Corrigendum

Corrigendum to "Deformation quantization in algebraic geometry" [Adv. Math. 198 (2005) 383–432]

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0. Introduction

This note contains a correction of the *proofs* of the main results of [3], namely Theorems 0.1 and 0.2. The results are correct as originally stated.

The mistake in my original proofs was discovered Michel Van den Bergh, and I thank him for calling my attention to it. The way to fix the proofs is essentially contained in his paper [2].

Let me begin by explaining the mistake. As can be seen in Example 0.1 below, the mistake itself is of a rather elementary nature, but it was obscured by the complicated context.

Suppose $\mathbb{K}$ is a field of characteristic 0, and X is a smooth separated n-dimensional scheme over $\mathbb{K}$. Recall that the coordinate bundle Coor X is an infinite dimensional bundle over X, with free action by the group $\mathrm{GL}_{n,\mathbb{K}}$. The quotient bundle is by definition

$$\mathrm{LCC}\, X := \mathrm{Coor}\, X / \mathrm{GL}_{n,\mathbb{K}},$$

and the projection $\pi_{\mathrm{gl}} : \mathrm{Coor}\, X \to \mathrm{LCC}\, X$ is a $\mathrm{GL}_{n,\mathbb{K}}$-torsor.

The erroneous (implicit) assertion in [3] is that the de Rham complexes satisfy

$$(\pi_{\mathrm{gl}*} \Omega_{\mathrm{Coor}\, X})^{\mathrm{GL}_n(\mathbb{K})} = \Omega_{\mathrm{LCC}\, X}.$$

From that I deduced (incorrectly, top of page 424) that the Maurer–Cartan form ω_{MC} is a global section of the sheaf

$$\Omega^1_{\mathrm{LCC}\, X} \, \widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC}\, X}} \, \pi_{\mathrm{lcc}}^{\widehat{*}} \big(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}^0_{\mathrm{poly},X} \big).$$

(This is false, as can be seen from [2, Lemma 6.5.1].) This led to many incorrect formulas in [3, Section 7].

The correct thing to do is to work with the infinitesimal action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$. For $v \in \mathfrak{g}$ one has the contraction (inner derivative) ι_v, which is a degree -1 derivation of the de Rham complex $\pi_{\mathrm{gl}*} \Omega_{\mathrm{Coor}\, X}$. Recall that the Lie derivative is $\mathrm{L}_v := \mathrm{d} \circ \iota_v + \iota_v \circ \mathrm{d}$. A local section $\omega \in \pi_{\mathrm{gl}*} \Omega_{\mathrm{Coor}\, X}$ is said to be $\mathfrak{g}$-invariant if $\iota_v(\omega) = \mathrm{L}_v(\omega) = 0$ for all $v \in \mathfrak{g}$. According to [2, Lemma 9.2.3] one has

$$(\pi_{\mathrm{gl}*} \Omega_{\mathrm{Coor}\, X})^{\mathfrak{g}} = \Omega_{\mathrm{LCC}\, X}.$$

It is worthwhile to note that in my incorrect proof there was no need to invoke Kontsevich's property (P5) from [1]. The correct proof does require property (P5)—cf. [2, Lemma 9.2.1].

Example 0.1. Here is a simplified example. Suppose G is the affine algebraic group $\mathrm{GL}_{1,\mathbb{K}} = \mathrm{Spec}\, \mathbb{K}[t, t^{-1}]$, and X is the variety G, with regular left action. The group of rational points is $G(\mathbb{K}) = \mathbb{K}^{\times}$. The action of G on X is free, the invariant ring is $\mathcal{O}(X)^{G(\mathbb{K})} = \mathbb{K}$, and the quotient is $X/G = \mathrm{Spec}\, \mathbb{K}$. For the de Rham complex

$$\Omega(X) = \mathcal{O}(X) \oplus \Omega^1(X) = \mathbb{K}[t, t^{-1}] \oplus \mathbb{K}[t, t^{-1}] \cdot \mathrm{d}t$$

we have $\Omega(X)^{G(\mathbb{K})} \neq \mathbb{K}$, since it contains $t^{-1}\, \mathrm{d}t$. But for the infinitesimal action of the Lie algebra $\mathfrak{g} := \mathfrak{gl}_1(\mathbb{K})$ it is easy to see that $\Omega(X)^{\mathfrak{g}} = \mathbb{K}$.

After some deliberation I decided that the best way to present the correction is by completely rewriting [3, Section 7]. This is Section 1 below. Section 2 contains some additional minor corrections to [3].

1. The global L_∞ quasi-isomorphism

This is a revised version of [3, Section 7]. In this section we prove the main results of the paper [3], namely Theorem 0.1 (which is repeated here as Corollary 1.19) and Theorem 0.2 (which is repeated here, with more details, as Theorem 1.2). Throughout $\mathbb{K}$ is a field containing $\mathbb{R}$, and X is a smooth irreducible separated n-dimensional scheme over $\mathbb{K}$. We use all notation, definitions and results of [3, Sections 1–6] freely. However, the bibliography references relate to the list at the end of this note.

Suppose $U = \{U_0, \ldots, U_m\}$ is an open covering of the scheme X, consisting of affine open sets, each admitting an étale coordinate system, namely an étale morphism $U_i \to \mathbf{A}^n_{\mathbb{K}}$. For every i let $\sigma_i : U_i \to \mathrm{LCC}\, X$ be the corresponding section of $\pi_{\mathrm{lcc}} : \mathrm{LCC}\, X \to X$, and let σ be the resulting simplicial section (see [3, Theorem 6.5]).

Let $\mathcal{M}$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules. The mixed resolution $\mathrm{Mix}_U(\mathcal{M})$ was defined in [3, Section 6]. For any integer i let

$$\mathrm{G}^i \mathrm{Mix}_U(\mathcal{M}) := \bigoplus_{j=i}^{\infty} \mathrm{Mix}_U^j(\mathcal{M}),$$

so $\{G^i \operatorname{Mix}_U(\mathcal{M})\}_{i \in \mathbb{Z}}$ is a descending filtration of $\operatorname{Mix}_U(\mathcal{M})$ by subcomplexes, with $G^i \operatorname{Mix}_U(\mathcal{M}) = \operatorname{Mix}_U(\mathcal{M})$ for $i \leqslant 0$, and $\bigcap_i G^i \operatorname{Mix}_U(\mathcal{M}) = 0$. Let

$$\operatorname{gr}_G^i \operatorname{Mix}_U(\mathcal{M}) := G^i \operatorname{Mix}_U(\mathcal{M})/G^{i+1} \operatorname{Mix}_U(\mathcal{M})$$

and $\operatorname{gr}_G \operatorname{Mix}_U(\mathcal{M}) := \bigoplus_i \operatorname{gr}_G^i \operatorname{Mix}_U(\mathcal{M})$.

By [3, Proposition 6.3], if $\mathcal{G}_X$ is either $\mathcal{T}_{\operatorname{poly},X}$ or $\mathcal{D}_{\operatorname{poly},X}$, then $\operatorname{Mix}_U(\mathcal{G}_X)$ is a sheaf of DG Lie algebras on X, and the inclusion

$$\eta_{\mathcal{G}} : \mathcal{G}_X \to \operatorname{Mix}_U(\mathcal{G}_X)$$

is a DG Lie algebra quasi-isomorphism.

Note that if $\phi : \operatorname{Mix}_U(\mathcal{M}) \to \operatorname{Mix}_U(\mathcal{N})$ is a homomorphism of complexes that respects the filtration $\{G^i \operatorname{Mix}_U\}$, then there exists an induced homomorphism of complexes

$$\operatorname{gr}_G(\phi) : \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{M}) \to \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{N}).$$

Suppose $\mathcal{G}$ and $\mathcal{H}$ are sheaves of DG Lie algebras on a topological space Y. An L_∞ morphism $\Psi : \mathcal{G} \to \mathcal{H}$ is a sequence of sheaf morphisms $\psi_j : \prod^j \mathcal{G} \to \mathcal{H}$, such that for every open set $V \subset Y$ the sequence $\{\Gamma(V, \psi_j)\}_{j \geqslant 1}$ is an L_∞ morphism $\Gamma(V, \mathcal{G}) \to \Gamma(V, \mathcal{H})$. If $\psi_1 : \mathcal{G} \to \mathcal{H}$ is a quasi-isomorphism then Ψ is called an L_∞ quasi-isomorphism.

Recall that there is a canonical quasi-isomorphism of complexes of $\mathcal{O}_X$-modules

$$\mathcal{U}_1 : \mathcal{T}_{\operatorname{poly},X} \to \mathcal{D}_{\operatorname{poly},X}. \tag{1.1}$$

According to [4, Theorem 4.17], the induced homomorphism

$$\operatorname{gr}_G\big(\operatorname{Mix}_U(\mathcal{U}_1)\big) : \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{T}_{\operatorname{poly},X}) \to \operatorname{gr}_G \operatorname{Mix}_U(\mathcal{D}_{\operatorname{poly},X})$$

is a quasi-isomorphism.

Theorem 1.2. *Let X be an irreducible smooth separated $\mathbb{K}$-scheme. Let $U = \{U_0, \ldots, U_m\}$ be an open covering of X consisting of affine open sets, each admitting an étale coordinate system, and let σ be the associated simplicial section of the bundle $\operatorname{LCC} X \to X$. Then there is an induced L_∞ quasi-isomorphism*

$$\Psi_{\sigma} = \{\Psi_{\sigma;j}\}_{j \geqslant 1} : \operatorname{Mix}_U(\mathcal{T}_{\operatorname{poly},X}) \to \operatorname{Mix}_U(\mathcal{D}_{\operatorname{poly},X}).$$

The homomorphism $\Psi_{\sigma;1}$ respects the filtration $\{G^i \operatorname{Mix}_U\}$, and

$$\operatorname{gr}_G(\Psi_{\sigma;1}) = \operatorname{gr}_G\big(\operatorname{Mix}_U(\mathcal{U}_1)\big).$$

Proof. Let Y be some $\mathbb{K}$-scheme, and denote by $\mathbb{K}_Y$ the constant sheaf. For any p we view Ω_Y^p as a discrete inv $\mathbb{K}_Y$-module, and we put on $\Omega_Y = \bigoplus_{p \in \mathbb{N}} \Omega_Y^p$ the direct sum dir-inv structure. So Ω_Y is a discrete (and hence complete) DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Y$.

We shall abbreviate $\mathcal{A} := \Omega_{\operatorname{Coor} X}$, so that $\mathcal{A}^0 = \mathcal{O}_{\operatorname{Coor} X}$ etc. As explained above, $\mathcal{A}$ is a DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_{\operatorname{Coor} X}$, with discrete (but not trivial) dir-inv module structure.

There are sheaves of DG Lie algebras $\mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$ and $\mathcal{A} \,\widehat{\otimes}\, \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$ on the scheme $\text{Coor } X$. The differentials are $d_{\text{for}} = d \otimes \mathbf{1}$ and $d_{\text{for}} + \mathbf{1} \otimes d_{\mathcal{D}}$ respectively. As explained just prior to [3, Theorem 3.16], $\mathcal{U}$ extends to a continuous $\mathcal{A}$-multilinear L_∞ morphism

$$\mathcal{U}_{\mathcal{A}} = \{\mathcal{U}_{\mathcal{A};j}\}_{j \geqslant 1} : \mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!]) \to \mathcal{A} \,\widehat{\otimes}\, \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$$

of sheaves of DG Lie algebras on $\text{Coor } X$.

The MC form $\omega := \omega_{\text{MC}}$ is a global section of $\mathcal{A}^1 \,\widehat{\otimes}\, \mathcal{T}^0_{\text{poly}}(\mathbb{K}[\![t]\!])$ satisfying the MC equation in the DG Lie algebra $\mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$. See [3, Proposition 5.9]. According to [3, Theorem 3.16], the global section $\omega' := \mathcal{U}_{\mathcal{A};1}(\omega) \in \mathcal{A}^1 \,\widehat{\otimes}\, \mathcal{D}^0_{\text{poly}}(\mathbb{K}[\![t]\!])$ is a solution of the MC equation in the DG Lie algebra $\mathcal{A} \,\widehat{\otimes}\, \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])$, and there is a continuous $\mathcal{A}$-multilinear L_∞ morphism

$$\mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j \geqslant 1} : \big(\mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])\big)_\omega \to \big(\mathcal{A} \,\widehat{\otimes}\, \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!])\big)_{\omega'}$$

between the twisted DG Lie algebras. The formula is

$$\mathcal{U}_{\mathcal{A},\omega;j}(\gamma_1 \cdots \gamma_j) = \sum_{k \geqslant 0} \frac{1}{(j+k)!} \mathcal{U}_{\mathcal{A};j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j) \tag{1.3}$$

for $\gamma_1, \ldots, \gamma_j \in \mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!])$. The two twisted DG Lie algebras have differentials $d_{\text{for}} + \text{ad}(\omega)$ and $d_{\text{for}} + \text{ad}(\omega') + \mathbf{1} \otimes d_{\mathcal{D}}$ respectively.

This sum in (1.3) is actually finite, the number of nonzero terms in it depending on the bidegree of $\gamma_1 \cdots \gamma_j$. Indeed, if $\gamma_1 \cdots \gamma_j \in \mathcal{A}^q \,\widehat{\otimes}\, \mathcal{T}^p_{\text{poly}}(\mathbb{K}[\![t]\!])$, then

$$\mathcal{U}_{\mathcal{A};j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j) \in \mathcal{A}^{q+k} \,\widehat{\otimes}\, \mathcal{D}^{p+1-j-k}_{\text{poly}}(\mathbb{K}[\![t]\!]), \tag{1.4}$$

which is zero for $k > p - j + 2$; see proof of [4, Theorem 3.23].

By [3, Theorem 5.6] (the universal Taylor expansions) there are canonical isomorphisms of graded Lie algebras in $\text{Dir Inv Mod } \mathbb{K}_{\text{Coor } X}$

$$\mathcal{A} \,\widehat{\otimes}\, \mathcal{T}_{\text{poly}}(\mathbb{K}[\![t]\!]) \cong \mathcal{A} \,\widehat{\otimes}_{\mathcal{A}^0}\, \pi^{\widehat{*}}_{\text{coor}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \tag{1.5}$$

and

$$\mathcal{A} \,\widehat{\otimes}\, \mathcal{D}_{\text{poly}}(\mathbb{K}[\![t]\!]) \cong \mathcal{A} \,\widehat{\otimes}_{\mathcal{A}^0}\, \pi^{\widehat{*}}_{\text{coor}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}). \tag{1.6}$$

[3, Proposition 5.8] tells us that

$$d_{\text{for}} + \text{ad}(\omega) = \nabla_{\mathcal{P}}$$

under these identifications. Therefore

$$\begin{aligned} \mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j \geqslant 1} : \mathcal{A} \,\widehat{\otimes}_{\mathcal{A}^0}\, &\pi^{\widehat{*}}_{\text{coor}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly},X}) \\ &\to \mathcal{A} \,\widehat{\otimes}_{\mathcal{A}^0}\, \pi^{\widehat{*}}_{\text{coor}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly},X}) \end{aligned} \tag{1.7}$$

is a continuous $\mathcal{A}$-multilinear L_∞ morphism between these DG Lie algebras, whose differentials are $\nabla_{\mathcal{P}}$ and $\nabla_{\mathcal{P}} + \mathbf{1} \otimes d_{\mathcal{D}}$ respectively. As in the proof of [3, Theorem 5.6], under the identifications (1.5) and (1.6) we have the equality

$$\mathcal{U}_{\mathcal{A};1} = \mathbf{1} \otimes \widehat{\pi_{\text{coor}}^*}(\mathbf{1} \otimes \mathcal{U}_1); \tag{1.8}$$

i.e., it is the pullback of the map (1.1).

Let us filter the DG algebra $\mathcal{A}$ by the descending filtration $\{G^i \mathcal{A}\}_{i \in \mathbb{Z}}$, where $G^i \mathcal{A} := \bigoplus_{j=i}^{\infty} \mathcal{A}^i$. The DG Lie algebras appearing in Eq. (1.7) inherit this filtration. From formulas (1.3) and (1.4) we see that the homomorphism of complexes $\mathcal{U}_{\mathcal{A},\omega;1}$ respects the filtration, and from (1.8) we see that

$$\text{gr}_{\text{G}}(\mathcal{U}_{\mathcal{A},\omega;1}) = \text{gr}_{\text{G}}(\mathcal{U}_{\mathcal{A};1}) = \mathbf{1} \otimes \widehat{\pi_{\text{coor}}^*}(\mathbf{1} \otimes \mathcal{U}_1).$$

Let $n := \dim X$. As noted earlier, the action of $\mathfrak{g} := \mathfrak{gl}_n(\mathbb{K})$ gives

$$(\pi_{\text{gl}*} \mathcal{A})^{\mathfrak{g}} = (\pi_{\text{gl}*} \Omega_{\text{Coor} X})^{\mathfrak{g}} = \Omega_{\text{LCC} X}.$$

According to [2, Lemma 9.2.1], the L_∞ morphism $\mathcal{U}_{\mathcal{A},\omega}$ commutes with the action of the Lie algebra $\mathfrak{g}$. Therefore $\mathcal{U}_{\mathcal{A},\omega}$ descends (i.e. restricts) to a continuous $\Omega_{\text{LCC} X}$-multilinear L_∞ morphism

$$\begin{aligned}
\mathcal{U}_{\mathcal{A},\omega}^{\mathfrak{g}} : \Omega_{\text{LCC} X} \, \widehat{\otimes}_{\mathcal{O}_{\text{LCC} X}} \, \widehat{\pi_{\text{lcc}}^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\text{poly}, X}) \\
\to \Omega_{\text{LCC} X} \, \widehat{\otimes}_{\mathcal{O}_{\text{LCC} X}} \, \widehat{\pi_{\text{lcc}}^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\text{poly}, X}).
\end{aligned} \tag{1.9}$$

The DG Lie algebras in formula (1.9) also have filtrations $\{G^j\}_{j \in \mathbb{Z}}$, the homomorphism $\mathcal{U}_{\mathcal{A},\omega;1}^{\mathfrak{g}}$ respects this filtration, and we now have

$$\text{gr}_{\text{G}}\big(\mathcal{U}_{\mathcal{A},\omega;1}^{\mathfrak{g}}\big) = \text{gr}_{\text{G}}\big(\mathcal{U}_{\mathcal{A};1}^{\mathfrak{g}}\big) = \mathbf{1} \otimes \widehat{\pi_{\text{lcc}}^*}(\mathbf{1} \otimes \mathcal{U}_1). \tag{1.10}$$

According to [3, Theorem 6.4] there are induced operators

$$\Psi_{\boldsymbol{\sigma};j} := \boldsymbol{\sigma}^*\big(\mathcal{U}_{\mathcal{A},\omega;j}^{\mathfrak{g}}\big) : \prod^j \text{Mix}_U(\mathcal{T}_{\text{poly}, X}) \to \text{Mix}_U(\mathcal{D}_{\text{poly}, X})$$

for $j \geqslant 1$. The L_∞ identities in [3, Definition 3.7], when applied to the L_∞ morphism $\mathcal{U}_{\mathcal{A},\omega}^{\mathfrak{g}}$, are of the form considered in [3, Theorem 6.4(iii)]. Therefore these identities are preserved by $\boldsymbol{\sigma}^*$, and we conclude that the sequence $\Psi_{\boldsymbol{\sigma}} = \{\Psi_{\boldsymbol{\sigma},j}\}_{j=1}^{\infty}$ is an L_∞ morphism. Furthermore, $\Psi_{\boldsymbol{\sigma};1}$ respects the filtration $\{G^i \text{Mix}_U\}$, and from (1.10) we get

$$\text{gr}_{\text{G}}(\Psi_{\boldsymbol{\sigma};1}) = \text{gr}_{\text{G}}\big(\boldsymbol{\sigma}^*\big(\mathcal{U}_{\mathcal{A};1}^{\mathfrak{g}}\big)\big) = \text{gr}_{\text{G}}\big(\text{Mix}_U(\mathcal{U}_1)\big). \tag{1.11}$$

According to [4, Theorem 4.17] the homomorphism $\text{gr}_{\text{G}}(\text{Mix}_U(\mathcal{U}_1))$ is a quasi-isomorphism. Since the complexes $\text{Mix}_U(\mathcal{T}_{\text{poly}, X})$ and $\text{Mix}_U(\mathcal{D}_{\text{poly}, X})$ are bounded below, and the filtration is nonnegative and exhaustive, it follows that $\Psi_{\boldsymbol{\sigma};1}$ is also a quasi-isomorphism. $\square$

Corollary 1.12. *Taking global sections in Theorem* 1.2 *we get an* L_∞ *quasi-isomorphism*

$$\Gamma(X, \Psi_\sigma) = \left\{\Gamma(X, \Psi_{\sigma;j})\right\}_{j \geqslant 1} : \Gamma\left(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X})\right) \to \Gamma\left(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})\right).$$

Proof. Theorem 1.2 tells us that $\Psi_{\sigma;1}$ is a quasi-isomorphisms of complexes of sheaves. By [3, Theorem 6.2] it follows that

$$\Gamma(X, \Psi_{\sigma;1}) : \Gamma\left(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X})\right) \to \Gamma\left(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})\right)$$

is a quasi-isomorphism. □

Corollary 1.13. *The data* (U, σ) *induce a bijection*

$$\mathrm{MC}(\Psi_\sigma) : \mathrm{MC}\left(\Gamma\left(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X})\right)[\![\hbar]\!]^+\right) \xrightarrow{\simeq} \mathrm{MC}\left(\Gamma\left(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})\right)[\![\hbar]\!]^+\right).$$

Proof. Use Corollary 1.12 and [3, Corollary 3.10]. □

Recall that $\mathcal{T}_{\mathrm{poly}}(X) = \Gamma(X, \mathcal{T}_{\mathrm{poly}, X})$ and $\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X) = \Gamma(X, \mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}})$, and the latter is the DG Lie algebra of global poly differential operators that vanish if one of their arguments is 1.

Suppose $f : X' \to X$ is an étale morphism. According to [4, Proposition 4.6] there are DG Lie algebra homomorphisms $f^* : \mathcal{T}_{\mathrm{poly}}(X) \to \mathcal{T}_{\mathrm{poly}}(X')$ and $f^* : \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X) \to \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X')$. These homomorphisms extend to formal coefficients, and we get functions

$$\mathrm{MC}(f^*) : \mathrm{MC}\left(\mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+\right) \to \mathrm{MC}\left(\mathcal{T}_{\mathrm{poly}}(X')[\![\hbar]\!]^+\right)$$

etc.

One says that X is a $\mathcal{D}$-*affine variety* if $\mathrm{H}^q(X, \mathcal{M}) = 0$ for every quasi-coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and every $q > 0$.

Theorem 1.14. *Let X be an irreducible smooth separated $\mathbb{K}$-scheme. Assume X is $\mathcal{D}$-affine. Then there is a canonical function*

$$Q : \mathrm{MC}\left(\mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+\right) \to \mathrm{MC}\left(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[\![\hbar]\!]^+\right)$$

called the quantization map. *It has the following properties*:

 (i) *The function Q preserves first order terms.*
 (ii) *The function Q respects étale morphisms. Namely if X' is another $\mathcal{D}$-affine scheme, with quantization map Q', and if $f : X' \to X$ is an étale morphism, then*

$$Q' \circ \mathrm{MC}(f^*) = \mathrm{MC}(f^*) \circ Q.$$

(iii) *If X is affine, then Q is bijective.*
(iv) *The function Q is characterized as follows. Choose an open covering $U = \{U_0, \ldots, U_m\}$ of X consisting of affine open sets, each admitting an étale coordinate system. Let σ be*

the associated simplicial section of the bundle LCC $X \to X$. *Then there is a commutative diagram*

$$
\begin{array}{ccc}
\mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+\big) & \xrightarrow{\ \ Q\ \ } & \mathrm{MC}\big(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[\![\hbar]\!]^+\big) \\
{\scriptstyle \mathrm{MC}(\eta_{\mathcal{T}})}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{MC}(\eta_{\mathcal{D}})} \\
\mathrm{MC}\big(\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big)[\![\hbar]\!]^+\big) & \xrightarrow{\ \mathrm{MC}(\Psi_{\sigma})\ } & \mathrm{MC}\big(\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)[\![\hbar]\!]^+\big)
\end{array}
$$

in which the arrows $\mathrm{MC}(\Psi_{\sigma})$ *and* $\mathrm{MC}(\eta_{\mathcal{D}})$ *are bijections. Here* Ψ_{σ} *is the* L_{∞} *quasi-isomorphism from Theorem* 1.2, *and* $\eta_{\mathcal{T}}, \eta_{\mathcal{D}}$ *are the inclusions of DG Lie algebras.*

Let us elaborate a bit on the statement above. It says that to any MC solution $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathcal{T}_{\mathrm{poly}}^1(X)[\![\hbar]\!]^+$ there corresponds an MC solution $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor},1}(X)[\![\hbar]\!]^+$. The element $\beta = Q(\alpha)$ is uniquely determined up to gauge equivalence by the group $\exp(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor},0}(X)[\![\hbar]\!]^+)$. Given any local sections $f, g \in \mathcal{O}_X$ one has

$$
\frac{1}{2}\big(\beta_1(f,g) - \beta_1(g,f)\big) = \alpha_1(f,g) \in \mathcal{O}_X. \tag{1.15}
$$

The quantization map Q can be calculated (at least in theory) using the collection of sections σ and the universal formulas for deformation in [3, Theorem 3.13].

We will need a lemma before proving the theorem.

Lemma 1.16. *Let* $f, g \in \mathcal{O}_X = \mathcal{D}_{\mathrm{poly},X}^{-1}$ *be local sections.*

(1) *For any* $\beta \in \mathrm{Mix}_{\boldsymbol{U}}^0(\mathcal{D}_{\mathrm{poly},X}^1)$ *one has*

$$
\big[[\beta, f], g\big] = \beta(g,f) - \beta(f,g) \in \mathrm{Mix}_{\boldsymbol{U}}^0(\mathcal{O}_X).
$$

(2) *For any* $\beta \in \mathrm{Mix}_{\boldsymbol{U}}^1(\mathcal{D}_{\mathrm{poly},X}^0) \oplus \mathrm{Mix}_{\boldsymbol{U}}^2(\mathcal{D}_{\mathrm{poly},X}^{-1})$ *one has* $[[\beta, f], g] = 0$.

(3) *Let* $\gamma \in \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})^0$, *and define* $\beta := (\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma)$. *Then* $[[\beta, f], g] = 0$.

Proof. (1) [3, Proposition 6.3] implies that the embedding [3, (6.1)]:

$$
\begin{aligned}
\mathrm{Mix}_{\boldsymbol{U}}&(\mathcal{D}_{\mathrm{poly},X}) \\
&\subset \bigoplus_{p,q,r} \prod_{j \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_j^m} g_{\boldsymbol{i}*} g_{\boldsymbol{i}}^{-1}\big(\Omega^q(\boldsymbol{\Delta}_{\mathbb{K}}^j) \,\widehat{\otimes}\, (\Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}^r)\big)
\end{aligned}
$$

is a DG Lie algebra homomorphism. So by continuity we might as well assume that $\beta = aD$ with $a \in \Omega_X^0 = \mathcal{O}_X$ and $D \in \mathcal{D}_{\mathrm{poly},X}^1$. Moreover, since the Lie bracket of $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}$ is Ω_X-bilinear, we may assume that $a = 1$, i.e. $\beta = D$. Now the assertion is clear from the definition of the Gerstenhaber Lie bracket; see [1, Section 3.4.2].

(2) Applying the same reduction as above, but with $D \in \mathcal{D}_{\mathrm{poly},X}^r$ and $r \in \{0, -1\}$, we get $[[D, f], g] \in \mathcal{D}_{\mathrm{poly},X}^{r-2} = 0$.

(3) By part (2) it suffices to show that $[[\beta, f], g] = 0$ for $\beta := d_{\mathcal{D}}(\gamma)$ and $\gamma \in \mathrm{Mix}_U^0(\mathcal{D}_{\mathrm{poly}, X}^0)$. As explained above we may further assume that $\gamma = D \in \mathcal{D}_{\mathrm{poly}, X}^0$. Now the formulas for $d_{\mathcal{D}}$ and $[-, -]$ in [1, Section 3.4.2] imply that $[[d_{\mathcal{D}}(D), f], g] = 0$. $\square$

Proof of Theorem 1.14. Step 1. Take an open covering U as in property (iv). Since the sheaves $\mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}, p}$ are quasi-coherent left $\mathcal{D}_X$-modules, it follows that $\mathrm{H}^q(X, \mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}, p}) = 0$ for all p and all $q > 0$. Therefore $\Gamma(X, \mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}}) = \mathrm{R}\Gamma(X, \mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}})$ in the derived category $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$. Now by [3, Theorem 3.12] the inclusion $\mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}} \to \mathcal{D}_{\mathrm{poly}, X}$ is a quasi-isomorphism, and by [3, Theorem 6.2(1)] the inclusion $\mathcal{D}_{\mathrm{poly}, X} \to \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})$ is a quasi-isomorphism. According to [3, Theorem 6.2(2)] we have $\Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})) = \mathrm{R}\Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X}))$. The conclusion is that

$$\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X) = \Gamma(X, \mathcal{D}_{\mathrm{poly}, X}^{\mathrm{nor}}) \to \Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X})) \tag{1.17}$$

is a quasi-isomorphism of complexes of $\mathbb{K}$-modules. But in view of [3, Proposition 6.3], this is in fact a quasi-isomorphism of DG Lie algebras.

From (1.17) we deduce that

$$\eta_{\mathcal{D}} : \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[\![\hbar]\!]^+ \to \Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X}))[\![\hbar]\!]^+$$

is a quasi-isomorphism of DG Lie algebras. Using [3, Corollary 3.10] we see that $\mathrm{MC}(\eta_{\mathcal{D}})$ is bijective. Therefore the diagram in property (iv) defines Q uniquely.

According to Corollary 1.13, the arrow marked $\mathrm{MC}(\Psi_\sigma)$ is a bijection. So we have established property (iv), except for the independence of the open covering.

Step 2. The left vertical arrow comes from the DG Lie algebra homomorphism

$$\eta_{\mathcal{T}} : \mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+ \to \Gamma(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X}))[\![\hbar]\!]^+,$$

which is a quasi-isomorphism when $\mathrm{H}^q(X, \mathcal{T}_{\mathrm{poly}, X}^p) = 0$ for all p and all $q > 0$. So in case X is affine, the quantization map Q is bijective. This establishes property (iii).

Step 3. Now suppose $U' = \{U_0', \ldots, U_{m'}'\}$ is another such affine open covering of X, with sections $\sigma_i' : U_i' \to \mathrm{LCC}\,X$. Without loss of generality we may assume that $m' \geqslant m$, and that $U_i' = U_i$ and $\sigma_i' = \sigma_i$ for all $i \leqslant m$. There is a morphism of simplicial schemes $f : U \to U'$, that is an open and closed embedding. Correspondingly there is a commutative diagram

$$
\begin{array}{ccc}
\mathrm{MC}(\mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+) & \xrightarrow{\;\;Q\;\;} & \mathrm{MC}(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[\![\hbar]\!]^+) \\
{\scriptstyle \mathrm{MC}(\eta_{\mathcal{T}})} \downarrow & & \downarrow {\scriptstyle \mathrm{MC}(\eta_{\mathcal{D}})} \\
\mathrm{MC}(\Gamma(X, \mathrm{Mix}_{U'}(\mathcal{T}_{\mathrm{poly}, X}))[\![\hbar]\!]^+) & \xrightarrow{\mathrm{MC}(\Psi_{\sigma'})} & \mathrm{MC}(\Gamma(X, \mathrm{Mix}_{U'}(\mathcal{D}_{\mathrm{poly}, X}))[\![\hbar]\!]^+) \\
{\scriptstyle \mathrm{MC}(f^*)} \downarrow & & \downarrow {\scriptstyle \mathrm{MC}(f^*)} \\
\mathrm{MC}(\Gamma(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly}, X}))[\![\hbar]\!]^+) & \xrightarrow{\mathrm{MC}(\Psi_\sigma)} & \mathrm{MC}(\Gamma(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly}, X}))[\![\hbar]\!]^+),
\end{array}
$$

where the vertical arrows on the right are bijections. We conclude that Q is independent of U and σ. This concludes the proof of property (iv).

Step 4. Suppose $f : X' \to X$ is an étale morphism. Then we can choose an affine open covering U' of X' that refines U in the obvious sense. Each of the open sets U_i' inherits an étale coordinate system, and hence a section $\sigma_i' : U_i' \to \mathrm{LCC}\, X'$. We get a commutative diagram

$$
\begin{array}{ccc}
\mathrm{MC}\big(\Gamma\big(X, \mathrm{Mix}_U(\mathcal{T}_{\mathrm{poly},X})\big)[\![\hbar]\!]^+\big) & \xrightarrow{\ \mathrm{MC}(\Psi_\sigma)\ } & \mathrm{MC}\big(\Gamma\big(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly},X})\big)[\![\hbar]\!]^+\big) \\
{\scriptstyle V\mathrm{MC}(f^*)}\Big\downarrow & & {\scriptstyle \mathrm{MC}(f^*)}\Big\downarrow \\
\mathrm{MC}\big(\Gamma\big(X', \mathrm{Mix}_{U'}(\mathcal{T}_{\mathrm{poly},X'})\big)[\![\hbar]\!]^+\big) & \xrightarrow{\ \mathrm{MC}(\Psi_{\sigma'})\ } & \mathrm{MC}\big(\Gamma\big(X', \mathrm{Mix}_{U'}(\mathcal{D}_{\mathrm{poly},X'})\big)[\![\hbar]\!]^+\big).
\end{array}
$$

This proves property (ii).

Step 5. Finally we must show that Q preserves first order terms, i.e., property (i). Let

$$
\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathcal{T}_{\mathrm{poly}}(X)^1[\![\hbar]\!]^+
$$

be an MC solution, and let

$$
\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)^1[\![\hbar]\!]^+
$$

be an MC solution such that $\beta = Q(\alpha)$ modulo gauge equivalence. This means that there exists some

$$
\gamma = \sum_{k \geqslant 1} \gamma_k \hbar^k \in \Gamma\big(X, \mathrm{Mix}_U(\mathcal{D}_{\mathrm{poly},X})\big)^0[\![\hbar]\!]^+
$$

such that

$$
\sum_{j \geqslant 1} \frac{1}{j!} \Psi_{\sigma;j}\big(\alpha^j\big) = \exp(\mathrm{af})\big(\exp(\gamma)\big)(\beta),
$$

with notation as in [3, Lemma 3.2]. Cf. [3, Theorem 3.8]. In the first order term (i.e., the coefficient of $\hbar^1$) of this equation we have

$$
\Psi_{\sigma;1}(\alpha_1) = \beta_1 - (\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1); \tag{1.18}
$$

see [3, Eq. (3.3)].

In order to apply Lemma 1.16(2), we are interested in the component of $\Psi_{\sigma;1}(\alpha_1)$ living in the summand $\mathrm{Mix}_U^0(\mathcal{D}_{\mathrm{poly},X}^1)$. But this is exactly

$$
\mathrm{gr}_\mathrm{G}^0(\Psi_{\sigma;1})(\alpha_1) \in \mathrm{gr}_\mathrm{G}^0 \mathrm{Mix}_U\big(\mathcal{D}_{\mathrm{poly},X}^1\big) = \mathrm{Mix}_U^0\big(\mathcal{D}_{\mathrm{poly},X}^1\big).
$$

Since according to Theorem 1.2 we have

$$
\mathrm{gr}_\mathrm{G}^0(\Psi_{\sigma;1}) = \mathrm{gr}_\mathrm{G}^0\big(\mathrm{Mix}_U(\mathcal{U}_1)\big),
$$

it follows that the component we are interested in is

$$\mathrm{gr}_G^0\big(\mathrm{Mix}_U(\mathcal{U}_1)\big)(\alpha_1) = \mathcal{U}_1(\alpha_1).$$

Now take any two local sections $f, g \in \mathcal{O}_X$. Using Lemma 1.16 we get

$$\big[[\Psi_{\sigma;1}(\alpha_1), f], g\big] = \big[[\mathcal{U}_1(\alpha_1), f], g\big] = \mathcal{U}_1(\alpha_1)(g, f) - \mathcal{U}_1(\alpha_1)(f, g) = -2\alpha_1(f, g),$$

$$\big[[\beta_1, f], g\big] = \beta_1(g, f) - \beta_1(f, g)$$

and

$$\big[[(\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1), f], g\big] = 0.$$

Combining these equations with Eq. (1.18) we see that Eq. (1.15) indeed holds. So the proof is done. □

Corollary 1.19. *Let X be an irreducible smooth separated $\mathbb{K}$-scheme. Assume X is $\mathcal{D}$-affine. Then the quantization map Q of Theorem 1.14 may be interpreted as a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \to \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}.$$

The quantization map Q preserves first order terms, and commutes with étale morphisms $f : X' \to X$. If X is affine then Q is bijective.

Proof. By definition the left side is $\mathrm{MC}(\mathcal{T}_{\mathrm{poly}}(X)[\![\hbar]\!]^+)$. On the other hand, according to [3, Theorem 1.13] every deformation quantization of $\mathcal{O}_X$ can be trivialized globally, and by [3, Proposition 1.14] any gauge equivalence between globally trivialized deformation quantizations is a global gauge equivalence. Hence the right side is $\mathrm{MC}(\mathcal{D}_{\mathrm{poly}}^{\mathrm{nor}}(X)[\![\hbar]\!]^+)$. □

2. Miscellaneous errors

Here is a list of minor errors in the paper [3].

(1) Section 3, bottom of page 395: the formula should be

$$\mathrm{af}(\gamma)(\omega) := [\gamma, \omega] - \mathrm{d}(\gamma) = \mathrm{ad}(\gamma)(\omega) - \mathrm{d}(\gamma) \in \mathfrak{m} \,\widehat{\otimes}\, \mathfrak{g}^1,$$

(2) Definition 5.2, page 411: the formula should be

$$\nabla_{\mathcal{P}} : \mathcal{P}_X \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{P}_X.$$

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