THE CONTINUOUS HOCHSCHILD COCHAIN COMPLEX OF A SCHEME (SURVEY)

AMNON YEKUTIELI

This is a revision of the notes to my lecture at the conference "Résolution des singularités et géométrie non commutative, à la Mémoire de Ruth Michler" held at CIRM, Luminy in July 2001. The title of the lecture was "Decomposition of the Hochschild complex of a scheme."

The work discussed in my lecture began when I visited Monique Lejeune-Jalabert in Grenoble in February 1997. We had been trying to understand a certain construction of Chern classes, and this led to a discussion of Hochschild complexes and formal schemes.

The main result stated in my lecture (decomposition in arbitrary characteristic, using minimal injective resolutions) was obtained a few months later, in September 1997. This was during a conference in Obergurgle. It was there, in Obergurgle 1997, that I met Ruth Michler.

Two months after the Luminy conference, in September 2001, Michel Van den Bergh discovered a serious gap in my proof of the decomposition. I could not repair the gap, and subsequently the paper had to be withdrawn for revision. The new version of the paper [Ye3] concentrates on the use of formal completions to study Hochschild cohomology, which I believe is interesting in itself. This revision is reflected in the present notes.

1. Hochschild Complex of a $\mathbb{K}$-Algebra

Let $\mathbb{K}$ be a commutative ring and A a commutative $\mathbb{K}$-algebra. The (unnormalized) bar resolution of A is the exact sequence

$$\cdots \to \mathcal{B}_2(A) \xrightarrow{\partial} \mathcal{B}_1(A) \xrightarrow{\partial} \mathcal{B}_0(A) \to A \to 0,$$

where

$$\mathcal{B}_q(A) := A^{\otimes(q+2)} = A \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} A.$$

The formula for ∂ is

$$\partial(a_0 \otimes \cdots \otimes a_{q+1}) := \sum_{i=0}^{q} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q+1}.$$

Write

$$A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A = \mathcal{B}_0(A).$$

Then $\mathcal{B}_q(A)$ is an A^{e}-module via

$$a_1 \otimes a_2 \mapsto a_1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_2,$$

and ∂ is A^{e}-linear.

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Since we will work with derived categories (with upper indices and differentials of degree $+1$) it will be convenient to write

$$\mathcal{B}^{-q}(A) := \mathcal{B}_q(A).$$

The Hochschild cochain complex of A is $\mathcal{C}^{\cdot}_{\mathrm{dual}}(A)$ where

$$\begin{aligned}\mathcal{C}^{q}_{\mathrm{dual}}(A) &:= \mathrm{Hom}_{A^{\mathrm{e}}}(\mathcal{B}_q(A), A) \\ &\cong \mathrm{Hom}_{\mathbb{K}}(A^{\otimes q}, A).\end{aligned}$$

In the literature usually one finds the notation $\mathcal{C}^{\cdot}(A)$ or $\mathcal{C}^{\cdot}(A, A)$ for this complex. The Hochschild cohomology of A is

$$\mathrm{HH}^i(A) := \mathrm{H}^i \mathcal{C}^{\cdot}_{\mathrm{dual}}(A).$$

It is well known that Hochschild cohomology controls deformations of A as associative algebra. More on that in the Appendix.

When $\mathbb{K}$ is a field, $\mathcal{B}^{\cdot}(A)$ is a free resolution of A as A^{e}-module. Hence

$$\begin{aligned}\mathcal{C}^{\cdot}_{\mathrm{dual}}(A) &= \mathrm{Hom}_{A^{\mathrm{e}}}(\mathcal{B}^{\cdot}(A), A) \\ &\cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, A)\end{aligned}$$

in derived category $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ of A^{e}-modules. In cohomology we get

$$\begin{aligned}\mathrm{HH}^i(A) &= \mathrm{H}^i \,\mathrm{Hom}_{A^{\mathrm{e}}}(\mathcal{B}^{\cdot}(A), A) \\ &\cong \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A).\end{aligned} \tag{1.1}$$

Now suppose $\mathbb{K}$ is a noetherian ring (e.g. a finitely generated algebra over a field or over $\mathbb{Z}$). Even though the Hochschild cohomology $\mathrm{H}^i\mathcal{C}^{\cdot}_{\mathrm{dual}}(A)$ still makes sense and has the same deformation meaning, we choose to concentrate on $\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A)$ as the definition of Hochschild cohomology. This definition will extend well to schemes. Also, as we shall see, the Ext definition coincides with continuous Hochschild cohomology, when A is smooth.

Let A be a smooth $\mathbb{K}$-algebra. By smooth I mean that A is finitely generated and formally smooth – i.e. it has the lifting property. (E.g. $A = \mathbb{K}[t_1, \cdots, t_n]$, the polynomial algebra.) This implies A is flat, and the module of differentials $\Omega^1_A = \Omega^1_{A/\mathbb{K}}$ is projective. Write

$$\mathcal{T}_{A/\mathbb{K}} := \mathrm{Der}_{\mathbb{K}}(A) = \mathrm{Hom}_A(\Omega^1_A, A)$$

and

$$\bigwedge^q \mathcal{T}_A := \bigwedge^q_A \mathcal{T}_{A/\mathbb{K}}.$$

Theorem 1.2 (Hochschild-Kostant-Rosenberg). *Suppose $\mathbb{K}$ is noetherian and A is a smooth $\mathbb{K}$-algebra. Then there is a canonical isomorphism*

$$\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A) \cong \bigwedge^i \mathcal{T}_A.$$

In the derived category $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ we have the complex of poly-tangents

$$\bigoplus\nolimits_q \Big(\bigwedge^q \mathcal{T}_A\Big)[-q] = \Big(\cdots \to 0 \to A \xrightarrow{0} \mathcal{T}_A \xrightarrow{0} \bigwedge^2 \mathcal{T}_A \to \cdots\Big)$$

with A in degree 0.

Because each A-module $\bigwedge^q \mathcal{T}_A$ is projective, an easy calculation (cf. [Ye3]) shows that the HKR Theorem implies:

Proposition 1.3. *There is an isomorphism*

$$\mathrm{RHom}_{A^{\mathrm{e}}}(A, A) \cong \bigoplus_q \big(\bigwedge^q \mathcal{T}_A\big)[-q]$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

2. The Hochschild Complex of a Scheme

$\mathbb{K}$ is still any noetherian ring. Let X be a separated finite type scheme over $\mathbb{K}$. For example, $X \subset \mathbf{P}^n_{\mathbb{K}}$, a subvariety of projective n-space over $\mathbb{K}$. Then $\mathcal{O}_X$ is a coherent $\mathcal{O}_{X^2}$-module via the diagonal embedding.

The naive attempt to define a Hochschild complex on X does not work (even if $\mathbb{K}$ is a field). The modules $\mathcal{C}^q_{\mathrm{dual}}(A)$ are not covariant in A. On the other hand the bar sheaves $\mathcal{B}^{-q}(X)$ on X, associated to the presheaves $U \mapsto \mathcal{B}^{-q}(A)$, are ill-behaved.

There is however a way to get around this problem, due to Swan [Sw], which gives a good definition of Hochschild cohomology of X.

Another option is to simply define the Hochschild complex of X to be

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X).$$

This is an object in the derived category $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_{X^2})$ of $\mathcal{O}_{X^2}$-modules. It has the benefit that on any affine open set

$$U = \mathrm{Spec}\, A \subset X$$

one has

$$\mathrm{H}^i(U, \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)) \cong \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A).$$

Furthermore:

Theorem 2.1 (Y)**.** *Let $\mathbb{K}$ be a noetherian ring. If X is flat finite type separated scheme over $\mathbb{K}$ then Swan's definition of the Hochschild cochain complex of X coincides with* $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$.

Here is a less obvious way to proceed. The bad behavior of the bar resolution $\mathcal{B}^{\cdot}(X)$ can be fixed by passing to the completion along the diagonal.

For $q \geq 0$ let $\mathfrak{X}^{q+2}$ be the completion of X^{q+2} along the diagonal embedding of X. It is a noetherian formal scheme (see [EGA I]). Define

$$\widehat{\mathcal{B}}^{-q}(X) := \mathcal{O}_{\mathfrak{X}^{q+2}};$$

it is a sheaf of $\mathcal{O}_{\mathfrak{X}^2}$-modules with an adic topology.

Example 2.2. Take

$$X := \mathbf{A}^1_{\mathbb{K}} = \mathrm{Spec}\, \mathbb{K}[t].$$

Then

$$X^2 = \mathrm{Spec}\, \mathbb{K}[t \otimes 1, 1 \otimes t].$$

Let

$$\tilde{\mathrm{d}}t := t \otimes 1 - 1 \otimes t$$

be the function defining the diagonal. The formal scheme $\mathfrak{X}^2$ is then

$$\mathfrak{X}^2 \cong \mathrm{Spf}\, \mathbb{K}[t][[\tilde{\mathrm{d}}t]].$$

For any affine open set $U = \mathrm{Spec}\, A \subset X$ one has

$$\Gamma(U, \widehat{\mathcal{B}}^0(X)) = \Gamma(U, \mathcal{O}_{\mathfrak{X}^2}) \cong A[[\tilde{\mathrm{d}}t]].$$

The sheaves of continuous Hochschild cochains of X are

$$\mathcal{C}^q_{\mathrm{cd}}(X) := \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{X^2}}(\widehat{\mathcal{B}}^{-q}(X), \mathcal{O}_X).$$

Here "cd" stands for continuous dual. The sheaves $\mathcal{C}^q_{\mathrm{cd}}(X)$ are quasi-coherent $\mathcal{O}_X$-modules. Furthermore on an affine open set $U = \operatorname{Spec} A \subset X$ one has

$$\Gamma(U, \mathcal{C}^q_{\mathrm{cd}}(X)) \cong \{ f \in \operatorname{Hom}_{\mathbb{K}}(A^{\otimes q}, A) \mid f \text{ is a differential operator in each factor} \}.$$

Thus $\mathcal{C}^{\cdot}_{\mathrm{cd}}(X)$ is also called the complex of poly-differential operators.

The next theorem has a "folklore" status; it was certainly known to Konstevich in some version. A proof can be found in [Ye3].

Theorem 2.3. *Let $\mathbb{K}$ be a noetherian ring. Assume X is a separated smooth scheme over $\mathbb{K}$. Then there is an isomorphism*

$$\mathcal{C}^{\cdot}_{\mathrm{cd}}(X) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$.

3. Decomposition in Characteristic 0

There is a standard homomorphism

$$\pi_{\mathrm{cd}} : \bigwedge^q \mathcal{T}_X \to \mathcal{C}^q_{\mathrm{cd}}(X).$$

Locally it is given by the formula

$$\begin{aligned} &\pi_{\mathrm{cd}}(v_1 \wedge \cdots \wedge v_q)(1 \otimes a_1 \otimes \cdots \otimes a_q \otimes 1) \\ &\quad = \sum_{\sigma \in \Sigma_q} \operatorname{sgn}(\sigma) v_{\sigma(1)}(a_1) \cdots v_{\sigma(q)}(a_q) \end{aligned}$$

for $v_i \in \mathcal{T}_A = \operatorname{Der}_{\mathbb{K}}(A)$ and $a_i \in A$, where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Theorem 3.1. *Let $\mathbb{K}$ be a noetherian ring and let X be a separated smooth scheme over $\mathbb{K}$ of relative dimension n. If $n!$ is invertible in $\mathbb{K}$ then*

$$\pi_{\mathrm{cd}} : \bigoplus_q \left(\bigwedge^q \mathcal{T}_X\right)[-q] \to \mathcal{C}^{\cdot}_{\mathrm{cd}}(X)$$

is a quasi-isomorphism.

This result (in characteristic 0) seems to have been known for some time. The quasi-isomorphism π_{cd}, up to rational constants, is the one underlying the Formality Theorem of Kontsevich (see Appendix). In fact Kontsevich [Ko] has a proof of this theorem when X a C^{∞} real manifold. The proof of Theorem 3.1 can be found in [Ye3].

A consequence is a Hodge-type decomposition globally:

Corollary 3.2. *With $\mathbb{K}$ and X as above there is a canonical decomposition*

$$\operatorname{Ext}^i_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus_q \mathrm{H}^{i-q}\left(X, \bigwedge^q \mathcal{T}_X\right).$$

The assumption on the characteristic is important.

Theorem 3.3 (Y). *Assume $\mathbb{K}$ is a Gorenstein noetherian ring of finite Krull dimension and X is smooth over $\mathbb{K}$ of relative dimension n. Then π_{cd} is a quasi-isomorphism iff $n!$ is invertible in $\mathcal{O}_X$.*

Here are a couple of questions.

Question 3.4. Suppose $\mathbb{K}$ has mixed characteristics and $n!$ is invertible in $\mathbb{K}$. Over $\mathbb{K} \otimes_{\mathbb{Z}} \mathbb{Q}$ there is a Hodge-type decomposition of the Hochschild complex due to Gerstenhaber-Schack [GS] and Loday. What is the connection to the decomposition we get over $\mathbb{K}$ in Theorem 3.1?

Question 3.5. Suppose $\mathbb{K}$ is a field of characteristic p, A is a smooth $\mathbb{K}$-algebra of dimension n and $n! \geq p > 0$. Find an explicit formula for a quasi-isomorphism

$$\phi : \bigoplus_q \big(\bigwedge^q \mathcal{T}_A\big)[-q] \to \mathcal{C}^{\cdot}_{\mathrm{cd}}(A)$$

representing the decomposition from Proposition 1.3.

Appendix A. Deformations

Here is an overview of the role of the Hochschild complex in deformation theory. This will help put the decomposition results in context.

The following approach to deformation problems is attributed to Deligne (cf. [GM]). Here $\mathbb{K}$ is a field of characteristic 0.

Let $\mathfrak{g}$ be a differential graded (DG) Lie algebra over $\mathbb{K}$. Thus $\mathfrak{g} = \bigoplus \mathfrak{g}^i$ is a complex equipped with a graded Lie bracket compatible with the differential. For $\lambda \in \mathfrak{g}^0$, the map $\rho(\lambda) : \mathfrak{g}^1 \to \mathfrak{g}^1$,

$$\rho(\lambda)(\alpha) = [\lambda, \alpha] - \mathrm{d}\lambda,$$

is an affine transformation of the vector space $\mathfrak{g}^1$. For any vector space V let us write $\mathrm{Aff}(V) = \mathrm{Gl}(V) \ltimes V$ for the group of affine transformations, and $\mathfrak{aff}(V) = \mathfrak{gl}(V) \oplus V$ for the Lie algebra. Then $\rho : \mathfrak{g}^0 \to \mathfrak{aff}(\mathfrak{g}^1)$ is a Lie algebra homomorphism.

If $\mathfrak{g}^0$ is nilpotent there is a nilpotent algebraic group G^0 (defined using the Campbell-Baker-Hausdorff formula), for which the exponential map $\exp : \mathfrak{g}^0 \to G^0$ is a bijection. Hence the Lie algebra homomorphism ρ induces a group homomorphism $G^0 \to \mathrm{Aff}(\mathfrak{g}^1)$. In other words, G^0 acts on $\mathfrak{g}^1$ by affine transformations. Define

$$\mathrm{MC}(\mathfrak{g}) := \{\alpha \in \mathfrak{g}^1 \mid \mathrm{d}\alpha + \tfrac{1}{2}[\alpha, \alpha] = 0\} / G^0,$$

the set of solutions of the Maurer-Cartan equation modulo gauge equivalence.

Suppose now that $\mathfrak{g}$ is any DG Lie algebra. Let R be an artinian local $\mathbb{K}$-algebra with maximal ideal $\mathfrak{m}$ and $R/\mathfrak{m} = \mathbb{K}$. Then $\mathfrak{m} \otimes \mathfrak{g}$ is a nilpotent DG Lie algebra. Define

$$\mathrm{Def}_{\mathfrak{g}}(R) := \mathrm{MC}(\mathfrak{m} \otimes \mathfrak{g}).$$

Many infinitesimal deformation problems are described by this construction.

Example A.1. Suppose A is a commutative $\mathbb{K}$-algebra. Define a DG Lie algebra $\mathfrak{g}$ as follows. As complex,

$$\mathfrak{g} := \mathcal{C}^{\cdot}_{\mathrm{dual}}(A)[1],$$

i.e.

$$\mathfrak{g}^i = \mathcal{C}^{i+1}_{\mathrm{dual}}(A) \cong \mathrm{Hom}_{\mathbb{K}}(A^{\otimes(i+1)}, A).$$

The Lie bracket is the Gerstenhaber bracket. Note that $\mathfrak{g}^1 = \mathrm{Hom}_{\mathbb{K}}(A^{\otimes 2}, A)$, which is where $\mathbb{K}$-algebra structures on the vector space A will live. And indeed, for any artinian $\mathbb{K}$-algebra R as above, the set $\mathrm{Def}_{\mathfrak{g}}(R)$ coincides with the set of flat associative R-algebras $\tilde{A}$, such that $\tilde{A} \otimes_R \mathbb{K} \cong A$, up to isomorphism.

If instead we take for $\mathfrak{g}$ the subalgebra

$$\mathcal{D}^{\cdot}_{A,\mathrm{poly}} := \mathcal{C}^{\cdot}_{\mathrm{cd}}(A)[1]$$

of poly-differential operators, then $\mathrm{Def}_{\mathfrak{g}}(R)$ describes deformations of A which depend differentially on A. Such a deformation will localize on $\operatorname{Spec} A$.

Example A.2. Suppose A is a smooth $\mathbb{K}$-algebra. A Poisson structure is a biderivation $\{-,-\} : A \otimes A \to A$ satisfying certain axioms. Let R be a local artinian $\mathbb{K}$-algebra. The set of equivalence classes of Poisson structures on $A \otimes R$ that vanish modulo $\mathfrak{m}$, up to gauge transformation, is isomorphic to $\mathrm{Def}_{\mathfrak{g}}(R)$, where $\mathfrak{g}$ is the DG Lie algebra

$$\mathfrak{g} = \mathcal{T}^{\cdot}_{A,\mathrm{poly}} := \bigoplus\nolimits_i \big(\bigwedge\nolimits^i \mathcal{T}_A\big)[i+1]$$

equipped with 0 differential and the Schouten-Nijenhuis bracket.

For other examples cf. [HS].

Define

$$U_1 : \mathfrak{g}_1 = \mathcal{T}^{\cdot}_{A,\mathrm{poly}} \to \mathfrak{g}_2 = \mathcal{D}^{\cdot}_{A,\mathrm{poly}}$$

by

$$U_1(\alpha) = \tfrac{1}{q!}\pi_{\mathrm{cd}}(\alpha), \quad \alpha \in \bigwedge\nolimits^q \mathcal{T}_A.$$

The quasi-isomorphism U_1 does not respect the Lie algebra structures, but in cohomology

$$\mathrm{H}(U_1) : \mathrm{H}\mathfrak{g}_1 \to \mathrm{H}\mathfrak{g}_2$$

is an isomorphism of graded Lie algebras.

The Kontsevich Formality Theorem [Ko] says that U_1 extends to an L_∞ morphism $\mathfrak{g}_1 \to \mathfrak{g}_2$. This means there is a sequence of morphisms $U_1, U_2, U_3 \ldots$ that constitute a "higher homotopy" map. It follows that for any artinian $\mathbb{K}$-algebra R, $\mathrm{Def}_{\mathfrak{g}_1}(R) \cong \mathrm{Def}_{\mathfrak{g}_2}(R)$. Taking $R = \mathbb{K}[\hbar]/(\hbar^i)$ and passing to the limit one gets

$$\mathrm{Def}_{\mathfrak{g}_1}(\mathbb{K}[[\hbar]]) \cong \mathrm{Def}_{\mathfrak{g}_2}(\mathbb{K}[[\hbar]]),$$

so that formal deformations of A over the power series ring $\mathbb{K}[[\hbar]]$ correspond bijectively (modulo gauge equivalence) to formal Poisson structures $\alpha(\hbar) = \sum_{i=1}^{\infty} \alpha_i \hbar^i$, $\alpha_i \in \mathfrak{g}_2^1 = \bigwedge^2 \mathcal{T}_A$. In particular, given any Poisson structure α_1 there exists a formal deformation of A, corresponding to $\alpha(\hbar) = \alpha_1 \hbar$.

Kontsevich's result was proved when A is the ring of C^∞ functions on a real manifold X. But it seems to hold also for a smooth algebraic variety X over a field $\mathbb{K}$ of characteristic 0. A similar line of reasoning is claimed in [Ko] to imply that the isomorphism

$$\bigoplus\nolimits_i \mathrm{Ext}^i_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus\nolimits_{i,q} \mathrm{H}^{i-q}\big(X, \bigwedge\nolimits^q \mathcal{T}_X\big)$$

sends the Yoneda product on the left side to the cup product on the right.

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Department of Mathematics, Ben Gurion University, Be'er Sheva 84105, ISRAEL
E-mail address: amyekut@math.bgu.ac.il
http://www.math.bgu.ac.il/∼amyekut