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# Deformation quantization in algebraic geometry

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Dedicated to Professor Michael Artin on the Occasion of his 70th Birthday

**Abstract**

We study deformation quantizations of the structure sheaf $\mathcal{O}_X$ of a smooth algebraic variety $X$ in characteristic 0. Our main result is that when $X$ is $\mathcal{D}$-affine, any formal Poisson structure on $X$ determines a deformation quantization of $\mathcal{O}_X$ (canonically, up to gauge equivalence). This is an algebro-geometric analogue of Kontsevich's celebrated result.


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## 0. Introduction

This article began with an attempt to understand the work of Kontsevich [Ko1,Ko3], Cattaneo–Felder–Tomassini [CFT] and Nest–Tsygan [NT] on deformation quantization of Poisson manifolds. Moreover, we tried to see to what extent the methods applied in the case of $\mathrm{C}^\infty$ manifolds can be carried over to the algebro-geometric case.

If $X$ is a $\mathrm{C}^\infty$ manifold with Poisson structure $\alpha$ then there is always a deformation quantization of the algebra of functions $\mathrm{C}^\infty(X)$ with first-order term $\alpha$. This was

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proved by Kontsevich [Ko1]. Furthermore, Kontsevich proved that such a deformation quantization is unique in a suitable sense.

If $X$ is either a complex analytic manifold or a smooth algebraic variety then one wants to deform the sheaf of functions $\mathcal{O}_X$. As might be expected there are potential obstructions, due to the lack of global (analytic or algebraic) functions and sections of bundles. The case of a complex analytic manifold with holomorphic symplectic structure was treated in [NT]. The algebraic case was studied in [Ko3], where several approaches were discussed. In the present paper we take a somewhat different direction than [Ko3].

First let us explain what we mean by deformation quantization in the context of algebraic geometry. Let $\mathbb{K}$ be a field of characteristic 0, and let $X$ be a smooth algebraic variety over $\mathbb{K}$. The tangent sheaf of $X$ is denoted by $\mathcal{T}_X$. Given an element $\alpha \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$ let $\{-,-\}_\alpha$ be the $\mathbb{K}$-bilinear sheaf morphism $\mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ defined by $\{f,g\}_\alpha := \langle \alpha, \mathrm{d}(f) \wedge \mathrm{d}(g) \rangle$ for local sections $f, g \in \mathcal{O}_X$. If $\{-,-\}_\alpha$ is a Lie bracket on $\mathcal{O}_X$ then it is called a *Poisson bracket*, $\alpha$ is called a *Poisson structure on X*, and the pair $(X, \alpha)$ is called a *Poisson variety*. It is known that $\alpha$ is a Poisson structure if and only if $[\alpha, \alpha] = 0$ for the Schouten–Nijenhuis bracket.

Let $\hbar$ be an indeterminate (the "Planck constant"). A *star product* on $\mathcal{O}_X[[\hbar]]$ is a $\mathbb{K}[[\hbar]]$-bilinear sheaf morphism

$$\star : \mathcal{O}_X[[\hbar]] \times \mathcal{O}_X[[\hbar]] \to \mathcal{O}_X[[\hbar]]$$

which makes $\mathcal{O}_X[[\hbar]]$ into a sheaf of associative unital $\mathbb{K}[[\hbar]]$-algebras. The unit element for $\star$ has to be $1 \in \mathcal{O}_X$, and for any local sections $f, g \in \mathcal{O}_X$ their product should satisfy $f \star g \equiv fg \bmod \hbar$. Furthermore, there is a differential condition: there should be a sequence of bi-differential operators $\beta_j : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$, such that

$$f \star g = fg + \sum_{j=1}^{\infty} \beta_j(f,g)\hbar^j \in \mathcal{O}_X[[\hbar]].$$

A *deformation quantization* of $\mathcal{O}_X$ is by definition a star product on $\mathcal{O}_X[[\hbar]]$.

Actually there is a more refined notion of deformation quantization, which has a local nature; see Section 1. In the body of the paper the deformation defined in the previous paragraph is referred to as a *globally trivialized deformation quantization*. However, according to Theorem 1.13, if $\mathrm{H}^1(X, \mathcal{D}_X) = 0$ then any deformation quantization is equivalent to a globally trivialized one. So for the purpose of the introduction (cf. Theorem 0.1 below) we might as well consider only globally trivialized deformation quantizations.

Suppose $\star$ is some star product on $\mathcal{O}_X[[\hbar]]$. Given two local sections $f, g \in \mathcal{O}_X$ define $\{f,g\}_\star \in \mathcal{O}_X$ to be the unique local section satisfying

$$2\hbar\{f,g\}_\star \equiv f \star g - g \star f \bmod \hbar^2.$$

This is a Poisson bracket on $\mathcal{O}_X$. Note that $\{f,g\}_\star = \frac{1}{2}\big(\beta_1(f,g) - \beta_1(g,f)\big)$.

Let $(X, \alpha)$ be a Poisson variety. A *deformation quantization of* $(X, \alpha)$ is a deformation quantization $\star$ such that $\{f, g\}_\star = \{f, g\}_\alpha$.

There is an obvious notion of *gauge equivalence* for deformation quantizations. First we need to define what is a gauge equivalence of $\mathcal{O}_X[[\hbar]]$. This is a $\mathbb{K}[[\hbar]]$-linear sheaf automorphism $\gamma : \mathcal{O}_X[[\hbar]] \xrightarrow{\simeq} \mathcal{O}_X[[\hbar]]$ of the following form: there is a sequence $\gamma_j : \mathcal{O}_X \to \mathcal{O}_X$ of differential operators, such that

$$\gamma(f) = f + \sum_{j=1}^{\infty} \gamma_j(f)\hbar^j$$

for all $f \in \mathcal{O}_X$; and also $\gamma(1) = 1$. Two star products $\star$ and $\star'$ on $\mathcal{O}_X[[\hbar]]$ are said to be gauge equivalent if there is some gauge equivalence $\gamma$ such that

$$f \star' g = \gamma^{-1}\big(\gamma(f) \star \gamma(g)\big)$$

for all $f, g \in \mathcal{O}_X$.

To state the main result of our paper we need the notion of formal Poisson structure on $X$. This is a series $\alpha = \sum_{k=1}^{\infty} \alpha_k \hbar^k \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)[[\hbar]]$ satisfying $[\alpha, \alpha] = 0$. For instance, if $\alpha_1$ is a Poisson structure then $\alpha := \alpha_1 \hbar$ is a formal Poisson structure. Two formal Poisson structure $\alpha$ and $\alpha'$ are called gauge equivalent if there is some $\gamma = \sum_{k=1}^{\infty} \gamma_k \hbar^k \in \Gamma(X, \mathcal{T}_X)[[\hbar]]$ such that $\alpha' = \exp(\mathrm{ad}(\gamma))(\alpha)$.

Recall that the variety $X$ is said to be *$\mathcal{D}$-affine* if $\mathrm{H}^i(X, \mathcal{M}) = 0$ for all quasi-coherent left $\mathcal{D}_X$-modules $\mathcal{M}$ and all $i > 0$. Here $\mathcal{D}_X$ is the sheaf of differential operators on $X$.

**Theorem 0.1.** *Let $X$ be a smooth algebraic variety over the field $\mathbb{K}$. Assume $X$ is $\mathcal{D}$-affine and $\mathbb{R} \subset \mathbb{K}$. Then there is a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \to \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

*called the* quantization map. *The map $Q$ preserves first-order terms, and commutes with étale morphisms $X' \to X$. If $X$ is affine then $Q$ is bijective. There is an explicit formula for $Q$.*

This is an algebraic analogue of [Ko1, Theorem 1.3]. Theorem 0.1 is repeated as Corollaries 7.12 and 7.13 in the body of the paper. Full details, including the explicit formula for the quantization map $Q$, are in Theorem 7.7. By "preserving first-order terms" we mean that given a formal Poisson structure $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$ and associated deformation quantization $Q(\alpha) = \star$, then $\{-,-\}_\star = \{-,-\}_{\alpha_1}$. If $f : X' \to X$ is an étale morphism then any formal Poisson structure $\alpha$ on $X$ can be pulled back to a formal Poisson structure $f^*(\alpha)$ on $X'$; and likewise any deformation quantization $\star$ on

$X$ can be pulled back to a deformation quantization $f^*(\star)$ on $X'$. The third assertion in Theorem 0.1 says that if $X'$ is also $\mathcal{D}$-affine then $Q(f^*(\alpha)) = f^*(Q(\alpha))$.

There are two important classes of varieties satisfying the conditions of Theorem 0.1. The first consists of all smooth affine varieties. Note that even if $X$ is affine,yet does not admit an étale morphism $X \to \mathbf{A}^n_{\mathbb{K}}$, the result is not trivial —since changes of coordinates have to be accounted for (cf. Corollary 3.24).

The second class of examples is that of the flag varieties $X = G/P$, where $G$ is a connected reductive algebraic group and $P$ is a parabolic subgroup. By the Beilinson–Bernstein Theorem the variety $X$ is $\mathcal{D}$-affine. This class of varieties includes the projective spaces $\mathbf{P}^n_{\mathbb{K}}$.

Here is an outline of the paper (with some of the features simplified). There are two important sheaves of DG Lie algebras on $X$: the sheaf of poly vector fields $\mathcal{T}_{\mathrm{poly},X}$, and the sheaf of polydifferential operators $\mathcal{D}_{\mathrm{poly},X}$ (see Section 3). Their global sections $\mathcal{T}_{\mathrm{poly}}(X) := \Gamma(X, \mathcal{T}_{\mathrm{poly},X})$ and $\mathcal{D}_{\mathrm{poly}}(X) := \Gamma(X, \mathcal{D}_{\mathrm{poly},X})$ control Poisson structures and deformation quantizations, respectively. If one could find an $\mathrm{L}_\infty$ quasi-isomorphism $\mathcal{T}_{\mathrm{poly}}(X) \to \mathcal{D}_{\mathrm{poly}}(X)$ this would imply Theorem 0.1. However, unless $X$ is affine and admits an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$, there is no reason why such a quasi-isomorphism should exist.

Imitating Fedosov [Fe] and Kontsevich [Ko1], we use formal geometry to solve the global problem. The adaptation of this theory to algebraic geometry is done in Section 4. There is an infinite dimensional bundle $\pi : \mathrm{LCC}\, X \to X$, which parameterizes formal coordinate systems on $X$ modulo linear change of coordinates. (In [Ko1] the notation for this bundle is $X^{\mathrm{aff}}$.) Let $\mathcal{P}_X$ be the sheaf of principal parts on $X$. The complete pullbacks $\widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X})$ and $\widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$ are sheaves of DG Lie algebras on $\mathrm{LCC}\, X$ (see Section 5). The universal deformation formulas of Kontsevich give rise to an $\mathrm{L}_\infty$ quasi-isomorphism

$$\widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}).$$

When $X$ is a $\mathrm{C}^\infty$ manifold the bundle $\mathrm{LCC}\, X$ has contractible fibers, and thus it has global $\mathrm{C}^\infty$ sections. This fact is crucial for Kontsevich's proof. However, in our algebraic setup there is no reason to assume that $\pi : \mathrm{LCC}\, X \to X$ has any global sections.

We discovered a way to get around the absence of global sections in the case of an algebraic variety: the idea is to use *simplicial sections*. This idea is inspired by a construction of Bott; see [Bo,HY]. A simplicial section $\boldsymbol{\sigma}$ of $\pi : \mathrm{LCC}\, X \to X$, based on an open covering $X = \bigcup U_{(i)}$, consists of a family of morphisms $\sigma_{\boldsymbol{i}} : \mathbf{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to \mathrm{LCC}\, X$, where $\boldsymbol{i} = (i_0, \ldots, i_q)$ is a multi-index; $\mathbf{\Delta}^q_{\mathbb{K}}$ is the $q$-dimensional geometric simplex; and $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$. The morphisms $\sigma_{\boldsymbol{i}}$ are required to be compatible with $\pi$ and to satisfy simplicial relations.

It is easy to show that sections of $\pi$ exist locally. Because of the particular geometry of the bundle $\mathrm{LCC}\, X$, if we take a sufficiently fine affine open covering $X = \bigcup U_{(i)}$, and choose a section $\sigma_{(i)} : U_{(i)} \to \mathrm{LCC}\, X$ for each $i$, then these sections can be extended to a simplicial section $\boldsymbol{\sigma}$. (See Fig. 2 for an illustration. The details of this construction are worked out in the companion paper [Ye4].)

In order to make use of the simplicial section we need *mixed resolutions*. The mixed resolution of a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ is a complex $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$, which combines a de Rham-type differential related to $\mathcal{P}_X$, called the Grothendieck connection, together with a Čech-simplicial-type differential related to the covering $\boldsymbol{U} = \{U_{(i)}\}$. (See Section 6 for a review of mixed resolutions.) We show that the inclusions $\mathcal{T}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$ and $\mathcal{D}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ are quasi-isomorphisms of sheaves of DG Lie algebras. We then prove the following result (which is Theorem 7.1 in the body of the paper):

**Theorem 0.2.** *Let $\mathbb{K}$ be a field containing $\mathbb{R}$, and let $X$ be a smooth $n$-dimensional algebraic variety over $\mathbb{K}$. Suppose $\boldsymbol{U} = \{U_{(i)}\}$ is an open covering of $X$, where each $U_{(i)}$ is affine and admits an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$. Let $\boldsymbol{\sigma}$ be the corresponding simplicial section of $\pi : \mathrm{LCC}\, X \to X$. Then there is an induced $\mathrm{L}_\infty$ quasi-isomorphism*

$$\Psi_{\boldsymbol{\sigma}} : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

*between sheaves of DG Lie algebras.*

We should point out that the construction of the $\mathrm{L}_\infty$ morphism $\Psi_{\boldsymbol{\sigma}}$ involves twisting, due to the presence of the Grothendieck connection in the mixed resolution $\mathrm{Mix}_{\boldsymbol{U}}(-)$. This sort of twisting is discussed in detail in the companion paper [Ye2].

Passing to global sections we obtain an $\mathrm{L}_\infty$ quasi-isomorphism

$$\Gamma(X, \Psi_{\boldsymbol{\sigma}}) : \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big).$$

There are DG Lie algebra homomorphisms

$$\mathcal{T}_{\mathrm{poly}}(X) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) \tag{0.3}$$

and

$$\mathcal{D}_{\mathrm{poly}}(X) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big). \tag{0.4}$$

Each sheaf $\mathcal{D}^p_{\mathrm{poly},X}$ is a quasi-coherent left $\mathcal{D}_X$-module. Hence if $X$ is $\mathcal{D}$-affine the homomorphism (0.4) is a quasi-isomorphism. By standard results of deformation theory (that are reviewed in Section 3) this implies the existence of the quantization map $Q$ in Theorem 0.1. In case $X$ is affine homomorphism (0.3) is also a quasi-isomorphism, and thus $Q$ is bijective.

An earlier version of this paper was much longer. The current version contains only the main results; auxiliary results were moved to the companion papers [Ye2,Ye3,Ye4].

Finally, let us mention several recent papers and surveys dealing with deformation quantization: [BK1,BK2,CDH,CF,CI,Do1,Do2,Ke].

## 1. Deformation quantizations of $\mathcal{O}_X$

Throughout the paper $\mathbb{K}$ is a field of characteristic 0. By default all algebras and schemes in the paper are over $\mathbb{K}$, and so are all morphisms. The symbol $\otimes$ denotes $\otimes_{\mathbb{K}}$. The letter $\hbar$ denotes an indeterminate, and $\mathbb{K}[[\hbar]]$ is the power series algebra.

Let $X$ be a smooth separated irreducible $n$-dimensional scheme over $\mathbb{K}$.

**Definition 1.1.** Let $U \subset X$ be an open set. A *star product* on $\mathcal{O}_U[[\hbar]]$ is a $\mathbb{K}[[\hbar]]$-bilinear sheaf morphism

$$\star : \mathcal{O}_U[[\hbar]] \times \mathcal{O}_U[[\hbar]] \to \mathcal{O}_U[[\hbar]]$$

satisfying the following conditions:

(i) The product $\star$ makes $\mathcal{O}_U[[\hbar]]$ into a sheaf of associative unital $\mathbb{K}[[\hbar]]$-algebras with unit $1 \in \mathcal{O}_U$.
(ii) There is a sequence $\beta_j : \mathcal{O}_U \times \mathcal{O}_U \to \mathcal{O}_U$ of bi-differential operators, such that for any two local sections $f, g \in \mathcal{O}_U$ one has

$$f \star g = fg + \sum_{j=1}^{\infty} \beta_j(f, g)\hbar^j.$$

Note that $f \star g \equiv fg \bmod \hbar$, and also $\beta_j(f, 1) = \beta_j(1, f) = 0$ for all $f$ and $j$.

**Definition 1.2.** Let $\mathcal{A}$ be a sheaf of $\hbar$-adically complete flat $\mathbb{K}[[\hbar]]$-algebras on $X$, and let $\psi : \mathcal{A}/\hbar\mathcal{A} \xrightarrow{\simeq} \mathcal{O}_X$ be an isomorphism of sheaves of $\mathbb{K}$-algebras. Let $U \subset X$ be an open set. A *differential trivialization of* $(\mathcal{A}, \psi)$ *on* $U$ is an isomorphism

$$\tau : \mathcal{O}_U[[\hbar]] \xrightarrow{\simeq} \mathcal{A}|_U$$

of sheaves of $\mathbb{K}[[\hbar]]$-modules satisfying the conditions below.

(i) Let $\star$ denote the product of $\mathcal{A}$. Then the $\mathbb{K}[[\hbar]]$-bilinear product $\star_\tau$ on $\mathcal{O}_U[[\hbar]]$, defined by

$$f \star_\tau g := \tau^{-1}\big(\tau(f) \star \tau(g)\big)$$

for local sections $f, g \in \mathcal{O}_U$, is a star product.
(ii) For any local section $f \in \mathcal{O}_U$ one has $(\psi \circ \tau)(f) = f$.

Condition (i) implies that $\tau(1_{\mathcal{O}_X}) = 1_{\mathcal{A}}$, where $1_{\mathcal{O}_X}$ and $1_{\mathcal{A}}$ are the unit elements of $\mathcal{O}_X$ and $\mathcal{A}$, respectively.

**Definition 1.3.** Let $U \subset X$ be an open set. A *gauge equivalence* of $\mathcal{O}_U[[\hbar]]$ is a $\mathbb{K}[[\hbar]]$-linear automorphism of sheaves

$$\gamma : \mathcal{O}_U[[\hbar]] \xrightarrow{\simeq} \mathcal{O}_U[[\hbar]]$$

satisfying these conditions:

(i) There is a sequence of differential operators $\gamma_k : \mathcal{O}_U \to \mathcal{O}_U$, such that for any local section $f \in \mathcal{O}_U$

$$\gamma(f) = f + \sum_{k=1}^{\infty} \gamma_k(f)\hbar^k.$$

(ii) $\gamma(1) = 1$.

Condition (ii) is equivalent to $\gamma_k(1) = 0$ for all $k$. The gauge equivalences of $\mathcal{O}_U[[\hbar]]$ form a group under composition.

**Definition 1.4.** Let $(\mathcal{A}, \psi)$ be as in Definition 1.2. A *differential structure* $\boldsymbol{\tau} = \{\tau_i\}$ on $(\mathcal{A}, \psi)$ consists of an open covering $X = \bigcup_i U_i$, and for every $i$ a differential trivialization $\tau_i : \mathcal{O}_{U_i}[[\hbar]] \xrightarrow{\simeq} \mathcal{A}|_{U_i}$ of $(\mathcal{A}, \psi)$ on $U_i$. The condition is that for any two indices $i, j$ the transition automorphism $\tau_j^{-1} \circ \tau_i$ of $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$ is a gauge equivalence.

**Example 1.5.** If $\mathcal{A}$ is commutative then automatically it has a differential structure $\boldsymbol{\tau} = \{\tau_i\}$, with the additional property that each differential trivialization $\tau_i : \mathcal{O}_{U_i}[[\hbar]] \xrightarrow{\simeq} \mathcal{A}|_{U_i}$ is an isomorphism of algebras. Here $\mathcal{O}_{U_i}[[\hbar]]$ is the usual power series algebra. Let us explain how this is done. Choose an affine open covering $X = \bigcup U_i$. For any $i$ let $C_i := \Gamma(U_i, \mathcal{O}_X)$. By formal smoothness of $\mathbb{K} \to C_i$ the isomorphism $\psi^{-1} : C_i \xrightarrow{\simeq} \Gamma(U_i, \mathcal{A})/(\hbar)$ lifts to an isomorphism of algebras $\tau_i : C_i[[\hbar]] \xrightarrow{\simeq} \Gamma(U_i, \mathcal{A})$. Due to commutativity the isomorphism $\tau_i$ sheafifies to a differential trivialization on $U_i$. Commutativity also implies that the transitions $\tau_j^{-1} \circ \tau_i$ are gauge equivalences. The differential structure $\boldsymbol{\tau}$ is unique up to gauge equivalence (see Definition 1.8 below). The first-order terms of the gauge equivalences $\tau_j^{-1} \circ \tau_i$ are derivations, and they give the deformation class of $\mathcal{A}$ in $\mathrm{H}^1(X, \mathcal{T}_X)$.

For a noncommutative algebra $\mathcal{A}$ it seems that *we must stipulate the existence of a differential structure*. Furthermore a given algebra $\mathcal{A}$ might have distinct differential structures. Thus we are led to the next definition.

**Definition 1.6.** A *deformation quantization of* $\mathcal{O}_X$ is the data $(\mathcal{A}, \psi, \boldsymbol{\tau})$, where $\mathcal{A}$ is a sheaf of $\hbar$-adically complete flat $\mathbb{K}[[\hbar]]$-algebras on $X$; $\psi : \mathcal{A}/\hbar\mathcal{A} \xrightarrow{\simeq} \mathcal{O}_X$ is an isomorphism of sheaves of $\mathbb{K}$-algebras; and $\boldsymbol{\tau}$ is a differential structure on $(\mathcal{A}, \psi)$.

If there is no danger of confusion we shall sometimes just say that $\mathcal{A}$ is a deformation quantization, keeping the rest of the data implicit.

**Example 1.7.** Let $Y$ be a smooth variety and $X := \mathrm{T}^*Y$ the cotangent bundle, with projection $\pi : X \to Y$. $X$ is a symplectic variety, so it has a nondegenerate Poisson structure $\alpha$. Let $\mathcal{B} := \bigoplus_{i=0}^{\infty} (\mathrm{F}_i \mathcal{D}_Y)\hbar^i \subset \mathcal{D}_Y[\hbar]$ be the Rees algebra of $\mathcal{D}_Y$ w.r.t. the order filtration $\{\mathrm{F}_i \mathcal{D}_Y\}$. So $\mathcal{B}/\hbar\mathcal{B} \cong \pi_* \mathcal{O}_X$ and $\mathcal{B}/(\hbar - 1)\mathcal{B} \cong \mathcal{D}_Y$. Define $\mathcal{B}_m := \mathcal{B}/\hbar^{m+1}\mathcal{B}$. Consider the sheaf of $\mathbb{K}[\hbar]$-algebras $\pi^{-1}\mathcal{B}_m$ on $X$. It can be localized to a sheaf of algebras $\mathcal{A}_m$ on $X$ such that $\mathcal{A}_m \cong \pi^* \mathcal{B}_m = \mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_Y} \pi^{-1}\mathcal{B}_m$ as left $\mathcal{O}_X$-modules. In particular $\mathcal{A}_0 \cong \mathcal{O}_X$. Let $\mathcal{A} := \lim_{\leftarrow m} \mathcal{A}_m$. Then $\mathcal{A}$ is a deformation quantization of $(X, \alpha)$. Note the similarity to microlocal differential operators [Sch].

**Definition 1.8.** Suppose $(\mathcal{A}, \psi, \boldsymbol{\tau})$ and $(\mathcal{A}', \psi', \boldsymbol{\tau}')$ are two deformation quantizations of $\mathcal{O}_X$. A *gauge equivalence*

$$\gamma : (\mathcal{A}, \psi, \boldsymbol{\tau}) \to (\mathcal{A}', \psi', \boldsymbol{\tau}')$$

is a isomorphism $\gamma : \mathcal{A} \xrightarrow{\simeq} \mathcal{A}'$ of sheaves of $\mathbb{K}[[\hbar]]$-algebras satisfying the following two conditions:

(i) One has $\psi = \psi' \circ \gamma : \mathcal{A} \to \mathcal{O}_X$.
(ii) Let $\{U_i\}$ and $\{U'_j\}$ be the open coverings associated to $\boldsymbol{\tau}$ and $\boldsymbol{\tau}'$, respectively. Then for any two indices $i, j$ the automorphism ${\tau'_j}^{-1} \circ \gamma \circ \tau_i$ of $\mathcal{O}_{U_i \cap U'_j}[[\hbar]]$ is a gauge equivalence, in the sense of Definition 1.3.

Let $\Omega^p_X = \Omega^p_{X/\mathbb{K}}$ be the sheaf of differentials of degree $p$, and let $\mathcal{T}_X = \mathcal{T}_{X/\mathbb{K}}$ be the tangent sheaf of $X$. For every $p \geqslant 0$ there is a canonical pairing

$$\langle -, - \rangle : (\textstyle\bigwedge^p_{\mathcal{O}_X} \mathcal{T}_X) \times \Omega^p_X \to \mathcal{O}_X.$$

**Definition 1.9.** (1) A *Poisson bracket* on $\mathcal{O}_X$ is a biderivation

$$\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$$

which makes $\mathcal{O}_X$ into a sheaf of Lie algebras.

(2) Let $\alpha \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$. Define a biderivation $\{-, -\}_\alpha$ by the formula

$$\{f, g\}_\alpha := \langle \alpha, \mathrm{d}(f) \wedge \mathrm{d}(g) \rangle$$

for local sections $f, g \in \mathcal{O}_X$. If $\{-, -\}_\alpha$ is a Poisson bracket then $\alpha$ is called a *Poisson structure*, and $(X, \alpha)$ is called a *Poisson variety*.

The next result is an easy calculation.

**Proposition 1.10.** *Let* $(\mathcal{A}, \psi, \boldsymbol{\tau})$ *be a deformation quantization of* $\mathcal{O}_X$, *and denote by* $\star$ *the multiplication of* $\mathcal{A}$. *Given two local sections* $f, g \in \mathcal{O}_X$ *choose liftings* $\tilde{f}, \tilde{g} \in \mathcal{A}$. *Then the formula*

$$\{f, g\}_{\mathcal{A}} := \psi\left(\frac{1}{2\hbar}(\tilde{f} \star \tilde{g} - \tilde{g} \star \tilde{f})\right) \in \mathcal{O}_X$$

*defines a Poisson bracket on* $\mathcal{O}_X$.

Suppose $\{U_i\}$ is the open covering associated with the differential trivialization $\boldsymbol{\tau}$, and for each $i$ the collection of bi-differential operators on $\mathcal{O}_{U_i}$ occurring in Definition 1.1 is $\{\beta_{i,j}\}_{j=1}^{\infty}$. Then for local sections $f, g \in \mathcal{O}_{U_i}$ one has

$$\{f, g\}_{\mathcal{A}} = \frac{1}{2}\left(\beta_{i,1}(f, g) - \beta_{i,1}(g, f)\right).$$

**Definition 1.11.** Let $\alpha$ be a Poisson structure on $X$. A *deformation quantization* of the Poisson variety $(X, \alpha)$ is a deformation quantization $(\mathcal{A}, \psi, \boldsymbol{\tau})$ of $\mathcal{O}_X$ such that the Poisson brackets satisfy

$$\{-,-\}_{\mathcal{A}} = \{-,-\}_{\alpha}.$$

**Definition 1.12.** A *globally trivialized deformation quantization of* $\mathcal{O}_X$ is a deformation quantization $(\mathcal{A}, \psi, \boldsymbol{\tau})$ in which the differential structure $\boldsymbol{\tau}$ consists of a single differential trivialization $\tau : \mathcal{O}_X[[\hbar]] \xrightarrow{\simeq} \mathcal{A}$.

In effect a globally trivialized deformation quantization of $\mathcal{O}_X$ is the same as a star product on $\mathcal{O}_X[[\hbar]]$; the correspondence is $\star \mapsto \star_\tau$ in the notation of Definition 1.2.

Let $\mathcal{D}_X$ be the sheaf of $\mathbb{K}$-linear differential operators on $X$.

**Theorem 1.13.** *Assume* $\mathrm{H}^1(X, \mathcal{D}_X) = 0$. *Then any deformation quantization* $(\mathcal{A}, \psi, \boldsymbol{\tau})$ *of* $\mathcal{O}_X$ *can be globally trivialized. Namely there is a globally trivialized deformation quantization* $(\mathcal{A}', \psi', \{\tau'\})$ *of* $\mathcal{O}_X$, *and a gauge equivalence* $(\mathcal{A}, \psi, \boldsymbol{\tau}) \to (\mathcal{A}', \psi', \{\tau'\})$.

**Proof.** We will take $(\mathcal{A}', \psi') := (\mathcal{A}, \psi)$, and produce a global differential trivialization $\tau'$.

By refining the open covering $\boldsymbol{U} = \{U_i\}$ associated with the differential structure $\boldsymbol{\tau} = \{\tau_i\}$ we may assume each of the open sets $U_i$ is affine. We may also assume that $\boldsymbol{U}$ is finite, say $\boldsymbol{U} = \{U_0, \ldots, U_m\}$. For any pair of indices $(i, j)$ let $\rho_{(i,j)} := \tau_j^{-1} \circ \tau_i$, which is a gauge equivalence of $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$. We are going to construct a gauge equivalence $\rho_i$ of $\mathcal{O}_{U_i}[[\hbar]]$, for every $i \in \{0, \ldots, m\}$, such that $\rho_{(i,j)} = \rho_j \circ \rho_i^{-1}$. Then the new differential structure $\boldsymbol{\tau}'$, defined by $\tau'_i := \tau_i \circ \rho_i$, will satisfy $\tau'^{-1}_j \circ \tau'_i = \mathbf{1}_{\mathcal{O}_{U_i \cap U_j}[[\hbar]]}$, the identity automorphism of $\mathcal{O}_{U_i \cap U_j}[[\hbar]]$. Therefore the various $\tau'_i$ can be glued to a global differential trivialization $\tau' : \mathcal{O}_X[[\hbar]] \xrightarrow{\simeq} \mathcal{A}$ as required.

Let $\mathcal{D}_X^{\mathrm{nor}}$ be the subsheaf of $\mathcal{D}_X$ consisting of operators that vanish on $1_{\mathcal{O}_X}$. This is the left ideal of $\mathcal{D}_X$ generated by the sheaf of derivation $\mathcal{T}_X$. There is a direct sum decomposition $\mathcal{D}_X = \mathcal{O}_X \oplus \mathcal{D}_X^{\mathrm{nor}}$ (as sheaves of left $\mathcal{O}_X$-modules), and therefore $\mathrm{H}^1(X, \mathcal{D}_X^{\mathrm{nor}}) = 0$.

Consider the sheaf of nonabelian groups $G$ on $X$ whose sections on an open set $U$ is the group of gauge equivalences of $\mathcal{O}_U[[\hbar]]$. Let $\mathcal{D}_X^{\mathrm{nor}}[[\hbar]]^+ := \hbar\mathcal{D}_X^{\mathrm{nor}}[[\hbar]]$. As sheaves of sets there is a canonical isomorphism $\mathcal{D}_X^{\mathrm{nor}}[[\hbar]]^+ \xrightarrow{\simeq} G$, whose formula is $\sum_{k=1}^{\infty} D_k\hbar^k \mapsto \mathbf{1}_{\mathcal{O}_X} + \sum_{k=1}^{\infty} D_k\hbar^k$. Define $G^k$ to be the subgroup of $G$ consisting of all equivalences congruent to $\mathbf{1}_{\mathcal{O}_X}$ modulo $\hbar^{k+1}$. Then each $G^k$ is a normal subgroup, and the map $\mathcal{D}_X^{\mathrm{nor}} \xrightarrow{\simeq} G^k/G^{k+1}$, $D \mapsto \mathbf{1}_{\mathcal{O}_X} + D\hbar^{k+1}$, is an isomorphism of sheaves of abelian groups. Moreover the conjugation action of $G$ on $G^k/G^{k+1}$ is trivial, so that $\gamma_1\gamma_2 = \gamma_2\gamma_1 \in G^k/G^{k+1}$ for every $\gamma_1 \in G^k$ and $\gamma_2 \in G$.

The gauge equivalences $\rho_i = \sum_{k=0}^{\infty} D_{i,k}\hbar^k$ will be defined by successive approximations; namely the differential operators $D_{i,k} \in \Gamma(U_i, \mathcal{D}_X^{\mathrm{nor}})$ shall be defined by recursion on $k$, simultaneously for all $i \in \{0, \dots, m\}$. For $k = 0$ we take $D_{i,0} := \mathbf{1}_{\mathcal{O}_{U_i}}$ of course. Now assume at the $k$th stage we have operators $\rho_i^{(k)} := \sum_{l=0}^{k} D_{i,l}\hbar^l$ which satisfy

$$\rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \equiv \rho_{(i,j)} \bmod \hbar^{k+1}.$$

This means that

$$\rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \in G^k.$$

By the properties of the group $G$ mentioned above the function $\{0, \dots, m\}^2 \to G^k/G^{k+1}$,

$$(i, j) \mapsto \rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \in \Gamma(U_i \cap U_j, G^k/G^{k+1}),$$

is a Čech 1-cocycle for the affine covering $\boldsymbol{U}$. Since $G^k/G^{k+1} \cong \mathcal{D}_X^{\mathrm{nor}}$, and we are given that $\mathrm{H}^1(X, \mathcal{D}_X^{\mathrm{nor}}) = 0$, it follows that there exists a 0-cochain $i \mapsto D_{i,k+1} \in \Gamma(U_i, \mathcal{D}_X^{\mathrm{nor}})$ such that

$$(\mathbf{1} + D_{j,k+1}\hbar^{k+1}) \circ (\mathbf{1} + D_{i,k+1}\hbar^{k+1})^{-1} \equiv \rho_j^{(k)} \circ (\rho_i^{(k)})^{-1} \circ (\rho_{(i,j)})^{-1} \bmod \hbar^{k+2}. \qquad \square$$

**Proposition 1.14.** *Let $\star$ and $\star'$ be two star products on $\mathcal{O}_X[[\hbar]]$. Consider the globally trivialized deformation quantizations $(\mathcal{A}, \psi, \boldsymbol{\tau})$ and $(\mathcal{A}', \psi, \boldsymbol{\tau})$, where $\mathcal{A} := (\mathcal{O}_X[[\hbar]], \star)$, $\mathcal{A}' := (\mathcal{O}_X[[\hbar]], \star')$, $\psi := \mathbf{1}_{\mathcal{O}_X}$ and $\boldsymbol{\tau} := \{\mathbf{1}_{\mathcal{O}_X[[\hbar]]}\}$. Then the deformation quantizations $(\mathcal{A}, \psi, \boldsymbol{\tau})$ and $(\mathcal{A}', \psi, \boldsymbol{\tau})$ are gauge equivalent, in the sense of Definition* 1.8, *iff there exists a gauge equivalence $\gamma$ of $\mathcal{O}_X[[\hbar]]$, in the sense of Definition* 1.3, *such that*

$$f \star' g = \gamma^{-1}\big(\gamma(f) \star \gamma(g)\big)$$

*for all local sections $f, g \in \mathcal{O}_X$.*

We leave out the easy proof.

## 2. Review of dir-inv modules

In this section we review the concept of dir-inv structure, which was introduced in [Ye2, Section 1]. A dir-inv structure is a generalization of adic topology, and it will turn out to be extremely useful in several places in the paper.

Let $C$ be a commutative $\mathbb{K}$-algebra. We denote by $\mathsf{Mod}\, C$ the category of $C$-modules.

**Definition 2.1.** (1) Let $M \in \mathsf{Mod}\, C$. An *inv module structure* on $M$ is an inverse system $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ of $C$-submodules of $M$. The pair $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is called an *inv $C$-module*.

(2) Let $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i \in \mathbb{N}})$ be two inv $C$-modules. A function $\phi : M \to N$ ($C$-linear or not) is said to be *continuous* if for every $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\mathrm{F}^{i'} M) \subset \mathrm{F}^i N$.

(3) Define $\mathsf{Inv}\,\mathsf{Mod}\, C$ to be the category whose objects are the inv $C$-modules, and whose morphisms are the continuous $C$-linear homomorphisms.

There is a full and faithful embedding of categories $\mathsf{Mod}\, C \hookrightarrow \mathsf{Inv}\,\mathsf{Mod}\, C$, $M \mapsto (M, \{\ldots, 0, 0\})$.

Recall that a directed set is a partially ordered set $J$ with the property that for any $j_1, j_2 \in J$ there exists $j_3 \in J$ such that $j_1, j_2 \leqslant j_3$.

**Definition 2.2.** (1) Let $M \in \mathsf{Mod}\, C$. A *dir-inv module structure* on $M$ is a direct system $\{\mathrm{F}_j M\}_{j \in J}$ of $C$-submodules of $M$, indexed by a nonempty directed set $J$, together with an inv module structure on each $\mathrm{F}_j M$, such that for every $j_1 \leqslant j_2$ the inclusion $\mathrm{F}_{j_1} M \hookrightarrow \mathrm{F}_{j_2} M$ is continuous. The pair $(M, \{\mathrm{F}_j M\}_{j \in J})$ is called a *dir-inv $C$-module*.

(2) Let $(M, \{\mathrm{F}_j M\})_{j \in J}$ and $(N, \{\mathrm{F}_k N\}_{k \in K})$ be two dir-inv $C$-modules. A function $\phi : M \to N$ ($C$-linear or not) is said to be *continuous* if for every $j \in J$ there exists $k \in K$ such that $\phi(\mathrm{F}_j M) \subset \mathrm{F}_k N$, and $\phi : \mathrm{F}_j M \to \mathrm{F}_k N$ is a continuous function between these two inv $C$-modules.

(3) Define $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$ to be the category whose objects are the dir-inv $C$-modules, and whose morphisms are the continuous $C$-linear homomorphisms.

An inv $C$-module $M$ can be endowed with a dir-inv module structure $\{\mathrm{F}_j M\}_{j \in J}$, where $J := \{0\}$ and $\mathrm{F}_0 M := M$. Thus we get a full and faithful embedding $\mathsf{Inv}\,\mathsf{Mod}\, C \hookrightarrow \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$.

Inv modules and dir-inv modules come in a few "flavors": trivial, discrete and complete. A *discrete inv module* is one which is isomorphic, in $\mathsf{Inv}\,\mathsf{Mod}\, C$, to an object of $\mathsf{Mod}\, C$ (via the canonical embedding above). A *complete inv module* is an inv module $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ such that the canonical map $M \to \lim_{\leftarrow i} M/\mathrm{F}^i M$ is bijective. A *discrete* (resp. *complete*) *dir-inv module* is one which is isomorphic, in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\, C$, to a dir-inv module $(M, \{\mathrm{F}_j M\}_{j \in J})$, where all the inv modules $\mathrm{F}_j M$ are discrete (resp. complete), and the canonical map $\lim_{j \to} \mathrm{F}_j M \to M$ in $\mathsf{Mod}\, C$ is bijective. A *trivial dir-inv module* is one which is isomorphic to an object of $\mathsf{Mod}\, C$. Discrete dir-inv modules are complete, but there are also other complete modules, as the next example shows.

**Example 2.3.** Assume $C$ is noetherian and $\mathfrak{c}$-adically complete for some ideal $\mathfrak{c}$. Let $M$ be a finitely generated $C$-module, and define $\mathrm{F}^i M := \mathfrak{c}^{i+1} M$. Then $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ is called the $\mathfrak{c}$-*adic inv structure*, and of course $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is a complete inv module. Next consider an arbitrary $C$-module $M$. We take $\{\mathrm{F}_j M\}_{j \in J}$ to be the collection of finitely generated $C$-submodules of $M$. This dir-inv module structure on $M$ is called the $\mathfrak{c}$-*adic dir-inv structure*. Again $(M, \{\mathrm{F}_j M\}_{j \in J})$ is a complete dir-inv $C$-module. Note that a finitely generated $C$-module $M$ is discrete as inv module iff $\mathfrak{c}^i M = 0$ for $i \gg 0$; and a $C$-module is discrete as dir-inv module iff it is a direct limit of discrete finitely generated modules.

The category $\mathsf{Dir\,Inv\,Mod}\, C$ is additive. Given a collection $\{M_k\}_{k \in K}$ of dir-inv modules, the direct sum $\bigoplus_{k \in K} M_k$ has a dir-inv module structure, making it into the coproduct of $\{M_k\}_{k \in K}$ in $\mathsf{Dir\,Inv\,Mod}\, C$. Note that if the index set $K$ is infinite and each $M_k$ is a nonzero discrete inv module, then $\bigoplus_{k \in K} M_k$ is a discrete dir-inv module which is not trivial. The tensor product $M \otimes_C N$ of two dir-inv modules is again a dir-inv module. There is a completion functor $M \mapsto \widehat{M}$. (Warning: if $M$ is complete then $\widehat{M} = M$, but it is not known if $\widehat{M}$ is complete for arbitrary $M$.) The completed tensor product is $M \widehat{\otimes}_C N := \widehat{M \otimes_C N}$. Completion commutes with direct sums: if $M \cong \bigoplus_{k \in K} M_k$ then $\widehat{M} \cong \bigoplus_{k \in K} \widehat{M}_k$.

A *graded dir-inv module* (or graded object in $\mathsf{Dir\,Inv\,Mod}\, C$) is a direct sum $M = \bigoplus_{k \in \mathbb{Z}} M_k$, where each $M_k$ is a dir-inv module. A *DG algebra* in $\mathsf{Dir\,Inv\,Mod}\, C$ is a graded dir-inv module $A = \bigoplus_{k \in \mathbb{Z}} A^k$, together with continuous $C$-(bi)linear functions $\mu : A \times A \to A$ and $\mathrm{d} : A \to A$, which make $A$ into a DG $C$-algebra. If $A$ is a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, and $\mathfrak{g}$ is a DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\, C$, then $A \widehat{\otimes}_C \mathfrak{g}$ is a DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\, C$.

Let $A$ be a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\, C$. A *DG A-module* in $\mathsf{Dir\,Inv\,Mod}\, C$ is a graded object $M$ in $\mathsf{Dir\,Inv\,Mod}\, C$, together with continuous $C$-(bi)linear functions $\mu : A \times M \to M$ and $\mathrm{d} : M \to M$, which make $M$ into a DG $A$-module in the usual sense. A *DG A-module Lie algebra* in $\mathsf{Dir\,Inv\,Mod}\, C$ is a DG Lie algebra $\mathfrak{g}$ in $\mathsf{Dir\,Inv\,Mod}\, C$, together with a continuous $C$-bilinear function $\mu : A \times \mathfrak{g} \to \mathfrak{g}$, such that $\mathfrak{g}$ becomes a DG $A$-module, and

$$[a_1 \gamma_1, a_2 \gamma_2] = (-1)^{i_2 j_1} a_1 a_2 \, [\gamma_1, \gamma_2]$$

for all $a_k \in A^{i_k}$ and $\gamma_k \in \mathfrak{g}^{j_k}$.

All the constructions above can be geometrized. Let $(Y, \mathcal{O})$ be a commutative ringed space over $\mathbb{K}$, i.e. $Y$ is a topological space, and $\mathcal{O}$ is a sheaf of commutative $\mathbb{K}$-algebras on $Y$. We denote by $\mathsf{Mod}\, \mathcal{O}$ the category of $\mathcal{O}$-modules on $Y$. Then we can talk about the category $\mathsf{Dir\,Inv\,Mod}\, \mathcal{O}$ of dir-inv $\mathcal{O}$-modules.

**Example 2.4.** Geometrizing Example 2.3, let $\mathfrak{X}$ be a noetherian formal scheme, with defining ideal $\mathcal{I}$. Then any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is an inv $\mathcal{O}_{\mathfrak{X}}$-module, with system of submodules $\{\mathcal{I}^{i+1}\mathcal{M}\}_{i\in\mathbb{N}}$, and $\mathcal{M}\cong\widehat{\mathcal{M}}$; cf. [EGA I]. We call an $\mathcal{O}_{\mathfrak{X}}$-module *dir-coherent* if it is a direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Any dir-coherent module is quasi-coherent, but it is not known if the converse is true. At any rate, a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is a dir-inv $\mathcal{O}_{\mathfrak{X}}$-module, where we take $\{\mathrm{F}_j\mathcal{M}\}_{j\in J}$ to be the collection of coherent submodules of $\mathcal{M}$. Any dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is then a complete dir-inv module. This dir-inv module structure on $\mathcal{M}$ is called the *$\mathcal{I}$-adic dir-inv structure*.

If $f:(Y',\mathcal{O}')\to(Y,\mathcal{O})$ is a morphism of ringed spaces and $\mathcal{M}\in\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then there is an obvious structure of dir-inv $\mathcal{O}'$-module on $f^*\mathcal{M}$, and we define $\widehat{f^*}\mathcal{M}:=\widehat{f^*\mathcal{M}}$. If $\mathcal{M}$ is a graded object in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then the inverse images $f^*\mathcal{M}$ and $\widehat{f^*}\mathcal{M}$ are graded objects in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. If $\mathcal{G}$ is an algebra (resp. a DG algebra) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then $f^*\mathcal{G}$ and $\widehat{f^*}\,\mathcal{G}$ are algebras (resp. DG algebras) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. Given $\mathcal{N}\in\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$ there is an obvious dir-inv $\mathcal{O}$-module structure on $f_*\mathcal{N}$.

## 3. Universal formulas for deformation quantization

In this section, as before, $\mathbb{K}$ is a field of characteristic 0.

From here to Corollary 3.10 we consider the following data. Let $\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}\mathfrak{g}^j$ be a DG Lie algebra over $\mathbb{K}$. We put on each $\mathfrak{g}^j$ the discrete inv $\mathbb{K}$-module structure, and $\mathfrak{g}$ is given the $\bigoplus$ dir-inv structure; so $\mathfrak{g}$ is a discrete, but possibly nontrivial, DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Let $A$ be noetherian commutative complete local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}$. We put on $A$ and $\mathfrak{m}$ the $\mathfrak{m}$-adic inv structures. For $i\geqslant 0$ let $A_i:=A/\mathfrak{m}^{i+1}$, which is an artinian local algebra with maximal ideal $\mathfrak{m}_i:=\mathfrak{m}/\mathfrak{m}^{i+1}$; so $A_i$ and $\mathfrak{m}_i$ are discrete inv modules. We obtain a new DG Lie algebra $A\mathbin{\widehat{\otimes}}_{\mathbb{K}}\mathfrak{g}=A\mathbin{\widehat{\otimes}}\mathfrak{g}=\bigoplus_{j\in\mathbb{Z}}A\mathbin{\widehat{\otimes}}\mathfrak{g}^j$, and there are related DG Lie algebras $\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}\subset A\mathbin{\widehat{\otimes}}\mathfrak{g}$ and $\mathfrak{m}_i\otimes\mathfrak{g}\subset A_i\otimes\mathfrak{g}$. Note that for every $j$ one has $A\mathbin{\widehat{\otimes}}\mathfrak{g}^j\cong\lim_{\leftarrow i}(A_i\otimes\mathfrak{g}^j)$ in $\mathsf{Inv\,Mod}\,\mathbb{K}$. In case $A=\mathbb{K}[[\hbar]]$ we shall also use the notation $\mathfrak{g}[[\hbar]]^+:=\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}$, namely $\mathfrak{g}[[\hbar]]^+=\bigoplus_j\hbar\mathfrak{g}^j[[\hbar]]$.

Recall the correspondence between finite-dimensional nilpotent Lie algebras and unipotent algebraic groups over $\mathbb{K}$ (see [Ho, Theorem XVI.4.2]). Given a nilpotent Lie algebra $\mathfrak{h}$ we denote by $\exp(\mathfrak{h})$ the corresponding group. This group has the same underlying scheme structure as $\mathfrak{h}$, and the product is according to the Campbell–Hausdorff formula. The assignment $\mathfrak{h}\mapsto\exp(\mathfrak{h})$ is functorial.

For any $i$ the Lie algebra $\mathfrak{m}_i\otimes\mathfrak{g}^0$ is a nilpotent, and in fact it is a direct limit of finite-dimensional nilpotent Lie algebras. Therefore we obtain a group $\exp(\mathfrak{m}_i\otimes\mathfrak{g}^0)$, which is a direct limit of unipotent groups. Passing to the inverse limit in $i$ we get a group $\exp(\mathfrak{m}\otimes\mathfrak{g}^0):=\lim_{\leftarrow i}\exp(\mathfrak{m}_i\otimes\mathfrak{g}^0)$.

Given a vector space $V$ over $\mathbb{K}$ let $\mathrm{Aff}(V):=\mathrm{GL}(V)\ltimes V$, the group of affine transformations. Its Lie algebra is $\mathfrak{aff}(V):=\mathfrak{gl}(V)\ltimes V$. If $V$ is finite dimensional then of course $\mathrm{Aff}(V)$ is an algebraic group; but we will be interested in $V:=\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}^1$.

For $\gamma\in\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}^0$ and $\omega\in\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}^1$ define

$$\mathrm{af}(\gamma)(\omega):=[\gamma,\omega]-\mathrm{d}(\gamma)=(\mathrm{ad}(\gamma)-\mathrm{d})(\omega)\in\mathfrak{m}\mathbin{\widehat{\otimes}}\mathfrak{g}^1,$$

where d and $[-,-]$ are the operations of the DG Lie algebra $\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}$. A calculation shows that this is a homomorphism of Lie algebras

$$\mathrm{af} : \mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^0 \to \mathfrak{aff}\left(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^1\right).$$

Recall that the *Maurer–Cartan equation* in $A \mathbin{\widehat{\otimes}} \mathfrak{g}$ is

$$\mathrm{d}(\omega) + \tfrac{1}{2}[\omega, \omega] = 0 \tag{3.1}$$

for $\omega \in A \mathbin{\widehat{\otimes}} \mathfrak{g}^1$.

**Lemma 3.2.** (1) *The Lie algebra homomorphism* af *integrates to a group homomorphism*

$$\exp(\mathrm{af}) : \exp(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^0) \to \mathrm{Aff}\left(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^1\right).$$

(2) *Assume* $\omega \in \mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^1$ *is a solution of the MC equation in* $\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}$, *and let* $\gamma \in \mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^0$. *Then* $\exp(\mathrm{af})(\exp(\gamma))(\omega)$ *is also a solution of the MC equation.*

**Proof.** We may assume that $\mathfrak{g} = \bigoplus_{j \geqslant 0} \mathfrak{g}^j$. First consider the nilpotent case. The DG Lie algebra $\mathfrak{m}_i \otimes \mathfrak{g}$ is the direct limit of sub DG Lie algebras $\mathfrak{h} = \bigoplus_{j \geqslant 0} \mathfrak{h}^j$, which are nilpotent, and each $\mathfrak{h}^j$ is a finite dimensional vector space. The arguments of [GM, Section 1.3] apply here, so we obtain a homomorphism of algebraic groups $\exp(\mathrm{af}) : \exp(\mathfrak{h}^0) \to \mathrm{Aff}(\mathfrak{h}^1)$, and $\exp(\mathfrak{h}^0)$ preserves the set of solutions of the MC equation in $\mathfrak{h}^1$. Passing to the direct limit over these subalgebras we get a homomorphism of groups $\exp(\mathrm{af}) : \exp(\mathfrak{m}_i \otimes \mathfrak{g}^0) \to \mathrm{Aff}(\mathfrak{m}_i \otimes \mathfrak{g}^1)$, and $\exp(\mathfrak{m}_i \otimes \mathfrak{g}^0)$ preserves the set of solutions of the MC equation in $\mathfrak{m}_i \otimes \mathfrak{g}^1$. Finally we pass to the inverse limit in $i$. □

The formula for $\exp(\mathrm{af})(\exp(\gamma))(\omega)$ is, according to [GM]:

$$\exp(\mathrm{af})(\exp(\gamma))(\omega) = \exp(\mathrm{ad}(\gamma))(\omega) + \frac{1 - \exp(\mathrm{ad}(\gamma))}{\mathrm{ad}(\gamma)}(\mathrm{d}(\gamma)). \tag{3.3}$$

On the right-hand side of the equation "exp" stands for the usual exponential power series $\exp(t) = \sum_{k=0}^{\infty} \frac{1}{k!} t^k$, and this makes sense because $\lim_{k \to \infty} \mathrm{ad}(\gamma)^k(\omega) = 0$ in the $\mathfrak{m}$-adic inv structure on $\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^1$.

**Definition 3.4.** Elements of the group $\exp(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}^0)$ are called *gauge equivalences*. We write

$$\mathrm{MC}(\mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}) := \frac{\{\text{solutions of the MC equation in } \mathfrak{m} \mathbin{\widehat{\otimes}} \mathfrak{g}\}}{\{\text{gauge equivalences}\}}.$$

**Lemma 3.5.** *The canonical projection*

$$\mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \to \lim_{\leftarrow i} \mathrm{MC}(\mathfrak{m}_i \otimes \mathfrak{g})$$

*is bijective.*

The easy proof is omitted.

**Remark 3.6.** Consider the super-commutative DG algebra $\Omega_{\mathbb{K}[t]} = \Omega^0_{\mathbb{K}[t]} \oplus \Omega^1_{\mathbb{K}[t]}$, where $\mathbb{K}[t]$ is the polynomial algebra in the variable $t$. There is an induced DG Lie algebra $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{g}$. For any $\lambda \in \mathbb{K}$ there is a DG Lie algebra homomorphism $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{g} \to \mathfrak{g}$, $t \mapsto \lambda$. Assume $A$ is artinian, and let $\omega_0$ and $\omega_1$ be two solutions of the MC equation in $\mathfrak{m} \otimes \mathfrak{g}$. According to [Ko1, 4.5.2(3)] the following conditions are equivalent:

(i) $\omega_0$ and $\omega_1$ are gauge equivalent, in the sense of Definition 3.4.
(ii) There is a solution $\omega(t)$ of the MC equation in the DG Lie algebra $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{m} \otimes \mathfrak{g}$, such that for $i \in \{0, 1\} \subset \mathbb{K}$, the specialization homomorphisms $\Omega_{\mathbb{K}[t]} \otimes \mathfrak{m} \otimes \mathfrak{g} \to \mathfrak{m} \otimes \mathfrak{g}$, $t \mapsto i$, send $\omega(t) \mapsto \omega_i$.

See also [Fu,Hi1]. We will not need these facts in our paper.

For a graded $\mathbb{K}$-module $M$ the expression $\bigwedge^i M$ denotes the $i$th super-exterior power.

**Definition 3.7.** Let $\mathfrak{g}$ and $\mathfrak{h}$ be two DG Lie algebras. An $\mathrm{L}_\infty$ *morphism* $\Psi : \mathfrak{g} \to \mathfrak{h}$ is a collection $\Psi = \{\psi_i\}_{i \geqslant 1}$ of $\mathbb{K}$-linear homomorphisms $\psi_i : \bigwedge^i \mathfrak{g} \to \mathfrak{h}$, each of them homogeneous of degree $1 - i$, satisfying

$$\begin{aligned} &\mathrm{d}\big(\psi_i(\gamma_1 \wedge \cdots \wedge \gamma_i)\big) - \sum_{k=1}^{i} \pm\psi_i\big(\gamma_1 \wedge \cdots \wedge \mathrm{d}(\gamma_k) \wedge \cdots \wedge \gamma_i\big) \\ &\quad = \frac{1}{2} \sum_{\substack{k,l \geqslant 1 \\ k+l=i}} \frac{1}{k!\,l!} \sum_{\sigma \in \Sigma_i} \pm\big[\psi_k(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)}), \psi_l(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(k+l)})\big] \\ &\qquad + \sum_{k<l} \pm\psi_{i-1}\big([\gamma_k, \gamma_l] \wedge \gamma_1 \wedge \cdots \wedge \gamma_i\big). \end{aligned}$$

Here $\gamma_k \in \mathfrak{g}$ are homogeneous elements, $\Sigma_i$ is the permutation group of $\{1, \ldots, i\}$, and the signs depend only on the indices, the permutations and the degrees of the elements $\gamma_k$. See [Ke, Section 6] or [CFT, Theorem 3.1] for the explicit signs.

An $\mathrm{L}_\infty$ morphism is a generalization of a DG Lie algebra homomorphism. Indeed, $\psi_1 : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of complexes of $\mathbb{K}$-modules, and $\mathrm{H}(\psi_1) : \mathrm{H}\mathfrak{g} \to \mathrm{H}\mathfrak{h}$ is a homomorphism of graded Lie algebras.

Suppose $\Psi = \{\psi_i\}_{i \geqslant 1} : \mathfrak{g} \to \mathfrak{h}$ is an $\mathrm{L}_\infty$ morphism. For every $i$ we can extend the $\mathbb{K}$-multilinear function $\psi_i : \prod^i \mathfrak{g} \to \mathfrak{h}$ uniquely to a continuous $A$-multilinear function $\psi_{A,i} : \prod^i (A \widehat{\otimes} \mathfrak{g}) \to A \widehat{\otimes} \mathfrak{h}$. These restrict to functions $\psi_{A,i} : \prod^i (\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \to \mathfrak{m} \widehat{\otimes} \mathfrak{h}$. Clearly $\Psi_A = \{\psi_{A,i}\} : \mathfrak{m} \widehat{\otimes} \mathfrak{g} \to \mathfrak{m} \widehat{\otimes} \mathfrak{h}$ is an $\mathrm{L}_\infty$ morphism; we call it the continuous $A$-multilinear extension of $\Psi$.

**Theorem 3.8** (*Kontsevich [Ko1, Section 4.4], Fukaya [Fu, Theorem 2.2.2]*). *Assume $A$ is artinian. Let $\Psi : \mathfrak{g} \to \mathfrak{h}$ be an $\mathrm{L}_\infty$ quasi-isomorphism. Then the function*

$$\omega \mapsto \sum_{j \geqslant 1} \frac{1}{j!} \psi_{A,j}(\omega^j) \tag{3.9}$$

*induces a bijection*

$$\mathrm{MC}(\Psi_A) : \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}) \xrightarrow{\simeq} \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{h}).$$

**Corollary 3.10.** *Let $(A, \mathfrak{m})$ be a complete noetherian local $\mathbb{K}$-algebra, and let $\Psi : \mathfrak{g} \to \mathfrak{h}$ be an $\mathrm{L}_\infty$ quasi-isomorphism between two discrete DG Lie algebras. Then function* (3.9) *induces a bijection*

$$\mathrm{MC}(\Psi_A) : \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{g}) \xrightarrow{\simeq} \mathrm{MC}(\mathfrak{m} \widehat{\otimes} \mathfrak{h}).$$

**Proof.** Use Theorem 3.8 and Lemma 3.5. □

Let $C$ be a commutative $\mathbb{K}$-algebra. The module of derivations of $C$ relative to $\mathbb{K}$ is denoted by $\mathcal{T}_{C/\mathbb{K}} = \mathcal{T}_C$. For $p \geqslant -1$ let $\mathcal{T}^p_{\mathrm{poly}}(C) := \bigwedge^{p+1}_C \mathcal{T}_C$, the $p$th exterior power. The direct sum $\mathcal{T}_{\mathrm{poly}}(C) := \bigoplus_{p \geqslant -1} \mathcal{T}^p_{\mathrm{poly}}(C)$ is a DG Lie algebra over $\mathbb{K}$ with trivial differential and with the Schouten–Nijenhuis Lie bracket (see [Ko2] for details).

For any $p \geqslant 0$ and $m \geqslant 0$ let $\mathrm{F}_m \mathcal{D}^p_{\mathrm{poly}}(C)$ be the set of $\mathbb{K}$-multilinear functions $\phi : C^{p+1} \to C$ that are differential operators of order $\leqslant m$ in each argument (in the sense of [EGA IV]). For $p = -1$ let $\mathrm{F}_m \mathcal{D}^{-1}_{\mathrm{poly}}(C) := C$. Define $\mathcal{D}^p_{\mathrm{poly}}(C) := \bigcup_{m \geqslant 0} \mathrm{F}_m \mathcal{D}^p_{\mathrm{poly}}(C)$ and $\mathcal{D}_{\mathrm{poly}}(C) := \bigoplus_{p \geqslant -1} \mathcal{D}^p_{\mathrm{poly}}(C)$. This is a subDG Lie algebra of the shifted Hochschild cochain complex of $C$, with shifted Hochschild differential and Gerstenhaber Lie bracket (see [Ko1]). We view $\mathcal{D}_{\mathrm{poly}}(C)$ as a left $C$-module by the rule $(c \cdot \phi)(c_1, \ldots, c_{p+1}) := c \cdot \phi(c_1, \ldots, c_{p+1})$.

For $p \geqslant 0$ define $\mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly}}(C)$ to be the subset of $\mathcal{D}^p_{\mathrm{poly}}(C)$ consisting of the polydifferential operators $\phi : C^{p+1} \to C$ such that $\phi(c_1, \ldots, c_{p+1}) = 0$ if $c_i = 1$ for some $i$. For $p = -1$ let $\mathcal{D}^{\mathrm{nor},-1}_{\mathrm{poly}}(C) := C$. Then $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) := \bigoplus_{p \geqslant -1} \mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly}}(C)$ is a subDG Lie algebra of $\mathcal{D}_{\mathrm{poly}}(C)$.

For any integer $p \geqslant 0$ there is a $C$-linear homomorphism

$$\mathcal{U}_1 : \mathcal{T}^p_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly}}(C)$$

with formula

$$\mathcal{U}_1(\xi_1 \wedge \cdots \wedge \xi_{p+1})(c_1, \ldots, c_{p+1}) := \frac{1}{(p+1)!} \sum_{\sigma \in \Sigma_{p+1}} \operatorname{sgn}(\sigma) \xi_{\sigma(1)}(c_1) \cdots \xi_{\sigma(p+1)}(c_{p+1}) \tag{3.11}$$

for elements $\xi_1, \ldots, \xi_{p+1} \in \mathcal{T}_C$ and $c_1, \ldots, c_{p+1} \in C$. For $p = -1$ the map $\mathcal{U}_1 : \mathcal{T}^{-1}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor},-1}_{\mathrm{poly}}(C)$ is the identity (of $C$).

The next result is a variant of the Hochschild–Kostant–Rosenberg Theorem. A slightly weaker result appeared in [Ye1]. See [Ko1] for the $\mathrm{C}^{\infty}$ version.

**Theorem 3.12** (*Yekutieli [Ye2, Corollary 4.12]*). *Suppose $C$ is a smooth $\mathbb{K}$-algebra. Then the homomorphism $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ and the inclusion $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$ are both quasi-isomorphisms of complexes of $C$-modules.*

Here is a slight modification of the celebrated result of Kontsevich, known as the *Kontsevich Formality Theorem* [Ko1, Theorem 6.4]. In the form below it is proved in [Ye2, Theorem 4.13].

**Theorem 3.13.** *Let $\mathbb{K}[\boldsymbol{t}] = \mathbb{K}[t_1, \ldots, t_n]$ be the polynomial algebra in $n$ variables, and assume that $\mathbb{R} \subset \mathbb{K}$. There is a collection of $\mathbb{K}$-linear homomorphisms*

$$\mathcal{U}_j : \bigwedge^j \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]),$$

*indexed by $j \in \{1, 2, \ldots\}$, satisfying the following conditions*:

(i) *The sequence $\mathcal{U} = \{\mathcal{U}_j\}$ is an $\mathrm{L}_\infty$-morphism $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$*;
(ii) *each $\mathcal{U}_j$ is a polydifferential operator of $\mathbb{K}[\boldsymbol{t}]$-modules*;
(iii) *each $\mathcal{U}_j$ is equivariant for the standard action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[\boldsymbol{t}]$*;
(iv) *the homomorphism $\mathcal{U}_1$ is given by Eq.* (3.11);
(v) *for any $j \geqslant 2$ and $\alpha_1, \ldots, \alpha_j \in \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ one has $\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0$*;
(vi) *for any $j \geqslant 2$, $\alpha_1 \in \mathfrak{gl}_n(\mathbb{K}) \subset \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ and $\alpha_2, \ldots, \alpha_j \in \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ one has $\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j) = 0$.*

**Remark 3.14.** Presumably the image of $\mathcal{U}_j$ is inside $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ for all $j$. However we did not verify this.

**Remark 3.15.** The methods of Tamarkin [Ta,Hi2], or suitable arithmetic considerations [Ko2], should make it possible to extend Theorem 3.13, and hence all results of our paper, to any field $\mathbb{K}$ of characteristic 0.

Consider the power series algebra $\mathbb{K}[[\boldsymbol{t}]] = \mathbb{K}[[t_1, \ldots, t_n]]$. As in Example 2.4, the $\mathbb{K}[[\boldsymbol{t}]]$-modules $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ have the $\boldsymbol{t}$-adic dir-inv structures. These are DG Lie algebras in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Because $\mathbb{K}[\boldsymbol{t}] \to \mathbb{K}[[\boldsymbol{t}]]$ is flat and $\boldsymbol{t}$-adically formally étale, it follows that there is an induced $\mathrm{L}_\infty$ morphism $\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Since each

$$\mathcal{U}_j : \prod^j \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) \to \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$$

is a polydifferential operator over $\mathbb{K}[[\boldsymbol{t}]]$, it is continuous for the dir-inv structures. See [Ye2, Proposition 4.6] for details and proofs.

Now suppose we are given a complete super-commutative associative unital DG algebra $A = \bigoplus_{i \geqslant 0} A^i$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Let

$$\mathcal{U}_{A;j} : \prod^j \big(A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big) \to A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$$

be the continuous $A$-multilinear extension of $\mathcal{U}_j$. It almost immediate from Definition 3.7 that $\mathcal{U}_A = \{\mathcal{U}_{A;j}\}_{j \geqslant 1}$ is an $\mathrm{L}_\infty$ morphism; see [Ye2, Proposition 3.25].

Let us recall the notion of twisting for a DG Lie algebra $\mathfrak{g}$. Suppose $\omega \in \mathfrak{g}^1$ is a solution of the MC equation (3.1). The twisted DG Lie algebra $\mathfrak{g}_\omega$ is the same graded Lie algebra, but the new differential is $\mathrm{d}_\omega := \mathrm{d} + \mathrm{ad}(\omega)$; i.e. $\mathrm{d}_\omega(\alpha) = \mathrm{d}(\alpha) + [\omega, \alpha]$.

**Theorem 3.16** (*Yekutieli [Ye2, Theorem 0.1]*). *Assume $\mathbb{R} \subset \mathbb{K}$. Let $A = \bigoplus_{i \geqslant 0} A^i$ be a complete super-commutative associative unital DG algebra in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, *and let $\omega \in A^1 \mathbin{\widehat{\otimes}} \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ be a solution of the Maurer–Cartan equation in $A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. For any element $\alpha \in \bigwedge^j \big(A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)$ define*

$$\mathcal{U}_{A,\omega;j}(\alpha) := \sum_{k \geqslant 0} \frac{1}{(j+k)!} \mathcal{U}_{A;j+k}(\omega^k \wedge \alpha) \in A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

*Let $\omega' := \mathcal{U}_{A;1}(\omega)$. Then $\omega'$ is a solution of the Maurer–Cartan equation in $A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$, and the sequence $\{\mathcal{U}_{A,\omega;j}\}_{j \geqslant 1}$ is a continuous $A$-multilinear* $\mathrm{L}_\infty$ *quasi-isomorphism*

$$\mathcal{U}_{A,\omega} : \big(A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_\omega \to \big(A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big)_{\omega'}.$$

The sum occurring in the definition of $\mathcal{U}_{A,\omega;j}(\alpha)$ is always finite (but the number of nonzero terms depends on the argument $\alpha$).

The group $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathbb{K}[[\boldsymbol{t}]]$ by linear change of coordinates. This is an action by $\mathbb{K}$-algebra automorphisms, and hence $\mathrm{GL}_n(\mathbb{K})$ acts on $\mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $\mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ by continuous DG Lie algebra automorphisms. Suppose we are given an action of $\mathrm{GL}_n(\mathbb{K})$ on $A$ by continuous unital DG algebra automorphisms. Then we obtain an action of $\mathrm{GL}_n(\mathbb{K})$ on $A \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ and $A \mathbin{\widehat{\otimes}} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ by continuous DG Lie algebra automorphisms.

**Proposition 3.17.** *Each operator* $\mathcal{U}_{A;j}$ *is* $\mathrm{GL}_n(\mathbb{K})$*-equivariant, i.e.* $\mathcal{U}_{A;j}(g(\alpha)) = g(\mathcal{U}_{A;j}(\alpha))$ *for any* $g \in \mathrm{GL}_n(\mathbb{K})$ *and* $\alpha \in \bigwedge^j \big(A \,\widehat{\otimes}\, \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])\big)$*. Moreover, if* $\omega$ *is* $\mathrm{GL}_n(\mathbb{K})$*-invariant, then each operator* $\mathcal{U}_{A,\omega;j}$ *is* $\mathrm{GL}_n(\mathbb{K})$*-equivariant.*

**Proof.** Using continuity and multilinearity we may assume that

$$\alpha = (a_1 \otimes \alpha_1) \wedge \cdots \wedge (a_j \otimes \alpha_j),$$

with $a_k \in A$ and $\alpha_k \in \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])$. Then

$$\begin{aligned} g\big(\mathcal{U}_{A;j}(\alpha)\big) &= \pm\, g\big(a_1 \cdots a_j \cdot \mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j)\big) \\ &= \pm\, g(a_1 \cdots a_j) \cdot g\big(\mathcal{U}_j(\alpha_1 \wedge \cdots \wedge \alpha_j)\big) \\ &=^{\diamond} \pm\, g(a_1 \cdots a_j) \cdot \mathcal{U}_j\big(g(\alpha_1 \wedge \cdots \wedge \alpha_j)\big) \\ &= \mathcal{U}_{A;j}\big(g(\alpha)\big), \end{aligned}$$

where the equality marked $\diamond$ is due to condition (iii) of Theorem 3.13.

We see that $g\big(\mathcal{U}_{A,\omega;j}(\alpha)\big) = \mathcal{U}_{A,g(\omega);j}\big(g(\alpha)\big)$. Hence the second assertion. $\square$

Let $X$ be a smooth irreducible separated $n$-dimensional $\mathbb{K}$-scheme.

**Proposition 3.18.** *There are sheaves of DG Lie algebras* $\mathcal{T}_{\mathrm{poly},X}$, $\mathcal{D}_{\mathrm{poly},X}$ *and* $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}$ *on* $X$*. As left* $\mathcal{O}_X$*-modules all three are quasi-coherent. The sheaves* $\mathcal{D}_{\mathrm{poly},X}$ *and* $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}$ *are quasi-coherent left* $\mathcal{D}_X$*-modules. For any affine open set* $U = \operatorname{Spec} C \subset X$ *one has* $\Gamma(U, \mathcal{T}_{\mathrm{poly},X}) = \mathcal{T}_{\mathrm{poly}}(C)$, $\Gamma(U, \mathcal{D}_{\mathrm{poly},X}) = \mathcal{D}_{\mathrm{poly}}(C)$ *and* $\Gamma(U, \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}) = \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ *as DG Lie algebras and as* $C$*-modules.*

**Proof.** Let $U' = \operatorname{Spec} C' \subset U$ be an open subset. Then $C \to C'$ is an étale ring homomorphism. According to [Ye2, Proposition 4.6] there are functorial DG Lie algebra homomorphisms $\mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{T}_{\mathrm{poly}}(C')$ and $\mathcal{D}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C')$ such that $C' \otimes_C \mathcal{T}_{\mathrm{poly}}(C) \cong \mathcal{T}_{\mathrm{poly}}(C')$ and $C' \otimes_C \mathcal{D}_{\mathrm{poly}}(C) \cong \mathcal{D}_{\mathrm{poly}}(C')$. Therefore we get quasi-coherent sheaves $\mathcal{T}_{\mathrm{poly},X}$ and $\mathcal{D}_{\mathrm{poly},X}$.

For any $i \in \{1, \ldots, p+1\}$ let $\varepsilon_i : \mathcal{D}^p_{\mathrm{poly},X} \to \mathcal{D}^{p-1}_{\mathrm{poly},X}$ be the map $\varepsilon(\phi)(f_1, \ldots, f_p) := \phi(f_1, \ldots, 1, \ldots, f_p)$,with 1 inserted at the $i$th position. This is an $\mathcal{O}_X$-linear homomorphism, and $\mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly},X} = \bigcap \operatorname{Ker}(\varepsilon_i)$. Thus $\mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly},X}$ is quasi-coherent.

The left $\mathcal{D}_X$-module structures on $\mathcal{D}_{\mathrm{poly},X}$ and $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}$ are by composition of operators. $\square$

Following Kontsevich we call $\mathcal{T}_{\mathrm{poly},X}$ the algebra of *polyvector fields* on $X$, and $\mathcal{D}_{\mathrm{poly},X}$ is called the algebra of *polydifferential operators*. The subalgebra $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}$ is called the algebra of *normalized polydifferential operators*.

Let us write $\mathcal{T}_{\mathrm{poly}}(X) = \Gamma(X, \mathcal{T}_{\mathrm{poly},X})$, the DG Lie algebra of global polyvector fields on $X$. We consider each $\mathcal{T}^p_{\mathrm{poly}}(X)$ as a discrete inv module, and $\mathcal{T}_{\mathrm{poly}}(X) = \bigoplus_p \mathcal{T}^p_{\mathrm{poly}}(X)$

gets the $\bigoplus$ dir-inv structure, so it is a discrete DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. Likewise we define $\mathcal{D}_{\mathrm{poly}}(X)$ and $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)$.

A series $\alpha = \sum_{k=1}^{\infty} \alpha_k \hbar^k \in \mathcal{T}^1_{\mathrm{poly}}(X)[[\hbar]]^+$ satisfying $[\alpha, \alpha] = 0$ is called a *formal Poisson structure* on $X$. Two formal Poisson structure $\alpha$ and $\alpha'$ are called gauge equivalent if there is some $\gamma = \sum_{k=1}^{\infty} \gamma_k \hbar^k \in \mathcal{T}^0_{\mathrm{poly}}(X)[[\hbar]]^+$ such that $\alpha' = \exp(\mathrm{ad}(\gamma))(\alpha)$. Thus the set $\mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^+\big)$ is the set of gauge equivalence classes of formal Poisson structures on $X$.

**Example 3.19.** Let $\alpha_1 \in \Gamma(X, \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X)$ be a Poisson structure on $X$ (Definition 1.9). Then $\alpha := \alpha_1 \hbar$ is a formal Poisson structure.

**Proposition 3.20.** *An element*

$$\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}^{\mathrm{nor},1}_{\mathrm{poly}}(X)[[\hbar]]^+$$

*is a solution of the Maurer–Cartan equation in* $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^+$ *iff the pairing*

$$(f, g) \mapsto f \star_\beta g := fg + \sum_{j=1}^{\infty} \beta_j(f, g) \hbar^j$$

*for local sections* $f, g \in \mathcal{O}_X$, *is a star product on* $\mathcal{O}_X[[\hbar]]$ (*see Definition* 1.1).

**Proof.** The assertion is actually local: it is enough to prove it for an affine open set $U = \operatorname{Spec} C \subset X$. Take $\beta \in \mathcal{D}^{\mathrm{nor},1}_{\mathrm{poly}}(C)[[\hbar]]^+$. We have to prove that $\star_\beta$ is an associative product on $C[[\hbar]]$ iff $\beta$ is a solution of the MC equation in $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)[[\hbar]]^+$. This assertion is made in [Ke, Corollary 4.5]. See also [Ko1, Section 4.6.2]. (For a nondifferential star product this is the original discovery of Gerstenhaber, see [Ge].) □

**Proposition 3.21.** *Under the identification*, *in Proposition* 3.20, *of solutions of the MC equation in* $\mathcal{D}^{\mathrm{nor},1}_{\mathrm{poly}}(X)[\hbar]]^+$ *with star products on* $\mathcal{O}_X[[\hbar]]$, *the notion of gauge equivalence in Definition* 3.4 *coincides with that in Proposition* 1.14.

**Proof.** Let $\beta$ and $\beta'$ be two solutions of the MC equation in $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^+$, and let $\star$ and $\star'$ be the corresponding star products on $\mathcal{O}_X[[\hbar]]$. Given $\gamma \in \mathcal{D}^{\mathrm{nor},0}_{\mathrm{poly}}(X)[[\hbar]]^+$, let $\exp(\gamma) := 1 + \gamma + \frac{1}{2}\gamma^2 + \cdots$ be the corresponding gauge equivalence of $\mathcal{O}_X[[\hbar]]$. As stated implicitly in [Ko1, Section 4.6.2; Ke, Chapter 2, Lemma 4.2 and Section 5.1], one has $\beta' = \exp(\mathrm{af})(\exp(\pm\gamma))(\beta)$ iff for all local sections $f, g \in \mathcal{O}_X$ one has

$$f \star' g = \exp(\gamma)^{-1}\big(\exp(\gamma)(f) \star \exp(\gamma)(g)\big). \tag{3.22}$$

(The reason for the sign ambiguity is that Refs. [Ko1,Ke,GM] are inconsistent with each other regarding signs, and we did not carry out this calculation ourselves.) □

An immediate consequence is:

**Corollary 3.23.** *The assignment* $\beta \mapsto \star_\beta$ *of Proposition* 3.20 *gives rise to a bijection from* $\mathrm{MC}\big(\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^+\big)$ *to set of gauge equivalence classes of globally trivialized deformation quantizations of* $\mathcal{O}_X$.

Here is a first approximation of Theorem 0.1.

**Corollary 3.24.** *Let* $X$ *be an* $n$*-dimensional affine scheme admitting an étale morphism* $X \to \mathbf{A}^n_{\mathbb{K}}$. *Then there is a bijection* $Q$ *as in Theorem* 0.1.

**Proof.** Write $X = \operatorname{Spec} C$ and $\mathbf{A}^n_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[\boldsymbol{t}]$. Because $\mathbb{K}[\boldsymbol{t}] \to C$ is an étale ring homomorphism, according to [Ye2, Proposition 4.6] we have $\mathcal{T}_{\mathrm{poly}}(C) = C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$ and $\mathcal{D}_{\mathrm{poly}}(C) = C \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[\boldsymbol{t}])$. By condition (ii) in Theorem 3.13 the universal operators $\mathcal{U}_j$ are polydifferential operators over $\mathbb{K}[\boldsymbol{t}]$, and hence according to [Ye2, Proposition 2.6] they extend to $C$-multilinear operators, giving an $\mathrm{L}_\infty$ morphism $\mathcal{U} : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$; and by [Ye2, Corollary 4.12] this is an $\mathrm{L}_\infty$ quasi-isomorphism. The inclusion $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) \to \mathcal{D}_{\mathrm{poly}}(C)$ is a DG Lie algebra quasi-isomorphism. Now use Corollaries 3.10 and 3.23. □

**Remark 3.25.** The method of $\mathrm{L}_\infty$ morphisms is suitable only for characteristic 0. For an approach in positive characteristic see [BK2].

## 4. Formal geometry—coordinate bundles, etc.

In this section we translate the notions of formal geometry (in the sense of Gelfand–Kazhdan [GK]; cf. [Ko1, Section 7]) to the language of algebraic geometry (schemes and sheaves). As before $\mathbb{K}$ is a field of characteristic 0, and $X$ is a smooth separated irreducible scheme over $\mathbb{K}$ of dimension $n$.

For a closed point $x \in X$ the residue field $\boldsymbol{k}(x)$ lifts uniquely into the complete local ring $\widehat{\mathcal{O}}_{X,x}$, and any choice of system of coordinates $\boldsymbol{t} = (t_1, \ldots, t_n)$ gives rise to an isomorphism of $\mathbb{K}$-algebras

$$\widehat{\mathcal{O}}_{X,x} \cong \boldsymbol{k}(x)[[\boldsymbol{t}]] = \boldsymbol{k}(x)[[t_1, \ldots, t_n]].$$

Of course the condition that an $n$-tuple of elements $\boldsymbol{t}$ in the maximal ideal $\mathfrak{m}_x$ is a system of coordinates is that their residue classes form a basis of the $\boldsymbol{k}(x)$-module $\mathfrak{m}_x/\mathfrak{m}_x^2$, the Zariski cotangent space.

Suppose $U$ is an open neighborhood of $x$ and $f \in \Gamma(U, \mathcal{O}_X)$. The Taylor expansion of $f$ at $x$ w.r.t. $\boldsymbol{t}$ is

$$f = \sum_{\boldsymbol{i} \in \mathbb{N}^n} a_{\boldsymbol{i}} \boldsymbol{t}^{\boldsymbol{i}} \in \widehat{\mathcal{O}}_{X,x},$$

where $a_{\boldsymbol{i}} \in \boldsymbol{k}(x)$ and $\boldsymbol{t}^{\boldsymbol{i}} := t_1^{i_1} \ldots t_n^{i_n}$. The coefficients are given by the usual formula

$$a_{\boldsymbol{i}} = \frac{1}{\boldsymbol{i}!} \left( \left( \frac{\partial}{\partial t_1} \right)^{i_1} \cdots \left( \frac{\partial}{\partial t_n} \right)^{i_n} f \right)(x),$$

where for any $g \in \widehat{\mathcal{O}}_{X,x}$ we write $g(x) \in \boldsymbol{k}(x)$ for its residue class.

The *jet bundle* of $X$ is an infinite dimensional scheme $\operatorname{Jet} X$, that comes with a projection $\pi_{\mathrm{jet}} : \operatorname{Jet} X \to X$. Given a closed point $x \in X$ the $\boldsymbol{k}(x)$-rational points of the fiber $\pi_{\mathrm{jet}}^{-1}(x)$ correspond to "jets of functions at $x$", namely to elements of the complete local ring $\widehat{\mathcal{O}}_{X,x}$. Here is a way to visualize such a fiber: choose a coordinate system $\boldsymbol{t}$. Then a jet is just the data $\{a_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{N}^n}$ of its Taylor coefficients. So set theoretically the fiber $\pi_{\mathrm{jet}}^{-1}(x)$ is just the set $\boldsymbol{k}(x)^{\mathbb{N}^n}$.

The naive description above does not make $\operatorname{Jet} X$ into a scheme. So let us try another approach. Consider the diagonal embedding $\Delta : X \to X^2 = X \times X$. Let $\mathcal{I}_{X,\mathrm{alg}}$ be the ideal sheaf $\operatorname{Ker}(\Delta^* : \mathcal{O}_{X^2} \to \mathcal{O}_X)$. The *sheaf of principal parts* of $X$ is

$$\mathcal{P}_X = \mathcal{P}_{X/\mathbb{K}} := \lim_{\leftarrow d} \mathcal{O}_{X^2} / \mathcal{I}_{X,\mathrm{alg}}^d$$

(cf. [EGA IV]). It is a sheaf of commutative rings, equipped with two ring homomorphisms $\mathrm{p}_1^*, \mathrm{p}_2^* : \mathcal{O}_X \to \mathcal{P}_X$, namely $\mathrm{p}_1^*(f) := f \otimes 1$ and $\mathrm{p}_2^*(f) := 1 \otimes f$. We consider $\mathcal{P}_X$ as a left $\mathcal{O}_X$-module via $\mathrm{p}_1^*$ and as a right $\mathcal{O}_X$-module via $\mathrm{p}_2^*$.

The sheaf of rings $\mathcal{P}_X$ can be thought of the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of the formal scheme $\mathfrak{X}$ which is the formal completion of $X^2$ along $\Delta(X)$. We denote by $\mathcal{I}_X$ the ideal $\operatorname{Ker}(\mathcal{P}_X \to \mathcal{O}_X)$; it is just the completion of the ideal $\mathcal{I}_{X,\mathrm{alg}}$. By default we shall consider $\mathcal{P}_X$ as an $\mathcal{O}_X$-algebra via $\mathrm{p}_1^*$.

**Proposition 4.1** (*Yekutieli [Ye1, Lemma 2.6]*). *Let $U \subset X$ be an open set admitting an étale morphism $U \to \mathbf{A}^n_{\mathbb{K}} = \operatorname{Spec} \mathbb{K}[s_1, \ldots, s_n]$. For $i = 1, \ldots, n$ define*

$$\tilde{s}_i := 1 \otimes s_i - s_i \otimes 1 \in \Gamma(U, \mathcal{I}).$$

*Then*

$$\mathcal{P}_X|_U \cong \mathcal{O}_U[[\tilde{s}_1, \ldots, \tilde{s}_n]]$$

*as sheaves of $\mathcal{O}_U$-algebras, either via $\mathrm{p}_1^*$ or via $\mathrm{p}_2^*$.*

**Definition 4.2.** Let $U \subset X$ be an open set.

(1) A *system of étale coordinates* on $U$ is a sequence $\boldsymbol{s} = (s_1, \ldots, s_n)$ of elements in $\Gamma(U, \mathcal{O}_X)$ s.t. the morphism $U \to \mathbf{A}^n_{\mathbb{K}}$ it determines is étale.
(2) A *system of formal coordinates* on $U$ is a sequence $\boldsymbol{t} = (t_1, \ldots, t_n)$ of elements in $\Gamma(U, \mathcal{I}_X)$ s.t. the homomorphism of sheaves of rings $\mathcal{O}_U[[\boldsymbol{t}]] \to \mathcal{P}_X|_U$ extending $\mathrm{p}_1^*$ is an isomorphism.

**Proposition 4.3.** *Given a closed point $x \in X$ one has a canonical isomorphism of $\mathcal{O}_X$-algebras* (*via* $\mathrm{p}_2^*$):

$$\boldsymbol{k}(x) \otimes_{\mathcal{O}_X} \mathcal{P}_X \cong \widehat{\mathcal{O}}_{X,x}.$$

*If $\boldsymbol{t} = (t_1, \ldots, t_n)$ is a system of formal coordinates on some neighborhood $U$ of $x$, and we let $t_i(x) := 1 \otimes t_i \in \widehat{\mathcal{O}}_{X,x}$ under the above isomorphism, then the sequence $\boldsymbol{t}(x) := (t_1(x), \ldots, t_n(x))$ is a system of coordinates in $\widehat{\mathcal{O}}_{X,x}$.*

The easy proof is left out.

**Example 4.4.** Assume $\boldsymbol{s}$ is a system of étale coordinates on $U$, and let $t_i := \tilde{s}_i$. By Proposition 4.1 the sequence $\boldsymbol{t} := (t_1, \ldots, t_n)$ is a system of formal coordinates on $U$. Given a closed point $x \in U$ we have $t_i(x) = s_i - s_i(x)$, where $s_i(x) \in \boldsymbol{k}(x) \subset \widehat{\mathcal{O}}_{X,x}$. The sequence $\boldsymbol{t}(x) = \big(t_1(x), \ldots, t_n(x)\big)$ is a system of coordinates in $\widehat{\mathcal{O}}_{X,x}$.

**Corollary 4.5.** *Let $U \subset X$ be an open set admitting a formal system of coordinates $\boldsymbol{t} \in \Gamma(U, \mathcal{I}_X)^{\times n}$. For $f \in \Gamma(U, \mathcal{O}_X)$ let us write*

$$\mathrm{p}_2^*(f) = \sum_{\boldsymbol{i}} \mathrm{p}_1^*(a_{\boldsymbol{i}})\, \boldsymbol{t}^{\boldsymbol{i}} \in \Gamma(U, \mathcal{P}_X)$$

*with $a_{\boldsymbol{i}} \in \Gamma(U, \mathcal{O}_X)$. Then under the isomorphism of Proposition* 4.3 *we recover the Taylor expansion at any closed point $x \in U$*:

$$f = \sum_{\boldsymbol{i}} a_{\boldsymbol{i}}(x)\, \boldsymbol{t}(x)^{\boldsymbol{i}} \in \widehat{\mathcal{O}}_{X,x}.$$

Again the easy proof is omitted.

The conclusion from Proposition 4.3 is that the sheaf of sections of the bundle $\operatorname{Jet} X$ should be $\mathcal{P}_X$. By the standard schematic formalism we deduce the defining formula

$$\operatorname{Jet} X := \operatorname{Spec}_X \mathrm{S}_{\mathcal{O}_X} \mathcal{D}_X,$$

where $\mathcal{D}_X = \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\mathcal{P}_X, \mathcal{O}_X)$, the sheaf of differential operators, is considered as a locally free left $\mathcal{O}_X$-module; $\mathrm{S}_{\mathcal{O}_X} \mathcal{D}_X$ is the symmetric algebra of the $\mathcal{O}_X$-module $\mathcal{D}_X$; and $\operatorname{Spec}_X$ refers to the relative spectrum over $X$ of a quasi-coherent $\mathcal{O}_X$-algebra.

If $U$ is a sufficiently small affine open set in $X$ admitting an étale coordinate system $\boldsymbol{s}$, then $\operatorname{Jet} U$ can be made more explicit. We know that

$$\Gamma(U, \mathcal{D}_X) = \bigoplus_{\boldsymbol{i} \in \mathbb{N}^n} \Gamma(U, \mathcal{O}_X) \left( \frac{\partial}{\partial s_1} \right)^{i_1} \cdots \left( \frac{\partial}{\partial s_n} \right)^{i_n},$$

so letting $\xi_{\boldsymbol{i}}$ be a commutative indeterminate we have

$$\pi_{\mathrm{jet}}^{-1}(U) \cong \operatorname{Jet} U \cong \operatorname{Spec} \Gamma(U, \mathcal{O}_X)[\{\xi_{\boldsymbol{i}}\}_{\boldsymbol{i} \in \mathbb{N}^n}].$$

The next geometric object we need is the *bundle of formal coordinate systems* of $X$, which we denote by $\operatorname{Coor} X$. (In [Ko1] the notation is $X^{\mathrm{coor}}$. However we feel that stylistically $\operatorname{Coor} X$ is better, since it resembles the usual bundle notation $\mathrm{T}X$ and $\mathrm{T}^*X$.) The scheme $\operatorname{Coor} X$ comes with a projection $\pi_{\mathrm{coor}} : \operatorname{Coor} X \to X$, and the fiber over a closed point $x \in X$ corresponds to the set of $\mathbb{K}$-algebra isomorphisms $\boldsymbol{k}(x)[[\boldsymbol{t}]] \xrightarrow{\simeq} \widehat{\mathcal{O}}_{X,x}$. We are going to make this more precise below.

Consider a morphism of schemes $f : Y \to X$. Unless $f$ is quasi-finite the sheaf $f^* \mathcal{P}_X$ is not particularly interesting, but its completion is. To make this work nicely we are going to use inv structures (see Section 2). Since $(X, \mathcal{P}_X)$ is a noetherian formal scheme, any coherent $\mathcal{P}_X$-module $\mathcal{M}$ has the $\mathcal{I}_X$-adic inv structure (see Example 2.4). Using the ring homomorphism $\mathrm{p}_1^* : \mathcal{O}_X \to \mathcal{P}_X$ the module $\mathcal{M}$ becomes an inv $\mathcal{O}_X$-module, and so the complete inverse image $\widehat{f^*} \mathcal{M}$ is defined. Taking $\mathcal{M} := \mathcal{P}_X$ we get an $\mathcal{O}_Y$-algebra $\widehat{f^*} \mathcal{P}_X$. By Proposition 4.1 we see that $\widehat{f^*} \mathcal{I}_X$ is a sheaf of ideals in $\widehat{f^*} \mathcal{P}_X$, and $\widehat{f^*} \mathcal{P}_X$ is $\widehat{f^*} \mathcal{I}_X$ -adically complete in the usual sense, namely $\widehat{f^*} \mathcal{P}_X \cong \lim_{\leftarrow m} \widehat{f^*} \mathcal{P}_X / (\widehat{f^*} \mathcal{I}_X)^m$.

Now we can state the geometric property that should characterize $\operatorname{Coor} X$. There should be a sequence $\boldsymbol{t} = (t_1, \ldots, t_n)$ of elements in $\Gamma(\operatorname{Coor} X, \widehat{\pi^*_{\mathrm{coor}}} \mathcal{I}_X)$ such that for any open set $U \subset X$, the assignment $\sigma \mapsto \sigma^*(\boldsymbol{t}) := (\sigma^*(t_1), \ldots, \sigma^*(t_n))$ shall be a bijection from the set of sections $\sigma : U \to \operatorname{Coor} X$ to the set of formal coordinate systems on $U$. Thus the sheaf of sections of $\operatorname{Coor} X$, let us call it $\mathcal{Q}$, has to be a subsheaf of $\mathcal{I}_X^{\times n} = \mathcal{I}_X \times \cdots \times \mathcal{I}_X$. The condition for an $n$-tuple $\boldsymbol{t}$ to be in $\mathcal{Q}$ is that under the composed map

$$\mathcal{I}_X^{\times n} \to \left( \mathcal{I}_X / \mathcal{I}_X^2 \right)^{\times n} = \left( \Omega_X^1 \right)^{\times n} \xrightarrow{\wedge} \Omega_X^n \tag{4.6}$$

one has $t_1 \wedge \cdots \wedge t_n \neq 0$. Note that $\mathcal{Q} = \lim_{\leftarrow m} \mathcal{Q}^m$, where $\mathcal{Q}^1$ is the sheaf of frames of $\Omega_X^1$.

We conclude that $\operatorname{Coor} X$ is a subscheme of $(\operatorname{Jet} X)^{\times n}$. Specifically, $\operatorname{Coor} X$ is an open subscheme of $\operatorname{Spec}_X \mathrm{S}_{\mathcal{O}_X}\big( (\mathcal{D}_X / \mathcal{O}_X)^{\oplus n} \big)$, where $\mathcal{D}_X / \mathcal{O}_X$ is viewed as a locally free left $\mathcal{O}_X$-module. Over any affine open set $U \subset X$ admitting an étale coordinate system, say $\boldsymbol{s}$, one has

$$\pi_{\mathrm{coor}}^{-1}(U) \cong \operatorname{Coor} U \cong \operatorname{Spec} \Gamma(U, \mathcal{O}_X)[\{\xi_{\boldsymbol{i},j}\}, d^{-1}]. \tag{4.7}$$

In this formula $\boldsymbol{i} = (i_1, \ldots, i_n)$ runs over $\mathbb{N}^n - \{(0, \ldots, 0)\}$ and $j$ runs over $\{1, \ldots, n\}$. The indeterminate $\xi_{\boldsymbol{i},j}$ corresponds to the DO $\frac{1}{\boldsymbol{i}!}(\frac{\partial}{\partial s_1})^{i_1} \cdots (\frac{\partial}{\partial s_n})^{i_n}$ in the $j$th copy of $\mathcal{D}_X$. The symbol $\boldsymbol{e}_i$ denotes the row whose only nonzero entry is 1 in the $i$th place, so $[\xi_{\boldsymbol{e}_i,j}]$ is the matrix whose $(i, j)$ entry is the indeterminate $\xi_{\boldsymbol{e}_i,j}$ corresponding to the DO $\frac{\partial}{\partial s_i}$ in the $j$th copy of $\mathcal{D}_X$. Finally $d := \det([\xi_{\boldsymbol{e}_i,j}])$.

The next results justify the heuristic considerations above.

**Theorem 4.8.** *Consider the functor $F : (\mathsf{Sch}/X)^{\mathrm{op}} \to \mathsf{Sets}$ defined as follows. For any $X$-scheme $Y$, with structural morphism $g : Y \to X$, we let $FY$ be the set of $\mathcal{O}_Y$-algebra isomorphisms $\phi : \mathcal{O}_Y[[\boldsymbol{t}]] \xrightarrow{\simeq} \widehat{g^*}\mathcal{P}_X$ such that $\phi(\mathcal{O}_Y[[\boldsymbol{t}]] \cdot \boldsymbol{t}) = \widehat{g^*}\mathcal{I}_X$. Then* Coor $X$ *is a fine moduli space for $F$, namely $F \cong \mathrm{Hom}_{\mathsf{Sch}/X}(-, \mathrm{Coor}\, X)$.*

**Proof.** Suppose we are given $g : Y \to X$ and $\phi : \mathcal{O}_Y[[\boldsymbol{t}]] \xrightarrow{\simeq} \widehat{g^*}\mathcal{P}_X$. Define $a_i := \phi(t_i) \in \Gamma(Y, \widehat{g^*}\mathcal{I}_X)$. These elements satisfy $a_1 \wedge \cdots \wedge a_n \neq 0$ like in equation (4.6). Each $a_i$ gives rise to an $\mathcal{O}_Y$-linear sheaf homomorphism $\mathcal{O}_Y \to \widehat{g^*}\mathcal{I}_X$. Since $\mathcal{I}_X/\mathcal{I}_X^m$ is a coherent locally free $\mathcal{O}_X$-module for every $m \geqslant 1$ we see that

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_Y}(\widehat{g^*}\mathcal{I}_X, \mathcal{O}_Y) \cong \mathcal{O}_Y \otimes_{g^{-1}\mathcal{O}_X} g^{-1}\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\mathcal{I}_X, \mathcal{O}_X) = g^*(\mathcal{D}_X/\mathcal{O}_X).$$

So after dualization, i.e. applying the functor $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_Y}(-, \mathcal{O}_Y)$, each $a_i$ gives a homomorphism of $\mathcal{O}_Y$-modules $g^*(\mathcal{D}_X/\mathcal{O}_X) \to \mathcal{O}_Y$. By adjunction we get $\mathcal{O}_X$-linear homomorphisms $\mathcal{D}_X/\mathcal{O}_X \to g_*\mathcal{O}_Y$, and therefore an $\mathcal{O}_X$-algebra homomorphism $\mathrm{S}_{\mathcal{O}_X}\big((\mathcal{D}_X/\mathcal{O}_X)^{\oplus n}\big) \to g_*\mathcal{O}_Y$. Passing to schemes we obtain a morphism of $X$-schemes

$$\tilde{\phi} : Y \to \mathrm{Spec}_X\, \mathrm{S}_{\mathcal{O}_X}\big((\mathcal{D}_X/\mathcal{O}_X)^{\oplus n}\big).$$

Because $a_1 \wedge \cdots \wedge a_n \neq 0$ this is actually a morphism $\tilde{\phi} : Y \to \mathrm{Coor}\, X$. The process we have described is reversible, and hence $FY \cong \mathrm{Hom}_{\mathsf{Sch}/X}(Y, \mathrm{Coor}\, X)$. $\square$

**Corollary 4.9.** *There is a canonical isomorphism of $\mathcal{O}_{\mathrm{Coor}\, X}$-algebras*

$$\mathcal{O}_{\mathrm{Coor}\, X}[[\boldsymbol{t}]] \cong \widehat{\pi^*_{\mathrm{coor}}}\,\mathcal{P}_X.$$

*This isomorphism has the following universal property: for any open set $U \subset X$ the assignment $\sigma \mapsto \sigma^*(\boldsymbol{t})$ is a bijection of sets*

$$\mathrm{Hom}_{\mathsf{Sch}/X}(U, \mathrm{Coor}\, X) \xrightarrow{\simeq} \{\text{formal coordinate systems on } U\}.$$

**Proof.** Applying the theorem to $Y := \mathrm{Coor}\, X$, $g := \pi_{\mathrm{coor}}$ and the identity morphism $\tilde{\phi}_0 : \mathrm{Coor}\, X \to \mathrm{Coor}\, X$, we obtain a canonical isomorphism $\phi_0 : \mathcal{O}_{\mathrm{Coor}\, X}[[\boldsymbol{t}]] \cong \widehat{\pi^*_{\mathrm{coor}}}\,\mathcal{P}_X$ with the desired universal property. $\square$

On $\operatorname{Coor} X$ we have a universal Taylor expansion:

**Corollary 4.10.** *Suppose $U \subset X$ is open and $f \in \Gamma(U, \mathcal{O}_X)$. Then there are functions $a_{\boldsymbol{i}} \in \Gamma(\pi_{\mathrm{coor}}^{-1}(U), \mathcal{O}_{\operatorname{Coor} X})$ s.t.*

$$\widehat{\pi_{\mathrm{coor}}^{*}}(\mathrm{p}_2^*(f)) = \sum_{\boldsymbol{i}\in\mathbb{N}^n} a_{\boldsymbol{i}}\, \boldsymbol{t}^{\boldsymbol{i}} \in \Gamma(\pi_{\mathrm{coor}}^{-1}(U), \widehat{\pi_{\mathrm{coor}}^{*}}\, \mathcal{P}_X),$$

*where $\boldsymbol{t}$ is the universal coordinate system in $\Gamma(\operatorname{Coor} X, \widehat{\pi_{\mathrm{coor}}^{*}}\, \mathcal{P}_X)$. Given a section $\sigma : U \to \operatorname{Coor} X$ we obtain a Taylor expansion*

$$\mathrm{p}_2^*(f) = \sum_{\boldsymbol{i}\in\mathbb{N}^n} \sigma^*(a_{\boldsymbol{i}})\, \sigma^*(\boldsymbol{t})^{\boldsymbol{i}} \in \Gamma(U, \mathcal{P}_X)$$

*as in Corollary* 4.5.

The proof is left to the reader.

Suppose $\boldsymbol{s} = (s_1, \ldots, s_n)$ is an étale coordinate system on an open set $U \subset X$. As before let $\tilde{s}_i := 1 \otimes s_i - s_i \otimes 1 \in \Gamma(U, \mathcal{P}_X)$, and define $\boldsymbol{s} := (\tilde{s}_1, \ldots, \tilde{s}_n)$, which is a formal coordinate system on $U$. Then on $\operatorname{Coor} U = \pi_{\mathrm{coor}}^{-1}(U)$ we have isomorphisms of $\mathcal{O}_{\operatorname{Coor} U}$-algebras

$$\mathcal{O}_{\operatorname{Coor} U}[[\boldsymbol{t}]] \cong (\widehat{\pi_{\mathrm{coor}}^{*}} \mathcal{P}_X)|_{\operatorname{Coor} U} \cong \mathcal{O}_{\operatorname{Coor} U}[[\boldsymbol{s}]].$$

Using the coordinate functions $\xi_{\boldsymbol{i},j} \in \Gamma(\operatorname{Coor} U, \mathcal{O}_{\operatorname{Coor} U})$ from formula (4.7) we then have

$$t_j = \sum_{\boldsymbol{i}} \xi_{\boldsymbol{i},j} \boldsymbol{s}^{\boldsymbol{i}}, \tag{4.11}$$

where the sum is on $\boldsymbol{i} \in \mathbb{N}^n - \{(0, \ldots, 0)\}$.

For $i \geqslant 1$ let $\operatorname{Coor}^i X$ be the bundle over $X$ parameterizing coordinate systems up to order $i$ (i.e. modulo order $\geqslant i + 1$). There are projections $\operatorname{Coor} X \to \operatorname{Coor}^i X \to \operatorname{Coor}^{i-1} X \to X$. The next theorem describes the geometry of these bundles.

Let $G(\mathbb{K})$ be the group of $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[[\boldsymbol{t}]]$. Then $G(\mathbb{K})$ is the group of $\mathbb{K}$-rational points of a pro-algebraic group $G = \mathrm{GL}_{n,\mathbb{K}} \ltimes N$, where $N$ is a pro-unipotent group. The action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathbb{K}[[\boldsymbol{t}]]$ is by linear change of coordinates; and $N(\mathbb{K})$ is the subgroup of $G(\mathbb{K})$ consisting of automorphisms that act trivially modulo $(\boldsymbol{t})^2$.

According to Corollary 4.9 there is a canonical embedding of $\mathbb{K}$-algebras

$$\mathbb{K}[[\boldsymbol{t}]] \hookrightarrow \Gamma(\operatorname{Coor} X, \widehat{\pi_{\mathrm{coor}}^{*}} \mathcal{P}_X). \tag{4.12}$$

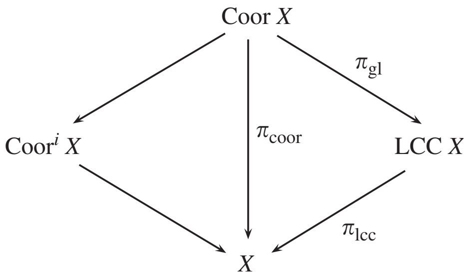


Fig. 1.

**Theorem 4.13.** (1) $\mathrm{Coor}\, X \cong \lim_{\leftarrow i} \mathrm{Coor}^i X$ *as schemes over* $X$.

(2) $\mathrm{Coor}\, X$ *is a* $G$*-torsor over* $X$. *The action of* $G$ *on* $\mathrm{Coor}\, X$ *is characterized by the fact that embedding* (4.12) *is* $G(\mathbb{K})$*-equivariant.*

(3) $\mathrm{Coor}^1 X$ *is a* $\mathrm{GL}_{n,\mathbb{K}}$*-torsor over* $X$, *and* $\mathrm{Coor}\, X$ *is a* $\mathrm{GL}_{n,\mathbb{K}}$*-equivariant* $N$*-torsor over* $\mathrm{Coor}^1 X$.

(4) *The geometric quotient* (*cf.* [GIT])

$$\mathrm{LCC}\, X := \mathrm{Coor}\, X / \mathrm{GL}_{n,\mathbb{K}}$$

*exists, with projection* $\pi_{\mathrm{gl}} : \mathrm{Coor}\, X \to \mathrm{LCC}\, X$, *and* $\mathrm{Coor}\, X$ *is a* $\mathrm{GL}_{n,\mathbb{K}}$*-torsor over* $\mathrm{LCC}\, X$.

(5) *Let* $U \subset X$ *be an affine open set admitting an étale coordinate system. Then all the torsors in parts* (2)–(4) *are trivial over* $U$ (*i.e. they admit sections*).

"LCC" stands for "linear coordinate classes". In [Ko1] the notation for $\mathrm{LCC}\, X$ is $X^{\mathrm{aff}}$. Note that the bundle $\mathrm{LCC}\, X$ has no group action; but locally, for $U$ as in part (5), there is a non-canonical isomorphism of schemes $\mathrm{LCC}\, U \cong N \times U$. $\mathrm{Coor}^1 X$ is the frame bundle of $\Omega^1_X$. The various bundles and projections are depicted in Fig. 1.

**Proof of Theorem 4.13.** (1) This is an immediate consequence of the moduli property of $\mathrm{Coor}\, X$ (see Theorem 4.8), and an analogous property of $\mathrm{Coor}^i X$.

(2) Given $g \in G(\mathbb{K})$ let us denote by $g(\boldsymbol{t})$ the sequence $(g(t_1), \ldots, g(t_n))$ in $\mathbb{K}[[\boldsymbol{t}]]$. By Theorem 4.8 there exists a unique $X$-morphism $\tilde{g} : \mathrm{Coor}\, X \to \mathrm{Coor}\, X$ such that the algebra homomorphism $\tilde{g}^* : \Gamma(\mathrm{Coor}, \widehat{\pi^*_{\mathrm{coor}} \mathcal{P}_X}) \to \Gamma(\mathrm{Coor}, \widehat{\pi^*_{\mathrm{coor}} \mathcal{P}_X})$ sends $\boldsymbol{t}$ to $g(\boldsymbol{t})$. We have to prove that $\tilde{g}$ is an automorphism, and that $g \mapsto \tilde{g}$ is a group homomorphism from $G(\mathbb{K})$ to $\mathrm{Aut}_{\mathsf{Sch}/X}(\mathrm{Coor}\, X)$.

Now via embedding (4.12), the homomorphism $\tilde{g}^*$ restricts to the automorphism $g$ on $\mathbb{K}[[\boldsymbol{t}]]$. If $g$ is the identity automorphism of $\mathbb{K}[[\boldsymbol{t}]]$, then by uniqueness $\tilde{g}$ has to be the identity automorphism of $\mathrm{Coor}\, X$. Next take two elements $g_1, g_2 \in G(\mathbb{K})$. Then

$$\begin{aligned} \widetilde{g_2 \circ g_1}^*(\boldsymbol{t}) &= (g_2 \circ g_1)(\boldsymbol{t}) = g_2(g_1(\boldsymbol{t})) = \tilde{g}_2^*(g_1(\boldsymbol{t})) = g_1(\tilde{g}_2^*(\boldsymbol{t})) \\ &= g_1(g_2(\boldsymbol{t})) = (\tilde{g}_1^* \circ \tilde{g}_2^*)(\boldsymbol{t}) = (\tilde{g}_2 \circ \tilde{g}_1)^*(\boldsymbol{t}). \end{aligned}$$

Thus indeed we have a group action.

Due to the moduli property this action becomes geometric, i.e. it is a morphism of schemes $G \times \operatorname{Coor} X \to \operatorname{Coor} X$. The explicit local description (4.7) shows that $\operatorname{Coor} X$ is in fact a $G$-torsor over $X$.

(3), (4) These are consequence of (2).

(5) Clear from formula (4.7). $\square$

## 5. Formal differential calculus

As before $\mathbb{K}$ is a field of characteristic 0, and $X$ is a smooth separated irreducible $n$-dimensional $\mathbb{K}$-scheme.

Recall the algebra homomorphism $\mathrm{p}_1^* : \mathcal{O}_X \to \mathcal{P}_X$. We define $\mathcal{T}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*)$ to be the sheaf of derivations of $\mathcal{P}_X$ relative to $\mathcal{O}_X$. Thus for any affine open set $U = \operatorname{Spec} C \subset X$, writing $\widehat{A} := \Gamma(U, \mathcal{P}_X)$, we have

$$\Gamma\big(U, \mathcal{T}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*)\big) = \mathcal{T}_{\widehat{A}/C} = \operatorname{Der}_C(\widehat{A}).$$

Similarly, we define $\mathcal{T}^i_{\mathrm{poly}}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*)$ and $\mathcal{D}^i_{\mathrm{poly}}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*)$.

**Lemma 5.1.** *Let $\mathcal{G}$ stand either for $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$, so that $\mathcal{G}_X = \mathcal{T}_{\mathrm{poly},X}$ etc.*

(1) *The graded left $\mathcal{O}_X$-module $\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$ is a DG Lie algebra in* $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}_X$. *The homomorphism $\mathcal{G}_X \to \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$ given by $\gamma \mapsto 1 \otimes \gamma$ is a DG Lie algebra homomorphism.*

(2) *There is a canonical isomorphism*

$$\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X \cong \mathcal{G}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*)$$

*of sheaves of DG Lie algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}_X$.

(3) *Suppose $f : Y \to X$ is a morphism of schemes. Then*

$$f^{\widehat{*}}\,\mathcal{G}(\mathcal{P}_X/\mathcal{O}_X; \mathrm{p}_1^*) \cong \mathcal{G}(f^{\widehat{*}}\mathcal{P}_X/\mathcal{O}_Y)$$

*as DG Lie algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}_Y$.

**Proof.** (1) Let $U \subset X$ be an affine open set, and define $C := \Gamma(U, \mathcal{O}_X)$ and $A := C \otimes C$. Let $\mathfrak{a} := \operatorname{Ker}(A \to C)$, and let $\widehat{A}$ be the $\mathfrak{a}$-adic completion of $A$. The left $C$-module $C \otimes \mathcal{G}(C)$ is a DG Lie algebra over $C$. When we consider $C \otimes \mathcal{G}(C)$ as an $A$-module, the bracket

$$[-,-] : (C \otimes \mathcal{G}(C)) \times (C \otimes \mathcal{G}(C)) \to C \otimes \mathcal{G}(C)$$

and the differential

$$\mathrm{d} : C \otimes \mathcal{G}(C) \to C \otimes \mathcal{G}(C)$$

are polydifferential operators, and hence they are continuous for the $\mathfrak{a}$-adic dir-inv structure (see [Ye2, Example 1.8]). So according to [Ye2, Proposition 2.3], $C \widehat{\otimes \mathcal{G}(C)}$ is a DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\, C$. But

$$\Gamma(U, \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \cong \widehat{A} \otimes_C \mathcal{G}(C) \cong C \widehat{\otimes \mathcal{G}(C)}.$$

(2), (3) By definition $\mathcal{G}(C) = \mathcal{G}(C/\mathbb{K})$. By base change there is an isomorphism $B \otimes \mathcal{G}(C/\mathbb{K}) \cong \mathcal{G}\big((C \otimes B)/B\big)$ for any $\mathbb{K}$-algebra $B$. □

**Definition 5.2.** Consider the de Rham differential $\mathrm{d} : \mathcal{O}_{X^2} \to \Omega^1_{X^2/X} = \mathrm{p}_1^* \Omega^1_X$ relative to the projection $\mathrm{p}_2 : X^2 \to X$. Passing to the completion along the diagonal we obtain the *Grothendieck connection*

$$\nabla_{\mathcal{P}} : \mathcal{P}_X \to \mathcal{P}_X \otimes_{\mathcal{O}_X} \Omega^1_X.$$

Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then the connection $\nabla_{\mathcal{P}}$ extends uniquely to a degree 1 endomorphism of the graded sheaf

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus\nolimits_{p \geqslant 0} \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}.$$

The formula is

$$\nabla_{\mathcal{P}}(\alpha \otimes a \otimes m) := \mathrm{d}(\alpha) \otimes a \otimes m + (-1)^p \alpha \wedge \nabla_{\mathcal{P}}(a) \otimes m$$

for local sections $\alpha \in \Omega^p_X$, $a \in \mathcal{P}_X$ and $m \in \mathcal{M}$. The connection is integrable, i.e. $\nabla_{\mathcal{P}} \circ \nabla_{\mathcal{P}} = 0$, and it makes $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ into a DG $\Omega_X$-module.

**Theorem 5.3** (*Yekutieli [Ye3, Theorem 4.4]*). *Let $X$ be a smooth $\mathbb{K}$-scheme and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then the map*

$$\mathcal{M} \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}, \ m \mapsto 1 \otimes 1 \otimes m$$

*is a quasi-isomorphism.*

Given any $\mathbb{K}$-scheme $Y$ let $\mathbb{K}_Y$ be the constant sheaf $\mathbb{K}$ on $Y$. We consider $\Omega^p_Y = \Omega^p_{Y/\mathbb{K}}$ as a discrete inv $\mathcal{O}_Y$-module, and $\Omega_Y = \bigoplus_{p \geqslant 0} \Omega^p_Y$ gets the $\bigoplus$ dir-inv structure. Thus $\Omega_Y$ is a discrete DG algebra in $\mathsf{Dir\,Inv\,Mod}\, \mathbb{K}_Y$. Note that if $Y$ is infinite dimensional then $\Omega_Y$ will be unbounded.

Suppose $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Then for any $p$ the sheaf $\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a dir-coherent $\mathcal{P}_X$-module (see Example 2.4), so it has the $\mathcal{I}_X$-adic dir-inv module structure. The connection

$$\nabla_{\mathcal{P}} : \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

is a differential operator of $\mathcal{P}_X$-modules (of order $\leqslant 1$), and therefore it is continuous for the dir-inv structures (see [Ye2, Proposition 2.3]). So in fact $\boldsymbol{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a DG $\boldsymbol{\Omega}_X$-module in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$.

Suppose $f : Y \to X$ is some morphism of schemes. The complete pullback $\widehat{f^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a dir-inv $\mathcal{O}_Y$-module. Moreover $\boldsymbol{\Omega}_Y \widehat{\otimes}_{\mathcal{O}_Y} \widehat{f^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a DG $\boldsymbol{\Omega}_Y$-module in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Y$. Its differential is also denoted by $\nabla_{\mathcal{P}}$. In particular, when $\mathcal{M} = \mathcal{O}_X$, we obtain a super-commutative associative unital DG algebra $\boldsymbol{\Omega}_Y \widehat{\otimes}_{\mathcal{O}_Y} \widehat{f^*}\mathcal{P}_X$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Y$. Its degree 0 component is $\widehat{f^*}\mathcal{P}_X$, which is a complete commutative algebra in $\mathsf{Inv\,Mod}\,\mathcal{O}_Y$. For details and proofs see [Ye2, Section 1].

**Proposition 5.4.** *Let $\mathcal{G}$ denote either $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$, so that $\mathcal{G}_X = \mathcal{T}_{\mathrm{poly},X}$, etc. Also let $\mathrm{d}_{\mathcal{G}}$ and $[-,-]_{\mathcal{G}}$ denote the differential and the bracket of $\mathcal{G}$.*

(1) *The graded sheaf $\boldsymbol{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$ is a DG $\boldsymbol{\Omega}_X$-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$. The differential is $\nabla_{\mathcal{P}} + \mathbf{1} \otimes \mathbf{1} \otimes \mathrm{d}_{\mathcal{G}}$, and the bracket is the continuous $\boldsymbol{\Omega}_X$-bilinear extension of $[-,-]_{\mathcal{G}}$.*
(2) *The canonical map $\mathcal{G}_X \to \boldsymbol{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$ is a DG Lie algebra quasi-isomorphism.*
(3) *Suppose $f : Y \to X$ is a morphism of schemes. Then*

$$\boldsymbol{\Omega}_Y \widehat{\otimes}_{\mathcal{O}_Y} \widehat{f^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X)$$

*is a DG $\boldsymbol{\Omega}_Y$-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Y$. The canonical map*

$$f^{-1}(\boldsymbol{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \to \boldsymbol{\Omega}_Y \widehat{\otimes}_{\mathcal{O}_Y} \widehat{f^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X)$$

*is a homomorphism of DG Lie algebras.*

The explicit formulas for part (1) are

$$\mathrm{d}(\alpha_1 \otimes 1 \otimes \gamma_1) := \mathrm{d}(\alpha_1) \otimes 1 \otimes \gamma_1 + (-1)^{i_1} \alpha_1 \otimes 1 \otimes \mathrm{d}_{\mathcal{G}}(\gamma_1)$$

and

$$[\alpha_1 \otimes 1 \otimes \gamma_1, \alpha_2 \otimes 1 \otimes \gamma_2] := (-1)^{j_1 i_2} (\alpha_1 \wedge \alpha_2) \otimes 1 \otimes [\gamma_1, \gamma_2]$$

for $\alpha_k \in \boldsymbol{\Omega}_X^{i_k}$ and $\gamma_k \in \mathcal{G}_X^{j_k}$.

**Proof.** (1) Using the notation of the proof of Lemma 5.1, $\boldsymbol{\Omega}_C \otimes \mathcal{G}(C)$ is a DG $\boldsymbol{\Omega}_C$-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$, with the $\mathfrak{a}$-adic dir-inv structure. Hence so is its completion

$$\boldsymbol{\Omega}_C \widehat{\otimes \mathcal{G}}(C) \cong \boldsymbol{\Omega}_C \otimes_C \widehat{A} \otimes_C \mathcal{G}(C) \cong \Gamma(U, \boldsymbol{\Omega}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X).$$

(2) In the proof of part (1) the inclusion $\mathcal{G}(C) \subset \boldsymbol{\Omega}_C \otimes \mathcal{G}(C)$ is a DG algebra homomorphism. According to [Ye3, Theorem 4.4] we have a quasi-isomorphism.

(3) This is by Yekutieli [Ye2, Proposition 1.22(2)]. $\square$

**Definition 5.5.** Let $\mathcal{G}$ denote either $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$, and let d be the de Rham differential on $\Omega_{\mathrm{Coor}\,X}$. Put on $\mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$ the $\boldsymbol{t}$-adic dir-inv structure. Define

$$\mathrm{d}_{\mathrm{for}} := \mathrm{d} \otimes \mathbf{1} : \Omega^{p}_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]]) \to \Omega^{p+1}_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]]).$$

According to [Ye2, Proposition 1.19],

$$\Omega_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]]) = \bigoplus\nolimits_{p,q} \Omega^{p}_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}^{q}(\mathbb{K}[[\boldsymbol{t}]])$$

is a DG Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_{\mathrm{Coor}\,X}$, with differential $\mathrm{d}_{\mathrm{for}} + \mathbf{1} \otimes \mathrm{d}_{\mathcal{G}}$. The explicit formula is

$$(\mathrm{d}_{\mathrm{for}} + \mathbf{1} \otimes \mathrm{d}_{\mathcal{G}})(\alpha \otimes \gamma) = \mathrm{d}(\alpha) \otimes \gamma + (-1)^{p} \alpha \otimes \mathrm{d}_{\mathcal{G}}(\gamma)$$

for $\alpha \in \Omega^{p}_{\mathrm{Coor}\,X}$ and $\gamma \in \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$.

**Theorem 5.6** (*Universal Taylor Expansion*). *Let $\mathcal{G}$ denote either $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$. There is a canonical isomorphism*

$$\Omega_{\mathrm{Coor}\,X} \widehat{\otimes}_{\mathcal{O}_{\mathrm{Coor}\,X}} \widehat{\pi^{*}_{\mathrm{coor}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) \cong \Omega_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$$

*of graded Lie algebras in* $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}_{\mathrm{Coor}\,X}$, *extending the isomorphism of Corollary* 4.9.

*Warning*: the isomorphism in the theorem does not respect the differentials; cf. Proposition 5.8 below.

**Proof of Theorem 5.6.** By Corollary 4.9 we know that $\widehat{\pi^{*}_{\mathrm{coor}}}\, \mathcal{P}_X \cong \mathcal{O}_{\mathrm{Coor}\,X}[[\boldsymbol{t}]]$ canonically as inv $\mathcal{O}_{\mathrm{Coor}\,X}$-algebras. Using Lemma 5.1 we then obtain isomorphisms of graded Lie algebras over $\mathcal{O}_{\mathrm{Coor}\,X}$

$$\begin{aligned}
\widehat{\pi^{*}_{\mathrm{coor}}}\, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X) &\cong \widehat{\pi^{*}_{\mathrm{coor}}}\, \mathcal{G}(\mathcal{P}_X / \mathcal{O}_X; \mathrm{p}_1^{*}) \\
&\cong \mathcal{G}(\widehat{\pi^{*}_{\mathrm{coor}}} \mathcal{P}_X / \mathcal{O}_{\mathrm{Coor}\,X}) \\
&\cong \mathcal{G}(\mathcal{O}_{\mathrm{Coor}\,X}[[\boldsymbol{t}]] / \mathcal{O}_{\mathrm{Coor}\,X}) \\
&\cong \mathcal{O}_{\mathrm{Coor}\,X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]]).
\end{aligned}$$

Finally we may apply the functor $\Omega_{\mathrm{Coor}\,X} \widehat{\otimes}_{\mathcal{O}_{\mathrm{Coor}\,X}} -$. $\square$

**Definition 5.7.** The *Maurer–Cartan form of X* is

$$\omega_{\mathrm{MC}} := \sum_{i=1}^{n} \nabla_{\mathcal{P}}(t_i) \cdot \frac{\partial}{\partial t_i} \in \Gamma\big(\mathrm{Coor}\, X, \Omega^1_{\mathrm{Coor}\, X} \widehat{\otimes}\, \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])\big),$$

where $\nabla_{\mathcal{P}}(t_i)$ is defined using the canonical isomorphism in Theorem 5.6.

The Lie algebra $\mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) = \mathcal{T}(\mathbb{K}[[\boldsymbol{t}]])$ is also a Lie subalgebra of $\mathcal{D}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) = \mathcal{D}(\mathbb{K}[[\boldsymbol{t}]])$. Keeping the notation of Theorem 5.6, for any local section $\alpha \in \Omega_{\mathrm{Coor}\, X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$ let

$$\mathrm{ad}(\omega_{\mathrm{MC}})(\alpha) := [\omega_{\mathrm{MC}}, \alpha].$$

The operation $\mathrm{ad}(\omega_{\mathrm{MC}})$ is $\mathbb{K}$-linear endomorphism of degree 1 of the graded sheaf $\Omega_{\mathrm{Coor}\, X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$.

**Proposition 5.8.** *Let* $\mathcal{G}$ *denote either* $\mathcal{T}_{\mathrm{poly}}$ *or* $\mathcal{D}_{\mathrm{poly}}$. *Under the isomorphism of Theorem* 5.6, *there is equality*

$$\nabla_{\mathcal{P}} = \mathrm{d}_{\mathrm{for}} + \mathrm{ad}(\omega_{\mathrm{MC}})$$

*as endomorphisms of* $\Omega_{\mathrm{Coor}\, X} \widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])$.

**Proof.** First we shall consider $\mathcal{G} = \mathcal{T}_{\mathrm{poly}}$. Let us write $\omega := \omega_{\mathrm{MC}}$, $\mathrm{d} := \mathrm{d}_{\mathrm{for}}$, $\nabla := \nabla_{\mathcal{P}}$ and $\partial_i := \frac{\partial}{\partial t_i}$. For any multi-index $\boldsymbol{j} = (j_1 < \cdots < j_q)$ let us write

$$\boldsymbol{\partial}_{\boldsymbol{j}} := \partial_{j_1} \wedge \cdots \wedge \partial_{j_q} \in \mathcal{T}^{q-1}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]).$$

Take a local section $\alpha \in \Omega^p_{\mathrm{Coor}\, X}$ and a multi-index $\boldsymbol{i} \in \mathbb{N}^n$, and consider $\alpha \otimes \boldsymbol{t}^{\boldsymbol{i}} \boldsymbol{\partial}_{\boldsymbol{j}} \in \Omega^p_{\mathrm{Coor}\, X} \widehat{\otimes}\, \mathcal{T}^{q-1}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. Then

$$\begin{aligned} \nabla(\alpha \otimes \boldsymbol{t}^{\boldsymbol{i}} \boldsymbol{\partial}_{\boldsymbol{j}}) &= \mathrm{d}(\alpha) \cdot \boldsymbol{t}^{\boldsymbol{i}} \cdot \boldsymbol{\partial}_{\boldsymbol{j}} \pm \alpha \cdot \nabla(\boldsymbol{t}^{\boldsymbol{i}}) \cdot \boldsymbol{\partial}_{\boldsymbol{j}} \pm \alpha \cdot \boldsymbol{t}^{\boldsymbol{i}} \cdot \nabla(\boldsymbol{\partial}_{\boldsymbol{j}}), \\ \mathrm{d}(\alpha \otimes \boldsymbol{t}^{\boldsymbol{i}} \boldsymbol{\partial}_{\boldsymbol{j}}) &= \mathrm{d}(\alpha) \cdot \boldsymbol{t}^{\boldsymbol{i}} \cdot \boldsymbol{\partial}_{\boldsymbol{j}}, \\ \mathrm{ad}(\omega)(\alpha \otimes \boldsymbol{t}^{\boldsymbol{i}} \boldsymbol{\partial}_{\boldsymbol{j}}) &= \pm \alpha \cdot \mathrm{ad}(\omega)(\boldsymbol{t}^{\boldsymbol{i}}) \cdot \boldsymbol{\partial}_{\boldsymbol{j}} \pm \alpha \cdot \boldsymbol{t}^{\boldsymbol{i}} \cdot \mathrm{ad}(\omega)(\boldsymbol{\partial}_{\boldsymbol{j}}). \end{aligned}$$

Now

$$\nabla(\boldsymbol{t}^{\boldsymbol{i}}) = \sum_k \nabla(t_k) \cdot \partial_k(\boldsymbol{t}^{\boldsymbol{i}}) = \mathrm{ad}(\omega)(\boldsymbol{t}^{\boldsymbol{i}}).$$

It remains to show that

$$\nabla(\boldsymbol{\partial}_{\boldsymbol{j}}) = \mathrm{ad}(\omega)(\boldsymbol{\partial}_{\boldsymbol{j}}).$$

Take any $\beta \in \Omega_{\mathrm{Coor}\, X} \mathbin{\widehat{\otimes}} \mathcal{T}^{q-1}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$, and write it as $\beta = \sum_{\boldsymbol{k}} \beta_{\boldsymbol{k}} \boldsymbol{\partial}_{\boldsymbol{k}}$, where the sum is over the multi-indices $\boldsymbol{k} = (k_1 < \cdots < k_q)$, and $\beta_{\boldsymbol{k}} \in \Omega_{\mathrm{Coor}\, X} \mathbin{\widehat{\otimes}} \mathbb{K}[[\boldsymbol{t}]]$. Then $[\beta, \boldsymbol{t}^{\boldsymbol{k}}] = \beta_{\boldsymbol{k}}$. We see that $\beta = 0$ iff $[\beta, \boldsymbol{t}^{\boldsymbol{k}}] = 0$ for all such $\boldsymbol{k}$. Therefore is suffices to prove that $[\nabla(\boldsymbol{\partial}_{\boldsymbol{j}}), \boldsymbol{t}^{\boldsymbol{k}}] = [\mathrm{ad}(\omega)(\boldsymbol{\partial}_{\boldsymbol{j}}), \boldsymbol{t}^{\boldsymbol{k}}]$ for any $\boldsymbol{k}$. Now $[\boldsymbol{\partial}_{\boldsymbol{j}}, \boldsymbol{t}^{\boldsymbol{k}}] \in \mathbb{K}$ (it is 0 or 1). Because $\nabla$ is a $\mathbb{K}$-linear derivation, we have

$$0 = \nabla([\boldsymbol{\partial}_{\boldsymbol{j}}, \boldsymbol{t}^{\boldsymbol{k}}]) = [\nabla(\boldsymbol{\partial}_{\boldsymbol{j}}), \boldsymbol{t}^{\boldsymbol{k}}] \pm [\boldsymbol{\partial}_{\boldsymbol{j}}, \nabla(\boldsymbol{t}^{\boldsymbol{k}})].$$

Likewise

$$0 = \mathrm{ad}(\omega)([\boldsymbol{\partial}_{\boldsymbol{j}}, \boldsymbol{t}^{\boldsymbol{k}}]) = [\mathrm{ad}(\omega)(\boldsymbol{\partial}_{\boldsymbol{j}}), \boldsymbol{t}^{\boldsymbol{k}}] \pm [\boldsymbol{\partial}_{\boldsymbol{j}}, \mathrm{ad}(\omega)(\boldsymbol{t}^{\boldsymbol{k}})].$$

And by definition of $\omega$ we have

$$\mathrm{ad}(\omega)(\boldsymbol{t}^{\boldsymbol{k}}) = \sum_{l} \nabla(t_l)\, \partial_l(\boldsymbol{t}^{\boldsymbol{k}}) = \nabla(\boldsymbol{t}^{\boldsymbol{k}}).$$

The case $\mathcal{G} = \mathcal{D}_{\mathrm{poly}}$ is handled similarly, using the basis

$$\boldsymbol{\partial}_{\boldsymbol{j}_1, \ldots, \boldsymbol{j}_q} := \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^{\boldsymbol{j}_1} \otimes \cdots \otimes \left(\frac{\partial}{\partial \boldsymbol{t}}\right)^{\boldsymbol{j}_q} \in \mathcal{D}^{q-1}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]),$$

see Eq. (4.3) in [Ye2]. □

**Proposition 5.9.** *The form* $\omega_{\mathrm{MC}}$ *satisfies the identity*

$$\mathrm{d}_{\mathrm{for}}(\omega_{\mathrm{MC}}) + \tfrac{1}{2}[\omega_{\mathrm{MC}}, \omega_{\mathrm{MC}}] = 0;$$

*namely it is a solution of the MC equation in the DG Lie algebra* $\Omega_{\mathrm{Coor}\, X} \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$ *with differential* $\mathrm{d}_{\mathrm{for}}$.

**Proof.** Let us write $\omega := \omega_{\mathrm{MC}}$, $\mathrm{d} := \mathrm{d}_{\mathrm{for}}$, $\nabla := \nabla_{\mathcal{P}}$, $\beta := \mathrm{d}(\omega) + \frac{1}{2}[\omega, \omega]$ and $\mathfrak{g} := \Omega_{\mathrm{Coor}\, X} \mathbin{\widehat{\otimes}} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]])$. As explained in the proof of the previous proposition, it suffices to show that $[\beta, t_i] = 0$ for all $i$.

By definition of $\omega$ we have

$$[\omega, t_i] = \nabla_{\mathcal{P}}(t_i). \tag{5.10}$$

Next we use the fact that d is an odd derivation of $\mathfrak{g}$ to obtain

$$\mathrm{d}([\omega, t_i]) = [\mathrm{d}(\omega), t_i] - [\omega, \mathrm{d}(t_i)].$$

But $\mathrm{d}(t_i) = 0$, so

$$[\mathrm{d}(\omega), t_i] = \mathrm{d}(\nabla_{\mathcal{P}}(t_i)). \tag{5.11}$$

The graded Jacobi identity in $\mathfrak{g}$ tells us that

$$[[\omega, \omega], t_i] + [[t_i, \omega], \omega] - [[\omega, t_i], \omega] = 0.$$

Hence $[[\omega, \omega], t_i] = 2[\omega, [\omega, t_i]]$, and plugging in (5.10) we arrive at

$$\tfrac{1}{2}[[\omega, \omega], t_i] = \mathrm{ad}(\omega)(\nabla_{\mathcal{P}}(t_i)). \tag{5.12}$$

Finally, combining (5.11), (5.12) and Proposition 5.8 we get

$$\begin{aligned}[][\beta, t_i] &= [\mathrm{d}(\omega), t_i] + \tfrac{1}{2}[[\omega, \omega], t_i] = \mathrm{d}(\nabla_{\mathcal{P}}(t_i)) + \mathrm{ad}(\omega)(\nabla_{\mathcal{P}}(t_i)) \\ &= (\mathrm{d} + \mathrm{ad}(\omega))(\nabla_{\mathcal{P}}(t_i)) = \nabla_{\mathcal{P}}(\nabla_{\mathcal{P}}(t_i)) = 0. \qquad \square\end{aligned}$$

According to Theorem 4.13(2) the group $G(\mathbb{K})$ of $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[[\boldsymbol{t}]]$ acts on the bundle $\mathrm{Coor}\, X$. Therefore for any open set $U \subset X$ this group acts on the algebra $\Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \widehat{\pi_{\mathrm{coor}}^{*}}\mathcal{P}_X\big)$. More generally, let $\mathcal{G}$ denote either $\mathcal{T}_{\mathrm{poly}}$ or $\mathcal{D}_{\mathrm{poly}}$. Let us introduce the temporary notation

$$\mathfrak{h}(U, \mathcal{G}) := \Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \Omega_{\mathrm{Coor}\, X} \,\widehat{\otimes}_{\mathcal{O}_{\mathrm{Coor}\, X}}\, \widehat{\pi_{\mathrm{coor}}^{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G})\big)$$

and

$$\mathfrak{h}'(U, \mathcal{G}) := \Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \Omega_{\mathrm{Coor}\, X} \,\widehat{\otimes}\, \mathcal{G}(\mathbb{K}[[\boldsymbol{t}]])\big).$$

These are graded Lie algebras. The group $G(\mathbb{K})$ acts on $\mathfrak{h}(U, \mathcal{G})$ via its geometric action on $\mathrm{Coor}\, X$. On the other hand there is an action of $G(\mathbb{K})$ on $\mathfrak{h}'(U, \mathcal{G})$ via its action on $\Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \Omega_{\mathrm{Coor}\, X}\big)$ and on $\mathbb{K}[[\boldsymbol{t}]]$.

**Proposition 5.13.** *The canonical isomorphism* $\mathfrak{h}(U, \mathcal{G}) \cong \mathfrak{h}'(U, \mathcal{G})$ *of Theorem* 5.6 *is* $G(\mathbb{K})$*-equivariant.*

**Proof.** By Theorem 4.13(2) the algebra isomorphism

$$\Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \mathcal{O}_{\mathrm{Coor}\, X}[[\boldsymbol{t}]]\big) \cong \Gamma\big(\pi_{\mathrm{coor}}^{-1}(U), \widehat{\pi_{\mathrm{coor}}^{*}}\mathcal{P}_X\big)$$

of Corollary 4.9 is $G(\mathbb{K})$-equivariant. Tracing the isomorphisms used in the proof of Theorem 5.6 we deduce the same for the isomorphism $\mathfrak{h}(U, \mathcal{G}) \cong \mathfrak{h}'(U, \mathcal{G})$. □

We view $\omega_{\mathrm{MC}}$ as an element of $\mathfrak{h}(X, \mathcal{T}_{\mathrm{poly}}) \cong \mathfrak{h}'(X, \mathcal{T}_{\mathrm{poly}})$. Due to Proposition 5.13 we can talk about *the* action of $G(\mathbb{K})$ on $\omega_{\mathrm{MC}}$. Recall that $\mathrm{GL}_n(\mathbb{K})$ sits inside $G(\mathbb{K})$ as the group of linear changes of coordinates.

**Proposition 5.14.** *The element* $\omega_{\mathrm{MC}}$ *is* $\mathrm{GL}_n(\mathbb{K})$*-invariant.*

**Proof.** Since the Grothendieck connection $\nabla_{\mathcal{P}}$ on $\mathfrak{h}(X, \mathcal{T}_{\mathrm{poly}})$ is induced from $X$, it commutes with the action of $G(\mathbb{K})$. Hence in particular $g(\nabla_{\mathcal{P}}(t_i)) = \nabla_{\mathcal{P}}(g(t_i))$ for any $g \in \mathrm{GL}_n(\mathbb{K})$ and $i \in \{1, \ldots, n\}$.

Fix such a matrix $g = [g_{i,j}]$. So $g_{i,j} \in \mathbb{K}$ and $g(t_i) = \sum_{j=1}^{n} g_{i,j} t_j$. Let $h = [h_{i,j}] := (g^{-1})^{\mathrm{t}}$, the transpose inverse matrix. Then in the induced action of $\mathrm{GL}_n(\mathbb{K})$ on $\mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[\boldsymbol{t}]]) = \mathcal{T}(\mathbb{K}[[\boldsymbol{t}]])$ we have $g(\frac{\partial}{\partial t_i}) = \sum_{j=1}^{n} h_{i,j} \frac{\partial}{\partial t_j}$. Thus

$$
\begin{aligned}
g(\omega_{\mathrm{MC}}) &= g\left(\sum_i \nabla_{\mathcal{P}}(t_i) \cdot \frac{\partial}{\partial t_i}\right) = \sum_i g\big(\nabla_{\mathcal{P}}(t_i)\big) \cdot g\left(\frac{\partial}{\partial t_i}\right) \\
&= \sum_i \nabla_{\mathcal{P}}\big(g(t_i)\big) \cdot g\left(\frac{\partial}{\partial t_i}\right) = \sum_{i,j,k} g_{i,j} h_{i,k} \big(\nabla_{\mathcal{P}}(t_j)\big) \cdot \frac{\partial}{\partial t_k} \\
&= \sum_j \nabla_{\mathcal{P}}(t_j) \cdot \frac{\partial}{\partial t_j} = \omega_{\mathrm{MC}}. \qquad \square
\end{aligned}
$$

**Remark 5.15.** The adjoint of $\omega_{\mathrm{MC}}$ is an element of $\Gamma\big(\mathrm{Coor}\, X, \mathcal{T}_{\mathrm{Coor}\, X} \widehat{\otimes} \widehat{\Omega}^1_{\mathbb{K}[[\boldsymbol{t}]]/\mathbb{K}}\big)$, and it gives rise to a Lie algebra homomorphism $\mathcal{T}_{\mathbb{K}[[\boldsymbol{t}]]} \to \Gamma(\mathrm{Coor}\, X, \mathcal{T}_{\mathrm{Coor}\, X})$. In this way $\mathcal{T}_{\mathbb{K}[[\boldsymbol{t}]]}$ acts infinitesimally on $\mathrm{Coor}\, X$. Now inside $\mathcal{T}_{\mathbb{K}[[\boldsymbol{t}]]} = \bigoplus_{i=1}^{n} \mathbb{K}[[\boldsymbol{t}]] \frac{\partial}{\partial t_i}$ there is a subalgebra $\mathfrak{g} := \bigoplus_{i,j} \mathbb{K}[[\boldsymbol{t}]]\, t_i \frac{\partial}{\partial t_j}$. The Lie algebra $\mathfrak{g}$ is the Lie algebra of the pro-algebraic group $G = \mathrm{Aut}(\mathbb{K}[[\boldsymbol{t}]])$, the group of $\mathbb{K}$-algebra automorphisms of $\mathbb{K}[[\boldsymbol{t}]]$. The infinitesimal action of $\mathfrak{g}$ on $\mathrm{Coor}\, X$ is the differential of the action of $G$ on $\mathrm{Coor}\, X$ (cf. Theorem 4.13). The action of $\mathcal{T}_{\mathbb{K}[[\boldsymbol{t}]]}$ on $\mathrm{Coor}\, X$ is the main feature of the Gelfand–Kazhdan formal geometry. However we do not use this action (at least not directly) in our paper.

## 6. Review of mixed resolutions

As always $\mathbb{K}$ is a field of characteristic 0. In this section we review the constructions and results of the paper [Ye3].

Let $\boldsymbol{\Delta}$ denote the category with set of objects the natural numbers. For any $p, q \in \mathbb{N}$ the set of morphisms in $\boldsymbol{\Delta}$ from $p$ to $q$ is the set $\boldsymbol{\Delta}_p^q$ of order preserving functions

$\alpha : \{0, \ldots, p\} \to \{0, \ldots, q\}$. Recall that a cosimplicial object in some category $\mathsf{C}$ is a functor $C : \mathbf{\Delta} \to \mathsf{C}$. Usually one writes $C^p$ instead of $C(p)$, and refers to the sequence $\{C^p\}_{p \geqslant 0}$ as a cosimplicial object (the morphisms remaining implicit). The category of cosimplicial objects in $\mathsf{C}$ is denoted by $\mathbf{\Delta}\mathsf{C}$.

We are interested in cosimplicial dir-inv $\mathbb{K}$-modules, i.e. in objects $M = \{M^p\}_{p \geqslant 0}$ in $\mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. As explained in [Ye3], there is a functor

$$\widehat{\tilde{\mathrm{N}}} : \mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K} \to \mathsf{DGMod}\,\mathbb{K},$$

the latter being the category of complexes of $\mathbb{K}$-modules. This is the *complete Thom–Sullivan normalization* functor, which is a generalization of constructions in [HS,HY]. By definition there is an embedding

$$\widehat{\tilde{\mathrm{N}}}^q M \subset \prod_{i=0}^{\infty} \left( \Omega^q(\mathbf{\Delta}^i_{\mathbb{K}}) \,\widehat{\otimes}\, M^i \right).$$

Here $\mathbf{\Delta}^i_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[t_0, \ldots, t_i]/(t_0 + \cdots + t_i - 1)$ is the $i$-dimensional geometric simplex, and $\Omega^q\left(\mathbf{\Delta}^i_{\mathbb{K}}\right) := \Gamma\left(\mathbf{\Delta}^i_{\mathbb{K}}, \Omega^q_{\mathbf{\Delta}^i_{\mathbb{K}}}\right)$ is a discrete inv $\mathbb{K}$-module. The differential $\partial : \widehat{\tilde{\mathrm{N}}}^q M \to \widehat{\tilde{\mathrm{N}}}^{q+1} M$ is induced from the de Rham differentials $\mathrm{d} : \Omega^q\left(\mathbf{\Delta}^i_{\mathbb{K}}\right) \to \Omega^{q+1}\left(\mathbf{\Delta}^i_{\mathbb{K}}\right)$.

Let $X$ be a separated smooth irreducible $n$-dimensional $\mathbb{K}$-scheme. Choose an affine open covering $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ of $X$. Given $\boldsymbol{i} = (i_0, \ldots, i_q) \in \mathbf{\Delta}^m_q$ let $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_m)}$, and let $g_{\boldsymbol{i}} : U_{\boldsymbol{i}} \to X$ be the inclusion. For a sheaf $\mathcal{M}$ on $X$ we write

$$\mathrm{C}^q(\boldsymbol{U}, \mathcal{M}) := \prod_{\boldsymbol{i} \in \mathbf{\Delta}^m_q} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\, \mathcal{M}.$$

The sequence $\{\mathrm{C}^q(\boldsymbol{U}, \mathcal{M})\}_{q \geqslant 0}$ is then a cosimplicial sheaf on $X$. This is a variant of the Čech resolution of $\mathcal{M}$.

Suppose $\mathcal{M}$ is a dir-inv $\mathbb{K}_X$-module, i.e. a sheaf of $\mathbb{K}$-modules on $X$ with a dir-inv structure. For any open set $V \subset X$ we then have a cosimplicial dir-inv $\mathbb{K}$-module $\left\{\Gamma\big(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M})\big)\right\}_{q \geqslant 0}$. Applying the functor $\widehat{\tilde{\mathrm{N}}}^q$ to it we obtain a $\mathbb{K}$-module $\widehat{\tilde{\mathrm{N}}}^q \Gamma\big(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$. It turns out that the presheaf $V \mapsto \widehat{\tilde{\mathrm{N}}}^q \Gamma\big(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$ is a sheaf, and we denote it by $\widehat{\tilde{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M})$. So there is a functor

$$\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, -) : \mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X \to \mathsf{DGMod}\,\mathbb{K}_X,$$

and there is a functorial homomorphism $\mathcal{M} \to \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. If $\mathcal{M}$ is a complete dir-inv module then according to [Ye3, Theorem 3.7] the homomorphism $\mathcal{M} \to \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is in fact a quasi-isomorphism. We call $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ the *commutative Čech resolution* of $\mathcal{M}$, since $\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$ is a super-commutative DG algebra.

Now suppose $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Let $p$ be some natural number. Then $\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a complete dir-inv $\mathcal{P}_X$-module with the $\mathcal{I}_X$-adic dir-inv structure. Define

$$\mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) := \widehat{\tilde{\mathrm{N}}}{}^q\mathrm{C}(\boldsymbol{U}, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

This is a sheaf on $X$, and there is an embedding of sheaves

$$\mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \subset \prod_{j \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}^m_j} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^j_{\mathbb{K}}) \,\widehat{\otimes}\, (\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})\right). \tag{6.1}$$

In addition to the differential $\partial : \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}^{p,q+1}_{\boldsymbol{U}}(\mathcal{M})$ there is a second differential $\nabla_{\mathcal{P}} : \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}^{p+1,q}_{\boldsymbol{U}}(\mathcal{M})$ coming from the connection $\nabla_{\mathcal{P}}$ of Definition 5.2. We now totalize

$$\mathrm{Mix}^{i}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus\nolimits_{p+q=i} \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M})$$

and let $\mathrm{d}_{\mathrm{mix}} := \partial + (-1)^q \nabla_{\mathcal{P}}$. This is the *mixed resolution* of $\mathcal{M}$, which is a functor

$$\mathrm{Mix}_{\boldsymbol{U}} : \mathsf{QCoh}\,\mathcal{O}_X \to \mathsf{DGMod}\,\mathbb{K}_X.$$

**Theorem 6.2** (*Yekutieli [Ye3, Theorem 4.15]*). *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module.*
(1) *There is a functorial quasi-isomorphism $\mathcal{M} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$.*
(2) *There is a functorial isomorphism $\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \cong \mathrm{R}\Gamma(X, \mathcal{M})$ in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.*

Of course the functor $\mathrm{Mix}_{\boldsymbol{U}}$ can be extended to bounded below complexes of quasi-coherent $\mathcal{O}_X$-modules, by totalizing.

The sheaves of DG Lie algebras $\mathcal{T}_{\mathrm{poly},X}$ and $\mathcal{D}_{\mathrm{poly},X}$ are bounded below complexes of quasi-coherent $\mathcal{O}_X$-modules, so the above theorem applies to them. In addition we have:

**Proposition 6.3.** *Let $\mathcal{G}_X$ stand for either $\mathcal{T}_{\mathrm{poly},X}$ or $\mathcal{D}_{\mathrm{poly},X}$. Then $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{G}_X)$ is a sheaf of DG Lie algebras, with differential*

$$\mathrm{d}_{\mathrm{mix}} + (-1)^i \mathrm{d}_{\mathcal{G}} : \mathrm{Mix}^i_{\boldsymbol{U}}\left(\mathcal{G}^j_X\right) \to \mathrm{Mix}^{i+1}_{\boldsymbol{U}}\left(\mathcal{G}^j_X\right) \oplus \mathrm{Mix}^i_{\boldsymbol{U}}\left(\mathcal{G}^{j+1}_X\right).$$

*The quasi-isomorphism $\mathcal{G}_X \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{G}_X)$ of Theorem 6.2(1) is a homomorphism of DG Lie algebras.*

**Proof.** By Proposition 5.4 the sheaf $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{G}_X$ is a DG $\Omega_X$-module Lie algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$. Now use [Ye3, Proposition 5.5]. □

Suppose $\pi : Z \to X$ is some morphism of schemes (possibly of infinite type). A *simplicial section* of $\pi$ based on the covering $\boldsymbol{U}$ is a collection of $X$-morphisms

$$\boldsymbol{\sigma} = \{\sigma_{\boldsymbol{i}} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to Z\}$$

indexed by $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$, $q \in \mathbb{N}$, which satisfies the simplicial relations (see [Ye3, Definition 6.1]).

The sheaf $\Omega^p_Z$ is considered as a discrete inv $\mathbb{K}_Z$-module, and $\Omega_Z = \bigoplus_p \Omega^p_Z$ has the $\bigoplus$ dir-inv structure. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ the graded sheaf $\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is then a DG $\Omega_Z$-module in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Z$, with differential $\nabla_{\mathcal{P}}$. See Section 2.

Let $A$ be an associative unital super-commutative DG $\mathbb{K}$-algebra. Consider a homogeneous $A$-multilinear function $\phi : M_1 \times \cdots \times M_r \to N$, where $M_1, \ldots, M_r, N$ are DG $A$-modules. There is an operation of composition for such functions: given functions $\psi_i : \prod_j L_{i,j} \to M_i$ the composition is $\phi \circ (\psi_1 \times \cdots \times \psi_r) : \prod_{i,j} L_{i,j} \to N$. There is also a summation operation: if $\phi_j : \prod_i M_i \to N$ are homogeneous of equal degree then so is their sum $\sum_j \phi_j$. Finally let $\phi \circ \mathrm{d} : \prod_i M_i \to N$ be the homogeneous $\mathbb{K}$-multilinear function

$$(\phi \circ \mathrm{d})(m_1, \ldots, m_r) := \sum_{i=1}^{r} \pm \phi(m_1, \ldots, \mathrm{d}(m_i), \ldots, m_r)$$

with Koszul signs.

**Theorem 6.4** (*Yekutieli [Ye3, Theorem 6.3]*). *Suppose $\boldsymbol{\sigma}$ is simplicial section of $\pi : Z \to X$ based on $\boldsymbol{U}$. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules, and let*

$$\phi : \prod_{i=1}^{r} \big(\Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)\big) \to \Omega_Z \widehat{\otimes}_{\mathcal{O}_Z} \widehat{\pi^*}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

*be a continuous $\Omega_Z$-multilinear sheaf morphism on $Z$ of degree $k$. Then there is an induced $\mathbb{K}$-multilinear sheaf morphism of degree $k$*

$$\boldsymbol{\sigma}^*(\phi) : \prod_{i=1}^{r} \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_i) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$$

*on $X$ with the following properties*:

(i) *The assignment $\phi \mapsto \boldsymbol{\sigma}^*(\phi)$ respects the operations of composition and summation.*
(ii) *If $\phi = \widehat{\pi^*}(\phi_0)$ for some continuous $\Omega_X$-multilinear morphism*

$$\phi_0 : \prod_{i=1}^{r} \big(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i\big) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

*then $\boldsymbol{\sigma}^*(\phi) = \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \phi_0)$.*

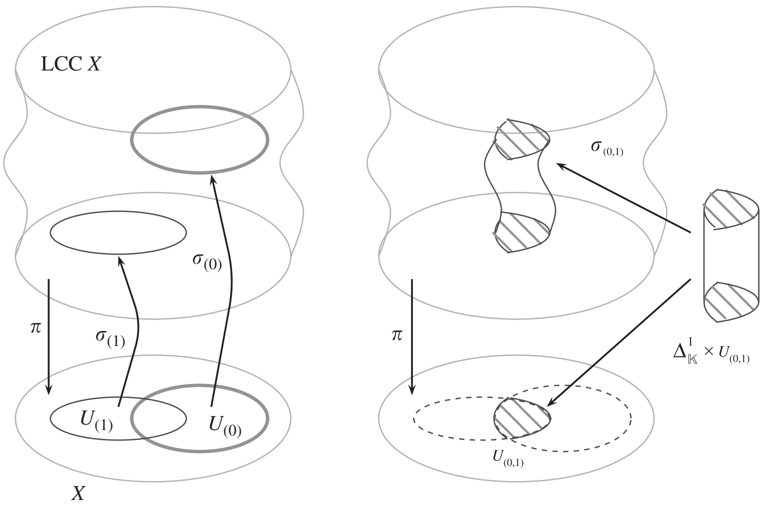


Fig. 2. Simplicial sections, $q = 1$. We start with sections over two open sets $U_{(0)}$ and $U_{(1)}$ in the left diagram; and we pass to a simplicial section $\sigma_{(0,1)}$ on the right.

(iii) *Suppose $\psi$ is another such $\Omega_Z$-multilinear sheaf morphism of degree $k+1$, and the equation $\nabla_{\mathcal{P}} \circ \phi - (-1)^k \phi \circ \nabla_{\mathcal{P}} = \psi$ holds. Then*

$$\mathrm{d}_{\mathrm{mix}} \circ \boldsymbol{\sigma}^*(\phi) - (-1)^k \boldsymbol{\sigma}^*(\phi) \circ \mathrm{d}_{\mathrm{mix}} = \boldsymbol{\sigma}^*(\psi).$$

We are interested in the bundle $\pi_{\mathrm{lcc}} : \mathrm{LCC}\, X \to X$.

**Theorem 6.5.** *Assume each affine open set $U_{(i)}$ admits an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$. Then there exist sections $\sigma_{(i)} : U_{(i)} \to \mathrm{LCC}\, X$, and furthermore they extend to a simplicial section $\boldsymbol{\sigma}$* of $\pi_{\mathrm{lcc}} : \mathrm{LCC}\, X \to X$.

**Proof.** By Theorem 4.13, $\pi_{\mathrm{coor}} : \mathrm{Coor}\, X \to X$ is a locally trivial $G$-torsor over $X$, and $G = \mathrm{GL}_{n,\mathbb{K}} \ltimes N$, where $N$ is a pro-unipotent group. By definition $\mathrm{LCC}\, X = \mathrm{Coor}\, X / \mathrm{GL}_{n,\mathbb{K}}$. According to Example 4.4 and Corollary 4.9, for any $i$ there is a section $\sigma_{(i)} : U_{(i)} \to \mathrm{Coor}\, X$. Now use [Ye4, Theorem 2.2]. □

Here is the idea behind the proof of [Ye4, Theorem 2.2]. There is an averaging process for sections of torsors under unipotent groups. The bundle $\mathrm{LCC}\, X$ is "almost" a torsor under the pro-unipotent group $N$. Given a multi-index $\boldsymbol{i} = (i_0, \ldots, i_q)$ the morphism $\sigma_{\boldsymbol{i}} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to \mathrm{LCC}\, X$ is then a family of weighted averages of the sections $\sigma_{(i_0)}, \ldots, \sigma_{(i_q)} : U_{\boldsymbol{i}} \to \mathrm{LCC}\, X$, parameterized by the simplex $\boldsymbol{\Delta}^q_{\mathbb{K}}$. See Fig. 2 for an illustration.

## 7. The global $\mathrm{L}_\infty$ quasi-isomorphism

In this section we prove the main result of the paper. Here again $X$ is a smooth irreducible separated $n$-dimensional scheme over the field $\mathbb{K}$, and also $\mathbb{R} \subset \mathbb{K}$.

Fix an open covering $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ of the scheme $X$ consisting of affine open sets, each admitting an étale morphism $U_{(i)} \to \mathbf{A}^n_{\mathbb{K}}$. For every $i$ let $\sigma_{(i)} : U_{(i)} \to \mathrm{LCC}\, X$ be the corresponding section of $\pi_{\mathrm{lcc}} : \mathrm{LCC}\, X \to X$, and let $\boldsymbol{\sigma}$ be the resulting simplicial section (see Theorem 6.5).

Let $\mathcal{M}$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules. The mixed resolution $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ was defined in Section 6. For any integer $i$ let $\mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus_{j=i}^{\infty} \mathrm{Mix}^j_{\boldsymbol{U}}(\mathcal{M})$, so $\{\mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\}_{i\in\mathbb{Z}}$ is a descending filtration of $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ by subcomplexes, with $\mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) = \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ for $i \ll 0$ and $\bigcap_i \mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) = 0$. Let

$$\mathrm{gr}^i_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) \;/\; \mathrm{G}^{i+1}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$$

and $\mathrm{gr}_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus_i \mathrm{gr}^i_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$.

By Proposition 6.3, if $\mathcal{G}_X$ is either $\mathcal{T}_{\mathrm{poly},X}$ or $\mathcal{D}_{\mathrm{poly},X}$, then $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{G}_X)$ is a sheaf of DG Lie algebras on $X$, and $\mathcal{G}_X \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{G}_X)$ is a DG Lie algebra quasi-isomorphism.

Note that if $\phi : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$ is a homomorphism of complexes that respects the filtration $\{\mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}\}$, then there exists an induced homomorphism of complexes

$$\mathrm{gr}_{\mathrm{G}}(\phi) : \mathrm{gr}_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{gr}_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N}).$$

Suppose $\mathcal{G}$ and $\mathcal{H}$ are sheaves of DG Lie algebras on a topological space $Y$. An $\mathrm{L}_\infty$ morphism $\Psi : \mathcal{G} \to \mathcal{H}$ is a sequence of sheaf morphisms $\psi_j : \prod^j \mathcal{G} \to \mathcal{H}$, such that for every open set $V \subset Y$ the sequence $\{\Gamma(V, \psi_j)\}_{j\geqslant 1}$ is an $\mathrm{L}_\infty$ morphism $\Gamma(V,\mathcal{G}) \to \Gamma(V,\mathcal{H})$. If $\psi_1 : \mathcal{G} \to \mathcal{H}$ is a quasi-isomorphism then $\Psi$ is called an $\mathrm{L}_\infty$ quasi-morphism.

**Theorem 7.1.** *Let $\boldsymbol{U}$ and $\boldsymbol{\sigma}$ be as above. Then there is an induced $\mathrm{L}_\infty$ quasi-isomorphism*

$$\Psi_{\boldsymbol{\sigma}} = \{\Psi_{\boldsymbol{\sigma};j}\}_{j\geqslant 1} : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}).$$

*The homomorphism $\Psi_{\boldsymbol{\sigma};1}$ respects the filtration $\{\mathrm{G}^i \mathrm{Mix}_{\boldsymbol{U}}\}$, and*

$$\mathrm{gr}_{\mathrm{G}}(\Psi_{\boldsymbol{\sigma};1}) = \mathrm{gr}_{\mathrm{G}}(\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{U}_1)).$$

**Proof.** Let $Y$ be some $\mathbb{K}$-scheme, and denote by $\mathbb{K}_Y$ the constant sheaf. For any $p$ we view $\Omega^p_Y$ as a discrete inv $\mathbb{K}_Y$-module, and we put on $\Omega_Y = \bigoplus_{p\in\mathbb{N}} \Omega^p_Y$ direct sum dir-inv structure. So $\Omega_Y$ is a discrete (and hence complete) DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_Y$.

We shall abbreviate $\mathcal{A} := \Omega_{\mathrm{Coor}\, X}$, so that $\mathcal{A}^0 = \mathcal{O}_{\mathrm{Coor}\, X}$, etc. As explained above, $\mathcal{A}$ is a DG algebra in $\mathsf{Dir\,Inv\,Mod}\, \mathbb{K}_{\mathrm{Coor}\, X}$, with discrete (but not trivial) dir-inv module structure.

There are sheaves of DG Lie algebras $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])$ and $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[t]])$ on the scheme Coor $X$. The differentials are $\mathrm{d}_{\mathrm{for}} = \mathrm{d} \otimes \mathbf{1}$ and $\mathrm{d}_{\mathrm{for}} + \mathbf{1} \otimes \mathrm{d}_{\mathcal{D}}$, respectively. As explained just prior to Theorem 3.16, there is a continuous $\mathcal{A}$-multilinear $\mathrm{L}_\infty$ morphism

$$\mathcal{U}_{\mathcal{A}} = \{\mathcal{U}_{\mathcal{A};j}\}_{j \geqslant 1} : \mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]]) \to \mathcal{A} \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[t]]).$$

The MC form $\omega := \omega_{\mathrm{MC}}$ is a global section of $\mathcal{A}^1 \widehat{\otimes} \mathcal{T}^0_{\mathrm{poly}}(\mathbb{K}[[t]])$ satisfying the MC equation in the DG Lie algebra $\mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])$. See Proposition 5.9. According to Theorem 3.16 the global section $\omega' := \mathcal{U}_{\mathcal{A};1}(\omega) \in \mathcal{A}^1 \widehat{\otimes} \mathcal{D}^0_{\mathrm{poly}}(\mathbb{K}[[t]])$ is a solution of the MC equation in the DG Lie algebra $\mathcal{A} \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[t]])$, and there is a continuous $\mathcal{A}$-multilinear $\mathrm{L}_\infty$ morphism

$$\mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j \geqslant 1} : \left(\mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])\right)_{\omega} \to \left(\mathcal{A} \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[t]])\right)_{\omega'}$$

between the twisted DG Lie algebras. The formula is

$$\mathcal{U}_{\mathcal{A},\omega;j}(\gamma_1 \cdots \gamma_j) = \sum_{k \geqslant 0} \frac{1}{(j+k)!} \mathcal{U}_{\mathcal{A};j+k}(\omega^k \gamma_1 \cdots \gamma_j) \tag{7.2}$$

for $\gamma_1, \ldots, \gamma_j \in \mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]])$. The two twisted DG Lie algebras have differentials $\mathrm{d}_{\mathrm{for}} + \mathrm{ad}(\omega)$ and $\mathrm{d}_{\mathrm{for}} + \mathrm{ad}(\omega') + \mathbf{1} \otimes \mathrm{d}_{\mathcal{D}}$, respectively.

By Theorem 5.6 (the universal Taylor expansions) there are canonical isomorphisms of graded Lie algebras in $\mathsf{Dir\,Inv\,Mod}\, \mathbb{K}_{\mathrm{Coor}\, X}$

$$\mathcal{A} \widehat{\otimes} \mathcal{T}_{\mathrm{poly}}(\mathbb{K}[[t]]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \widehat{\pi^*_{\mathrm{coor}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X})$$

and

$$\mathcal{A} \widehat{\otimes} \mathcal{D}_{\mathrm{poly}}(\mathbb{K}[[t]]) \cong \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \widehat{\pi^*_{\mathrm{coor}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}).$$

Proposition 5.8 tells us that

$$\mathrm{d}_{\mathrm{for}} + \mathrm{ad}(\omega) = \nabla_{\mathcal{P}}$$

under these identifications. Therefore

$$\mathcal{U}_{\mathcal{A},\omega} : \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \widehat{\pi^*_{\mathrm{coor}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \mathcal{A} \widehat{\otimes}_{\mathcal{A}^0} \widehat{\pi^*_{\mathrm{coor}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$$

is a continuous $\mathcal{A}$-multilinear $\mathrm{L}_\infty$ morphism between these DG Lie algebras, whose differentials are $\nabla_{\mathcal{P}}$ and $\nabla_{\mathcal{P}} + \mathbf{1} \otimes \mathrm{d}_{\mathcal{D}}$, respectively.

By Propositions 5.13 and 5.14 the form $\omega$ is $\mathrm{GL}_n(\mathbb{K})$-invariant. So according to Proposition 3.17 each of the operators $\mathcal{U}_{\mathcal{A};j}$ and $\mathcal{U}_{\mathcal{A},\omega;j}$ is $\mathrm{GL}_n(\mathbb{K})$-equivariant. We conclude that $\omega$ is a global section of

$$\Omega^1_{\mathrm{LCC}\,X}\,\widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC}\,X}}\,\widehat{\pi^*_{\mathrm{lcc}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}^0_{\mathrm{poly},X}),$$

and the operators $\mathcal{U}_{\mathcal{A};j}$ and $\mathcal{U}_{\mathcal{A},\omega;j}$ descend to continuous $\Omega_{\mathrm{LCC}\,X}$-multilinear operators

$$\mathcal{U}_{\mathcal{A};j},\ \mathcal{U}_{\mathcal{A},\omega;j} : \prod^j \big(\Omega_{\mathrm{LCC}\,X}\,\widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC}\,X}}\,\widehat{\pi^*_{\mathrm{lcc}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X})\big)$$
$$\to \Omega_{\mathrm{LCC}\,X}\,\widehat{\otimes}_{\mathcal{O}_{\mathrm{LCC}\,X}}\,\widehat{\pi^*_{\mathrm{lcc}}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$$

satisfying formula (7.2). The sequence $\mathcal{U}_{\mathcal{A},\omega} = \{\mathcal{U}_{\mathcal{A},\omega;j}\}_{j\geqslant 1}$ is an $\mathrm{L}_\infty$ morphism.

According to Theorem 6.4 there are induced operators

$$\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};j}),\ \boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A},\omega;j}) : \prod^j \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}).$$

The $\mathrm{L}_\infty$ identities in Definition 3.7, when applied to the $\mathrm{L}_\infty$ morphism $\mathcal{U}_{\mathcal{A},\omega}$, are of the form considered in Theorem 6.4(iii). Therefore these identities are preserved by $\boldsymbol{\sigma}^*$, and we conclude that the sequence $\{\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A},\omega;j})\}_{j\geqslant 1}$ is an $\mathrm{L}_\infty$ morphism. There is a global section $\boldsymbol{\sigma}^*(\omega) \in \mathrm{Mix}^1_{\boldsymbol{U}}(\mathcal{T}^0_{\mathrm{poly},X})$, and the formula

$$\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A},\omega;j})(\gamma_1\cdots\gamma_j) = \sum_{k\geqslant 0} \frac{1}{(j+k)!}\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};j+k})\big(\boldsymbol{\sigma}^*(\omega)^k\gamma_1\cdots\gamma_j\big) \tag{7.3}$$

holds for local sections $\gamma_1,\ldots,\gamma_j \in \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$. This sum is finite, the number of nonzero terms in it depending on the bidegree of $\gamma_1\cdots\gamma_j$. Indeed, if $\gamma_1\cdots\gamma_j \in \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{T}^p_{\mathrm{poly},X})$ then

$$\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};j+k})\big(\boldsymbol{\sigma}^*(\omega)^k\gamma_1\cdots\gamma_j\big) \in \mathrm{Mix}^{q+k}_{\boldsymbol{U}}(\mathcal{D}^{p+1-j-k}_{\mathrm{poly},X}), \tag{7.4}$$

which is zero for $k > p - j + 2$; see proof of [Ye2, Theorem 3.23].

Finally we define $\Psi_{\boldsymbol{\sigma};j} := \boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A},\omega;j})$. The collection $\Psi_{\boldsymbol{\sigma}} = \{\Psi_{j,\boldsymbol{\sigma}}\}_{j=1}^{\infty}$ is then an $\mathrm{L}_\infty$ morphism. From Eq. (7.4) we see that $\Psi_{\boldsymbol{\sigma};1}$ respects the filtration $\{\mathrm{G}^i\,\mathrm{Mix}_{\boldsymbol{U}}\}$, and according to Eq. (7.3) we see that

$$\mathrm{gr}_{\mathrm{G}}(\Psi_{\boldsymbol{\sigma};1}) = \mathrm{gr}_{\mathrm{G}}(\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};1})) = \mathrm{gr}_{\mathrm{G}}(\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{U}_1)).$$

According to [Ye3, Theorem 4.17] the homomorphism $\mathrm{gr}_{\mathrm{G}}\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{U}_1)$ is a quasi-isomorphism. Since the complexes $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$ and $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ are bounded below, and the filtration is nonnegative and exhaustive, it follows that $\Psi_{\boldsymbol{\sigma};1}$ is also a quasi-isomorphism. □

**Corollary 7.5.** *Taking global sections in Theorem* 7.1 *we get an* $\mathrm{L}_\infty$ *quasi-isomorphism*

$$\Gamma(X, \Psi_{\boldsymbol{\sigma}}) = \{\Gamma(X, \Psi_{\boldsymbol{\sigma}; j})\}_{j \geqslant 1} : \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big).$$

**Proof.** By Theorem 6.2 the homomorphism

$$\Gamma\left(X, \mathrm{Mix}_{\boldsymbol{U}}(\Psi_{\sigma;1})\right) : \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)$$

is a quasi-isomorphism. Cf. [Ye3, Theorem 0.1]. □

**Corollary 7.6.** *The data* $(\boldsymbol{U}, \boldsymbol{\sigma})$ *induces a bijection*

$$\begin{aligned} \mathrm{MC}(\Psi_{\boldsymbol{\sigma}}) : \ & \mathrm{MC}\Big(\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big)[[\hbar]]^+\Big) \\ & \xrightarrow{\simeq} \mathrm{MC}\Big(\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)[[\hbar]]^+\Big). \end{aligned}$$

**Proof.** Use Corollaries 7.5 and 3.10. □

Recall that $\mathcal{T}_{\mathrm{poly}}(X) = \Gamma(X, \mathcal{T}_{\mathrm{poly},X})$ and $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X) = \Gamma(X, \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X})$; and the latter is the DG Lie algebra of global polydifferential operators that vanish if one of their arguments is 1.

**Theorem 7.7.** *Assume* $\mathrm{H}^q(X, \mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly},X}) = 0$ *for all* $p$ *and all* $q > 0$. *Then there is a canonical function*

$$Q : \mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^+\big) \xrightarrow{\simeq} \mathrm{MC}\big(\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^+\big)$$

*preserving first-order terms. If moreover* $\mathrm{H}^q(X, \mathcal{T}^p_{\mathrm{poly},X}) = 0$ *for all* $p$ *and all* $q > 0$, *then* $Q$ *is bijective. The function* $Q$ *is called the* quantization map, *and it is characterized as follows. Choose an open covering* $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ *of* $X$ *consisting of affine open sets, each admitting an étale morphism* $U_{(i)} \to \mathbf{A}^n_{\mathbb{K}}$. *Let* $\boldsymbol{\sigma}$ *be the associated simplicial section of* $\mathrm{LCC}\, X \to X$. *Then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^+\big) & \xrightarrow{Q} & \mathrm{MC}\big(\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^+\big) \\ {\scriptstyle \mathrm{MC(inc)}}\downarrow & & {\scriptstyle \mathrm{MC(inc)}}\downarrow \\ \mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}))[[\hbar]]^+\big) & \xrightarrow{\mathrm{MC}(\Psi_{\boldsymbol{\sigma}})} & \mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}))[[\hbar]]^+\big) \end{array} \tag{7.8}$$

*in which the right vertical arrow is bijective. Here* $\Psi_{\boldsymbol{\sigma}}$ *is the* $\mathrm{L}_\infty$ *quasi-isomorphism from Theorem* 7.1, *and* "inc" *denotes the various inclusions of DG Lie algebras.*

Let us elaborate a bit on the statement above. It says that to any formal Poisson structure $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j \in \mathcal{T}^1_{\mathrm{poly}}(X)[[\hbar]]^+$ there corresponds a star product $\star_\beta$, with $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}^{\mathrm{nor},1}_{\mathrm{poly}}(X)[[\hbar]]^+$ (cf. Proposition 3.20). The element $\beta = Q(\alpha)$ is uniquely determined up to gauge equivalence by $\exp(\mathcal{D}^{\mathrm{nor},0}_{\mathrm{poly}}(X)[[\hbar]]^+)$. Given any local sections $f, g \in \mathcal{O}_X$ one has

$$\frac{1}{2}\big(\beta_1(f,g) - \beta_1(g,f)\big) = \{f,g\}_{\alpha_1}.$$

The quantization map $Q$ can be calculated (at least in theory) using the collection of sections $\boldsymbol{\sigma}$ and the universal formulas for deformation in Theorem 3.13.

We will need a lemma before proving the theorem.

**Lemma 7.9.** *Let $f, g \in \mathcal{O}_X = \mathcal{D}^{-1}_{\mathrm{poly},X}$ be local sections.*

(1) *For any $\beta \in \mathrm{Mix}^0_{\boldsymbol{U}}(\mathcal{D}^1_{\mathrm{poly},X})$ one has*

$$[[\beta, f], g] = \beta(g,f) - \beta(f,g) \in \mathrm{Mix}^0_{\boldsymbol{U}}(\mathcal{O}_X).$$

(2) *For any $\beta \in \mathrm{Mix}^1_{\boldsymbol{U}}(\mathcal{D}^0_{\mathrm{poly},X}) \oplus \mathrm{Mix}^2_{\boldsymbol{U}}(\mathcal{D}^{-1}_{\mathrm{poly},X})$ one has $[[\beta, f], g] = 0$.*

(3) *Let $\gamma \in \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})^0$, and define $\beta := (\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma)$. Then $[[\beta, f], g] = 0$.*

**Proof.** (1) Proposition 6.3 implies that the embedding (6.1):

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}) \subset \bigoplus_{p,q,r} \prod_{j \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}^m_j} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \big(\Omega^q(\boldsymbol{\Delta}^j_{\mathbb{K}}) \hat{\otimes} (\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}^r_{\mathrm{poly},X})\big)$$

is a DG Lie algebra homomorphism. So by continuity we might as well assume that $\beta = aD$ with $a \in \Omega^0_X = \mathcal{O}_X$ and $D \in \mathcal{D}^1_{\mathrm{poly},X}$. Moreover, since the Lie bracket of $\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}$ is $\Omega_X$-bilinear, we may assume that $a = 1$, i.e. $\beta = D$. Now the assertion is clear from the definition of the Gerstenhaber Lie bracket, see [Ko1, Section 3.4.2].

(2) Applying the same reduction as above, but with $D \in \mathcal{D}^r_{\mathrm{poly},X}$ and $r \in \{0, -1\}$, we get $[[D, f], g] \in \mathcal{D}^{r-2}_{\mathrm{poly},X} = 0$.

(3) By part (2) it suffices to show that $[[\beta, f], g] = 0$ for $\beta := \mathrm{d}_{\mathcal{D}}(\gamma)$ and $\gamma \in \mathrm{Mix}^0_{\boldsymbol{U}}(\mathcal{D}^0_{\mathrm{poly},X})$. As explained above we may further assume that $\gamma = D \in \mathcal{D}^0_{\mathrm{poly},X}$. Now the formulas for $\mathrm{d}_{\mathcal{D}}$ and $[-,-]$ in [Ko1, Section 3.4.2] imply that $[[\mathrm{d}_{\mathcal{D}}(D), f], g] = 0$. □

**Proof of Theorem 7.7.** We are given that $\mathrm{H}^q(X, \mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly},X}) = 0$ for all $p$ and all $q > 0$; and therefore $\Gamma(X, \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}) = \mathrm{R}\Gamma(X, \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X})$ in the derived category $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

Now by Theorem 3.12 the inclusion $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X} \to \mathcal{D}_{\mathrm{poly},X}$ is a quasi-isomorphism, and by Theorem 6.2(1) the inclusion $\mathcal{D}_{\mathrm{poly},X} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ is a quasi-isomorphism. According to Theorem 6.2(2) we have $\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big) = \mathrm{R}\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)$. The conclusion is that

$$\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X) = \Gamma(X, \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly},X}) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big) \tag{7.10}$$

is a quasi-isomorphism of complexes of $\mathbb{K}$-modules. But in view of Proposition 6.3 this is also a homomorphism of DG Lie algebras.

From (7.10) we deduce that

$$\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^{+} \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})\big)[[\hbar]]^{+}$$

is a quasi-isomorphism of DG Lie algebras. Using Corollary 3.10 we see that the right vertical arrow in diagram (7.8) is bijective. Therefore this diagram defines $Q$ uniquely.

According to Corollary 7.6 the bottom arrow in diagram (7.8) is a bijection. The left vertical arrow comes from the DG Lie algebra homomorphism

$$\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^{+} \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})\big)[[\hbar]]^{+},$$

which is a quasi-isomorphism when $\mathrm{H}^{q}(X, \mathcal{T}^{p}_{\mathrm{poly},X}) = 0$ for all $p$ and all $q > 0$. So in case of this further vanishing of cohomology the map $Q$ is bijective.

Now suppose $\boldsymbol{U}' = \{U'_{(0)}, \ldots, U'_{(m')}\}$ is another such covering of $X$, with sections $\sigma'_{(i)} : U'_{(i)} \to \mathrm{LCC}\, X$. Without loss of generality we may assume that $m' \geqslant m$, and that $U'_{(i)} = U_{(i)}$ and $\sigma'_{(i)} = \sigma_{(i)}$ for all $i \leqslant m$. There is a morphism of simplicial schemes $f : \boldsymbol{U} \to \boldsymbol{U}'$, that is an open and closed embedding. Correspondingly there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^{+}\big) & \xrightarrow{Q} & \mathrm{MC}\big(\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^{+}\big) \\
{\scriptstyle \mathrm{MC}(\mathrm{inc})}\downarrow & & \downarrow{\scriptstyle \mathrm{MC}(\mathrm{inc})} \\
\mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}'}(\mathcal{T}_{\mathrm{poly},X}))[[\hbar]]^{+}\big) & \xrightarrow{\mathrm{MC}(\Psi_{\boldsymbol{\sigma}'})} & \mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}'}(\mathcal{D}_{\mathrm{poly},X}))[[\hbar]]^{+}\big) \\
{\scriptstyle \mathrm{MC}(f^{*})}\downarrow & & \downarrow{\scriptstyle \mathrm{MC}(f^{*})} \\
\mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}))[[\hbar]]^{+}\big) & \xrightarrow{\mathrm{MC}(\Psi_{\boldsymbol{\sigma}})} & \mathrm{MC}\big(\Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}))[[\hbar]]^{+}\big),
\end{array}$$

where the vertical arrows on the right are bijections. We conclude that $Q$ is independent of $\boldsymbol{U}$ and $\boldsymbol{\sigma}$.

Finally we must show that $Q$ preserves first-order terms. Let $\alpha = \sum_{j=1}^{\infty} \alpha_j \hbar^j$ be a formal Poisson structure, and let $\beta = \sum_{j=1}^{\infty} \beta_j \hbar^j \in \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)^{1}[[\hbar]]^{+}$ be a solution of the MC equation, such that $\beta = Q(\alpha)$ modulo gauge equivalence. This means that there exists some

$$\gamma = \sum_{k \geqslant 1} \gamma_k \hbar^k \in \Gamma(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X}))^0[[\hbar]]^+$$

such that

$$\sum_{j \geqslant 1} \frac{1}{j!} \Psi_{\boldsymbol{\sigma};j}(\alpha^j) = \exp(\mathrm{af})(\exp(\gamma))(\beta),$$

with notation as in Lemma 3.2. In the first-order term (i.e. the coefficient of $\hbar^1$) of this equation we have

$$\Psi_{\boldsymbol{\sigma};1}(\alpha_1) = \beta_1 - (\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1); \tag{7.11}$$

see Eq. (3.3). Now by definition (see proof of Theorem 7.1)

$$\Psi_{\boldsymbol{\sigma};1}(\alpha_1) = \boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A},\omega;1})(\alpha_1) = \sum_{k \geqslant 0} \frac{1}{(1+k)!} \boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};1+k})\big(\boldsymbol{\sigma}^*(\omega_{\mathrm{MC}})^k \alpha_1\big),$$

and the component in $\Gamma(X, \mathrm{Mix}^0_{\boldsymbol{U}}(\mathcal{D}^1_{\mathrm{poly},X}))[[\hbar]]^+$ is the summand with $k = 0$, namely $\boldsymbol{\sigma}^*(\mathcal{U}_{\mathcal{A};1})(\alpha_1) = \mathcal{U}_1(\alpha_1)$. Using Lemma 7.9 we get

$$[[\Psi_{\boldsymbol{\sigma};1}(\alpha_1), f], g] = [[\mathcal{U}_1(\alpha_1), f], g] = \mathcal{U}_1(\alpha_1)(g, f) - \mathcal{U}_1(\alpha_1)(f, g) = -2\{f, g\}_{\alpha_1},$$

$$[[\beta_1, f], g] = \beta_1(g, f) - \beta_1(f, g)$$

and

$$[[(\mathrm{d}_{\mathrm{mix}} + \mathrm{d}_{\mathcal{D}})(\gamma_1), f], g] = 0$$

for every local sections $f, g \in \mathcal{O}_X$. Combining these equations with Eq. (7.11) the proof is done. □

One says that $X$ is a *$\mathcal{D}$-affine variety* if $\mathrm{H}^q(X, \mathcal{M}) = 0$ for every quasi-coherent left $\mathcal{D}_X$-module $\mathcal{M}$ and every $q > 0$.

**Corollary 7.12.** *Assume $X$ is $\mathcal{D}$-affine. Then the quantization map $Q$ of Theorem* 7.7 *may be interpreted as a canonical function*

$$Q : \frac{\{\text{formal Poisson structures on } X\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{deformation quantizations of } \mathcal{O}_X\}}{\text{gauge equivalence}}$$

*preserving first-order terms. If $X$ is affine then $Q$ is bijective.*

**Proof.** By definition the left-hand side is $\mathrm{MC}\big(\mathcal{T}_{\mathrm{poly}}(X)[[\hbar]]^{+}\big)$. On the other hand, according to Theorem 1.13 every deformation quantization of $\mathcal{O}_X$ can be trivialized globally, and by Proposition 1.14 any gauge equivalence between globally trivialized deformation quantizations is a global gauge equivalence. Hence the right-hand side is $\mathrm{MC}\big(\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X)[[\hbar]]^{+}\big)$. Since each $\mathcal{D}^{\mathrm{nor},p}_{\mathrm{poly},X}$ is a quasi-coherent left $\mathcal{D}_X$-module, and each $\mathcal{T}^{p}_{\mathrm{poly},X}$ is a quasi-coherent $\mathcal{O}_X$-module, we can apply Theorem 7.7. □

Suppose $f : X' \to X$ is an étale morphism. According to [Ye2, Proposition 4.6] there are DG Lie algebra homomorphisms $f^* : \mathcal{T}_{\mathrm{poly}}(X) \to \mathcal{T}_{\mathrm{poly}}(X')$ and $f^* : \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X) \to \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(X')$. Given a formal Poisson structure $\alpha$ on $X$ we then obtain a formal Poisson structure $f^*(\alpha)$ on $X'$. Similarly a star product $\star$ on $\mathcal{O}_X[[\hbar]]$ induces a star product $f^*(\star)$ on $\mathcal{O}_{X'}[[\hbar]]$,

**Corollary 7.13.** *The quantization map $Q$ respects étale morphisms. Namely if $X$ and $X'$ are $\mathcal{D}$-affine schemes and $f : X' \to X$ is an étale morphism, then for any formal Poisson structure $\alpha$ on $X$ one has $Q(f^*(\alpha)) = f^*(Q(\alpha))$.*

**Proof.** This is clear from the proof of Theorem 7.7. □

## 8. Complements and remarks

Suppose $C$ is some smooth commutative $\mathbb{K}$-algebra, where $\mathbb{K}$ is a field containing $\mathbb{R}$. It is conceivable to look for a star product on $C[[\hbar]]$ that is *nondifferential*. Namely, a $\mathbb{K}[[\hbar]]$-bilinear, associative, unital multiplication $\star$ on $C[[\hbar]]$ of the form

$$f \star g = fg + \sum_{k=1}^{\infty} \beta_k(f, g)\hbar^k,$$

where the normalized $\mathbb{K}$-bilinear functions $\beta_k : C^2 \to C$ are not necessarily bi-differential operators. Indeed, classically this was the type of deformation that had been considered (cf. [Ge]). There is a corresponding notion of nondifferential gauge equivalence, via an automorphism $\gamma = \mathbf{1}_C + \sum_{k=1}^{\infty} \gamma_k \hbar^k$ of $C[[\hbar]]$ with $\gamma_k : C \to C$ normalized $\mathbb{K}$-linear functions.

**Proposition 8.1.** *Let $C$ be a smooth $\mathbb{K}$-algebra. Then the obvious function*

$$\frac{\{\text{star products on } C[[\hbar]]\}}{\text{gauge equivalence}} \to \frac{\{\text{nondifferential star products on } C[[\hbar]]\}}{\text{nondifferential gauge equivalence}}$$

*is bijective.*

**Proof.** Let us denote by $\mathcal{G}(C)$ the shifted full Hochschild cochain complex of $C$, and let $\mathcal{G}^{\mathrm{nor}}(C)$ be the subcomplex of normalized cochains. It is a well-known fact that the inclusion $\mathcal{G}^{\mathrm{nor}}(C) \hookrightarrow \mathcal{G}(C)$ is a quasi-isomorphism (it is an immediate consequence of [ML, Corollary X.2.2]). By Yekutieli [Ye1, Lemma 4.3] the $C$-linear map $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{G}(C)$ is a quasi-isomorphism, and by Theorem 3.12 the map $\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly}}(C) \to \mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C)$ is a quasi-isomorphism. The upshot is that the inclusion $\mathcal{D}^{\mathrm{nor}}_{\mathrm{poly}}(C) \hookrightarrow \mathcal{G}^{\mathrm{nor}}(C)$ is a quasi-isomorphism of DG Lie algebras. Now we can use Propositions 3.20 and 3.21, as well as their "classical" nondifferential variants (see proof of [Ke, Corollary 4.5]). □

Combining Proposition 8.1 with Corollary 7.12 (applied to $X := \operatorname{Spec} C$) we obtain:

**Corollary 8.2.** *Let $C$ be a smooth $\mathbb{K}$-algebra. Then there is a canonical bijection of sets*

$$Q : \frac{\{\text{formal Poisson structures on } C\}}{\text{gauge equivalence}} \xrightarrow{\simeq} \frac{\{\text{nondifferential star products on } C[[\hbar]]\}}{\text{nondifferential gauge equivalence}}$$

*preserving first-order terms.*

**Question 8.3.** In case $X$ is affine and admits an étale morphism $X \to \mathbf{A}^n_{\mathbb{K}}$, how are the deformation quantizations of Corollaries 7.12 and 3.24 related?

**Remark 8.4.** The methods of this paper, combined with the ideas of [CFT], can be used to prove the following result. Suppose $\mathbb{R} \subset \mathbb{K}$ and $\mathrm{H}^2(X, \mathcal{O}_X) = 0$. Let $\alpha$ be any Poisson structure on $X$. Then the Poisson variety $(X, \alpha)$ admits a deformation quantization, in the sense of Definition 1.11.

**Question 8.5.** Given a smooth scheme $X$, is it possible to determine which Poisson structures on $X$ can be quantized? The papers [NT,BK1] say that for a symplectic structure to be quantizable there are cohomological obstructions. Can anything like that be done for a degenerate Poisson structure?

**Remark 8.6.** Artin worked out a noncommutative deformation theory for schemes that goes step by step, from $\mathbb{K}[\hbar]/(\hbar^m)$ to $\mathbb{K}[\hbar]/(\hbar^{m+1})$; see [Ar1,Ar2]. The first-order data is a Poisson structure, and at each step there are well-defined obstructions to the process. Presumably Artin's deformations are deformation quantizations in the sense of Definition 1.6, namely they admit differential structures; but this requires a proof.

In the case of the projective plane $\mathbf{P}^2$ and a nonzero Poisson structure $\alpha$, the zero locus of $\alpha$ is a cubic divisor $E$. Assume $E$ is smooth. Artin asserts (private communication) that a particular deformation of $\mathcal{O}_{\mathbf{P}^2}$ with first-order term $\alpha$ lifts to a deformation of the homogeneous coordinate ring $B := \bigoplus_{i \geqslant 0} \Gamma(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(i))$. Namely there is a graded $\mathbb{K}[[\hbar]]$-algebra structure on $\bigoplus_{i \geqslant 0} B_i[[\hbar]]$, say $\star$, such that $a \star b \equiv ab \bmod \hbar$, and $a \star b - b \star a \equiv 2\hbar\{a, b\}_\alpha \bmod \hbar^2$, for all $a, b \in B$. Moreover, after tensoring with the field $\mathbb{K}((\hbar))$ this should be a three-dimensional Sklyanin algebra, presumably with associated elliptic curve $\mathbb{K}((\hbar)) \times_{\mathbb{K}} E$.

The above should be compared to [Ko3, Section 3].

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