau_i\}$  on  $(\mathcal{A}, \psi)$  consists of an open covering  $X = \bigcup_i U_i$ , and for every  $i$  a differential trivialization  $\tau_i : \mathcal{O}_{U_i}[[\hbar]] \xrightarrow{\cong} \mathcal{A}|_{U_i}$  of  $(\mathcal{A}, \psi)$  on  $U_i$ . The condition is that for any two indices  $i, j$  the transition automorphism  $\tau_j^{-1} \circ \tau_i$  of  $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$  is a gauge equivalence.

**Example 1.5.** If  $\mathcal{A}$  is commutative then automatically it has a differential structure  $\tau = \{\tau_i\}$ , with the additional property that each differential trivialization  $\tau_i : \mathcal{O}_{U_i}[[\hbar]] \xrightarrow{\cong} \mathcal{A}|_{U_i}$  is an isomorphism of algebras. Here  $\mathcal{O}_{U_i}[[\hbar]]$  is the usual power series algebra. Let us explain how this is done. Choose an affine open covering  $X = \bigcup U_i$ . For any  $i$  let  $C_i := \Gamma(U_i, \mathcal{O}_X)$ . By formal smoothness of  $\mathbb{K} \rightarrow C_i$  the isomorphism  $\psi^{-1} : C_i \xrightarrow{\cong} \Gamma(U_i, \mathcal{A})/(\hbar)$  lifts to an isomorphism of algebras  $\tau_i : C_i[[\hbar]] \xrightarrow{\cong} \Gamma(U_i, \mathcal{A})$ . Due to commutativity the isomorphism  $\tau_i$  sheafifies to a differential trivialization on  $U_i$ . Commutativity also implies that the transitions  $\tau_j^{-1} \circ \tau_i$  are gauge equivalences. The differential structure  $\tau$  is unique up to gauge equivalence (see Definition 1.8 below). The first order terms of the gauge equivalences  $\tau_j^{-1} \circ \tau_i$  are derivations, and they give the deformation class of  $\mathcal{A}$  in  $H^1(X, \mathcal{T}_X)$ .

For a noncommutative algebra  $\mathcal{A}$  it seems that *we must stipulate the existence of a differential structure*. Furthermore a given algebra  $\mathcal{A}$  might have distinct differential structures. Thus we are led to the next definition.

**Definition 1.6.** A *deformation quantization* of  $\mathcal{O}_X$  is the data  $(\mathcal{A}, \psi, \tau)$ , where  $\mathcal{A}$  is a sheaf of  $\hbar$ -adically complete flat  $\mathbb{K}[[\hbar]]$ -algebras on  $X$ ;  $\psi : \mathcal{A}/\hbar\mathcal{A} \xrightarrow{\cong} \mathcal{O}_X$  is an isomorphism of sheaves of  $\mathbb{K}$ -algebras; and  $\tau$  is a differential structure on  $(\mathcal{A}, \psi)$ .

If there is no danger of confusion we shall sometimes just say that  $\mathcal{A}$  is a deformation quantization, keeping the rest of the data implicit.

**Example 1.7.** Let  $Y$  be a smooth variety and  $X := T^*Y$  the cotangent bundle, with projection  $\pi : X \rightarrow Y$ .  $X$  is a symplectic variety, so it has a non-degenerate Poisson structure  $\alpha$ . Let  $\mathcal{B} := \bigoplus_{i=0}^{\infty} (F_i \mathcal{D}_Y) \hbar^i \subset \mathcal{D}_Y[[\hbar]]$  be the Rees algebra of  $\mathcal{D}_Y$  w.r.t. the order filtration  $\{F_i \mathcal{D}_Y\}$ . So  $\mathcal{B}/\hbar\mathcal{B} \cong \pi_* \mathcal{O}_X$  and  $\mathcal{B}/(\hbar-1)\mathcal{B} \cong \mathcal{D}_Y$ . Define  $\mathcal{B}_m := \mathcal{B}/\hbar^{m+1}\mathcal{B}$ . Consider the sheaf of  $\mathbb{K}[[\hbar]]$ -algebras  $\pi^{-1}\mathcal{B}_m$  on  $X$ . It can be localized to a sheaf of algebras  $\mathcal{A}_m$  on  $X$  such that  $\mathcal{A}_m \cong \pi^*\mathcal{B}_m = \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{B}_m$  as left  $\mathcal{O}_X$ -modules. In particular  $\mathcal{A}_0 \cong \mathcal{O}_X$ . Let  $\mathcal{A} := \varprojlim_m \mathcal{A}_m$ . Then  $\mathcal{A}$  is a deformation quantization of  $(X, \alpha)$ . Note the similarity to microlocal differential operators [Sch].

**Definition 1.8.** Suppose  $(\mathcal{A}, \psi, \tau)$  and  $(\mathcal{A}', \psi', \tau')$  are two deformation quantizations of  $\mathcal{O}_X$ . A *gauge equivalence*

$$\gamma : (\mathcal{A}, \psi, \tau) \rightarrow (\mathcal{A}', \psi', \tau')$$

is a isomorphism  $\gamma : \mathcal{A} \xrightarrow{\cong} \mathcal{A}'$  of sheaves of  $\mathbb{K}[[\hbar]]$ -algebras satisfying the following two conditions:

- (i) One has  $\psi = \psi' \circ \gamma : \mathcal{A} \rightarrow \mathcal{O}_X$ .
- (ii) Let  $\{U_i\}$  and  $\{U'_j\}$  be the open coverings associated to  $\tau$  and  $\tau'$  respectively. Then for any two indices  $i, j$  the automorphism  $\tau'_j{}^{-1} \circ \gamma \circ \tau_i$  of  $\mathcal{O}_{U_i \cap U'_j}[[\hbar]]$  is a gauge equivalence, in the sense of Definition 1.3.

Let  $\Omega_X^p = \Omega_{X/\mathbb{K}}^p$  be the sheaf of differentials of degree  $p$ , and let  $\mathcal{T}_X = \mathcal{T}_{X/\mathbb{K}}$  be the tangent sheaf of  $X$ . For every  $p \geq 0$  there is a canonical pairing

$$\langle -, - \rangle : (\bigwedge_{\mathcal{O}_X}^p \mathcal{T}_X) \times \Omega_X^p \rightarrow \mathcal{O}_X.$$

**Definition 1.9.** (1) A *Poisson bracket* on  $\mathcal{O}_X$  is a biderivation

$$\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$$

which makes  $\mathcal{O}_X$  into a sheaf of Lie algebras.

- (2) Let  $\alpha \in \Gamma(X, \bigwedge_{\mathcal{O}_X}^2 \mathcal{T}_X)$ . Define a biderivation  $\{-, -\}_\alpha$  by the formula

$$\{f, g\}_\alpha := \langle \alpha, d(f) \wedge dg \rangle$$

for local sections  $f, g \in \mathcal{O}_X$ . If  $\{-, -\}_\alpha$  is a Poisson bracket then  $\alpha$  is called a *Poisson structure*, and  $(X, \alpha)$  is called a *Poisson variety*.

The next result is an easy calculation.

**Proposition 1.10.** Let  $(\mathcal{A}, \psi, \tau)$  be a deformation quantization of  $\mathcal{O}_X$ , and denote by  $\star$  the multiplication of  $\mathcal{A}$ . Given two local sections  $f, g \in \mathcal{O}_X$  choose liftings  $\tilde{f}, \tilde{g} \in \mathcal{A}$ . Then the formula

$$\{f, g\}_\mathcal{A} := \psi(\hbar^{-1}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f})) \in \mathcal{O}_X$$

defines a Poisson bracket on  $\mathcal{O}_X$ .

Suppose  $\{U_i\}$  is the open covering associated with the differential trivialization  $\tau$ , and for each  $i$  the collection of bi-differential operators on  $\mathcal{O}_{U_i}$  occurring in Definition 1.1 is  $\{\beta_{i,j}\}_{j=1}^\infty$ . Then for local sections  $f, g \in \mathcal{O}_{U_i}$  one has

$$\{f, g\}_\mathcal{A} = \beta_{i,1}(f, g) - \beta_{i,1}(g, f).$$

**Definition 1.11.** Let  $\alpha$  be a Poisson structure on  $X$ . A *deformation quantization* of the Poisson variety  $(X, \alpha)$  is a deformation quantization  $(\mathcal{A}, \psi, \tau)$  of  $\mathcal{O}_X$  such that the Poisson brackets satisfy

$$\{-, -\}_\mathcal{A} = 2\{-, -\}_\alpha.$$

**Definition 1.12.** A *globally trivialized deformation quantization* of  $\mathcal{O}_X$  is a deformation quantization  $(\mathcal{A}, \psi, \tau)$  in which the differential structure  $\tau$  consists of a single differential trivialization  $\tau : \mathcal{O}_X[[\hbar]] \xrightarrow{\cong} \mathcal{A}$ .

In effect a globally trivialized deformation quantization of  $\mathcal{O}_X$  is the same as a star product on  $\mathcal{O}_X[[\hbar]]$ ; the correspondence is  $\star \mapsto \star_\tau$  in the notation of Definition 1.2.

Let  $\mathcal{D}_X$  be the sheaf of  $\mathbb{K}$ -linear differential operators on  $X$ .

**Theorem 1.13.** *Assume  $H^1(X, \mathcal{D}_X) = 0$ . Then any deformation quantization  $(\mathcal{A}, \psi, \tau)$  of  $\mathcal{O}_X$  can be globally trivialized. Namely there is a globally trivialized deformation quantization  $(\mathcal{A}', \psi', \{\tau'\})$  of  $\mathcal{O}_X$ , and a gauge equivalence  $(\mathcal{A}, \psi, \tau) \rightarrow (\mathcal{A}', \psi', \{\tau'\})$ .*

*Proof.* We will take  $(\mathcal{A}', \psi') := (\mathcal{A}, \psi)$ , and produce a global differential trivialization  $\tau'$ .

By refining the open covering  $\mathcal{U} = \{U_i\}$  associated with the differential structure  $\tau = \{\tau_i\}$  we may assume each of the open sets  $U_i$  is affine. We may also assume that  $\mathcal{U}$  is finite, say  $\mathcal{U} = \{U_0, \dots, U_m\}$ . For any pair of indices  $(i, j)$  let  $\rho_{(i,j)} := \tau_j^{-1} \circ \tau_i$ , which is a gauge equivalence of  $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$ . We are going to construct a gauge equivalence  $\rho_i$  of  $\mathcal{O}_{U_i}[[\hbar]]$ , for every  $i \in \{0, \dots, m\}$ , such that  $\rho_{(i,j)} = \rho_j \circ \rho_i^{-1}$ . Then the new differential structure  $\tau'$ , defined by  $\tau'_i := \tau_i \circ \rho_i$ , will satisfy  $\tau'^{-1}_j \circ \tau'_i = \mathbf{1}_{\mathcal{O}_{U_i \cap U_j}[[\hbar]]}$ , the identity automorphism of  $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$ . Therefore the various  $\tau'_i$  can be glued to a global differential trivialization  $\tau' : \mathcal{O}_X[[\hbar]] \xrightarrow{\cong} \mathcal{A}$  as required.

Let  $\mathcal{D}_X^{\text{nor}}$  be the subsheaf of  $\mathcal{D}_X$  consisting of operators that vanish on  $\mathbf{1}_{\mathcal{O}_X}$ . This is the left ideal of  $\mathcal{D}_X$  generated by the sheaf of derivation  $\mathcal{T}_X$ . There is a direct sum decomposition  $\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{D}_X^{\text{nor}}$  (as sheaves of left  $\mathcal{O}_X$ -modules), and therefore  $H^1(X, \mathcal{D}_X^{\text{nor}}) = 0$ .

Consider the sheaf of nonabelian groups  $G$  on  $X$  whose sections on an open set  $U$  is the group of gauge equivalences of  $\mathcal{O}_U[[\hbar]]$ . Let  $\mathcal{D}_X^{\text{nor}}[[\hbar]]^+ := \hbar \mathcal{D}_X^{\text{nor}}[[\hbar]]$ . As sheaves of sets there is a canonical isomorphism  $\mathcal{D}_X^{\text{nor}}[[\hbar]]^+ \xrightarrow{\cong} G$ , whose formula is  $\sum_{k=1}^{\infty} D_k \hbar^k \mapsto \mathbf{1}_{\mathcal{O}_X} + \sum_{k=1}^{\infty} D_k \hbar^k$ . Define  $G^k$  to be the subgroup of  $G$  consisting of all equivalences congruent to  $\mathbf{1}_{\mathcal{O}_X}$  modulo  $\hbar^{k+1}$ . Then each  $G^k$  is a normal subgroup, and the map  $\mathcal{D}_X^{\text{nor}} \xrightarrow{\cong} G^k / G^{k+1}$ ,  $D \mapsto \mathbf{1}_{\mathcal{O}_X} + D \hbar^{k+1}$ , is an isomorphism of sheaves of abelian groups. Moreover the conjugation action of  $G$  on  $G^k / G^{k+1}$  is trivial, so that  $\gamma_1 \gamma_2 = \gamma_2 \gamma_1 \in G^k / G^{k+1}$  for every  $\gamma_1 \in G^k$  and  $\gamma_2 \in G$ .

The gauge equivalences  $\rho_i = \sum_{k=0}^{\infty} \rho_{i,k} \hbar^k$  will be defined by successive approximations; namely the differential operators  $\rho_{i,k} \in \Gamma(U_i, \mathcal{D}_X^{\text{nor}})$  shall be defined by recursion on  $k$ , simultaneously for all  $i \in \{0, \dots, m\}$ . For  $k=0$  we take  $\rho_{i,0} := \mathbf{1}_{\mathcal{O}_{U_i}}$  of course. Now assume at the  $k$ -th stage we have operators  $\rho_i^{(k)} := \sum_{l=0}^k \rho_{i,l} \hbar^l$  which satisfy

$$\rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \equiv \rho_{(i,j)} \pmod{\hbar^{k+1}}.$$

This means that

$$\rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \in G^k.$$

By the properties of the group  $G$  mentioned above the function  $\{0, \dots, m\}^2 \rightarrow G^k / G^{k+1}$ ,

$$(i, j) \mapsto \rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \in \Gamma(U_i \cap U_j, G^k / G^{k+1}),$$

is a Čech 1-cocycle for the affine covering  $\mathcal{U}$ . Since  $G^k / G^{k+1} \cong \mathcal{D}_X^{\text{nor}}$ , and we are given that  $H^1(X, \mathcal{D}_X^{\text{nor}}) = 0$ , it follows that there exists a 0-cochain  $i \mapsto \rho_{i,k+1} \in \Gamma(U_i, \mathcal{D}_X^{\text{nor}})$  such that

$$(\mathbf{1} + \rho_{j,k+1} \hbar^{k+1}) \circ (\mathbf{1} + \rho_{i,k+1} \hbar^{k+1})^{-1} \equiv \rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \pmod{\hbar^{k+2}}.$$

□



**Proposition 1.14.** *Let  $\star$  and  $\star'$  be two star products on  $\mathcal{O}_X[[\hbar]]$ . Consider the globally trivialized deformation quantizations  $(\mathcal{A}, \psi, \tau)$  and  $(\mathcal{A}', \psi, \tau)$ , where  $\mathcal{A} := (\mathcal{O}_X[[\hbar]], \star)$ ,  $\mathcal{A}' := (\mathcal{O}_X[[\hbar]], \star')$ ,  $\psi := \mathbf{1}_{\mathcal{O}_X}$  and  $\tau := \{\mathbf{1}_{\mathcal{O}_X[[\hbar]]}\}$ . Then the deformation quantizations  $(\mathcal{A}, \psi, \tau)$  and  $(\mathcal{A}', \psi, \tau)$  are gauge equivalent, in the sense of Definition 1.8, iff there exists a gauge equivalence  $\gamma$  of  $\mathcal{O}_X[[\hbar]]$ , in the sense of Definition 1.3, such that*

$$f \star' g = \gamma^{-1}(\gamma(f) \star \gamma(g))$$

for all local sections  $f, g \in \mathcal{O}_X$ .

We leave out the easy proof.

## 2. REVIEW OF DIR-INV MODULES

In this section we review the concept of dir-inv structure, which was introduced in [Ye2, Section 1]. A dir-inv structure is a generalization of adic topology, and it will turn out to be extremely useful in several places in the paper.

Let  $C$  be a commutative  $\mathbb{K}$ -algebra. We denote by  $\text{Mod } C$  the category of  $C$ -modules.

**Definition 2.1.** (1) Let  $M \in \text{Mod } C$ . An *inv module structure* on  $M$  is an inverse system  $\{F^i M\}_{i \in \mathbb{N}}$  of  $C$ -submodules of  $M$ . The pair  $(M, \{F^i M\}_{i \in \mathbb{N}})$  is called an *inv  $C$ -module*.  
 (2) Let  $(M, \{F^i M\}_{i \in \mathbb{N}})$  and  $(N, \{F^i N\}_{i \in \mathbb{N}})$  be two inv  $C$ -modules. A function  $\phi : M \rightarrow N$  ( $C$ -linear or not) is said to be *continuous* if for every  $i \in \mathbb{N}$  there exists  $i' \in \mathbb{N}$  such that  $\phi(F^{i'} M) \subset F^i N$ .  
 (3) Define  $\text{Inv Mod } C$  to be the category whose objects are the inv  $C$ -modules, and whose morphisms are the continuous  $C$ -linear homomorphisms.

There is a full and faithful embedding of categories  $\text{Mod } C \hookrightarrow \text{Inv Mod } C$ ,  $M \mapsto (M, \{\dots, 0, 0\})$ .

Recall that a directed set is a partially ordered set  $J$  with the property that for any  $j_1, j_2 \in J$  there exists  $j_3 \in J$  such that  $j_1, j_2 \leq j_3$ .

**Definition 2.2.** (1) Let  $M \in \text{Mod } C$ . A *dir-inv module structure* on  $M$  is a direct system  $\{F_j M\}_{j \in J}$  of  $C$ -submodules of  $M$ , indexed by a nonempty directed set  $J$ , together with an inv module structure on each  $F_j M$ , such that for every  $j_1 \leq j_2$  the inclusion  $F_{j_1} M \hookrightarrow F_{j_2} M$  is continuous. The pair  $(M, \{F_j M\}_{j \in J})$  is called a *dir-inv  $C$ -module*.  
 (2) Let  $(M, \{F_j M\}_{j \in J})$  and  $(N, \{F_k N\}_{k \in K})$  be two dir-inv  $C$ -modules. A function  $\phi : M \rightarrow N$  ( $C$ -linear or not) is said to be *continuous* if for every  $j \in J$  there exists  $k \in K$  such that  $\phi(F_j M) \subset F_k N$ , and  $\phi : F_j M \rightarrow F_k N$  is a continuous function between these two inv  $C$ -modules.  
 (3) Define  $\text{Dir Inv Mod } C$  to be the category whose objects are the dir-inv  $C$ -modules, and whose morphisms are the continuous  $C$ -linear homomorphisms.

An inv  $C$ -module  $M$  can be endowed with a dir-inv module structure  $\{F_j M\}_{j \in J}$ , where  $J := \{0\}$  and  $F_0 M := M$ . Thus we get a full and faithful embedding  $\text{Inv Mod } C \hookrightarrow \text{Dir Inv Mod } C$ .

Inv modules and dir-inv modules come in a few “flavors”: trivial, discrete and complete. A *discrete inv module* is one which is isomorphic, in  $\text{Inv Mod } C$ , to an

object of  $\text{Mod } C$  (via the canonical embedding above). A *complete inv module* is an inv module  $(M, \{F^i M\}_{i \in \mathbb{N}})$  such that the canonical map  $M \rightarrow \lim_{\leftarrow i} M/F^i M$  is bijective. A *discrete* (resp. *complete*) *dir-inv module* is one which is isomorphic, in  $\text{Dir Inv Mod } C$ , to a dir-inv module  $(M, \{F_j M\}_{j \in J})$ , where all the inv modules  $F_j M$  are discrete (resp. complete), and the canonical map  $\lim_{j \rightarrow} F_j M \rightarrow M$  in  $\text{Mod } C$  is bijective. A *trivial dir-inv module* is one which is isomorphic to an object of  $\text{Mod } C$ . Discrete dir-inv modules are complete, but there are also other complete modules, as the next example shows.

**Example 2.3.** Assume  $C$  is noetherian and  $\mathfrak{c}$ -adically complete for some ideal  $\mathfrak{c}$ . Let  $M$  be a finitely generated  $C$ -module, and define  $F^i M := \mathfrak{c}^{i+1} M$ . Then  $\{F^i M\}_{i \in \mathbb{N}}$  is called the  *$\mathfrak{c}$ -adic inv structure*, and of course  $(M, \{F^i M\}_{i \in \mathbb{N}})$  is a complete inv module. Next consider an arbitrary  $C$ -module  $M$ . We take  $\{F_j M\}_{j \in J}$  to be the collection of finitely generated  $C$ -submodules of  $M$ . This dir-inv module structure on  $M$  is called the  *$\mathfrak{c}$ -adic dir-inv structure*. Again  $(M, \{F_j M\}_{j \in J})$  is a complete dir-inv  $C$ -module. Note that a finitely generated  $C$ -module  $M$  is discrete as inv module iff  $\mathfrak{c}^i M = 0$  for  $i \gg 0$ ; and a  $C$ -module is discrete as dir-inv module iff it is a direct limit of discrete finitely generated modules.

The category  $\text{Dir Inv Mod } C$  is additive. Given a collection  $\{M_k\}_{k \in K}$  of dir-inv modules, the direct sum  $\bigoplus_{k \in K} M_k$  has a dir-inv module structure, making it into the coproduct of  $\{M_k\}_{k \in K}$  in  $\text{Dir Inv Mod } C$ . Note that if the index set  $K$  is infinite and each  $M_k$  is a nonzero discrete inv module, then  $\bigoplus_{k \in K} M_k$  is a discrete dir-inv module which is not trivial. The tensor product  $M \otimes_C N$  of two dir-inv modules is again a dir-inv module. There is a completion functor  $M \mapsto \widehat{M}$ . (Warning: if  $M$  is complete then  $\widehat{M} = M$ , but it is not known if  $\widehat{M}$  is complete for arbitrary  $M$ .) The completed tensor product is  $M \widehat{\otimes}_C N := \widehat{M \otimes_C N}$ . Completion commutes with direct sums: if  $M \cong \bigoplus_{k \in K} M_k$  then  $\widehat{M} \cong \bigoplus_{k \in K} \widehat{M}_k$ .

A *graded dir-inv module* (or graded object in  $\text{Dir Inv Mod } C$ ) is a direct sum  $M = \bigoplus_{k \in \mathbb{Z}} M_k$ , where each  $M_k$  is a dir-inv module. A *DG algebra* in  $\text{Dir Inv Mod } C$  is a graded dir-inv module  $A = \bigoplus_{k \in \mathbb{Z}} A^k$ , together with continuous  $C$ -(bi)linear functions  $\mu : A \times A \rightarrow A$  and  $d : A \rightarrow A$ , which make  $A$  into a DG  $C$ -algebra. If  $A$  is a super-commutative associative unital DG algebra in  $\text{Dir Inv Mod } C$ , and  $\mathfrak{g}$  is a DG Lie Algebra in  $\text{Dir Inv Mod } C$ , then  $A \widehat{\otimes}_C \mathfrak{g}$  is a DG Lie Algebra in  $\text{Dir Inv Mod } C$ .

Let  $A$  be a super-commutative associative unital DG algebra in  $\text{Dir Inv Mod } C$ . A *DG  $A$ -module* in  $\text{Dir Inv Mod } C$  is a graded object  $M$  in  $\text{Dir Inv Mod } C$ , together with continuous  $C$ -(bi)linear functions  $\mu : A \times M \rightarrow M$  and  $d : M \rightarrow M$ , which make  $M$  into a DG  $A$ -module in the usual sense. A *DG  $A$ -module Lie algebra* in  $\text{Dir Inv Mod } C$  is a DG Lie algebra  $\mathfrak{g}$  in  $\text{Dir Inv Mod } C$ , together with a continuous  $C$ -bilinear function  $\mu : A \times \mathfrak{g} \rightarrow \mathfrak{g}$ , such that  $\mathfrak{g}$  becomes a DG  $A$ -module, and

$$[a_1 \gamma_1, a_2 \gamma_2] = (-1)^{i_2 j_1} a_1 a_2 [\gamma_1, \gamma_2]$$

for all  $a_k \in A^{i_k}$  and  $\gamma_k \in \mathfrak{g}^{j_k}$ .

All the constructions above can be geometrized. Let  $(Y, \mathcal{O})$  be a commutative ringed space over  $\mathbb{K}$ , i.e.  $Y$  is a topological space, and  $\mathcal{O}$  is a sheaf of commutative  $\mathbb{K}$ -algebras on  $Y$ . We denote by  $\text{Mod } \mathcal{O}$  the category of  $\mathcal{O}$ -modules on  $Y$ . Then we can talk about the category  $\text{Dir Inv Mod } \mathcal{O}$  of dir-inv  $\mathcal{O}$ -modules.

**Example 2.4.** Geometrizing Example 2.3, let  $\mathfrak{X}$  be a noetherian formal scheme, with defining ideal  $\mathcal{I}$ . Then any coherent  $\mathcal{O}_{\mathfrak{X}}$ -module  $\mathcal{M}$  is an inv  $\mathcal{O}_{\mathfrak{X}}$ -module,

with system of submodules  $\{\mathcal{I}^{i+1}\mathcal{M}\}_{i \in \mathbb{N}}$ , and  $\mathcal{M} \cong \widehat{\mathcal{M}}$ ; cf. [EGA I]. We call an  $\mathcal{O}_{\mathfrak{X}}$ -module *dir-coherent* if it is the direct limit of coherent  $\mathcal{O}_{\mathfrak{X}}$ -modules. Any dir-coherent module is quasi-coherent, but it is not known if the converse is true. At any rate, a dir-coherent  $\mathcal{O}_{\mathfrak{X}}$ -module  $\mathcal{M}$  is a dir-inv  $\mathcal{O}_{\mathfrak{X}}$ -module, where we take  $\{\mathbb{F}_j\mathcal{M}\}_{j \in J}$  to be the collection of coherent submodules of  $\mathcal{M}$ . Any dir-coherent  $\mathcal{O}_{\mathfrak{X}}$ -module is then a complete dir-inv module. This dir-inv module structure on  $\mathcal{M}$  is called the  *$\mathcal{I}$ -adic dir-inv structure*.

If  $f : (Y', \mathcal{O}') \rightarrow (Y, \mathcal{O})$  is a morphism of ringed spaces and  $\mathcal{M} \in \text{Dir Inv Mod } \mathcal{O}$ , then there is an obvious structure of dir-inv  $\mathcal{O}'$ -module on  $f^*\mathcal{M}$ , and we define  $f^*\mathcal{M} := \widehat{f^*\mathcal{M}}$ . If  $\mathcal{M}$  is a graded object in  $\text{Dir Inv Mod } \mathcal{O}$ , then the inverse images  $f^*\mathcal{M}$  and  $\widehat{f^*\mathcal{M}}$  are graded objects in  $\text{Dir Inv Mod } \mathcal{O}'$ . If  $\mathcal{G}$  is an algebra (resp. a DG algebra) in  $\text{Dir Inv Mod } \mathcal{O}$ , then  $f^*\mathcal{G}$  and  $\widehat{f^*\mathcal{G}}$  are algebras (resp. DG algebras) in  $\text{Dir Inv Mod } \mathcal{O}'$ . Given  $\mathcal{N} \in \text{Dir Inv Mod } \mathcal{O}'$  there is an obvious dir-inv  $\mathcal{O}$ -module structure on  $f_*\mathcal{N}$ .

### 3. UNIVERSAL FORMULAS FOR DEFORMATION QUANTIZATION

In this section, as before,  $\mathbb{K}$  is a field of characteristic 0.

From here to Corollary 3.10 we consider the following data. Let  $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}^j$  be a DG Lie algebra over  $\mathbb{K}$ . We put on each  $\mathfrak{g}^j$  the discrete inv  $\mathbb{K}$ -module structure, and  $\mathfrak{g}$  is given the  $\bigoplus$  dir-inv structure; so  $\mathfrak{g}$  is a discrete, but possibly nontrivial, DG Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}$ . Let  $A$  be noetherian commutative complete local  $\mathbb{K}$ -algebra with maximal ideal  $\mathfrak{m}$ . We put on  $A$  and  $\mathfrak{m}$  the  $\mathfrak{m}$ -adic inv structures. For  $i \geq 0$  let  $A_i := A/\mathfrak{m}^{i+1}$ , which is an artinian local algebra with maximal ideal  $\mathfrak{m}_i := \mathfrak{m}/\mathfrak{m}^{i+1}$ ; so  $A_i$  and  $\mathfrak{m}_i$  are discrete inv modules. We obtain a new DG Lie algebra  $A \widehat{\otimes}_{\mathbb{K}} \mathfrak{g} = A \widehat{\otimes} \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} A \widehat{\otimes} \mathfrak{g}^j$ , and there are related DG Lie algebras  $\mathfrak{m} \widehat{\otimes} \mathfrak{g} \subset A \widehat{\otimes} \mathfrak{g}$  and  $\mathfrak{m}_i \otimes \mathfrak{g} \subset A_i \otimes \mathfrak{g}$ . Note that for every  $j$  one has  $A \widehat{\otimes} \mathfrak{g}^j \cong \lim_{\leftarrow i} (A_i \otimes \mathfrak{g}^j)$  in  $\text{Inv Mod } \mathbb{K}$ . In case  $A = \mathbb{K}[[\hbar]]$  we shall also use the notation  $\mathfrak{g}[[\hbar]]^+ := \mathfrak{m} \widehat{\otimes} \mathfrak{g}$ , namely  $\mathfrak{g}[[\hbar]]^+ = \bigoplus_j \hbar \mathfrak{g}^j[[\hbar]]$ .

Recall the correspondence between finite dimensional nilpotent Lie algebras and unipotent algebraic groups over  $\mathbb{K}$  (see [Ho, Theorem XVI.4.2]). Given a nilpotent Lie algebra  $\mathfrak{h}$  we denote by  $\exp(\mathfrak{h})$  the corresponding group. This group has the same underlying scheme structure as  $\mathfrak{h}$ , and the product is according to the Campbell-Hausdorff formula. The assignment  $\mathfrak{h} \mapsto \exp(\mathfrak{h})$  is functorial.

For any  $i$  the Lie algebra  $\mathfrak{m}_i \otimes \mathfrak{g}^0$  is a nilpotent, and in fact it is a direct limit of finite dimensional nilpotent Lie algebras. Therefore we obtain a group  $\exp(\mathfrak{m}_i \otimes \mathfrak{g}^0)$ , which is a direct limit of unipotent groups. Passing to the inverse limit in  $i$  we get a group  $\exp(\mathfrak{m} \otimes \mathfrak{g}^0) := \lim_{\leftarrow i} \exp(\mathfrak{m}_i \otimes \mathfrak{g}^0)$ .

Given a vector space  $V$  over  $\mathbb{K}$  let  $\text{Aff}(V) := \text{GL}(V) \ltimes V$ , the group of affine transformations. Its Lie algebra is  $\mathfrak{aff}(V) := \mathfrak{gl}(V) \ltimes V$ . If  $V$  is finite dimensional then of course  $\text{Aff}(V)$  is an algebraic group; but we will be interested in  $V := \mathfrak{m} \widehat{\otimes} \mathfrak{g}^1$ .

For  $\gamma \in \mathfrak{m} \widehat{\otimes} \mathfrak{g}^0$  and  $\omega \in \mathfrak{m} \widehat{\otimes} \mathfrak{g}^1$  define

$$\text{af}(\gamma)(\omega) := [\gamma, \omega] - \text{d}(\gamma) = (\text{ad}(\gamma) - \text{d})(\omega) \in \mathfrak{m} \widehat{\otimes} \mathfrak{g}^1,$$

where  $\text{d}$  and  $[-, -]$  are the operations of the DG Lie algebra  $\mathfrak{m} \widehat{\otimes} \mathfrak{g}$ . A calculation shows that this is a homomorphism of Lie algebras

$$\text{af} : \mathfrak{m} \widehat{\otimes} \mathfrak{g}^0 \rightarrow \mathfrak{aff}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}^1).$$

Recall that the *Maurer-Cartan equation* in  $A \widehat{\otimes} \mathfrak{g}$  is

$$(3.1) \quad d(\omega) + \frac{1}{2}[\omega, \omega] = 0$$

for  $\omega \in A \widehat{\otimes} \mathfrak{g}^1$ .

**Lemma 3.2.** (1) *The Lie algebra homomorphism  $\text{af}$  integrates to a group homomorphism*

$$\exp(\text{af}) : \exp(\mathfrak{m} \widehat{\otimes} \mathfrak{g}^0) \rightarrow \text{Aff}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}^1).$$

(2) *Assume  $\omega \in \mathfrak{m} \widehat{\otimes} \mathfrak{g}^1$  is a solution of the MC equation in  $\mathfrak{m} \widehat{\otimes} \mathfrak{g}$ , and let  $\gamma \in \mathfrak{m} \widehat{\otimes} \mathfrak{g}^0$ . Then  $\exp(\text{af})(\exp(\gamma))(\omega)$  is also a solution of the MC equation.*

*Proof.* We may assume that  $\mathfrak{g} = \bigoplus_{j \geq 0} \mathfrak{g}^j$ . First consider the nilpotent case. The DG Lie algebra  $\mathfrak{m}_i \otimes \mathfrak{g}$  is the direct limit of sub DG Lie algebras  $\mathfrak{h} = \bigoplus_{j \geq 0} \mathfrak{h}^j$ , which are nilpotent, and each  $\mathfrak{h}^j$  is a finite dimensional vector space. The arguments of [GM, Section 1.3] apply here, so we obtain a homomorphism of algebraic groups  $\exp(\text{af}) : \exp(\mathfrak{h}^0) \rightarrow \text{Aff}(\mathfrak{h}^1)$ , and  $\exp(\mathfrak{h}^0)$  preserves the set of solutions of the MC equation in  $\mathfrak{h}^1$ . Passing to the direct limit over these subalgebras we get a homomorphism of groups  $\exp(\text{af}) : \exp(\mathfrak{m}_i \otimes \mathfrak{g}^0) \rightarrow \text{Aff}(\mathfrak{m}_i \otimes \mathfrak{g}^1)$ , and  $\exp(\mathfrak{m}_i \otimes \mathfrak{g}^0)$  preserves the set of solutions of the MC equation in  $\mathfrak{m}_i \otimes \mathfrak{g}^1$ . Finally we pass to the inverse limit in  $i$ .  $\square$

The formula for  $\exp(\text{af})(\exp(\gamma))(\omega)$  is, according to [GM]:

$$(3.3) \quad \exp(\text{af})(\exp(\gamma))(\omega) = \exp(\text{ad}(\gamma))(\omega) + \frac{1 - \exp(\text{ad}(\gamma))}{\text{ad}(\gamma)}(d(\gamma)).$$

On the right side of the equation “exp” stands for the usual exponential power series  $\exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k$ , and this makes sense because  $\lim_{k \rightarrow \infty} \text{ad}(\gamma)^k(\omega) = 0$  in the  $\mathfrak{m}$ -adic inv structure on  $\mathfrak{m} \widehat{\otimes} \mathfrak{g}^1$ .

**Definition 3.4.** Elements of the group  $\exp(\mathfrak{m} \widehat{\otimes} \mathfrak{g}^0)$  are called *gauge equivalences*. We write

$$\text{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) := \frac{\{\text{solutions of the MC equation in } \mathfrak{m} \widehat{\otimes} \mathfrak{g}\}}{\{\text{gauge equivalences}\}}.$$

**Lemma 3.5.** *The canonical projection*

$$\text{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \rightarrow \lim_{\leftarrow i} \text{MC}(\mathfrak{m}_i \otimes \mathfrak{g})$$

*is bijective.*

The easy proof is omitted.

**Remark 3.6.** Consider the super-commutative DG algebra  $\Omega_{\mathbb{K}[t]} = \Omega_{\mathbb{K}[t]}^0 \oplus \Omega_{\mathbb{K}[t]}^1$ , where  $\mathbb{K}[t]$  is the polynomial algebra in the variable  $t$ . There is an induced DG Lie algebra  $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{g}$ . For any  $\lambda \in \mathbb{K}$  there is a DG Lie algebra homomorphism  $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$ ,  $t \mapsto \lambda$ . Assume  $A$  is artinian, and let  $\omega_0$  and  $\omega_1$  be two solutions of the MC equation in  $\mathfrak{m} \otimes \mathfrak{g}$ . According to [Ko1, 4.5.2(3)] the following conditions are equivalent:

- (i)  $\omega_0$  and  $\omega_1$  are gauge equivalent, in the sense of Definition 3.4.
- (ii) There is a solution  $\omega(t)$  of the MC equation in the DG Lie algebra  $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{m} \otimes \mathfrak{g}$ , such that for  $i \in \{0, 1\} \subset \mathbb{K}$ , the specialization homomorphisms  $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{m} \otimes \mathfrak{g} \rightarrow \mathfrak{m} \otimes \mathfrak{g}$ ,  $t \mapsto i$ , send  $\omega(t) \mapsto \omega_i$ .

See also [Fu] and [Hi1]. We will not need these facts in our paper.

For a graded  $\mathbb{K}$ -module  $M$  the expression  $\bigwedge^i M$  denotes the  $i$ -th super-exterior power.

**Definition 3.7.** Let  $\mathfrak{g}$  and  $\mathfrak{h}$  be two DG Lie algebras. An  $L_\infty$  morphism  $\Psi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a collection  $\Psi = \{\psi_i\}_{i \geq 1}$  of  $\mathbb{K}$ -linear homomorphisms  $\psi_i : \bigwedge^i \mathfrak{g} \rightarrow \mathfrak{h}$ , each of them homogeneous of degree  $1 - i$ , satisfying

$$\begin{aligned} d(\psi_i(\gamma_1 \wedge \dots \wedge \gamma_i)) - \sum_{k=1}^i \pm \psi_i(\gamma_1 \wedge \dots \wedge d(\gamma_k) \wedge \dots \wedge \gamma_i) = \\ \frac{1}{2} \sum_{\substack{k,l \geq 1 \\ k+l=i}} \frac{1}{k!l!} \sum_{\sigma \in \Sigma_i} \pm [\psi_k(\gamma_{\sigma(1)} \wedge \dots \wedge \gamma_{\sigma(k)}), \psi_l(\gamma_{\sigma(k+1)} \wedge \dots \wedge \gamma_{\sigma(k+l)})] \\ + \sum_{k < l} \pm \psi_{i-1}([\gamma_k, \gamma_l] \wedge \gamma_1 \wedge \dots \wedge \gamma_i). \end{aligned}$$

Here  $\gamma_k \in \mathfrak{g}$  are homogeneous elements,  $\Sigma_i$  is the permutation group of  $\{1, \dots, i\}$ , and the signs depend only on the indices, the permutations and the degrees of the elements  $\gamma_k$ . See [Ke, Section 6] or [CFT, Theorem 3.1] for the explicit signs.

An  $L_\infty$  morphism is a generalization of a DG Lie algebra homomorphism. Indeed,  $\psi_1 : \mathfrak{g} \rightarrow \mathfrak{h}$  is a homomorphism of complexes of  $\mathbb{K}$ -modules, and  $H(\psi_1) : H\mathfrak{g} \rightarrow H\mathfrak{h}$  is a homomorphism of graded Lie algebras.

Suppose  $\Psi = \{\psi_i\}_{i \geq 1} : \mathfrak{g} \rightarrow \mathfrak{h}$  is an  $L_\infty$  morphism. For every  $i$  we can extend the  $\mathbb{K}$ -multilinear function  $\psi_i : \prod^i \mathfrak{g} \rightarrow \mathfrak{h}$  uniquely to a continuous  $A$ -multilinear function  $\psi_{A,i} : \prod^i (A \widehat{\otimes} \mathfrak{g}) \rightarrow A \widehat{\otimes} \mathfrak{h}$ . These restrict to functions  $\psi_{A,i} : \prod^i (\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \rightarrow \mathfrak{m} \widehat{\otimes} \mathfrak{h}$ . Clearly  $\Psi_A = \{\psi_{A,i}\} : \mathfrak{m} \widehat{\otimes} \mathfrak{g} \rightarrow \mathfrak{m} \widehat{\otimes} \mathfrak{h}$  is an  $L_\infty$  morphism; we call it the continuous  $A$ -multilinear extension of  $\Psi$ .

**Theorem 3.8** ([Ko1, Section 4.4], [Fu, Theorem 2.2.2]). *Assume  $A$  is artinian. Let  $\Psi : \mathfrak{g} \rightarrow \mathfrak{h}$  be an  $L_\infty$  quasi-isomorphism. Then the function*

$$(3.9) \quad \omega \mapsto \sum_{j \geq 1} \frac{1}{j!} \psi_{A,j}(\omega^j)$$

*induces a bijection*

$$\mathrm{MC}(\Psi_A) : \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \xrightarrow{\cong} \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{h}).$$

**Corollary 3.10.** *Let  $(A, \mathfrak{m})$  be a complete noetherian local  $\mathbb{K}$ -algebra, and let  $\Psi : \mathfrak{g} \rightarrow \mathfrak{h}$  be an  $L_\infty$  quasi-isomorphism between two discrete DG Lie algebras. Then the function (3.9) induces a bijection*

$$\mathrm{MC}(\Psi_A) : \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \xrightarrow{\cong} \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{h}).$$

*Proof.* Use Theorem 3.8 and Lemma 3.5. □

Let  $C$  be a commutative  $\mathbb{K}$ -algebra. The module of derivations of  $C$  relative to  $\mathbb{K}$  is denoted by  $\mathcal{T}_{C/\mathbb{K}} = \mathcal{T}_C$ . For  $p \geq -1$  let  $\mathcal{T}_{\mathrm{poly}}^p(C) := \bigwedge_C^{p+1} \mathcal{T}_C$ , the  $p$ -th exterior power. The direct sum  $\mathcal{T}_{\mathrm{poly}}(C) := \bigoplus_{p \geq -1} \mathcal{T}_{\mathrm{poly}}^p(C)$  is a DG Lie algebra over  $\mathbb{K}$  with trivial differential and with the Schouten-Nijenhuis Lie bracket (see [Ko2] for details).

For any  $p \geq 0$  and  $m \geq 0$  let  $F_m \mathcal{D}_{\text{poly}}^p(C)$  be the set of  $\mathbb{K}$ -multilinear functions  $\phi : C^{p+1} \rightarrow C$  that are differential operators of order  $\leq m$  in each argument (in the sense of [EGA IV]). For  $p = -1$  let  $F_m \mathcal{D}_{\text{poly}}^{-1}(C) := C$ . Define  $\mathcal{D}_{\text{poly}}^p(C) := \bigcup_{m \geq 0} F_m \mathcal{D}_{\text{poly}}^p(C)$  and  $\mathcal{D}_{\text{poly}}(C) := \bigoplus_{p \geq -1} \mathcal{D}_{\text{poly}}^p(C)$ . This is a sub DG Lie algebra of the shifted Hochschild cochain complex of  $C$ , with shifted Hochschild differential and Gerstenhaber Lie bracket (see [Ko1]). We view  $\mathcal{D}_{\text{poly}}(C)$  as a left  $C$ -module by the rule  $(c \cdot \phi)(c_1, \dots, c_{p+1}) := c \cdot \phi(c_1, \dots, c_{p+1})$ .

For  $p \geq 0$  define  $\mathcal{D}_{\text{poly}}^{\text{nor}, p}(C)$  to be the subset of  $\mathcal{D}_{\text{poly}}^p(C)$  consisting of the poly differential operators  $\phi : C^{p+1} \rightarrow C$  such that  $\phi(c_1, \dots, c_{p+1}) = 0$  if  $c_i = 1$  for some  $i$ . For  $p = -1$  let  $\mathcal{D}_{\text{poly}}^{\text{nor}, -1}(C) := C$ . Then  $\mathcal{D}_{\text{poly}}^{\text{nor}}(C) := \bigoplus_{p \geq -1} \mathcal{D}_{\text{poly}}^{\text{nor}, p}(C)$  is a sub DG Lie algebra of  $\mathcal{D}_{\text{poly}}(C)$ .

For any integer  $p \geq 0$  there is a  $C$ -linear homomorphism

$$\mathcal{U}_1 : \mathcal{T}_{\text{poly}}^p(C) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}, p}(C)$$

with formula

$$(3.11) \quad \mathcal{U}_1(\xi_1 \wedge \dots \wedge \xi_{p+1})(c_1, \dots, c_{p+1}) := \frac{1}{(p+1)!} \sum_{\sigma \in \Sigma_{p+1}} \text{sgn}(\sigma) \xi_{\sigma(1)}(c_1) \cdots \xi_{\sigma(p+1)}(c_{p+1})$$

for elements  $\xi_1, \dots, \xi_{p+1} \in \mathcal{T}_C$  and  $c_1, \dots, c_{p+1} \in C$ . For  $p = -1$  the map  $\mathcal{U}_1 : \mathcal{T}_{\text{poly}}^{-1}(C) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}, -1}(C)$  is the identity (of  $C$ ).

The next result is a variant of the Hochschild-Kostant-Rosenberg Theorem. A slightly weaker result appeared in [Ye1]. See [Ko1] for the  $C^\infty$  version.

**Theorem 3.12** ([Ye2, Corollary 4.12]). *Suppose  $C$  is a smooth  $\mathbb{K}$ -algebra. Then the homomorphism  $\mathcal{U}_1 : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}}(C)$  and the inclusion  $\mathcal{D}_{\text{poly}}^{\text{nor}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C)$  are both quasi-isomorphisms of complexes of  $C$ -modules.*

Here is a slight modification of the celebrated result of Kontsevich, known as the *Kontsevich Formality Theorem* [Ko1, Theorem 6.4]. In the form below it is proved in [Ye2, Theorem 4.13].

**Theorem 3.13.** *Let  $\mathbb{K}[\mathbf{t}] = \mathbb{K}[t_1, \dots, t_n]$  be the polynomial algebra in  $n$  variables, and assume that  $\mathbb{R} \subset \mathbb{K}$ . There is a collection of  $\mathbb{K}$ -linear homomorphisms*

$$\mathcal{U}_j : \bigwedge^j \mathcal{T}_{\text{poly}}(\mathbb{K}[\mathbf{t}]) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{K}[\mathbf{t}]),$$

indexed by  $j \in \{1, 2, \dots\}$ , satisfying the following conditions.

- (i) *The sequence  $\mathcal{U} = \{\mathcal{U}_j\}$  is an  $L_\infty$ -morphism  $\mathcal{T}_{\text{poly}}(\mathbb{K}[\mathbf{t}]) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{K}[\mathbf{t}])$ .*
- (ii) *Each  $\mathcal{U}_j$  is a poly differential operator of  $\mathbb{K}[\mathbf{t}]$ -modules.*
- (iii) *Each  $\mathcal{U}_j$  is equivariant for the standard action of  $\text{GL}_n(\mathbb{K})$  on  $\mathbb{K}[\mathbf{t}]$ .*
- (iv) *The homomorphism  $\mathcal{U}_1$  is given by equation (3.11).*
- (v) *For any  $j \geq 2$  and  $\alpha_1, \dots, \alpha_j \in \mathcal{T}_{\text{poly}}^0(\mathbb{K}[\mathbf{t}])$  one has  $\mathcal{U}_j(\alpha_1 \wedge \dots \wedge \alpha_j) = 0$ .*
- (vi) *For any  $j \geq 2$ ,  $\alpha_1 \in \mathfrak{gl}_n(\mathbb{K}) \subset \mathcal{T}_{\text{poly}}^0(\mathbb{K}[\mathbf{t}])$  and  $\alpha_2, \dots, \alpha_j \in \mathcal{T}_{\text{poly}}(\mathbb{K}[\mathbf{t}])$  one has  $\mathcal{U}_j(\alpha_1 \wedge \dots \wedge \alpha_j) = 0$ .*

**Remark 3.14.** Presumably the image of  $\mathcal{U}_j$  is inside  $\mathcal{D}_{\text{poly}}^{\text{nor}}(\mathbb{K}[\mathbf{t}])$  for all  $j$ . However we did not verify this.

**Remark 3.15.** The methods of Tamarkin [Ta, Hi2], or suitable arithmetic considerations [Ko2], should make it possible to extend Theorem 3.13, and hence all results of our paper, to any field  $\mathbb{K}$  of characteristic 0.

Consider the power series algebra  $\mathbb{K}[[\mathbf{t}]] = \mathbb{K}[[t_1, \dots, t_n]]$ . As in Example 2.4, the  $\mathbb{K}[[\mathbf{t}]]$ -modules  $\mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  and  $\mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  have the  $\mathbf{t}$ -adic dir-inv structures. These are DG Lie algebras in  $\text{Dir Inv Mod } \mathbb{K}$ . Because  $\mathbb{K}[\mathbf{t}] \rightarrow \mathbb{K}[[\mathbf{t}]]$  is flat and  $\mathbf{t}$ -adically formally étale, it follows that there is an induced  $L_\infty$  morphism  $\mathcal{U} : \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ . Since each

$$\mathcal{U}_j : \prod^j \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \rightarrow \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$$

is a poly differential operator over  $\mathbb{K}[[\mathbf{t}]]$ , it is continuous for the dir-inv structures. See [Ye2, Proposition 4.6] for details and proofs.

Now suppose we are given a complete super-commutative associative unital DG algebra  $A = \bigoplus_{i \geq 0} A^i$  in  $\text{Dir Inv Mod } \mathbb{K}$ . Let

$$\mathcal{U}_{A;j} : \prod^j (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])) \rightarrow A \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$$

be the continuous  $A$ -multilinear extension of  $\mathcal{U}_j$ . It almost immediate from Definition 3.7 that  $\mathcal{U}_A = \{\mathcal{U}_{A;j}\}_{j \geq 1}$  is an  $L_\infty$  morphism; see [Ye2, Proposition 3.25].

Let's recall the notion of twisting for a DG Lie algebra  $\mathfrak{g}$ . Suppose  $\omega \in \mathfrak{g}^1$  is a solution of the MC equation (3.1). The twisted DG Lie algebra  $\mathfrak{g}_\omega$  is the same graded Lie algebra, but the new differential is  $d_\omega := d + \text{ad}(\omega)$ ; i.e.  $d_\omega(\alpha) = d(\alpha) + [\omega, \alpha]$ .

**Theorem 3.16** ([Ye2, Theorem 0.1]). *Assume  $\mathbb{R} \subset \mathbb{K}$ . Let  $A = \bigoplus_{i \geq 0} A^i$  be a complete super-commutative associative unital DG algebra in  $\text{Dir Inv Mod } \mathbb{K}$ , and let  $\omega \in A^1 \widehat{\otimes} \mathcal{T}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]])$  be a solution of the Maurer-Cartan equation in  $A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ . For any element  $\alpha \in \bigwedge^j (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))$  define*

$$\mathcal{U}_{A,\omega;j}(\alpha) := \sum_{k \geq 0} \frac{1}{(j+k)!} (\mathcal{U}_{A;j+k})(\omega^k \wedge \alpha) \in A \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]).$$

Let  $\omega' := \mathcal{U}_{A;1}(\omega)$ . Then  $\omega'$  is a solution of the Maurer-Cartan equation in  $A \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ , and the sequence  $\{\mathcal{U}_{A,\omega;j}\}_{j \geq 1}$  is a continuous  $A$ -multilinear  $L_\infty$  quasi-isomorphism

$$\mathcal{U}_{A,\omega} : (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))_\omega \rightarrow (A \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))_{\omega'}.$$

The sum occurring in the definition of  $\mathcal{U}_{A,\omega;j}(\alpha)$  is always finite (but the number of nonzero terms depends on the argument  $\alpha$ ).

The group  $\text{GL}_n(\mathbb{K})$  acts on  $\mathbb{K}[[\mathbf{t}]]$  by linear change of coordinates. This is an action by  $\mathbb{K}$ -algebra automorphisms, and hence  $\text{GL}_n(\mathbb{K})$  acts on  $\mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  and  $\mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  by continuous DG Lie algebra automorphisms. Suppose we are given an action of  $\text{GL}_n(\mathbb{K})$  on  $A$  by continuous unital DG algebra automorphisms. Then we obtain an action of  $\text{GL}_n(\mathbb{K})$  on  $A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  and  $A \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  by continuous DG Lie algebra automorphisms.

**Proposition 3.17.** *Each operator  $\mathcal{U}_{A;j}$  is  $\text{GL}_n(\mathbb{K})$ -equivariant, i.e.  $\mathcal{U}_{A;j}(g(\alpha)) = g(\mathcal{U}_{A;j}(\alpha))$  for any  $g \in \text{GL}_n(\mathbb{K})$  and  $\alpha \in \bigwedge^j (A \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))$ . Moreover, if  $\omega$  is  $\text{GL}_n(\mathbb{K})$ -invariant, then each operator  $\mathcal{U}_{A,\omega;j}$  is  $\text{GL}_n(\mathbb{K})$ -equivariant.*

*Proof.* Using continuity and multilinearity we may assume that

$$\alpha = (a_1 \otimes \alpha_1) \wedge \cdots \wedge (a_j \otimes \alpha_j),$$

with  $a_k \in A$  and  $\alpha_k \in \mathcal{T}_{\text{poly}}(\mathbb{K}[[\hbar]])$ . Then

$$\begin{aligned} g(\mathcal{U}_{A;j}(\alpha)) &= \pm g(a_1 \cdots a_j \cdot \mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j)) \\ &= \pm g(a_1 \cdots a_j) \cdot g(\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j)) \\ &= \diamond \pm g(a_1 \cdots a_j) \cdot \mathcal{U}_j(g(\alpha_1 \wedge \cdots \wedge \alpha_j)) \\ &= \mathcal{U}_{A;j}(g(\alpha)), \end{aligned}$$

where the equality marked  $\diamond$  is due to condition (iii) of Theorem 3.13.

We see that  $g(\mathcal{U}_{A,\omega;j}(\alpha)) = \mathcal{U}_{A,g(\omega);j}(g(\alpha))$ . Hence the second assertion.  $\square$

Let  $X$  be a smooth irreducible separated  $n$ -dimensional  $\mathbb{K}$ -scheme.

**Proposition 3.18.** *There are sheaves of DG Lie algebras  $\mathcal{T}_{\text{poly},X}$ ,  $\mathcal{D}_{\text{poly},X}$  and  $\mathcal{D}_{\text{poly},X}^{\text{nor}}$  on  $X$ . As left  $\mathcal{O}_X$ -modules all three are quasi-coherent. The sheaves  $\mathcal{D}_{\text{poly},X}$  and  $\mathcal{D}_{\text{poly},X}^{\text{nor}}$  are quasi-coherent left  $\mathcal{D}_X$ -modules. For any affine open set  $U = \text{Spec } C \subset X$  one has  $\Gamma(U, \mathcal{T}_{\text{poly},X}) = \mathcal{T}_{\text{poly}}(C)$ ,  $\Gamma(U, \mathcal{D}_{\text{poly},X}) = \mathcal{D}_{\text{poly}}(C)$  and  $\Gamma(U, \mathcal{D}_{\text{poly},X}^{\text{nor}}) = \mathcal{D}_{\text{poly}}^{\text{nor}}(C)$  as DG Lie algebras and as  $C$ -modules.*

*Proof.* Let  $U' = \text{Spec } C' \subset U$  be an open subset. Then  $C \rightarrow C'$  is an étale ring homomorphism. According to [Ye2, Proposition 4.6] there are functorial DG Lie algebra homomorphisms  $\mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{T}_{\text{poly}}(C')$  and  $\mathcal{D}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C')$  such that  $C' \otimes_C \mathcal{T}_{\text{poly}}(C) \cong \mathcal{T}_{\text{poly}}(C')$  and  $C' \otimes_C \mathcal{D}_{\text{poly}}(C) \cong \mathcal{D}_{\text{poly}}(C')$ . Therefore we get quasi-coherent sheaves  $\mathcal{T}_{\text{poly},X}$  and  $\mathcal{D}_{\text{poly},X}$ .

For any  $i \in \{1, \dots, p+1\}$  let  $\epsilon_i : \mathcal{D}_{\text{poly},X}^p \rightarrow \mathcal{D}_{\text{poly},X}^{p-1}$  be the map  $\epsilon(\phi)(f_1, \dots, f_p) := \phi(f_1, \dots, 1, \dots, f_p)$ , with 1 inserted at the  $i$ -th position. This is an  $\mathcal{O}_X$ -linear homomorphism, and  $\mathcal{D}_{\text{poly},X}^{\text{nor},p} = \bigcap \text{Ker}(\epsilon_i)$ . Thus  $\mathcal{D}_{\text{poly},X}^{\text{nor},p}$  is quasi-coherent.

The left  $\mathcal{D}_X$ -module structures on  $\mathcal{D}_{\text{poly},X}$  and  $\mathcal{D}_{\text{poly},X}^{\text{nor}}$  are by composition of operators.  $\square$

Following Kontsevich we call  $\mathcal{T}_{\text{poly},X}$  the algebra of *poly vector fields* on  $X$ , and  $\mathcal{D}_{\text{poly},X}$  is called the algebra of *poly differential operators*. The subalgebra  $\mathcal{D}_{\text{poly},X}^{\text{nor}}$  is called the algebra of *normalized poly differential operators*.

Let us write  $\mathcal{T}_{\text{poly}}(X) = \Gamma(X, \mathcal{T}_{\text{poly},X})$ , the DG Lie algebra of global poly vector fields on  $X$ . We consider each  $\mathcal{T}_{\text{poly}}^p(X)$  as a discrete inv module, and  $\mathcal{T}_{\text{poly}}(X) = \bigoplus_p \mathcal{T}_{\text{poly}}^p(X)$  gets the  $\bigoplus$  dir-inv structure, so it is a discrete DG Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}$ . Likewise we define  $\mathcal{D}_{\text{poly}}(X)$  and  $\mathcal{D}_{\text{poly}}^{\text{nor}}(X)$ .

A series  $\alpha = \sum_{k=1}^{\infty} \alpha_k \hbar^k \in \mathcal{T}_{\text{poly}}^1(X)[[\hbar]]^+$  satisfying  $[\alpha, \alpha] = 0$  is called a *formal Poisson structure* on  $X$ . Two formal Poisson structure  $\alpha$  and  $\alpha'$  are called gauge equivalent if there is some  $\gamma = \sum_{k=1}^{\infty} \gamma_k \hbar^k \in \mathcal{T}_{\text{poly}}^0(X)[[\hbar]]^+$  such that  $\alpha' = \exp(\text{ad}(\gamma))(\alpha)$ . Thus the set  $\text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+)$  is the set of gauge equivalence classes of formal Poisson structures on  $X$ .

**Example 3.19.** Let  $\alpha_1 \in \Gamma(X, \bigwedge_{\mathcal{O}_X}^2 \mathcal{T}_X)$  be a Poisson structure on  $X$  (Definition 1.9). Then  $\alpha := \alpha_1 \hbar$  is a formal Poisson structure.

**Proposition 3.20.** *An element*

$$\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\text{poly}}^{\text{nor},1}(X)[[\hbar]]^+$$



is a solution of the Maurer-Cartan equation in  $\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+$  iff the pairing

$$(f, g) \mapsto f \star_{\beta} g := fg + \sum_{j=1}^{\infty} \beta_j(f, g) \hbar^j,$$

for local sections  $f, g \in \mathcal{O}_X$ , is a star product on  $\mathcal{O}_X[[\hbar]]$  (see Definition 1.1).

*Proof.* The assertion is actually local: it is enough to prove it for an affine open set  $U = \text{Spec } C \subset X$ . Take  $\beta \in \mathcal{D}_{\text{poly}}^{\text{nor},1}(C)[[\hbar]]^+$ . We have to prove that  $\star_{\beta}$  is an associative product on  $C[[\hbar]]$  iff  $\beta$  is a solution of the MC equation in  $\mathcal{D}_{\text{poly}}^{\text{nor}}(C)[[\hbar]]^+$ . This assertion is made in [Ke, Corollary 4.5]. See also [Ko1, Section 4.6.2]. (For a non-differential star product this is the original discovery of Gerstenhaber, see [Ge].)  $\square$

**Proposition 3.21.** *Under the identification, in Proposition 3.20, of solutions of the MC equation in  $\mathcal{D}_{\text{poly}}^{\text{nor},1}(X)[\hbar]^+$  with star products on  $\mathcal{O}_X[[\hbar]]$ , the notion of gauge equivalence in Definition 3.4 coincides with that in Proposition 1.14.*

*Proof.* Let  $\beta$  and  $\beta'$  be two solutions of the MC equation in  $\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+$ , and let  $\star$  and  $\star'$  be the corresponding star products on  $\mathcal{O}_X[[\hbar]]$ . Given  $\gamma \in \mathcal{D}_{\text{poly}}^{\text{nor},0}(X)[[\hbar]]^+$ , let  $\exp(\gamma) := 1 + \gamma + \frac{1}{2}\gamma^2 + \dots$  be the corresponding gauge equivalence of  $\mathcal{O}_X[[\hbar]]$ . As stated implicitly in [Ko1, Section 4.6.2] and [Ke, Ch. 2, Lemma 4.2 and Section 5.1], one has  $\beta' = \exp(\text{af})(\exp(\pm\gamma))(\beta)$  iff for all local sections  $f, g \in \mathcal{O}_X$  one has

$$(3.22) \quad f \star' g = \exp(\gamma)^{-1}(\exp(\gamma)(f) \star \exp(\gamma)(g)).$$

(The reason for the sign ambiguity is that the references [Ko1], [Ke] and [GM] are inconsistent with each other regarding signs, and we did not carry out this calculation ourselves.)  $\square$

An immediate consequence is:

**Corollary 3.23.** *The assignment  $\beta \mapsto \star_{\beta}$  of Proposition 3.20 gives rise to a bijection from  $\text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+)$  to set of gauge equivalence classes of globally trivialized deformation quantizations of  $\mathcal{O}_X$ .*

Here is a first approximation of Theorem 0.1.

**Corollary 3.24.** *Let  $X$  be an  $n$ -dimensional affine scheme admitting an étale morphism  $X \rightarrow \mathbf{A}_{\mathbb{K}}^n$ . Then there is a bijection  $Q$  as in Theorem 0.1.*

*Proof.* Write  $X = \text{Spec } C$  and  $\mathbf{A}_{\mathbb{K}}^n = \text{Spec } \mathbb{K}[\mathbf{t}]$ . Because  $\mathbb{K}[\mathbf{t}] \rightarrow C$  is an étale ring homomorphism, according to [Ye2, Proposition 4.6] we have  $\mathcal{T}_{\text{poly}}(C) = C \otimes_{\mathbb{K}[\mathbf{t}]} \mathcal{T}_{\text{poly}}(\mathbb{K}[\mathbf{t}])$  and  $\mathcal{D}_{\text{poly}}(C) = C \otimes_{\mathbb{K}[\mathbf{t}]} \mathcal{D}_{\text{poly}}(\mathbb{K}[\mathbf{t}])$ . By condition (ii) in Theorem 3.13 the universal operators  $\mathcal{U}_j$  are poly differential operators over  $\mathbb{K}[\mathbf{t}]$ , and hence according to [Ye2, Proposition 2.6] they extend to  $C$ -multilinear operators, giving an  $L_{\infty}$  morphism  $\mathcal{U} : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C)$ ; and by [Ye2, Corollary 4.12] this is an  $L_{\infty}$  quasi-isomorphism. The inclusion  $\mathcal{D}_{\text{poly}}^{\text{nor}}(C) \rightarrow \mathcal{D}_{\text{poly}}(C)$  is a DG Lie algebra quasi-isomorphism. Now use Corollaries 3.10 and 3.23.  $\square$

**Remark 3.25.** The method of  $L_{\infty}$  morphisms is suitable only for characteristic 0. For an approach in positive characteristic see [BK2].

## 4. FORMAL GEOMETRY – COORDINATE BUNDLES ETC.

In this section we translate the notions of formal geometry (in the sense of Gelfand-Kazhdan [GK]; cf. [Ko1, Section 7]) to the language of algebraic geometry (schemes and sheaves). As before  $\mathbb{K}$  is a field of characteristic 0, and  $X$  is a smooth separated irreducible scheme over  $\mathbb{K}$  of dimension  $n$ .

For a closed point  $x \in X$  the residue field  $\mathbf{k}(x)$  lifts uniquely into the complete local ring  $\widehat{\mathcal{O}}_{X,x}$ , and any choice of system of coordinates  $\mathbf{t} = (t_1, \dots, t_n)$  gives rise to an isomorphism of  $\mathbb{K}$ -algebras

$$\widehat{\mathcal{O}}_{X,x} \cong \mathbf{k}(x)[[\mathbf{t}]] = \mathbf{k}(x)[[t_1, \dots, t_n]].$$

Of course the condition that an  $n$ -tuple of elements  $\mathbf{t}$  in the maximal ideal  $\mathfrak{m}_x$  is a system of coordinates is that their residue classes form a basis of the  $\mathbf{k}(x)$ -module  $\mathfrak{m}_x/\mathfrak{m}_x^2$ , the Zariski cotangent space.

Suppose  $U$  is an open neighborhood of  $x$  and  $f \in \Gamma(U, \mathcal{O}_X)$ . The Taylor expansion of  $f$  at  $x$  w.r.t.  $\mathbf{t}$  is

$$f = \sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}} \mathbf{t}^{\mathbf{i}} \in \widehat{\mathcal{O}}_{X,x},$$

where  $a_{\mathbf{i}} \in \mathbf{k}(x)$  and  $\mathbf{t}^{\mathbf{i}} := t_1^{i_1} \dots t_n^{i_n}$ . The coefficients are given by the usual formula

$$a_{\mathbf{i}} = \frac{1}{\mathbf{i}!} \left( \left( \frac{\partial}{\partial t_1} \right)^{i_1} \dots \left( \frac{\partial}{\partial t_n} \right)^{i_n} f \right)(x),$$

where for any  $g \in \widehat{\mathcal{O}}_{X,x}$  we write  $g(x) \in \mathbf{k}(x)$  for its residue class.

The *jet bundle* of  $X$  is an infinite dimensional scheme  $\text{Jet } X$ , that comes with a projection  $\pi_{\text{jet}} : \text{Jet } X \rightarrow X$ . Given a closed point  $x \in X$  the  $\mathbf{k}(x)$ -rational points of the fiber  $\pi_{\text{jet}}^{-1}(x)$  correspond to “jets of functions at  $x$ ”, namely to elements of the complete local ring  $\widehat{\mathcal{O}}_{X,x}$ . Here is a way to visualize such a fiber: choose a coordinate system  $\mathbf{t}$ . Then a jet is just the data  $\{a_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{N}^n}$  of its Taylor coefficients. So set theoretically the fiber  $\pi_{\text{jet}}^{-1}(x)$  is just the set  $\mathbf{k}(x)^{\mathbb{N}^n}$ .

The naive description above does not make  $\text{Jet } X$  into a scheme. So let us try another approach. Consider the diagonal embedding  $\Delta : X \rightarrow X^2 = X \times X$ . Let  $\mathcal{I}_{X,\text{alg}}$  be the ideal sheaf  $\text{Ker}(\Delta^* : \mathcal{O}_{X^2} \rightarrow \mathcal{O}_X)$ . The *sheaf of principal parts* of  $X$  is

$$\mathcal{P}_X = \mathcal{P}_{X/\mathbb{K}} := \lim_{\leftarrow d} \mathcal{O}_{X^2} / \mathcal{I}_{X,\text{alg}}^d$$

(cf. [EGA IV]). It is a sheaf of commutative rings, equipped with two ring homomorphisms  $p_1^*, p_2^* : \mathcal{O}_X \rightarrow \mathcal{P}_X$ , namely  $p_1^*(f) := f \otimes 1$  and  $p_2^*(f) := 1 \otimes f$ . We consider  $\mathcal{P}_X$  as a left  $\mathcal{O}_X$ -module via  $p_1^*$  and as a right  $\mathcal{O}_X$ -module via  $p_2^*$ .

The sheaf of rings  $\mathcal{P}_X$  can be thought of the the structure sheaf  $\mathcal{O}_{\mathfrak{X}}$  of the formal scheme  $\mathfrak{X}$  which is the formal completion of  $X^2$  along  $\Delta(X)$ . We denote by  $\mathcal{I}_X$  the ideal  $\text{Ker}(\mathcal{P}_X \rightarrow \mathcal{O}_X)$ ; it is just the completion of the ideal  $\mathcal{I}_{X,\text{alg}}$ . By default we shall consider  $\mathcal{P}_X$  as an  $\mathcal{O}_X$ -algebra via  $p_1^*$ .

**Proposition 4.1** ([Ye1, Lemma 2.6]). *Let  $U \subset X$  be an open set admitting an étale morphism  $U \rightarrow \mathbf{A}_{\mathbb{K}}^n = \text{Spec } \mathbb{K}[s_1, \dots, s_n]$ . For  $i = 1, \dots, n$  define*

$$\tilde{s}_i := 1 \otimes s_i - s_i \otimes 1 \in \Gamma(U, \mathcal{I}).$$

*Then*

$$\mathcal{P}_X|_U \cong \mathcal{O}_U[[\tilde{s}_1, \dots, \tilde{s}_n]]$$

*as sheaves of  $\mathcal{O}_U$ -algebras, either via  $p_1^*$  or via  $p_2^*$ .*

**Definition 4.2.** Let  $U \subset X$  be an open set.

- (1) A *system of étale coordinates* on  $U$  is a sequence  $\mathbf{s} = (s_1, \dots, s_n)$  of elements in  $\Gamma(U, \mathcal{O}_X)$  s.t. the morphism  $U \rightarrow \mathbf{A}_{\mathbb{K}}^n$  it determines is étale.
- (2) A *system of formal coordinates* on  $U$  is a sequence  $\mathbf{t} = (t_1, \dots, t_n)$  of elements in  $\Gamma(U, \mathcal{I}_X)$  s.t. the homomorphism of sheaves of rings  $\mathcal{O}_U[[\mathbf{t}]] \rightarrow \mathcal{P}_X|_U$  extending  $\mathfrak{p}_1^*$  is an isomorphism.

**Proposition 4.3.** *Given a closed point  $x \in X$  one has a canonical isomorphism of  $\mathcal{O}_X$ -algebras (via  $\mathfrak{p}_2^*$ ):*

$$\mathbf{k}(x) \otimes_{\mathcal{O}_X} \mathcal{P}_X \cong \widehat{\mathcal{O}}_{X,x}.$$

If  $\mathbf{t} = (t_1, \dots, t_n)$  is a system of formal coordinates on some neighborhood  $U$  of  $x$ , and we let  $t_i(x) := 1 \otimes t_i \in \widehat{\mathcal{O}}_{X,x}$  under the above isomorphism, then the sequence  $\mathbf{t}(x) := (t_1(x), \dots, t_n(x))$  is a system of coordinates in  $\widehat{\mathcal{O}}_{X,x}$ .

The easy proof is left out.

**Example 4.4.** Assume  $\mathbf{s}$  is a system of étale coordinates on  $U$ , and let  $t_i := \tilde{s}_i$ . By Proposition 4.1 the sequence  $\mathbf{t} := (t_1, \dots, t_n)$  is a system of formal coordinates on  $U$ . Given a closed point  $x \in U$  we have  $t_i(x) = s_i - s_i(x)$ , where  $s_i(x) \in \mathbf{k}(x) \subset \widehat{\mathcal{O}}_{X,x}$ . The sequence  $\mathbf{t}(x) = (t_1(x), \dots, t_n(x))$  is a system of coordinates in  $\widehat{\mathcal{O}}_{X,x}$ .

**Corollary 4.5.** *Let  $U \subset X$  be an open set admitting a formal system of coordinates  $\mathbf{t} \in \Gamma(U, \mathcal{I}_X)^{\times n}$ . For  $f \in \Gamma(U, \mathcal{O}_X)$  let us write*

$$\mathfrak{p}_2^*(f) = \sum_i \mathfrak{p}_1^*(a_i) \mathbf{t}^i \in \Gamma(U, \mathcal{P}_X)$$

with  $a_i \in \Gamma(U, \mathcal{O}_X)$ . Then under the isomorphism of Proposition 4.3 we recover the Taylor expansion at any closed point  $x \in U$ :

$$f = \sum_i a_i(x) \mathbf{t}(x)^i \in \widehat{\mathcal{O}}_{X,x}.$$

Again the easy proof is omitted.

The conclusion from Proposition 4.3 is that the sheaf of sections of the bundle  $\text{Jet } X$  should be  $\mathcal{P}_X$ . By the standard schematic formalism we deduce the defining formula

$$\text{Jet } X := \text{Spec}_X \mathcal{S}_{\mathcal{O}_X} \mathcal{D}_X,$$

where  $\mathcal{D}_X = \mathcal{H}om_{\mathcal{O}_X}^{\text{cont}}(\mathcal{P}_X, \mathcal{O}_X)$ , the sheaf of differential operators, is considered as a locally free left  $\mathcal{O}_X$ -module;  $\mathcal{S}_{\mathcal{O}_X} \mathcal{D}_X$  is the symmetric algebra of the  $\mathcal{O}_X$ -module  $\mathcal{D}_X$ ; and  $\text{Spec}_X$  refers to the relative spectrum over  $X$  of a quasi-coherent  $\mathcal{O}_X$ -algebra.

If  $U$  is a sufficiently small affine open set in  $X$  admitting an étale coordinate system  $\mathbf{s}$ , then  $\text{Jet } U$  can be made more explicit. We know that

$$\Gamma(U, \mathcal{D}_X) = \bigoplus_{\mathbf{i} \in \mathbb{N}^n} \Gamma(U, \mathcal{O}_X) \left( \frac{\partial}{\partial s_1} \right)^{i_1} \cdots \left( \frac{\partial}{\partial s_n} \right)^{i_n},$$

so letting  $\xi_i$  be a commutative indeterminate we have

$$\pi_{\text{jet}}^{-1}(U) \cong \text{Jet } U \cong \text{Spec } \Gamma(U, \mathcal{O}_X)[\{\xi_i\}_{i \in \mathbb{N}^n}].$$

The next geometric object we need is the *bundle of formal coordinate systems* of  $X$ , which we denote by  $\text{Coor } X$ . (In [Ko1] the notation is  $X^{\text{coor}}$ . However we feel that stylistically  $\text{Coor } X$  is better, since it resembles the usual bundle notation  $\text{TX}$

and  $T^*X$ .) The scheme  $\text{Coor } X$  comes with a projection  $\pi_{\text{coor}} : \text{Coor } X \rightarrow X$ , and the fiber over a closed point  $x \in X$  corresponds to the set of  $\mathbb{K}$ -algebra isomorphisms  $\mathbf{k}(x)[[\mathbf{t}]] \xrightarrow{\cong} \widehat{\mathcal{O}}_{X,x}$ . We are going to make this more precise below.

Consider a morphism of schemes  $f : Y \rightarrow X$ . Unless  $f$  is quasi-finite the sheaf  $f^*\mathcal{P}_X$  is not particularly interesting, but its completion is. To make this work nicely we are going to use inv structures (see Section 2). Since  $(X, \mathcal{P}_X)$  is a noetherian formal scheme, any coherent  $\mathcal{P}_X$ -module  $\mathcal{M}$  has the  $\mathcal{I}_X$ -adic inv structure (see Example 2.4). Using the ring homomorphism  $\text{p}_1^* : \mathcal{O}_X \rightarrow \mathcal{P}_X$  the module  $\mathcal{M}$  becomes an inv  $\mathcal{O}_X$ -module, and so the complete inverse image  $f^*\mathcal{M}$  is defined. Taking  $\mathcal{M} := \mathcal{P}_X$  we get an  $\mathcal{O}_Y$ -algebra  $f^*\mathcal{P}_X$ . By Proposition 4.1 we see that  $f^*\mathcal{I}_X$  is a sheaf of ideals in  $f^*\mathcal{P}_X$ , and  $f^*\mathcal{P}_X$  is  $f^*\mathcal{I}_X$ -adically complete in the usual sense, namely  $f^*\mathcal{P}_X \cong \varprojlim_m f^*\mathcal{P}_X/(f^*\mathcal{I}_X)^m$ .

Now we can state the geometric property that should characterize  $\text{Coor } X$ . There should be a sequence  $\mathbf{t} = (t_1, \dots, t_n)$  of elements in  $\Gamma(\text{Coor } X, \pi_{\text{coor}}^*\mathcal{I}_X)$  such that for any open set  $U \subset X$ , the assignment  $\sigma \mapsto \sigma^*(\mathbf{t}) := (\sigma^*(t_1), \dots, \sigma^*(t_n))$  shall be a bijection from the set of sections  $\sigma : U \rightarrow \text{Coor } X$  to the set of formal coordinate systems on  $U$ . Thus the sheaf of sections of  $\text{Coor } X$ , let us call it  $\mathcal{Q}$ , has to be a subsheaf of  $\mathcal{I}_X^{\times n} = \mathcal{I}_X \times \dots \times \mathcal{I}_X$ . The condition for an  $n$ -tuple  $\mathbf{t}$  to be in  $\mathcal{Q}$  is that under the composed map

$$(4.6) \quad \mathcal{I}_X^{\times n} \rightarrow (\mathcal{I}_X/\mathcal{I}_X^2)^{\times n} = (\Omega_X^1)^{\times n} \xrightarrow{\wedge} \Omega_X^n$$

one has  $t_1 \wedge \dots \wedge t_n \neq 0$ . Note that  $\mathcal{Q} = \varprojlim_m \mathcal{Q}^m$ , where  $\mathcal{Q}^1$  is the sheaf of frames of  $\Omega_X^1$ .

We conclude that  $\text{Coor } X$  is a subscheme of  $(\text{Jet } X)^{\times n}$ . Specifically,  $\text{Coor } X$  is an open subscheme of  $\text{Spec}_X \text{S}_{\mathcal{O}_X}((\mathcal{D}_X/\mathcal{O}_X)^{\oplus n})$ , where  $\mathcal{D}_X/\mathcal{O}_X$  is viewed as a locally free left  $\mathcal{O}_X$ -module. Over any affine open set  $U \subset X$  admitting an étale coordinate system, say  $\mathbf{s}$ , one has

$$(4.7) \quad \pi_{\text{coor}}^{-1}(U) \cong \text{Coor } U \cong \text{Spec } \Gamma(U, \mathcal{O}_X)[\{\xi_{i,j}\}, d^{-1}].$$

In this formula  $\mathbf{i} = (i_1, \dots, i_n)$  runs over  $\mathbb{N}^n - \{(0, \dots, 0)\}$  and  $j$  runs over  $\{1, \dots, n\}$ . The indeterminate  $\xi_{i,j}$  corresponds to the DO  $\frac{1}{\mathbf{i}!} (\frac{\partial}{\partial s_1})^{i_1} \dots (\frac{\partial}{\partial s_n})^{i_n}$  in the  $j$ -th copy of  $\mathcal{D}_X$ . The symbol  $\mathbf{e}_i$  denotes the row whose only nonzero entry is 1 in the  $i$ -th place, so  $[\xi_{\mathbf{e}_i, j}]$  is the matrix whose  $(i, j)$  entry is the indeterminate  $\xi_{\mathbf{e}_i, j}$  corresponding to the DO  $\frac{\partial}{\partial s_i}$  in the  $j$ -th copy of  $\mathcal{D}_X$ . Finally  $d := \det([\xi_{\mathbf{e}_i, j}])$ .

The next results justify the heuristic considerations above.

**Theorem 4.8.** *Consider the functor  $F : (\text{Sch}/X)^{\text{op}} \rightarrow \text{Sets}$  defined as follows. For any  $X$ -scheme  $Y$ , with structural morphism  $g : Y \rightarrow X$ , we let  $FY$  be the set of  $\mathcal{O}_Y$ -algebra isomorphisms  $\phi : \mathcal{O}_Y[[\mathbf{t}]] \xrightarrow{\cong} g^*\mathcal{P}_X$  such that  $\phi(\mathcal{O}_Y[[\mathbf{t}]] \cdot \mathbf{t}) = g^*\mathcal{I}_X$ . Then  $\text{Coor } X$  is a fine moduli space for  $F$ , namely  $F \cong \text{Hom}_{\text{Sch}/X}(-, \text{Coor } X)$ .*

*Proof.* Suppose we are given  $g : Y \rightarrow X$  and  $\phi : \mathcal{O}_Y[[\mathbf{t}]] \xrightarrow{\cong} g^*\mathcal{P}_X$ . Define  $a_i := \phi(t_i) \in \Gamma(Y, g^*\mathcal{I}_X)$ . These elements satisfy  $a_1 \wedge \dots \wedge a_n \neq 0$  like in equation (4.6). Each  $a_i$  gives rise to an  $\mathcal{O}_Y$ -linear sheaf homomorphism  $\mathcal{O}_Y \rightarrow g^*\mathcal{I}_X$ . Since  $\mathcal{I}_X/\mathcal{I}_X^m$  is a coherent locally free  $\mathcal{O}_X$ -module for every  $m \geq 1$  we see that

$$\mathcal{H}om_{\mathcal{O}_Y}^{\text{cont}}(g^*\mathcal{I}_X, \mathcal{O}_Y) \cong \mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\mathcal{H}om_{\mathcal{O}_X}^{\text{cont}}(\mathcal{I}_X, \mathcal{O}_X) = g^*(\mathcal{D}_X/\mathcal{O}_X).$$

So after dualization, i.e. applying the functor  $\mathcal{H}om_{\mathcal{O}_Y}^{\text{cont}}(-, \mathcal{O}_Y)$ , each  $a_i$  gives a homomorphism of  $\mathcal{O}_Y$ -modules  $g^*(\mathcal{D}_X/\mathcal{O}_X) \rightarrow \mathcal{O}_Y$ . By adjunction we get  $\mathcal{O}_X$ -linear homomorphisms  $\mathcal{D}_X/\mathcal{O}_X \rightarrow g_*\mathcal{O}_Y$ , and therefore an  $\mathcal{O}_X$ -algebra homomorphism

$S_{\mathcal{O}_X}((\mathcal{D}_X/\mathcal{O}_X)^{\oplus n}) \rightarrow g_*\mathcal{O}_Y$ . Passing to schemes we obtain a morphism of  $X$ -schemes

$$\tilde{\phi} : Y \rightarrow \mathrm{Spec}_X S_{\mathcal{O}_X}((\mathcal{D}_X/\mathcal{O}_X)^{\oplus n}).$$

Because  $a_1 \wedge \cdots \wedge a_n \neq 0$  this is actually a morphism  $\tilde{\phi} : Y \rightarrow \mathrm{Coor} X$ . The process we have described is reversible, and hence  $FY \cong \mathrm{Hom}_{\mathrm{Sch}/X}(Y, \mathrm{Coor} X)$ .  $\square$

**Corollary 4.9.** *There is a canonical isomorphism of  $\mathcal{O}_{\mathrm{Coor} X}$ -algebras*

$$\mathcal{O}_{\mathrm{Coor} X}[[\mathbf{t}]] \cong \pi_{\mathrm{coor}}^* \mathcal{P}_X.$$

*This isomorphism has the following universal property: for any open set  $U \subset X$  the assignment  $\sigma \mapsto \sigma^*(\mathbf{t})$  is a bijection of sets*

$$\mathrm{Hom}_{\mathrm{Sch}/X}(U, \mathrm{Coor} X) \xrightarrow{\cong} \{\text{formal coordinate systems on } U\}.$$

*Proof.* Applying the theorem to  $Y := \mathrm{Coor} X$ ,  $g := \pi_{\mathrm{coor}}$  and the identity morphism  $\tilde{\phi}_0 : \mathrm{Coor} X \rightarrow \mathrm{Coor} X$ , we obtain a canonical isomorphism  $\phi_0 : \mathcal{O}_{\mathrm{Coor} X}[[\mathbf{t}]] \cong \pi_{\mathrm{coor}}^* \mathcal{P}_X$  with the desired universal property.  $\square$

On  $\mathrm{Coor} X$  we have a universal Taylor expansion:

**Corollary 4.10.** *Suppose  $U \subset X$  is open and  $f \in \Gamma(U, \mathcal{O}_X)$ . Then there are functions  $a_i \in \Gamma(\pi_{\mathrm{coor}}^{-1}(U), \mathcal{O}_{\mathrm{Coor} X})$  s.t.*

$$\pi_{\mathrm{coor}}^*(p_2^*(f)) = \sum_{i \in \mathbb{N}^n} a_i \mathbf{t}^i \in \Gamma(\pi_{\mathrm{coor}}^{-1}(U), \pi_{\mathrm{coor}}^* \mathcal{P}_X),$$

*where  $\mathbf{t}$  is the universal coordinate system in  $\Gamma(\mathrm{Coor} X, \pi_{\mathrm{coor}}^* \mathcal{P}_X)$ . Given a section  $\sigma : U \rightarrow \mathrm{Coor} X$  we obtain a Taylor expansion*

$$p_2^*(f) = \sum_{i \in \mathbb{N}^n} \sigma^*(a_i) \sigma^*(\mathbf{t})^i \in \Gamma(U, \mathcal{P}_X)$$

*as in Corollary 4.5.*

The proof is left to the reader.

Suppose  $\mathbf{s} = (s_1, \dots, s_n)$  is an étale coordinate system on an open set  $U \subset X$ . As before let  $\tilde{s}_i := 1 \otimes s_i - s_i \otimes 1 \in \Gamma(U, \mathcal{P}_X)$ , and define  $\tilde{\mathbf{s}} := (\tilde{s}_1, \dots, \tilde{s}_n)$ , which is a formal coordinate system on  $U$ . Then on  $\mathrm{Coor} U = \pi_{\mathrm{coor}}^{-1}(U)$  we have isomorphisms of  $\mathcal{O}_{\mathrm{Coor} U}$ -algebras

$$\mathcal{O}_{\mathrm{Coor} U}[[\mathbf{t}]] \cong (\pi_{\mathrm{coor}}^* \mathcal{P}_X)|_{\mathrm{Coor} U} \cong \mathcal{O}_{\mathrm{Coor} U}[[\tilde{\mathbf{s}}]].$$

Using the coordinate functions  $\xi_{i,j} \in \Gamma(\mathrm{Coor} U, \mathcal{O}_{\mathrm{Coor} U})$  from formula (4.7) we then have

$$(4.11) \quad t_j = \sum_i \xi_{i,j} \mathbf{s}^i,$$

where the sum is on  $\mathbf{i} \in \mathbb{N}^n - \{(0, \dots, 0)\}$ .

For  $i \geq 1$  let  $\mathrm{Coor}^i X$  be the bundle over  $X$  parameterizing coordinate systems up to order  $i$  (i.e. modulo order  $\geq i+1$ ). There are projections  $\mathrm{Coor} X \rightarrow \mathrm{Coor}^i X \rightarrow \mathrm{Coor}^{i-1} X \rightarrow X$ . The next theorem describes the geometry of these bundles.

Let  $G(\mathbb{K})$  be the group of  $\mathbb{K}$ -algebra automorphisms of  $\mathbb{K}[[\mathbf{t}]]$ . Then  $G(\mathbb{K})$  is the group of  $\mathbb{K}$ -rational points of a pro-algebraic group  $G = \mathrm{GL}_{n,\mathbb{K}} \ltimes N$ , where  $N$  is a pro-unipotent group. The action of  $\mathrm{GL}_n(\mathbb{K})$  on  $\mathbb{K}[[\mathbf{t}]]$  is by linear change of coordinates; and  $N(\mathbb{K})$  is the subgroup of  $G(\mathbb{K})$  consisting of automorphisms that act trivially modulo  $(\mathbf{t})^2$ .

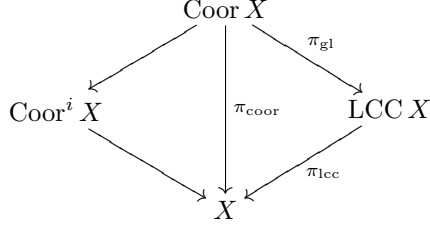


FIGURE 1

According to Corollary 4.9 there is a canonical embedding of  $\mathbb{K}$ -algebras

$$(4.12) \quad \mathbb{K}[[\mathbf{t}]] \hookrightarrow \Gamma(\text{Coor } X, \pi_{\text{coor}}^* \mathcal{P}_X).$$

- Theorem 4.13.** (1)  $\text{Coor } X \cong \lim_{\leftarrow i} \text{Coor}^i X$  as schemes over  $X$ .  
(2)  $\text{Coor } X$  is a  $G$ -torsor over  $X$ . The action of  $G$  on  $\text{Coor } X$  is characterized by the fact that the embedding (4.12) is  $G(\mathbb{K})$ -equivariant.  
(3)  $\text{Coor}^1 X$  is a  $\text{GL}_{n, \mathbb{K}}$ -torsor over  $X$ , and  $\text{Coor } X$  is a  $\text{GL}_{n, \mathbb{K}}$ -equivariant  $N$ -torsor over  $\text{Coor}^1 X$ .  
(4) The geometric quotient (cf. [GIT])

$$\text{LCC } X := \text{Coor } X / \text{GL}_{n, \mathbb{K}}$$

exists, with projection  $\pi_{\text{gl}} : \text{Coor } X \rightarrow \text{LCC } X$ , and  $\text{Coor } X$  is a  $\text{GL}_{n, \mathbb{K}}$ -torsor over  $\text{LCC } X$ .

- (5) Let  $U \subset X$  be an affine open set admitting an étale coordinate system. Then all the torsors in parts (2-4) are trivial over  $U$  (i.e. they admit sections).

“LCC” stands for “linear coordinate classes”. In [Ko1] the notation for  $\text{LCC } X$  is  $X^{\text{aff}}$ . Note that the bundle  $\text{LCC } X$  has no group action; but locally, for  $U$  as in part (5), there’s a non-canonical isomorphism of schemes  $\text{LCC } U \cong N \times U$ .  $\text{Coor}^1 X$  is the frame bundle of  $\Omega_X^1$ . The various bundles and projections are depicted in Figure 1.

*Proof.* (1) This is an immediate consequence of the moduli property of  $\text{Coor } X$  (see Theorem 4.8), and an analogous property of  $\text{Coor}^i X$ .

(2) Given  $g \in G(\mathbb{K})$  let us denote by  $g(\mathbf{t})$  the sequence  $(g(t_1), \dots, g(t_n))$  in  $\mathbb{K}[[\mathbf{t}]]$ . By Theorem 4.8 there exists a unique  $X$ -morphism  $\tilde{g} : \text{Coor } X \rightarrow \text{Coor } X$  such that the algebra homomorphism  $\tilde{g}^* : \Gamma(\text{Coor}, \pi_{\text{coor}}^* \mathcal{P}_X) \rightarrow \Gamma(\text{Coor}, \pi_{\text{coor}}^* \mathcal{P}_X)$  sends  $\mathbf{t}$  to  $g(\mathbf{t})$ . We have to prove that  $\tilde{g}$  is an automorphism, and that  $g \mapsto \tilde{g}$  is a group homomorphism from  $G(\mathbb{K})$  to  $\text{Aut}_{\text{Sch}/X}(\text{Coor } X)$ .

Now via the embedding (4.12), the homomorphism  $\tilde{g}^*$  restricts to the automorphism  $g$  on  $\mathbb{K}[[\mathbf{t}]]$ . If  $g$  is the identity automorphism of  $\mathbb{K}[[\mathbf{t}]]$ , then by uniqueness  $\tilde{g}$  has to be the identity automorphism of  $\text{Coor } X$ . Next take two elements  $g_1, g_2 \in G(\mathbb{K})$ . Then

$$\begin{aligned} \widetilde{g_2 \circ g_1}^*(\mathbf{t}) &= (g_2 \circ g_1)(\mathbf{t}) = g_2(g_1(\mathbf{t})) = \tilde{g}_2^*(g_1(\mathbf{t})) = g_1(\tilde{g}_2^*(\mathbf{t})) \\ &= g_1(g_2(\mathbf{t})) = (\tilde{g}_1^* \circ \tilde{g}_2^*)(\mathbf{t}) = (\tilde{g}_2 \circ \tilde{g}_1)^*(\mathbf{t}). \end{aligned}$$

Thus indeed we have a group action.

Due to the moduli property this action becomes geometric, i.e. it is a morphism of schemes  $G \times \text{Coor } X \rightarrow \text{Coor } X$ . The explicit local description (4.7) shows that  $\text{Coor } X$  is in fact a  $G$ -torsor over  $X$ .

(3, 4) These are consequence of (2).

(5) Clear from formula (4.7).  $\square$

## 5. FORMAL DIFFERENTIAL CALCULUS

As before  $\mathbb{K}$  is a field of characteristic 0, and  $X$  is a smooth separated irreducible  $n$ -dimensional  $\mathbb{K}$ -scheme.

Recall the algebra homomorphism  $p_1^* : \mathcal{O}_X \rightarrow \mathcal{P}_X$ . We define  $\mathcal{T}(\mathcal{P}_X/\mathcal{O}_X; p_1^*)$  to be the sheaf of derivations of  $\mathcal{P}_X$  relative to  $\mathcal{O}_X$ . Thus for any affine open set  $U = \text{Spec } C \subset X$ , writing  $\widehat{A} := \Gamma(U, \mathcal{P}_X)$ , we have

$$\Gamma(U, \mathcal{T}(\mathcal{P}_X/\mathcal{O}_X; p_1^*)) = \mathcal{T}_{\widehat{A}/C} = \text{Der}_C(\widehat{A}).$$

Similarly we define  $\mathcal{T}_{\text{poly}}^i(\mathcal{P}_X/\mathcal{O}_X; p_1^*)$  and  $\mathcal{D}_{\text{poly}}^i(\mathcal{P}_X/\mathcal{O}_X; p_1^*)$ .

**Lemma 5.1.** *Let  $\mathcal{G}$  stand either for  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ , so that  $\mathcal{G}_X = \mathcal{T}_{\text{poly}, X}$  etc.*

- (1) *The graded left  $\mathcal{O}_X$ -module  $\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$  is a DG Lie algebra in  $\text{Dir Inv Mod } \mathcal{O}_X$ . The homomorphism  $\mathcal{G}_X \rightarrow \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$  given by  $\gamma \mapsto 1 \otimes \gamma$  is a DG Lie algebra homomorphism.*
- (2) *There is a canonical isomorphism*

$$\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X \cong \mathcal{G}(\mathcal{P}_X/\mathcal{O}_X; p_1^*)$$

*of sheaves of DG Lie algebras in  $\text{Dir Inv Mod } \mathcal{O}_X$ .*

- (3) *Suppose  $f : Y \rightarrow X$  is a morphism of schemes. Then*

$$f^* \mathcal{G}(\mathcal{P}_X/\mathcal{O}_X; p_1^*) \cong \mathcal{G}(f^* \mathcal{P}_X/\mathcal{O}_Y)$$

*as DG Lie algebras in  $\text{Dir Inv Mod } \mathcal{O}_Y$ .*

*Proof.* (1) Let  $U \subset X$  be an affine open set, and define  $C := \Gamma(U, \mathcal{O}_X)$  and  $A := C \otimes C$ . Let  $\mathfrak{a} := \text{Ker}(A \rightarrow C)$ , and let  $\widehat{A}$  be the  $\mathfrak{a}$ -adic completion of  $A$ . The left  $C$ -module  $C \otimes \mathcal{G}(C)$  is a DG Lie algebra over  $C$ . When we consider  $C \otimes \mathcal{G}(C)$  as an  $A$ -module, the bracket

$$[-, -] : (C \otimes \mathcal{G}(C)) \times (C \otimes \mathcal{G}(C)) \rightarrow C \otimes \mathcal{G}(C)$$

and the differential

$$d : C \otimes \mathcal{G}(C) \rightarrow C \otimes \mathcal{G}(C)$$

are poly differential operators, and hence they are continuous for the  $\mathfrak{a}$ -adic dir-inv structure (see [Ye2, Example 1.8]). So according to [Ye2, Proposition 2.3],  $C \otimes \widehat{\mathcal{G}(C)}$  is a DG Lie algebra in  $\text{Dir Inv Mod } C$ . But

$$\Gamma(U, \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \cong \widehat{A} \otimes_C \mathcal{G}(C) \cong \widehat{C \otimes \mathcal{G}(C)}.$$

(2, 3) By definition  $\mathcal{G}(C) = \mathcal{G}(C/\mathbb{K})$ . By base change there is an isomorphism  $B \otimes \mathcal{G}(C/\mathbb{K}) \cong \mathcal{G}((C \otimes B)/B)$  for any  $\mathbb{K}$ -algebra  $B$ .  $\square$

**Definition 5.2.** Consider the de Rham differential  $d : \mathcal{O}_{X^2} \rightarrow \Omega_{X^2/X}^1 = p_1^* \Omega_X^1$  relative to the projection  $p_2 : X^2 \rightarrow X$ . Passing to the completion along the diagonal we obtain the *Grothendieck connection*

$$\nabla_{\mathcal{P}} : \mathcal{P}_X \rightarrow \mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega_X^1.$$

Let  $\mathcal{M}$  be an  $\mathcal{O}_X$ -module. Then the connection  $\nabla_{\mathcal{P}}$  extends uniquely to a degree 1 endomorphism of the graded sheaf

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus_{p \geq 0} \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

The formula is

$$\nabla_{\mathcal{P}}(\alpha \otimes a \otimes m) := d(\alpha) \otimes a \otimes m + (-1)^p \alpha \wedge \nabla_{\mathcal{P}}(a) \otimes m$$

for local sections  $\alpha \in \Omega_X^p$ ,  $a \in \mathcal{P}_X$  and  $m \in \mathcal{M}$ . The connection is integrable, i.e.  $\nabla_{\mathcal{P}} \circ \nabla_{\mathcal{P}} = 0$ , and it makes  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$  into a DG  $\Omega_X$ -module.

**Theorem 5.3** ([Ye3, Theorem 4.4]). *Let  $X$  be a smooth  $\mathbb{K}$ -scheme and let  $\mathcal{M}$  be an  $\mathcal{O}_X$ -module. Then the map*

$$\mathcal{M} \rightarrow \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \quad m \mapsto 1 \otimes 1 \otimes m$$

*is a quasi-isomorphism.*

Given any  $\mathbb{K}$ -scheme  $Y$  let  $\mathbb{K}_Y$  be the constant sheaf  $\mathbb{K}$  on  $Y$ . We consider  $\Omega_Y^p = \Omega_{Y/\mathbb{K}}^p$  as a discrete inv  $\mathcal{O}_Y$ -module, and  $\Omega_Y = \bigoplus_{p \geq 0} \Omega_Y^p$  gets the  $\bigoplus$  dir-inv structure. Thus  $\Omega_Y$  is a discrete DG algebra in  $\text{Dir Inv Mod } \mathbb{K}_Y$ . Note that if  $Y$  is infinite dimensional then  $\Omega_Y$  will be unbounded.

Suppose  $\mathcal{M}$  is a quasi-coherent  $\mathcal{O}_X$ -module. Then for any  $p$  the sheaf  $\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$  is a dir-coherent  $\mathcal{P}_X$ -module (see Example 2.4), so it has the  $\mathcal{I}_X$ -adic dir-inv module structure. The connection

$$\nabla_{\mathcal{P}} : \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \rightarrow \Omega_X^{p+1} \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

is a differential operator of  $\mathcal{P}_X$ -modules (of order  $\leq 1$ ), and therefore it is continuous for the dir-inv structures (see [Ye2, Proposition 2.3]). So in fact  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$  is a DG  $\Omega_X$ -module in  $\text{Dir Inv Mod } \mathbb{K}_X$ .

Suppose  $f : Y \rightarrow X$  is some morphism of schemes. The complete pullback  $f^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$  is a dir-inv  $\mathcal{O}_Y$ -module. Moreover  $\Omega_Y \widehat{\otimes}_{\mathcal{O}_Y} f^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$  is a DG  $\Omega_Y$ -module in  $\text{Dir Inv Mod } \mathbb{K}_Y$ . Its differential is also denoted by  $\nabla_{\mathcal{P}}$ . In particular, when  $\mathcal{M} = \mathcal{O}_X$ , we obtain a super-commutative associative unital DG algebra  $\Omega_Y \widehat{\otimes}_{\mathcal{O}_Y} f^* \mathcal{P}_X$  in  $\text{Dir Inv Mod } \mathbb{K}_Y$ . Its degree 0 component is  $f^* \mathcal{P}_X$ , which is a complete commutative algebra in  $\text{Inv Mod } \mathcal{O}_Y$ . For details and proofs see [Ye2, Section 1].

**Proposition 5.4.** *Let  $\mathcal{G}$  denote either  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ , so that  $\mathcal{G}_X = \mathcal{T}_{\text{poly}, X}$  etc. Also let  $d_{\mathcal{G}}$  and  $[-, -]_{\mathcal{G}}$  denote the differential and the bracket of  $\mathcal{G}$ .*

- (1) *The graded sheaf  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$  is a DG  $\Omega_X$ -module Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}_X$ . The differential is  $\nabla_{\mathcal{P}} + \mathbf{1} \otimes \mathbf{1} \otimes d_{\mathcal{G}}$ , and the bracket is the continuous  $\Omega_X$ -bilinear extension of  $[-, -]_{\mathcal{G}}$ .*
- (2) *The canonical map  $\mathcal{G}_X \rightarrow \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$  is a DG Lie algebra quasi-isomorphism.*



(3) Suppose  $f : Y \rightarrow X$  is a morphism of schemes. Then

$$\Omega_Y \widehat{\otimes}_{\mathcal{O}_Y} f^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X)$$

is a DG  $\Omega_Y$ -module Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}_Y$ . The canonical map

$$f^{-1}(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \rightarrow \Omega_Y \widehat{\otimes}_{\mathcal{O}_Y} f^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X)$$

is a homomorphism of DG Lie algebras.

The explicit formulas for part (1) are

$$d(\alpha_1 \otimes 1 \otimes \gamma_1) := d(\alpha_1) \otimes 1 \otimes \gamma_1 + (-1)^{i_1} \alpha_1 \otimes 1 \otimes d_{\mathcal{G}}(\gamma_1)$$

and

$$[\alpha_1 \otimes 1 \otimes \gamma_1, \alpha_2 \otimes 1 \otimes \gamma_2] := (-1)^{j_1 i_2} (\alpha_1 \wedge \alpha_2) \otimes 1 \otimes [\gamma_1, \gamma_2]$$

for  $\alpha_k \in \Omega_X^{i_k}$  and  $\gamma_k \in \mathcal{G}_X^{j_k}$ .

*Proof.* (1) Using the notation of the proof of Lemma 5.1,  $\Omega_C \otimes \mathcal{G}(C)$  is a DG  $\Omega_C$ -module Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}$ , with the  $\mathfrak{a}$ -adic dir-inv structure. Hence so is its completion

$$\Omega_C \widehat{\otimes} \widehat{\mathcal{G}}(C) \cong \Omega_C \otimes_C \widehat{A} \otimes_C \mathcal{G}(C) \cong \Gamma(U, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X).$$

(2) In the the proof of part (1) the inclusion  $\mathcal{G}(C) \subset \Omega_C \otimes \mathcal{G}(C)$  is a DG algebra homomorphism. According to [Ye3, Theorem 3.4] it is a quasi-isomorphism.

(3) This is by [Ye2, Proposition 1.22(2)].  $\square$

**Definition 5.5.** Let  $\mathcal{G}$  denote either  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ , and let  $d$  be the de Rham differential on  $\Omega_{\text{Coor } X}$ . Put on  $\mathcal{G}(\mathbb{K}[[\mathfrak{t}]])$  the  $\mathfrak{t}$ -adic dir-inv structure. Define

$$d_{\text{for}} := d \otimes \mathbf{1} : \Omega_{\text{Coor } X}^p \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathfrak{t}]]) \rightarrow \Omega_{\text{Coor } X}^{p+1} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathfrak{t}]]).$$

According to [Ye2, Proposition 1.19],

$$\Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathfrak{t}]]) = \bigoplus_{p,q} \Omega_{\text{Coor } X}^p \widehat{\otimes} \mathcal{G}^q(\mathbb{K}[[\mathfrak{t}]])$$

is a DG Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}_{\text{Coor } X}$ , with differential  $d_{\text{for}} + \mathbf{1} \otimes d_{\mathcal{G}}$ . The explicit formula is

$$(d_{\text{for}} + \mathbf{1} \otimes d_{\mathcal{G}})(\alpha \otimes \gamma) = d(\alpha) \otimes \gamma + (-1)^p \alpha \otimes d_{\mathcal{G}}(\gamma)$$

for  $\alpha \in \Omega_{\text{Coor } X}^p$  and  $\gamma \in \mathcal{G}(\mathbb{K}[[\mathfrak{t}]])$ .

**Theorem 5.6** (Universal Taylor Expansion). *Let  $\mathcal{G}$  denote either  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ . There is a canonical isomorphism*

$$\Omega_{\text{Coor } X} \widehat{\otimes}_{\mathcal{O}_{\text{Coor } X}} \widehat{\pi}_{\text{Coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \cong \Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathfrak{t}]])$$

of graded Lie algebras in  $\text{Dir Inv Mod } \mathcal{O}_{\text{Coor } X}$ , extending the isomorphism of Corollary 4.9.

Warning: the isomorphism in the theorem does not respect the differentials; cf. Proposition 5.8 below.

*Proof.* By Corollary 4.9 we know that  $\pi_{\text{Coor}}^* \mathcal{P}_X \cong \mathcal{O}_{\text{Coor } X}[[\mathbf{t}]]$  canonically as inv  $\mathcal{O}_{\text{Coor } X}$ -algebras. Using Lemma 5.1 we then obtain isomorphisms of graded Lie algebras over  $\mathcal{O}_{\text{Coor } X}$

$$\begin{aligned} \pi_{\text{Coor}}^* (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) &\cong \pi_{\text{Coor}}^* \mathcal{G}(\mathcal{P}_X / \mathcal{O}_X; \mathfrak{p}_1^*) \\ &\cong \mathcal{G}(\pi_{\text{Coor}}^* \mathcal{P}_X / \mathcal{O}_{\text{Coor } X}) \\ &\cong \mathcal{G}(\mathcal{O}_{\text{Coor } X}[[\mathbf{t}]] / \mathcal{O}_{\text{Coor } X}) \\ &\cong \mathcal{O}_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathbf{t}]]) . \end{aligned}$$

Finally we may apply the functor  $\Omega_{\text{Coor } X} \widehat{\otimes}_{\mathcal{O}_{\text{Coor } X}} -$ . □

**Definition 5.7.** The *Maurer-Cartan form* of  $X$  is

$$\omega_{\text{MC}} := \sum_{i=1}^n \nabla_{\mathcal{P}}(t_i) \cdot \frac{\partial}{\partial t_i} \in \Gamma(\text{Coor } X, \Omega_{\text{Coor } X}^1 \widehat{\otimes} \mathcal{T}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]]) ,)$$

where  $\nabla_{\mathcal{P}}(t_i)$  is defined using the canonical isomorphism in Theorem 5.6.

The Lie algebra  $\mathcal{T}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]]) = \mathcal{T}(\mathbb{K}[[\mathbf{t}]])$  is also a Lie subalgebra of  $\mathcal{D}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]]) = \mathcal{D}(\mathbb{K}[[\mathbf{t}]])$ . Keeping the notation of Theorem 5.6, for any local section  $\alpha \in \Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathbf{t}]])$  let

$$\text{ad}(\omega_{\text{MC}})(\alpha) := [\omega_{\text{MC}}, \alpha] .$$

The operation  $\text{ad}(\omega_{\text{MC}})$  is  $\mathbb{K}$ -linear endomorphism of degree 1 of the graded sheaf  $\Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathbf{t}]])$ .

**Proposition 5.8.** *Let  $\mathcal{G}$  denote either  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ . Under the isomorphism of Theorem 5.6, there is equality*

$$\nabla_{\mathcal{P}} = d_{\text{for}} + \text{ad}(\omega_{\text{MC}})$$

as endomorphisms of  $\Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathbf{t}]])$ .

*Proof.* First we shall consider  $\mathcal{G} = \mathcal{T}_{\text{poly}}$ . Let's write  $\omega := \omega_{\text{MC}}$ ,  $d := d_{\text{for}}$ ,  $\nabla := \nabla_{\mathcal{P}}$  and  $\partial_i := \frac{\partial}{\partial t_i}$ . For any multi-index  $\mathbf{j} = (j_1 < \dots < j_q)$  let's write

$$\partial_{\mathbf{j}} := \partial_{j_1} \wedge \dots \wedge \partial_{j_q} \in \mathcal{T}_{\text{poly}}^{q-1}(\mathbb{K}[[\mathbf{t}]]) .$$

Take a local section  $\alpha \in \Omega_{\text{Coor } X}^p$  and a multi-index  $\mathbf{i} \in \mathbb{N}^n$ , and consider  $\alpha \otimes \mathbf{t}^{\mathbf{i}} \partial_{\mathbf{j}} \in \Omega_{\text{Coor } X}^p \widehat{\otimes} \mathcal{T}_{\text{poly}}^{q-1}(\mathbb{K}[[\mathbf{t}]])$ . Then

$$\begin{aligned} \nabla(\alpha \otimes \mathbf{t}^{\mathbf{i}} \partial_{\mathbf{j}}) &= d(\alpha) \cdot \mathbf{t}^{\mathbf{i}} \cdot \partial_{\mathbf{j}} \pm \alpha \cdot \nabla(\mathbf{t}^{\mathbf{i}}) \cdot \partial_{\mathbf{j}} \pm \alpha \cdot \mathbf{t}^{\mathbf{i}} \cdot \nabla(\partial_{\mathbf{j}}) \\ d(\alpha \otimes \mathbf{t}^{\mathbf{i}} \partial_{\mathbf{j}}) &= d(\alpha) \cdot \mathbf{t}^{\mathbf{i}} \cdot \partial_{\mathbf{j}} \\ \text{ad}(\omega)(\alpha \otimes \mathbf{t}^{\mathbf{i}} \partial_{\mathbf{j}}) &= \pm \alpha \cdot \text{ad}(\omega)(\mathbf{t}^{\mathbf{i}}) \cdot \partial_{\mathbf{j}} \pm \alpha \cdot \mathbf{t}^{\mathbf{i}} \cdot \text{ad}(\omega)(\partial_{\mathbf{j}}) . \end{aligned}$$

Now

$$\nabla(\mathbf{t}^{\mathbf{i}}) = \sum_k \nabla(t_k) \cdot \partial_k(\mathbf{t}^{\mathbf{i}}) = \text{ad}(\omega)(\mathbf{t}^{\mathbf{i}}) .$$

It remains to show that

$$\nabla(\partial_{\mathbf{j}}) = \text{ad}(\omega)(\partial_{\mathbf{j}}) .$$

Take any  $\beta \in \Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{T}_{\text{poly}}^{q-1}(\mathbb{K}[[\mathbf{t}]])$ , and write it as  $\beta = \sum_{\mathbf{k}} \beta_{\mathbf{k}} \partial_{\mathbf{k}}$ , where the sum is over the multi-indices  $\mathbf{k} = (k_1 < \dots < k_q)$ , and  $\beta_{\mathbf{k}} \in \Omega_{\text{Coor } X} \widehat{\otimes} \mathbb{K}[[\mathbf{t}]]$ . Then  $[\beta, \mathbf{t}^{\mathbf{k}}] = \beta_{\mathbf{k}}$ . We see that  $\beta = 0$  iff  $[\beta, \mathbf{t}^{\mathbf{k}}] = 0$  for all such  $\mathbf{k}$ . Therefore it suffices to

prove that  $[\nabla(\partial_j), \mathbf{t}^k] = [\text{ad}(\omega)(\partial_j), \mathbf{t}^k]$  for any  $k$ . Now  $[\partial_j, \mathbf{t}^k] \in \mathbb{K}$  (it is 0 or 1). Because  $\nabla$  is a  $\mathbb{K}$ -linear derivation, we have

$$0 = \nabla([\partial_j, \mathbf{t}^k]) = [\nabla(\partial_j), \mathbf{t}^k] \pm [\partial_j, \nabla(\mathbf{t}^k)].$$

Likewise

$$0 = \text{ad}(\omega)([\partial_j, \mathbf{t}^k]) = [\text{ad}(\omega)(\partial_j), \mathbf{t}^k] \pm [\partial_j, \text{ad}(\omega)(\mathbf{t}^k)].$$

And by definition of  $\omega$  we have

$$\text{ad}(\omega)(\mathbf{t}^k) = \sum_l \nabla(t_l) \partial_l(\mathbf{t}^k) = \nabla(\mathbf{t}^k).$$

The case  $\mathcal{G} = \mathcal{D}_{\text{poly}}$  is handled similarly, using the basis

$$\partial_{j_1, \dots, j_q} := \left(\frac{\partial}{\partial t}\right)^{j_1} \otimes \dots \otimes \left(\frac{\partial}{\partial t}\right)^{j_q} \in \mathcal{D}_{\text{poly}}^{q-1}(\mathbb{K}[[\mathbf{t}]]),$$

see equation [Ye2, equation 4.3].  $\square$

**Proposition 5.9.** *The form  $\omega_{\text{MC}}$  satisfies the identity*

$$d_{\text{for}}(\omega_{\text{MC}}) + \frac{1}{2}[\omega_{\text{MC}}, \omega_{\text{MC}}] = 0;$$

*namely it is a solution of the MC equation in the DG Lie algebra  $\Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  with differential  $d_{\text{for}}$ .*

*Proof.* Let's write  $\omega := \omega_{\text{MC}}$ ,  $d := d_{\text{for}}$ ,  $\nabla := \nabla_{\mathcal{P}}$ ,  $\beta := d(\omega) + \frac{1}{2}[\omega, \omega]$  and  $\mathfrak{g} := \Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ . As explained in the proof of the previous proposition, it suffices to show that  $[\beta, t_i] = 0$  for all  $i$ .

By definition of  $\omega$  we have

$$(5.10) \quad [\omega, t_i] = \nabla_{\mathcal{P}}(t_i).$$

Next we use the fact that  $d$  is an odd derivation of  $\mathfrak{g}$  to obtain

$$d([\omega, t_i]) = [d(\omega), t_i] - [\omega, d(t_i)].$$

But  $d(t_i) = 0$ , so

$$(5.11) \quad [d(\omega), t_i] = d(\nabla_{\mathcal{P}}(t_i)).$$

The graded Jacobi identity in  $\mathfrak{g}$  tells us that

$$[[\omega, \omega], t_i] + [[t_i, \omega], \omega] - [[\omega, t_i], \omega] = 0.$$

Hence  $[[\omega, \omega], t_i] = 2[\omega, [\omega, t_i]]$ , and plugging in (5.10) we arrive at

$$(5.12) \quad \frac{1}{2}[[\omega, \omega], t_i] = \text{ad}(\omega)(\nabla_{\mathcal{P}}(t_i)).$$

Finally, combining (5.11), (5.12) and Proposition 5.8 we get

$$\begin{aligned} [\beta, t_i] &= [d(\omega), t_i] + \frac{1}{2}[[\omega, \omega], t_i] = d(\nabla_{\mathcal{P}}(t_i)) + \text{ad}(\omega)(\nabla_{\mathcal{P}}(t_i)) \\ &= (d + \text{ad}(\omega))(\nabla_{\mathcal{P}}(t_i)) = \nabla_{\mathcal{P}}(\nabla_{\mathcal{P}}(t_i)) = 0. \end{aligned}$$

$\square$

According to Theorem 4.13(2) the group  $G(\mathbb{K})$  of  $\mathbb{K}$ -algebra automorphisms of  $\mathbb{K}[[\mathbf{t}]]$  acts on the bundle  $\text{Coor } X$ . Therefore for any open set  $U \subset X$  this group acts on the algebra  $\Gamma(\pi_{\text{Coor}}^{-1}(U), \pi_{\text{Coor}}^* \mathcal{P}_X)$ . More generally, let  $\mathcal{G}$  denote either  $\mathcal{T}_{\text{poly}}$  or  $\mathcal{D}_{\text{poly}}$ . Let's introduce the temporary notation

$$\mathfrak{h}(U, \mathcal{G}) := \Gamma(\pi_{\text{Coor}}^{-1}(U), \Omega_{\text{Coor } X} \widehat{\otimes}_{\mathcal{O}_{\text{Coor } X}} \pi_{\text{Coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}))$$

and

$$\mathfrak{h}'(U, \mathcal{G}) := \Gamma(\pi_{\text{Coor}}^{-1}(U), \Omega_{\text{Coor } X} \widehat{\otimes} \mathcal{G}(\mathbb{K}[[\mathbf{t}]])).$$

These are graded Lie algebras. The group  $G(\mathbb{K})$  acts on  $\mathfrak{h}(U, \mathcal{G})$  via its geometric action on  $\text{Coor } X$ . On the other hand there is an action of  $G(\mathbb{K})$  on  $\mathfrak{h}'(U, \mathcal{G})$  via its action on  $\Gamma(\pi_{\text{Coor}}^{-1}(U), \Omega_{\text{Coor } X})$  and on  $\mathbb{K}[[\mathbf{t}]]$ .

**Proposition 5.13.** *The canonical isomorphism  $\mathfrak{h}(U, \mathcal{G}) \cong \mathfrak{h}'(U, \mathcal{G})$  of Theorem 5.6 is  $G(\mathbb{K})$ -equivariant.*

*Proof.* By Theorem 4.13(2) the algebra isomorphism

$$\Gamma(\pi_{\text{Coor}}^{-1}(U), \mathcal{O}_{\text{Coor } X}[[\mathbf{t}]]) \cong \Gamma(\pi_{\text{Coor}}^{-1}(U), \pi_{\text{Coor}}^* \mathcal{P}_X)$$

of Corollary 4.9 is  $G(\mathbb{K})$ -equivariant. Tracing the isomorphisms used in the proof of Theorem 5.6 we deduce the same for the isomorphism  $\mathfrak{h}(U, \mathcal{G}) \cong \mathfrak{h}'(U, \mathcal{G})$ .  $\square$

We view  $\omega_{\text{MC}}$  as an element of  $\mathfrak{h}(X, \mathcal{T}_{\text{poly}}) \cong \mathfrak{h}'(X, \mathcal{T}_{\text{poly}})$ . Due to Proposition 5.13 we can talk about *the* action of  $G(\mathbb{K})$  on  $\omega_{\text{MC}}$ . Recall that  $\text{GL}_n(\mathbb{K})$  sits inside  $G(\mathbb{K})$  as the group of linear changes of coordinates.

**Proposition 5.14.** *The element  $\omega_{\text{MC}}$  is  $\text{GL}_n(\mathbb{K})$ -invariant.*

*Proof.* Since the Grothendieck connection  $\nabla_{\mathcal{P}}$  on  $\mathfrak{h}(X, \mathcal{T}_{\text{poly}})$  is induced from  $X$ , it commutes with the action of  $G(\mathbb{K})$ . Hence in particular  $g(\nabla_{\mathcal{P}}(t_i)) = \nabla_{\mathcal{P}}(g(t_i))$  for any  $g \in \text{GL}_n(\mathbb{K})$  and  $i \in \{1, \dots, n\}$ .

Fix such a matrix  $g = [g_{i,j}]$ . So  $g_{i,j} \in \mathbb{K}$  and  $g(t_i) = \sum_{j=1}^n g_{i,j} t_j$ . Let  $h = [h_{i,j}] := (g^{-1})^t$ , the transpose inverse matrix. Then in the induced action of  $\text{GL}_n(\mathbb{K})$  on  $\mathcal{T}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]]) = \mathcal{T}(\mathbb{K}[[\mathbf{t}]])$  we have  $g(\frac{\partial}{\partial t_i}) = \sum_{j=1}^n h_{i,j} \frac{\partial}{\partial t_j}$ . Thus

$$\begin{aligned} g(\omega_{\text{MC}}) &= g\left(\sum_i \nabla_{\mathcal{P}}(t_i) \cdot \frac{\partial}{\partial t_i}\right) = \sum_i g(\nabla_{\mathcal{P}}(t_i)) \cdot g\left(\frac{\partial}{\partial t_i}\right) \\ &= \sum_i \nabla_{\mathcal{P}}(g(t_i)) \cdot g\left(\frac{\partial}{\partial t_i}\right) = \sum_{i,j,k} g_{i,j} h_{i,k} (\nabla_{\mathcal{P}}(t_j)) \cdot \frac{\partial}{\partial t_k} \\ &= \sum_j \nabla_{\mathcal{P}}(t_j) \cdot \frac{\partial}{\partial t_j} = \omega_{\text{MC}}. \end{aligned}$$

$\square$

**Remark 5.15.** The adjoint of  $\omega_{\text{MC}}$  is an element of  $\Gamma(\text{Coor } X, \mathcal{T}_{\text{Coor } X} \widehat{\otimes} \widehat{\Omega}_{\mathbb{K}[[\mathbf{t}]]/\mathbb{K}}^1)$ , and it gives rise to a Lie algebra homomorphism  $\mathcal{T}_{\mathbb{K}[[\mathbf{t}]}} \rightarrow \Gamma(\text{Coor } X, \mathcal{T}_{\text{Coor } X})$ . In this way  $\mathcal{T}_{\mathbb{K}[[\mathbf{t}]}}$  acts infinitesimally on  $\text{Coor } X$ . Now inside  $\mathcal{T}_{\mathbb{K}[[\mathbf{t}]}} = \bigoplus_{i=1}^n \mathbb{K}[[\mathbf{t}]] \frac{\partial}{\partial t_i}$  there is a subalgebra  $\mathfrak{g} := \bigoplus_{i,j} \mathbb{K}[[\mathbf{t}]] t_i \frac{\partial}{\partial t_j}$ . The Lie algebra  $\mathfrak{g}$  is the Lie algebra of the pro-algebraic group  $G = \text{Aut}(\mathbb{K}[[\mathbf{t}]])$ , the group of  $\mathbb{K}$ -algebra automorphisms of  $\mathbb{K}[[\mathbf{t}]]$ . The infinitesimal action of  $\mathfrak{g}$  on  $\text{Coor } X$  is the differential of the action of  $G$  on  $\text{Coor } X$  (cf. Theorem 4.13). The action of  $\mathcal{T}_{\mathbb{K}[[\mathbf{t}]}}$  on  $\text{Coor } X$  is the main feature of the Gelfand-Kazhdan formal geometry. However we do not use this action (at least not directly) in our paper.

## 6. REVIEW OF MIXED RESOLUTIONS

As always  $\mathbb{K}$  is a field of characteristic 0. In this section we review the constructions and results of the paper [Ye3].

Let  $\mathbf{\Delta}$  denote the category with set of objects the natural numbers. For any  $p, q \in \mathbb{N}$  the set of morphisms in  $\mathbf{\Delta}$  from  $p$  to  $q$  is the set  $\mathbf{\Delta}_p^q$  of order preserving functions  $\alpha : \{0, \dots, p\} \rightarrow \{0, \dots, q\}$ . Recall that a cosimplicial object in some category  $\mathbf{C}$  is a functor  $C : \mathbf{\Delta} \rightarrow \mathbf{C}$ . Usually one writes  $C^p$  instead of  $C(p)$ , and refers to the sequence  $\{C^p\}_{p \geq 0}$  as a cosimplicial object (the morphisms remaining implicit). The category of cosimplicial objects in  $\mathbf{C}$  is denoted by  $\mathbf{\Delta C}$ .

We are interested in cosimplicial dir-inv  $\mathbb{K}$ -modules, i.e. in objects  $M = \{M^p\}_{p \geq 0}$  in  $\mathbf{\Delta Dir Inv Mod } \mathbb{K}$ . As explained in [Ye3], there is a functor

$$\widehat{\mathbf{N}} : \mathbf{\Delta Dir Inv Mod } \mathbb{K} \rightarrow \mathbf{DGMod } \mathbb{K},$$

the latter being the category of complexes of  $\mathbb{K}$ -modules. This is the *complete Thom-Sullivan normalization* functor, which is a generalization of constructions in [HS] and [HY]. By definition there is an embedding

$$\widehat{\mathbf{N}}^q M \subset \prod_{i=0}^{\infty} (\Omega^q(\mathbf{\Delta}_{\mathbb{K}}^i) \widehat{\otimes} M^i).$$

Here  $\mathbf{\Delta}_{\mathbb{K}}^i := \text{Spec } \mathbb{K}[t_0, \dots, t_i]/(t_0 + \dots + t_i - 1)$  is the  $i$ -dimensional geometric simplex, and  $\Omega^q(\mathbf{\Delta}_{\mathbb{K}}^i) := \Gamma(\mathbf{\Delta}_{\mathbb{K}}^i, \Omega_{\mathbf{\Delta}_{\mathbb{K}}^i}^q)$  is a discrete inv  $\mathbb{K}$ -module. The differential  $\partial : \widehat{\mathbf{N}}^q M \rightarrow \widehat{\mathbf{N}}^{q+1} M$  is induced from the de Rham differentials  $d : \Omega^q(\mathbf{\Delta}_{\mathbb{K}}^i) \rightarrow \Omega^{q+1}(\mathbf{\Delta}_{\mathbb{K}}^i)$ .

Let  $X$  be a separated smooth irreducible  $n$ -dimensional  $\mathbb{K}$ -scheme. Choose an affine open covering  $\mathbf{U} = \{U_{(0)}, \dots, U_{(m)}\}$  of  $X$ . Given  $\mathbf{i} = (i_0, \dots, i_q) \in \mathbf{\Delta}_q^m$  let  $U_{\mathbf{i}} := U_{(i_0)} \cap \dots \cap U_{(i_m)}$ , and let  $g_{\mathbf{i}} : U_{\mathbf{i}} \rightarrow X$  be the inclusion. For a sheaf  $\mathcal{M}$  on  $X$  we write

$$C^q(\mathbf{U}, \mathcal{M}) := \prod_{\mathbf{i} \in \mathbf{\Delta}_q^m} g_{\mathbf{i}*} g_{\mathbf{i}}^{-1} \mathcal{M}.$$

The sequence  $\{C^q(\mathbf{U}, \mathcal{M})\}_{q \geq 0}$  is then a cosimplicial sheaf on  $X$ . This is a variant of the Čech resolution of  $\mathcal{M}$ .

Suppose  $\mathcal{M}$  is a dir-inv  $\mathbb{K}_X$ -module, i.e. a sheaf of  $\mathbb{K}$ -modules on  $X$  with a dir-inv structure. For any open set  $V \subset X$  we then have a cosimplicial dir-inv  $\mathbb{K}$ -module  $\{\Gamma(V, C^q(\mathbf{U}, \mathcal{M}))\}_{q \geq 0}$ . Applying the functor  $\widehat{\mathbf{N}}^q$  to it we obtain a  $\mathbb{K}$ -module  $\widehat{\mathbf{N}}^q \Gamma(V, C(\mathbf{U}, \mathcal{M}))$ . It turns out that the presheaf  $V \mapsto \widehat{\mathbf{N}}^q \Gamma(V, C(\mathbf{U}, \mathcal{M}))$  is a sheaf, and we denote it by  $\widehat{\mathbf{N}}^q C(\mathbf{U}, \mathcal{M})$ . So there is a functor

$$\widehat{\mathbf{N}} C(\mathbf{U}, -) : \mathbf{Dir Inv Mod } \mathbb{K}_X \rightarrow \mathbf{DGMod } \mathbb{K}_X,$$

and there is a functorial homomorphism  $\mathcal{M} \rightarrow \widehat{\mathbf{N}} C(\mathbf{U}, \mathcal{M})$ . If  $\mathcal{M}$  is a complete dir-inv module then according to [Ye3, Theorem 3.7] the homomorphism  $\mathcal{M} \rightarrow \widehat{\mathbf{N}} C(\mathbf{U}, \mathcal{M})$  is in fact a quasi-isomorphism. We call  $\widehat{\mathbf{N}} C(\mathbf{U}, \mathcal{M})$  the *commutative Čech resolution* of  $\mathcal{M}$ , since  $\widehat{\mathbf{N}} C(\mathbf{U}, \mathcal{O}_X)$  is a super-commutative DG algebra.

Now suppose  $\mathcal{M}$  is a quasi-coherent  $\mathcal{O}_X$ -module. Let  $p$  be some natural number. Then  $\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$  is a complete dir-inv  $\mathcal{P}_X$ -module with the  $\mathcal{I}_X$ -adic dir-inv

structure. Define

$$\text{Mix}_{\mathcal{U}}^{p,q}(\mathcal{M}) := \widehat{\mathbb{N}}^q \mathbb{C}(\mathcal{U}, \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

This is a sheaf on  $X$ , and there is an embedding of sheaves

$$(6.1) \quad \text{Mix}_{\mathcal{U}}^{p,q}(\mathcal{M}) \subset \prod_{j \in \mathbb{N}} \prod_{i \in \Delta_j^m} g_{i*} g_i^{-1} (\Omega^q(\Delta_{\mathbb{K}}^j) \widehat{\otimes} (\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})).$$

In addition to the differential  $\partial : \text{Mix}_{\mathcal{U}}^{p,q}(\mathcal{M}) \rightarrow \text{Mix}_{\mathcal{U}}^{p,q+1}(\mathcal{M})$  there is a second differential  $\nabla_{\mathcal{P}} : \text{Mix}_{\mathcal{U}}^{p,q}(\mathcal{M}) \rightarrow \text{Mix}_{\mathcal{U}}^{p+1,q}(\mathcal{M})$  coming from the connection  $\nabla_{\mathcal{P}}$  of Definition 5.2. We now totalize

$$\text{Mix}_{\mathcal{U}}^i(\mathcal{M}) := \bigoplus_{p+q=i} \text{Mix}_{\mathcal{U}}^{p,q}(\mathcal{M})$$

and let  $d_{\text{mix}} := \partial + (-1)^q \nabla_{\mathcal{P}}$ . This is the *mixed resolution* of  $\mathcal{M}$ , which is a functor

$$\text{Mix}_{\mathcal{U}} : \text{QCoh } \mathcal{O}_X \rightarrow \text{DGMod } \mathbb{K}_X.$$

**Theorem 6.2** ([Ye3, Theorem 4.14]). *Let  $\mathcal{M}$  be a quasi-coherent  $\mathcal{O}_X$ -module.*

- (1) *There is a functorial quasi-isomorphism  $\mathcal{M} \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{M})$ .*
- (2) *There is a functorial isomorphism  $\Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{M})) \cong \text{R}\Gamma(X, \mathcal{M})$  in  $\text{D}(\text{Mod } \mathbb{K})$ .*

Of course the functor  $\text{Mix}_{\mathcal{U}}$  can be extended to bounded below complexes of quasi-coherent  $\mathcal{O}_X$ -modules, by totalizing.

The sheaves of DG Lie algebras  $\mathcal{T}_{\text{poly},X}$  and  $\mathcal{D}_{\text{poly},X}$  are bounded below complexes of quasi-coherent  $\mathcal{O}_X$ -modules, so the above theorem applies to them. In addition we have:

**Proposition 6.3.** *Let  $\mathcal{G}_X$  stand for either  $\mathcal{T}_{\text{poly},X}$  or  $\mathcal{D}_{\text{poly},X}$ . Then  $\text{Mix}_{\mathcal{U}}(\mathcal{G}_X)$  is a sheaf of DG Lie algebras, with differential*

$$d_{\text{mix}} + (-1)^i d_{\mathcal{G}} : \text{Mix}_{\mathcal{U}}^i(\mathcal{G}_X^j) \rightarrow \text{Mix}_{\mathcal{U}}^{i+1}(\mathcal{G}_X^j) \oplus \text{Mix}_{\mathcal{U}}^i(\mathcal{G}_X^{j+1}).$$

*The quasi-isomorphism  $\mathcal{G}_X \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{G}_X)$  of Theorem 6.2(1) is a homomorphism of DG Lie algebras.*

*Proof.* By Proposition 5.4 the sheaf  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$  is a DG  $\Omega_X$ -module Lie algebra in  $\text{Dir Inv Mod } \mathbb{K}_X$ . Now use [Ye3, Proposition 5.5].  $\square$

Suppose  $\pi : Z \rightarrow X$  is some morphism of schemes (possibly of infinite type). A *simplicial section* of  $\pi$  based on the covering  $\mathcal{U}$  is a collection of  $X$ -morphisms

$$\sigma = \{\sigma_i : \Delta_{\mathbb{K}}^q \times U_i \rightarrow Z\}$$

indexed by  $i \in \Delta_q^m$ ,  $q \in \mathbb{N}$ , which satisfies the simplicial relations (see [Ye3, Definition 6.1]).

The sheaf  $\Omega_Z^p$  is considered as a discrete inv  $\mathbb{K}_Z$ -module, and  $\Omega_Z = \bigoplus_p \Omega_Z^p$  has the  $\bigoplus$  dir-inv structure. Given a quasi-coherent  $\mathcal{O}_X$ -module  $\mathcal{M}$  the graded sheaf  $\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^* (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$  is then a DG  $\Omega_Z$ -module in  $\text{Dir Inv Mod } \mathbb{K}_Z$ , with differential  $\nabla_{\mathcal{P}}$ . See Section 2.

Let  $A$  be an associative unital super-commutative DG  $\mathbb{K}$ -algebra. Consider a homogeneous  $A$ -multilinear function  $\phi : M_1 \times \cdots \times M_r \rightarrow N$ , where  $M_1, \dots, M_r, N$  are DG  $A$ -modules. There is an operation of composition for such functions: given functions  $\psi_i : \prod_j L_{i,j} \rightarrow M_i$  the composition is  $\phi \circ (\psi_1 \times \cdots \times \psi_r) : \prod_{i,j} L_{i,j} \rightarrow N$ . There is also a summation operation: if  $\phi_j : \prod_i M_i \rightarrow N$  are homogeneous of

equal degree then so is their sum  $\sum_j \phi_j$ . Finally let  $\phi \circ d : \prod_i M_i \rightarrow N$  be the homogeneous  $\mathbb{K}$ -multilinear function

$$(\phi \circ d)(m_1, \dots, m_r) := \sum_{i=1}^r \pm \phi(m_1, \dots, d(m_i), \dots, m_r)$$

with Koszul signs.

**Theorem 6.4** ([Ye3, Theorem 6.3]). *Suppose  $\sigma$  is simplicial section of  $\pi : Z \rightarrow X$  based on  $U$ . Let  $\mathcal{M}_1, \dots, \mathcal{M}_r, \mathcal{N}$  be quasi-coherent  $\mathcal{O}_X$ -modules, and let*

$$\phi : \prod_{i=1}^r (\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)) \rightarrow \Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \pi^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous  $\Omega_Z$ -multilinear sheaf morphism on  $Z$  of degree  $k$ . Then there is an induced  $\mathbb{K}$ -multilinear sheaf morphism of degree  $k$

$$\sigma^*(\phi) : \prod_{i=1}^r \text{Mix}_U(\mathcal{M}_i) \rightarrow \text{Mix}_U(\mathcal{N})$$

on  $X$  with the following properties.

- (i) The assignment  $\phi \mapsto \sigma^*(\phi)$  respects the operations of composition and summation.
- (ii) If  $\phi = \pi^*(\phi_0)$  for some continuous  $\Omega_X$ -multilinear morphism

$$\phi_0 : \prod_{i=1}^r (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \rightarrow \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

then  $\sigma^*(\phi) = \widehat{\text{NC}}(U, \phi_0)$ .

- (iii) Suppose  $\psi$  is another such  $\Omega_Z$ -multilinear sheaf morphism of degree  $k+1$ , and the equation  $\nabla_{\mathcal{P}} \circ \phi - (-1)^k \phi \circ \nabla_{\mathcal{P}} = \psi$  holds. Then

$$d_{\text{mix}} \circ \sigma^*(\phi) - (-1)^k \sigma^*(\phi) \circ d_{\text{mix}} = \sigma^*(\psi).$$

We are interested in the bundle  $\pi_{\text{lcc}} : \text{LCC } X \rightarrow X$ .

**Theorem 6.5.** *Assume each affine open set  $U_{(i)}$  admits an étale morphism to  $\mathbf{A}_{\mathbb{K}}^n$ . Then there exist sections  $\sigma_{(i)} : U_{(i)} \rightarrow \text{LCC } X$ , and furthermore they extend to a simplicial section  $\sigma$  of  $\pi_{\text{lcc}} : \text{LCC } X \rightarrow X$ .*

*Proof.* By Theorem 4.13,  $\pi_{\text{coor}} : \text{Coor } X \rightarrow X$  is a locally trivial  $G$ -torsor over  $X$ , and  $G = \text{GL}_{n, \mathbb{K}} \ltimes N$ , where  $N$  is a pro-unipotent group. By definition  $\text{LCC } X = \text{Coor } X / \text{GL}_{n, \mathbb{K}}$ . According to Example 4.4 and Corollary 4.9, for any  $i$  there is a section  $\sigma_{(i)} : U_{(i)} \rightarrow \text{Coor } X$ . Now use [Ye4, Theorem 2.2].  $\square$

Here is the idea behind the proof of [Ye4, Theorem 2.2]. There is an averaging process for sections of torsors under unipotent groups. The bundle  $\text{LCC } X$  is “almost” a torsor under the pro-unipotent group  $N$ . Given a multi-index  $\mathbf{i} = (i_0, \dots, i_q)$  the morphism  $\sigma_{\mathbf{i}} : \Delta_{\mathbb{K}}^q \times U_{\mathbf{i}} \rightarrow \text{LCC } X$  is then a family of weighted averages of the sections  $\sigma_{(i_0)}, \dots, \sigma_{(i_q)} : U_{\mathbf{i}} \rightarrow \text{LCC } X$ , parameterized by the simplex  $\Delta_{\mathbb{K}}^q$ . See Figure 2 for an illustration.

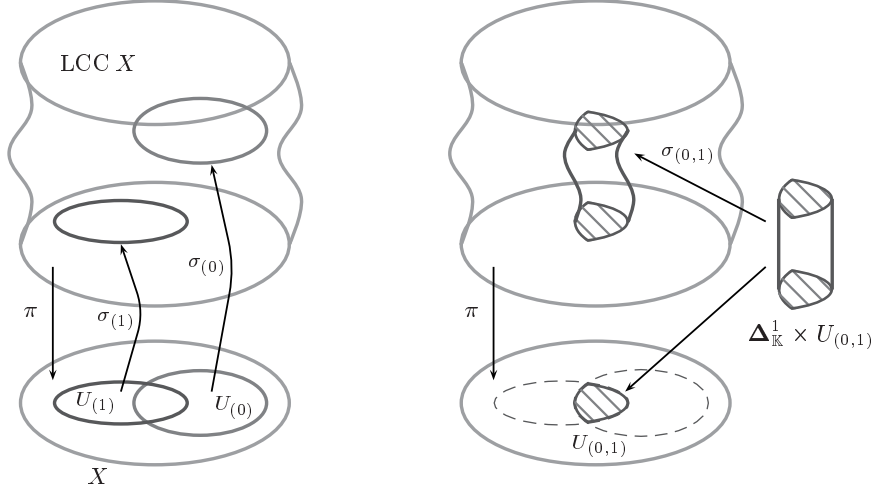


FIGURE 2. Simplicial sections,  $q = 1$ . We start with sections over two open sets  $U_{(0)}$  and  $U_{(1)}$  in the left diagram; and we pass to a simplicial section  $\sigma_{(0,1)}$  on the right.

## 7. THE GLOBAL $L_\infty$ QUASI-ISOMORPHISM

In this section we prove the main result of the paper. Here again  $X$  is a smooth irreducible separated  $n$ -dimensional scheme over the field  $\mathbb{K}$ , and also  $\mathbb{R} \subset \mathbb{K}$ .

Fix an open covering  $\mathbf{U} = \{U_{(0)}, \dots, U_{(m)}\}$  of the scheme  $X$  consisting of affine open sets, each admitting an étale morphism  $U_{(i)} \rightarrow \mathbf{A}_{\mathbb{K}}^n$ . For every  $i$  let  $\sigma_{(i)} : U_{(i)} \rightarrow \text{LCC } X$  be the corresponding section of  $\pi_{\text{lcc}} : \text{LCC } X \rightarrow X$ , and let  $\sigma$  be the resulting simplicial section (see Theorem 6.5).

Let  $\mathcal{M}$  be a bounded below complex of quasi-coherent  $\mathcal{O}_X$ -modules. The mixed resolution  $\text{Mix}_{\mathbf{U}}(\mathcal{M})$  was defined in Section 6. For any integer  $i$  let  $G^i \text{Mix}_{\mathbf{U}}(\mathcal{M}) := \bigoplus_{j=i}^{\infty} \text{Mix}_{\mathbf{U}}^j(\mathcal{M})$ , so  $\{G^i \text{Mix}_{\mathbf{U}}(\mathcal{M})\}_{i \in \mathbb{Z}}$  is a descending filtration of  $\text{Mix}_{\mathbf{U}}(\mathcal{M})$  by subcomplexes, with  $G^i \text{Mix}_{\mathbf{U}}(\mathcal{M}) = \text{Mix}_{\mathbf{U}}(\mathcal{M})$  for  $i \ll 0$  and  $\bigcap_i G^i \text{Mix}_{\mathbf{U}}(\mathcal{M}) = 0$ . Let

$$\text{gr}_G^i \text{Mix}_{\mathbf{U}}(\mathcal{M}) := G^i \text{Mix}_{\mathbf{U}}(\mathcal{M}) / G^{i+1} \text{Mix}_{\mathbf{U}}(\mathcal{M})$$

and  $\text{gr}_G \text{Mix}_{\mathbf{U}}(\mathcal{M}) := \bigoplus_i \text{gr}_G^i \text{Mix}_{\mathbf{U}}(\mathcal{M})$ .

By Proposition 6.3, if  $\mathcal{G}_X$  is either  $\mathcal{T}_{\text{poly}, X}$  or  $\mathcal{D}_{\text{poly}, X}$ , then  $\text{Mix}_{\mathbf{U}}(\mathcal{G}_X)$  is a sheaf of DG Lie algebras on  $X$ , and  $\mathcal{G}_X \rightarrow \text{Mix}_{\mathbf{U}}(\mathcal{G}_X)$  is a DG Lie algebra quasi-isomorphism.

Note that if  $\phi : \text{Mix}_{\mathbf{U}}(\mathcal{M}) \rightarrow \text{Mix}_{\mathbf{U}}(\mathcal{N})$  is a homomorphism of complexes that respects the filtration  $\{G^i \text{Mix}_{\mathbf{U}}\}$ , then there exists an induced homomorphism of complexes

$$\text{gr}_G(\phi) : \text{gr}_G \text{Mix}_{\mathbf{U}}(\mathcal{M}) \rightarrow \text{gr}_G \text{Mix}_{\mathbf{U}}(\mathcal{N}).$$

Suppose  $\mathcal{G}$  and  $\mathcal{H}$  are sheaves of DG Lie algebras on a topological space  $Y$ . An  $L_\infty$  morphism  $\Psi : \mathcal{G} \rightarrow \mathcal{H}$  is a sequence of sheaf morphisms  $\psi_j : \prod^j \mathcal{G} \rightarrow \mathcal{H}$ , such that for every open set  $V \subset Y$  the sequence  $\{\Gamma(V, \psi_j)\}_{j \geq 1}$  is an  $L_\infty$  morphism



$\Gamma(V, \mathcal{G}) \rightarrow \Gamma(V, \mathcal{H})$ . If  $\psi_1 : \mathcal{G} \rightarrow \mathcal{H}$  is a quasi-isomorphism then  $\Psi$  is called an  $L_\infty$  quasi-morphism.

**Theorem 7.1.** *Let  $\mathcal{U}$  and  $\sigma$  be as above. Then there is an induced  $L_\infty$  quasi-isomorphism*

$$\Psi_\sigma = \{\Psi_{\sigma;j} : \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly},X}) \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X})\}.$$

The homomorphism  $\Psi_{\sigma;1}$  respects the filtration  $\{G^i \text{Mix}_{\mathcal{U}}\}$ , and

$$\text{gr}_G(\Psi_{\sigma;1}) = \text{gr}_G(\text{Mix}_{\mathcal{U}}(\mathcal{U}_1)).$$

*Proof.* Let  $Y$  be some  $\mathbb{K}$ -scheme, and denote by  $\mathbb{K}_Y$  the constant sheaf. For any  $p$  we view  $\Omega_Y^p$  as a discrete inv  $\mathbb{K}_Y$ -module, and we put on  $\Omega_Y = \bigoplus_{p \in \mathbb{N}} \Omega_Y^p$  direct sum dir-inv structure. So  $\Omega_Y$  is a discrete (and hence complete) DG algebra in  $\text{Dir Inv Mod } \mathbb{K}_Y$ .

We shall abbreviate  $\mathcal{A} := \Omega_{\text{Coor } X}$ , so that  $\mathcal{A}^0 = \mathcal{O}_{\text{Coor } X}$  etc. As explained above,  $\mathcal{A}$  is a DG algebra in  $\text{Dir Inv Mod } \mathbb{K}_{\text{Coor } X}$ , with discrete (but not trivial) dir-inv module structure.

There are sheaves of DG Lie algebras  $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  and  $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$  on the scheme  $\text{Coor } X$ . The differentials are  $d_{\text{for}} = d \otimes \mathbf{1}$  and  $d_{\text{for}} + \mathbf{1} \otimes d_{\mathcal{D}}$  respectively. As explained just prior to Theorem 3.16, there is a continuous  $\mathcal{A}$ -multilinear  $L_\infty$  morphism

$$\mathcal{U}_{\mathcal{A}} = \{\mathcal{U}_{\mathcal{A};j} : \mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \rightarrow \mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])\}.$$

The MC form  $\omega := \omega_{\text{MC}}$  is a global section of  $\mathcal{A}^1 \widehat{\otimes} \mathcal{T}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]])$  satisfying the MC equation in the DG Lie algebra  $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ . See Proposition 5.9. According to Theorem 3.16 the global section  $\omega' := \mathcal{U}_{\mathcal{A};1}(\omega) \in \mathcal{A}^1 \widehat{\otimes} \mathcal{D}_{\text{poly}}^0(\mathbb{K}[[\mathbf{t}]])$  is a solution of the MC equation in the DG Lie algebra  $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ , and there is a continuous  $\mathcal{A}$ -multilinear  $L_\infty$  morphism

$$\mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j} : (\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))_\omega \rightarrow (\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]))_{\omega'}\}$$

between the twisted DG Lie algebras. The formula is

$$(7.2) \quad \mathcal{U}_{\mathcal{A},\omega;j}(\gamma_1 \cdots \gamma_j) = \sum_{k \geq 0} \frac{1}{(j+k)!} \mathcal{U}_{\mathcal{A};j+k}(\omega^k \gamma_1 \cdots \gamma_j)$$

for  $\gamma_1, \dots, \gamma_j \in \mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]])$ . The two twisted DG Lie algebras have differentials  $d_{\text{for}} + \text{ad}(\omega)$  and  $d_{\text{for}} + \text{ad}(\omega') + \mathbf{1} \otimes d_{\mathcal{D}}$  respectively.

By Theorem 5.6 (the universal Taylor expansions) there are canonical isomorphisms of graded Lie algebras in  $\text{Dir Inv Mod } \mathbb{K}_{\text{Coor } X}$

$$\mathcal{A} \widehat{\otimes} \mathcal{T}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X})$$

and

$$\mathcal{A} \widehat{\otimes} \mathcal{D}_{\text{poly}}(\mathbb{K}[[\mathbf{t}]]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}).$$

Proposition 5.8 tells us that

$$d_{\text{for}} + \text{ad}(\omega) = \nabla_{\mathcal{P}}$$

under these identifications. Therefore

$$\mathcal{U}_{\mathcal{A},\omega} : \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \rightarrow \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \pi_{\text{coor}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X})$$

is a continuous  $\mathcal{A}$ -multilinear  $L_\infty$  morphism between these DG Lie algebras, whose differentials are  $\nabla_{\mathcal{P}}$  and  $\nabla_{\mathcal{P}} + \mathbf{1} \otimes d_{\mathcal{D}}$  respectively.

By Propositions 5.13 and 5.14 the form  $\omega$  is  $\mathrm{GL}_n(\mathbb{K})$ -invariant. So according to Proposition 3.17 each of the operators  $\mathcal{U}_{A;j}$  and  $\mathcal{U}_{A,\omega;j}$  is  $\mathrm{GL}_n(\mathbb{K})$ -equivariant. We conclude that  $\omega$  is a global section of

$$\Omega_{\mathrm{LCC} X}^1 \widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC} X}} \pi_{\mathrm{lcc}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}^0),$$

and the operators  $\mathcal{U}_{A;j}$  and  $\mathcal{U}_{A,\omega;j}$  descend to continuous  $\Omega_{\mathrm{LCC} X}$ -multilinear operators

$$\begin{aligned} \mathcal{U}_{A;j}, \mathcal{U}_{A,\omega;j} &: \prod^j (\Omega_{\mathrm{LCC} X} \widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC} X}} \pi_{\mathrm{lcc}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X})) \\ &\rightarrow \Omega_{\mathrm{LCC} X} \widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC} X}} \pi_{\mathrm{lcc}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}) \end{aligned}$$

satisfying formula (7.2). The sequence  $\mathcal{U}_{A,\omega} = \{\mathcal{U}_{A,\omega;j}\}_{j \geq 1}$  is an  $L_\infty$  morphism.

According to Theorem 6.4 there are induced operators

$$\sigma^*(\mathcal{U}_{A;j}), \sigma^*(\mathcal{U}_{A,\omega;j}) : \prod^j \mathrm{Mix}_{\mathcal{U}}(\mathcal{T}_{\mathrm{poly},X}) \rightarrow \mathrm{Mix}_{\mathcal{U}}(\mathcal{D}_{\mathrm{poly},X}).$$

The  $L_\infty$  identities in Definition 3.7, when applied to the  $L_\infty$  morphism  $\mathcal{U}_{A,\omega}$ , are of the form considered in Theorem 6.4(iii). Therefore these identities are preserved by  $\sigma^*$ , and we conclude that the sequence  $\{\sigma^*(\mathcal{U}_{A,\omega;j})\}_{j \geq 1}$  is an  $L_\infty$  morphism. There's a global section  $\sigma^*(\omega) \in \mathrm{Mix}_{\mathcal{U}}^1(\mathcal{T}_{\mathrm{poly},X}^0)$ , and the formula

$$(7.3) \quad \sigma^*(\mathcal{U}_{A,\omega;j})(\gamma_1 \cdots \gamma_j) = \sum_{k \geq 0} \frac{1}{(j+k)!} \sigma^*(\mathcal{U}_{A;j+k})(\sigma^*(\omega)^k \gamma_1 \cdots \gamma_j)$$

holds for local sections  $\gamma_1, \dots, \gamma_j \in \mathrm{Mix}_{\mathcal{U}}(\mathcal{T}_{\mathrm{poly},X})$ . This sum is finite, the number of nonzero terms in it depending on the bidegree of  $\gamma_1 \cdots \gamma_j$ . Indeed, if  $\gamma_1 \cdots \gamma_j \in \mathrm{Mix}_{\mathcal{U}}^q(\mathcal{T}_{\mathrm{poly},X}^p)$  then

$$(7.4) \quad \sigma^*(\mathcal{U}_{A;j+k})(\sigma^*(\omega)^k \gamma_1 \cdots \gamma_j) \in \mathrm{Mix}_{\mathcal{U}}^{q+k}(\mathcal{D}_{\mathrm{poly},X}^{p+1-j-k}),$$

which is zero for  $k > p - j + 2$ ; see proof of [Ye2, Theorem 3.23].

Finally we define  $\Psi_{\sigma;j} := \sigma^*(\mathcal{U}_{A,\omega;j})$ . The collection  $\Psi_\sigma = \{\Psi_{\sigma;j}\}_{j=1}^\infty$  is then an  $L_\infty$  morphism. From equation (7.4) we see that  $\Psi_{\sigma;1}$  respects the filtration  $\{G^i \mathrm{Mix}_{\mathcal{U}}\}$ , and according to equation (7.3) we see that

$$\mathrm{gr}_G(\Psi_{\sigma;1}) = \mathrm{gr}_G(\sigma^*(\mathcal{U}_{A;1})) = \mathrm{gr}_G(\mathrm{Mix}_{\mathcal{U}}(\mathcal{U}_1)).$$

According to Theorems 3.12 and 6.2 the homomorphism  $\mathrm{Mix}_{\mathcal{U}}(\mathcal{U}_1)$  is a quasi-isomorphism. Since the complexes  $\mathrm{Mix}_{\mathcal{U}}(\mathcal{T}_{\mathrm{poly},X})$  and  $\mathrm{Mix}_{\mathcal{U}}(\mathcal{D}_{\mathrm{poly},X})$  are bounded below, and the filtration is nonnegative and exhaustive, it follows that  $\Psi_{\sigma;1}$  is also a quasi-isomorphism.  $\square$

**Corollary 7.5.** *Taking global sections in Theorem 7.1 we get an  $L_\infty$  quasi-isomorphism*

$$\Gamma(X, \Psi_\sigma) = \{\Gamma(X, \Psi_{\sigma;j})\}_{j \geq 1} : \Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\mathcal{T}_{\mathrm{poly},X})) \rightarrow \Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\mathcal{D}_{\mathrm{poly},X})).$$

*Proof.* By Theorem 6.2 the homomorphism

$$\Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\mathcal{U}_1)) : \Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\mathcal{T}_{\mathrm{poly},X})) \rightarrow \Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\mathcal{D}_{\mathrm{poly},X}))$$

is a quasi-isomorphism, and by the interaction with the filtration  $\{G^i \mathrm{Mix}_{\mathcal{U}}\}$  we see that  $\Gamma(X, \mathrm{Mix}_{\mathcal{U}}(\Psi_{\sigma;1}))$  is also a quasi-isomorphism.  $\square$

**Corollary 7.6.** *The data  $(\mathbf{U}, \sigma)$  induces a bijection*

$$\begin{aligned} \text{MC}(\Psi_\sigma) : \text{MC}\left(\Gamma(X, \text{Mix}_{\mathbf{U}}(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+\right) \\ \xrightarrow{\cong} \text{MC}\left(\Gamma(X, \text{Mix}_{\mathbf{U}}(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+\right). \end{aligned}$$

*Proof.* Use Corollaries 7.5 and 3.10.  $\square$

Recall that  $\mathcal{T}_{\text{poly}}(X) = \Gamma(X, \mathcal{T}_{\text{poly}, X})$  and  $\mathcal{D}_{\text{poly}}^{\text{nor}}(X) = \Gamma(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}})$ ; and the latter is the DG Lie algebra of global poly differential operators that vanish if one of their arguments is 1.

**Theorem 7.7.** *Assume  $H^q(X, \mathcal{D}_{\text{poly}, X}^{\text{nor}, p}) = 0$  for all  $p$  and all  $q > 0$ . Then there is a canonical function*

$$Q : \text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+) \xrightarrow{\cong} \text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+)$$

*preserving first order terms. If moreover  $H^q(X, \mathcal{T}_{\text{poly}, X}^p) = 0$  for all  $p$  and all  $q > 0$ , then  $Q$  is bijective. The function  $Q$  is called the quantization map, and it is characterized as follows. Choose an open covering  $\mathbf{U} = \{U_{(0)}, \dots, U_{(m)}\}$  of  $X$  consisting of affine open sets, each admitting an étale morphism  $U_{(i)} \rightarrow \mathbf{A}_{\mathbb{K}}^n$ . Let  $\sigma$  be the associated simplicial section of  $\text{LCC } X \rightarrow X$ . Then there is a commutative diagram*

$$(7.8) \quad \begin{array}{ccc} \text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+) & \xrightarrow{Q} & \text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+) \\ \text{MC}(\text{inc}) \downarrow & & \text{MC}(\text{inc}) \downarrow \\ \text{MC}(\Gamma(X, \text{Mix}_{\mathbf{U}}(\mathcal{T}_{\text{poly}, X}))[[\hbar]]^+) & \xrightarrow{\text{MC}(\Psi_\sigma)} & \text{MC}(\Gamma(X, \text{Mix}_{\mathbf{U}}(\mathcal{D}_{\text{poly}, X}))[[\hbar]]^+) \end{array}$$

*in which the right vertical arrow is bijective. Here  $\Psi_\sigma$  is the  $L_\infty$  quasi-isomorphism from Theorem 7.1, and “inc” denotes the various inclusions of DG Lie algebras.*

Let’s elaborate a bit on the statement above. It says that to any formal Poisson structure  $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathcal{T}_{\text{poly}}^1(X)[[\hbar]]^+$  there corresponds a star product  $\star_\beta$ , with  $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\text{poly}}^{\text{nor}, 1}(X)[[\hbar]]^+$  (cf. Proposition 3.20). The element  $\beta = Q(\alpha)$  is uniquely determined up to gauge equivalence by  $\exp(\mathcal{D}_{\text{poly}}^{\text{nor}, 0}(X)[[\hbar]]^+)$ . Given any local sections  $f, g \in \mathcal{O}_X$  one has

$$\beta_1(f, g) - \beta_1(g, f) = 2\{f, g\}_{\alpha_1}.$$

The quantization map  $Q$  can be calculated (at least in theory) using the collection of sections  $\sigma$  and the universal formulas for deformation in Theorem 3.13.

We’ll need a lemma before proving the theorem.

**Lemma 7.9.** *Let  $f, g \in \mathcal{O}_X = \mathcal{D}_{\text{poly}, X}^{-1}$  be local sections.*

(1) *For any  $\beta \in \text{Mix}_{\mathbf{U}}^0(\mathcal{D}_{\text{poly}, X}^1)$  one has*

$$[[\beta, f], g] = \beta(g, f) - \beta(f, g) \in \text{Mix}_{\mathbf{U}}^0(\mathcal{O}_X).$$

(2) *For any  $\beta \in \text{Mix}_{\mathbf{U}}^1(\mathcal{D}_{\text{poly}, X}^0) \oplus \text{Mix}_{\mathbf{U}}^2(\mathcal{D}_{\text{poly}, X}^{-1})$  one has  $[[\beta, f], g] = 0$ .*

(3) *Let  $\gamma \in \text{Mix}_{\mathbf{U}}(\mathcal{D}_{\text{poly}, X}^0)$ , and define  $\beta := (d_{\text{mix}} + d_{\mathcal{D}})(\gamma)$ . Then  $[[\beta, f], g] = 0$ .*

*Proof.* (1) Proposition 6.3 implies that the embedding (6.1):

$$\text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X}) \subset \bigoplus_{p,q,r} \prod_{j \in \mathbb{N}} \prod_{i \in \Delta_j^m} g_{i*} g_i^{-1} (\Omega^q(\Delta_{\mathbb{K}}^j) \widehat{\otimes} (\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}^r))$$

is a DG Lie algebra homomorphism. So by continuity we might as well assume that  $\beta = aD$  with  $a \in \Omega_X^0 = \mathcal{O}_X$  and  $D \in \mathcal{D}_{\text{poly},X}^1$ . Moreover, since the Lie bracket of  $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}$  is  $\Omega_X$ -bilinear, we may assume that  $a = 1$ , i.e.  $\beta = D$ . Now the assertion is clear from the definition of the Gerstenhaber Lie bracket, see [Ko1, Section 3.4.2].

(2) Applying the same reduction as above, but with  $D \in \mathcal{D}_{\text{poly},X}^r$  and  $r \in \{0, -1\}$ , we get  $[[D, f], g] \in \mathcal{D}_{\text{poly},X}^{r-2} = 0$ .

(3) By part (2) it suffices to show that  $[[\beta, f], g] = 0$  for  $\beta := d_{\mathcal{D}}(\gamma)$  and  $\gamma \in \text{Mix}_{\mathcal{U}}^0(\mathcal{D}_{\text{poly},X}^0)$ . As explained above we may further assume that  $\gamma = D \in \mathcal{D}_{\text{poly},X}^0$ . Now the formulas for  $d_{\mathcal{D}}$  and  $[-, -]$  in [Ko1, Section 3.4.2] imply that  $[[d_{\mathcal{D}}(D), f], g] = 0$ .  $\square$

*Proof of Theorem 7.7.* We are given that  $H^q(X, \mathcal{D}_{\text{poly},X}^{\text{nor},p}) = 0$  for all  $p$  and all  $q > 0$ ; and therefore  $\Gamma(X, \mathcal{D}_{\text{poly},X}^{\text{nor}}) = \text{R}\Gamma(X, \mathcal{D}_{\text{poly},X}^{\text{nor}})$  in the derived category  $\text{D}(\text{Mod } \mathbb{K})$ . Now by Theorem 3.12 the inclusion  $\mathcal{D}_{\text{poly},X}^{\text{nor}} \rightarrow \mathcal{D}_{\text{poly},X}$  is a quasi-isomorphism, and by Theorem 6.2(1) the inclusion  $\mathcal{D}_{\text{poly},X} \rightarrow \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X})$  is a quasi-isomorphism. According to Theorem 6.2(2) we have  $\Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X})) = \text{R}\Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X}))$ . The conclusion is that

$$(7.10) \quad \mathcal{D}_{\text{poly}}^{\text{nor}}(X) = \Gamma(X, \mathcal{D}_{\text{poly},X}^{\text{nor}}) \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X}))$$

is a quasi-isomorphism of complexes of  $\mathbb{K}$ -modules. But in view of Proposition 6.3 this is also a homomorphism of DG Lie algebras.

From (7.10) we deduce that

$$\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+ \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{D}_{\text{poly},X}))[[\hbar]]^+$$

is a quasi-isomorphism of DG Lie algebras. Using Corollary 3.10 we see that the right vertical arrow in the diagram (7.8) is bijective. Therefore this diagram defines  $Q$  uniquely.

According to Corollary 7.6 the bottom arrow in diagram (7.8) is a bijection. The left vertical arrow comes from the DG Lie algebra homomorphism

$$\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+ \rightarrow \Gamma(X, \text{Mix}_{\mathcal{U}}(\mathcal{T}_{\text{poly},X}))[[\hbar]]^+,$$

which is a quasi-isomorphism when  $H^q(X, \mathcal{T}_{\text{poly},X}^p) = 0$  for all  $p$  and all  $q > 0$ . So in case of this further vanishing of cohomology the map  $Q$  is bijective.

Now suppose  $\mathcal{U}' = \{U'_{(0)}, \dots, U'_{(m')}\}$  is another such covering of  $X$ , with sections  $\sigma'_{(i)} : U'_{(i)} \rightarrow \text{LCC } X$ . Without loss of generality we may assume that  $m' \geq m$ , and that  $U'_{(i)} = U_{(i)}$  and  $\sigma'_{(i)} = \sigma_{(i)}$  for all  $i \leq m$ . There is a morphism of simplicial schemes  $f : \mathcal{U} \rightarrow \mathcal{U}'$ , that is an open and closed embedding. Correspondingly there

is a commutative diagram

$$\begin{array}{ccc}
 \mathrm{MC}(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^+) & \xrightarrow{Q} & \mathrm{MC}(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[[\hbar]]^+) \\
 \mathrm{MC}(\mathrm{inc}) \downarrow & & \mathrm{MC}(\mathrm{inc}) \downarrow \\
 \mathrm{MC}(\Gamma(X, \mathrm{Mix}_{U'}(\mathcal{T}_{\mathrm{poly}, X}))[[\hbar]]^+) & \xrightarrow{\mathrm{MC}(\Psi_{\sigma'})} & \mathrm{MC}(\Gamma(X, \mathrm{Mix}_{U'}(\mathcal{D}_{\mathrm{poly}, X}))[[\hbar]]^+) \\
 \mathrm{MC}(f^*) \downarrow & & \mathrm{MC}(f^*) \downarrow \\
 \mathrm{MC}(\Gamma(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X}))[[\hbar]]^+) & \xrightarrow{\mathrm{MC}(\Psi_{\sigma})} & \mathrm{MC}(\Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X}))[[\hbar]]^+),
 \end{array}$$

where the vertical arrows on the right are bijections. We conclude that  $Q$  is independent of  $U$  and  $\sigma$ .

Finally we must show that  $Q$  preserves first order terms. Let  $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$  be a formal Poisson structure, and let  $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)^1[[\hbar]]^+$  be a solution of the MC equation, such that  $\beta = Q(\alpha)$  modulo gauge equivalence. This means that there exists some

$$\gamma = \sum_{k \geq 1} \gamma_k \hbar^k \in \Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X}))^0[[\hbar]]^+$$

such that

$$\sum_{j \geq 1} \frac{1}{j!} \Psi_{\sigma; j}(\alpha^j) = \exp(\mathrm{af})(\exp(\gamma))(\beta),$$

with notation as in Lemma 3.2. In the first order term (i.e. the coefficient of  $\hbar^1$ ) of this equation we have

$$(7.11) \quad \Psi_{\sigma; 1}(\alpha_1) = \beta_1 - (\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1);$$

see equation (3.3). Now by definition (see proof of Theorem 7.1)

$$\Psi_{\sigma; 1}(\alpha_1) = \sigma^*(\mathcal{U}_{A, \omega; 1}) = \sum_{k \geq 0} \frac{1}{(1+k)!} \sigma^*(\mathcal{U}_{A; 1+k}) (\sigma^*(\omega_{\mathrm{MC}})^k \alpha_1),$$

and the component in  $\Gamma(X, \mathrm{Mix}_U^0(\mathcal{D}_{\mathrm{poly}, X}^1)[[\hbar]]^+)$  is the summand with  $k = 0$ , namely  $\sigma^*(\mathcal{U}_{A; 1})(\alpha_1) = \mathcal{U}_1(\alpha_1)$ . Using Lemma 7.9 we get

$$\begin{aligned}
 [[\Psi_{\sigma; 1}(\alpha_1), f], g] &= [[\mathcal{U}_1(\alpha_1), f], g] = \mathcal{U}_1(\alpha_1)(g, f) - \mathcal{U}_1(\alpha_1)(f, g) = -2\{f, g\}_{\alpha_1}, \\
 [[\beta_1, f], g] &= \beta_1(g, f) - \beta_1(f, g)
 \end{aligned}$$

and

$$[[\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}}(\gamma_1), f], g] = 0$$

for every local sections  $f, g \in \mathcal{O}_X$ . Combining these equations with equation (7.11) the proof is done.  $\square$

One says that  $X$  is a  $\mathcal{D}$ -affine variety if  $\mathrm{H}^q(X, \mathcal{M}) = 0$  for every quasi-coherent left  $\mathcal{D}_X$ -module  $\mathcal{M}$  and every  $q > 0$ .

**Corollary 7.12.** *Assume  $X$  is  $\mathcal{D}$ -affine. Then the quantization map  $Q$  of Theorem 7.7 may be interpreted as a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

preserving first order terms. If  $X$  is affine then  $Q$  is bijective.

*Proof.* By definition the left side is  $\text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+)$ . On the other hand, according to Theorem 1.13 every deformation quantization of  $\mathcal{O}_X$  can be trivialized globally, and by Proposition 1.14 any gauge equivalence between globally trivialized deformation quantizations is a global gauge equivalence. Hence the right side is  $\text{MC}(\mathcal{D}_{\text{poly}}^{\text{nor}}(X)[[\hbar]]^+)$ . Since each  $\mathcal{D}_{\text{poly},X}^{\text{nor},p}$  is a quasi-coherent left  $\mathcal{D}_X$ -module, and each  $\mathcal{T}_{\text{poly},X}^p$  is a quasi-coherent  $\mathcal{O}_X$ -module, we can apply Theorem 7.7.  $\square$

Suppose  $f : X' \rightarrow X$  is an étale morphism. According to [Ye2, Proposition 4.6] there are DG Lie algebra homomorphisms  $f^* : \mathcal{T}_{\text{poly}}(X) \rightarrow \mathcal{T}_{\text{poly}}(X')$  and  $f^* : \mathcal{D}_{\text{poly}}^{\text{nor}}(X) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}}(X')$ . Given a formal Poisson structure  $\alpha$  on  $X$  we then obtain a formal Poisson structure  $f^*(\alpha)$  on  $X'$ . Similarly a star product  $\star$  on  $\mathcal{O}_X[[\hbar]]$  induces a star product  $f^*(\star)$  on  $\mathcal{O}_{X'}[[\hbar]]$ ,

**Corollary 7.13.** *The quantization map  $Q$  respects étale morphisms. Namely if  $X$  and  $X'$  are  $\mathcal{D}$ -affine schemes and  $f : X' \rightarrow X$  is an étale morphism, then for any formal Poisson structure  $\alpha$  on  $X$  one has  $Q(f^*(\alpha)) = f^*(Q(\alpha))$ .*

*Proof.* This is clear from the proof of Theorem 7.7.  $\square$

## 8. COMPLEMENTS AND REMARKS

Suppose  $C$  is some smooth commutative  $\mathbb{K}$ -algebra, where  $\mathbb{K}$  is a field containing  $\mathbb{R}$ . It is conceivable to look for a star product on  $C[[\hbar]]$  that is *non-differential*. Namely, a  $\mathbb{K}[[\hbar]]$ -bilinear, associative, unital multiplication  $\star$  on  $C[[\hbar]]$  of the form

$$f \star g = fg + \sum_{k=1}^{\infty} \beta_k(f, g) \hbar^k,$$

where the normalized  $\mathbb{K}$ -bilinear functions  $\beta_k : C^2 \rightarrow C$  are not necessarily bi-differential operators. Indeed, classically this was the type of deformation that had been considered (cf. [Ge]). There is a corresponding notion of non-differential gauge equivalence, via an automorphism  $\gamma = \mathbf{1}_C + \sum_{k=1}^{\infty} \gamma_k \hbar^k$  of  $C[[\hbar]]$  with  $\gamma_k : C \rightarrow C$  normalized  $\mathbb{K}$ -linear functions.

**Proposition 8.1.** *Let  $C$  be a smooth  $\mathbb{K}$ -algebra. Then the obvious function*

$$\frac{\{\text{star products on } C[[\hbar]]\}}{\text{gauge equivalence}} \rightarrow \frac{\{\text{non-differential star products on } C[[\hbar]]\}}{\text{non-differential gauge equivalence}}$$

*is bijective.*

*Proof.* Let us denote by  $\mathcal{G}(C)$  the shifted full Hochschild cochain complex of  $C$ , and let  $\mathcal{G}^{\text{nor}}(C)$  be the subcomplex of normalized cochains. It is a well-known fact that the inclusion  $\mathcal{G}^{\text{nor}}(C) \hookrightarrow \mathcal{G}(C)$  is a quasi-isomorphism (it is an immediate consequence of [ML, Corollary X.2.2]). By [Ye1, Lemma 4.3] the  $C$ -linear map  $\mathcal{U}_1 : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{G}(C)$  is a quasi-isomorphism, and by Theorem 3.12 the map  $\mathcal{U}_1 : \mathcal{T}_{\text{poly}}(C) \rightarrow \mathcal{D}_{\text{poly}}^{\text{nor}}(C)$  is a quasi-isomorphism. The upshot is that the inclusion  $\mathcal{D}_{\text{poly}}^{\text{nor}}(C) \hookrightarrow \mathcal{G}^{\text{nor}}(C)$  is a quasi-isomorphism of DG Lie algebras. Now we can use Propositions 3.20 and 3.21, as well as their “classical” non-differential variants (see proof of [Ke, Corollary 4.5]).  $\square$

Combining Proposition 8.1 with Corollary 7.12 (applied to  $X := \text{Spec } C$ ) we obtain:

**Corollary 8.2.** *Let  $C$  be a smooth  $\mathbb{K}$ -algebra. Then there is a canonical bijection of sets*

$$Q : \frac{\{\text{formal Poisson structures on } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{non-differential star products on } C[[\hbar]]\}}{\text{non-differential gauge equivalence}}$$

*preserving first order terms.*

**Question 8.3.** In case  $X$  is affine and admits an étale morphism  $X \rightarrow \mathbf{A}_{\mathbb{K}}^n$ , how are the the deformation quantizations of Corollary 7.12 Corollary 3.24 related?

**Remark 8.4.** The methods of this paper, combined with the ideas of [CFT], can be used to prove the following result. Suppose  $\mathbb{R} \subset \mathbb{K}$  and  $H^2(X, \mathcal{O}_X) = 0$ . Let  $\alpha$  be any Poisson structure on  $X$ . Then the Poisson variety  $(X, \alpha)$  admits a deformation quantization, in the sense of Definition 1.11.

**Question 8.5.** Given a smooth scheme  $X$ , is it possible to determine which Poisson structures on  $X$  can be quantized? The papers [NT] and [BK1] say that for a symplectic structure to be quantizable there are cohomological obstructions. Can anything like that be done for a degenerate Poisson structure?

**Remark 8.6.** Artin worked out a noncommutative deformation theory for schemes that goes step by step, from  $\mathbb{K}[\hbar]/(\hbar^m)$  to  $\mathbb{K}[\hbar]/(\hbar^{m+1})$ ; see [Ar1] and [Ar2]. The first order data is a Poisson structure, and at each step there are well defined obstructions to the process. Presumably Artin's deformations are deformation quantizations in the sense of Definition 1.6, namely they admit differential structures; but this requires a proof.

In the case of the projective plane  $\mathbf{P}^2$  and a nonzero Poisson structure  $\alpha$ , the zero locus of  $\alpha$  is a cubic divisor  $E$ . Assume  $E$  is smooth. Artin asserts (private communication) that a particular deformation of  $\mathcal{O}_{\mathbf{P}^2}$  with first order term  $\alpha$  lifts to a deformation of the homogeneous coordinate ring  $B := \bigoplus_{i \geq 0} \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(i))$ . Namely there is a graded  $\mathbb{K}[[\hbar]]$ -algebra structure on  $\bigoplus_{i \geq 0} B_i[[\hbar]]$ , say  $\star$ , such that  $a \star b \equiv ab \pmod{\hbar}$ , and  $a \star b - b \star a \equiv 2\hbar\{a, b\}_{\alpha} \pmod{\hbar^2}$ , for all  $a, b \in B$ . Moreover, after tensoring with the field  $\mathbb{K}((\hbar))$  this should be a three dimensional Sklyanin algebra, presumably with associated elliptic curve  $\mathbb{K}((\hbar)) \times_{\mathbb{K}} E$ .

The above should be compared to [Ko3, Section 3].

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