THE CONTINUOUS HOCHSCHILD COCHAIN COMPLEX OF A SCHEME

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Abstract. Let X be a separated finite type scheme over a noetherian base ring $\mathbb{K}$. There is a complex $\widehat{\mathcal{C}}^{\cdot}(X)$ of topological $\mathcal{O}_X$-modules, called the complete Hochschild chain complex of X. To any $\mathcal{O}_X$-module $\mathcal{M}$ – not necessarily quasi-coherent – we assign the complex $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M})$ of continuous Hochschild cochains with values in $\mathcal{M}$. Our first main result is that when X is smooth over $\mathbb{K}$ there is a functorial isomorphism

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M})$$

in the derived category $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_{X^2})$, where $X^2 := X \times_{\mathbb{K}} X$.

The second main result is that if X is smooth of relative dimension n and $n!$ is invertible in $\mathbb{K}$, then the standard maps $\pi : \widehat{\mathcal{C}}^{-q}(X) \to \Omega^q_{X/\mathbb{K}}$ induce a quasi-isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}\big(\bigoplus\nolimits_q \Omega^q_{X/\mathbb{K}}[q], \mathcal{M}\big) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}).$$

When $\mathcal{M} = \mathcal{O}_X$ this is the quasi-isomorphism underlying the Kontsevich Formality Theorem.

Combining the two results above we deduce a decomposition of the global Hochschild cohomology

$$\mathrm{Ext}^i_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \cong \bigoplus\nolimits_q \mathrm{H}^{i-q}\big(X, \big(\textstyle\bigwedge^q_{\mathcal{O}_X} \mathcal{T}_{X/\mathbb{K}}\big) \otimes_{\mathcal{O}_X} \mathcal{M}\big),$$

where $\mathcal{T}_{X/\mathbb{K}}$ is the relative tangent sheaf.

0. Introduction and Statement of Results

Let $\mathbb{K}$ be a noetherian commutative ring and X a separated $\mathbb{K}$-scheme of finite type. The diagonal morphism $\Delta : X \to X^2 = X \times_{\mathbb{K}} X$ is then a closed embedding. This allows us to identify the category $\mathsf{Mod}\,\mathcal{O}_X$ of $\mathcal{O}_X$-modules with its image inside $\mathsf{Mod}\,\mathcal{O}_{X^2}$ under the functor Δ_*.

We shall use derived categories freely in this paper, following the reference [RD].

Definition 0.1 (Hochschild Cohomology, First Definition)**.**

1. Given an $\mathcal{O}_X$-module $\mathcal{M}$ the Hochschild cochain complex of X with values in $\mathcal{M}$ is $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \in \mathsf{D}(\mathsf{Mod}\,\mathcal{O}_{X^2})$
2. The qth Hochschild cohomology of X with values in $\mathcal{M}$ is

$$\mathrm{Ext}^q_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) = \mathrm{H}^q\big(X^2, \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M})\big).$$

This definition of Hochschild cohomology was considered by Kontsevich [Ko] and Swan [Sw] among others. We observe that if $\mathbb{K}$ is a field, A is a commutative $\mathbb{K}$-algebra, $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A$, $X := \operatorname{Spec} A$, M is an A-module and $\mathcal{M}$ is the quasi-coherent

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$\mathcal{O}_X$-module associated to M, then $\operatorname{Ext}^q_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \cong \operatorname{Ext}^q_{A^e}(A, M) \cong \mathrm{HH}^q(A, M)$ is the usual Hochschild cohomology. This partly justifies the definition. As we shall see, Definition 0.1 agrees with two other plausible definitions of Hochschild cohomology of a scheme.

In Section 1 we introduce the complex $\widehat{\mathcal{C}}^{\cdot}(X)$ of complete Hochschild chains of X. For any q the sheaf $\widehat{\mathcal{C}}^{-q}(X) = \widehat{\mathcal{C}}_q(X)$ is a topological $\mathcal{O}_X$-module. (Note the unusual indexing, due to our use of derived categories.) If $q < 0$ then $\widehat{\mathcal{C}}_q(X) = 0$, whereas for any $q \geq 0$ and any affine open set $U = \operatorname{Spec} A \subset X$ the group of sections $\Gamma(U, \widehat{\mathcal{C}}_q(X))$ is an adic completion of the usual module of Hochschild chains $\mathcal{C}_q(A) = A^{\otimes(q+2)} \otimes_{A^e} A$. The coboundary operator $\partial : \widehat{\mathcal{C}}^{-q}(X) \to \widehat{\mathcal{C}}^{-q+1}(X)$ is continuous.

Definition 0.2 (Hochschild Cohomology, Second Definition)**.**

(1) Given an $\mathcal{O}_X$-module $\mathcal{M}$ the continuous Hochschild cochain complex of X with values in $\mathcal{M}$ is $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M})$, where $\mathcal{M}$ has the discrete topology.

(2) In the special case $\mathcal{M} = \mathcal{O}_X$ we write
$$\mathcal{C}^q_{\mathrm{cd}}(X) := \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{O}_X).$$

(3) The qth Hochschild cohomology of X with values in $\mathcal{M}$ is
$$\mathrm{H}^q\big(X, \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M})\big).$$

It turns out that on any open set U as above we get
$$\Gamma(U, \mathcal{C}^q_{\mathrm{cd}}(X)) \cong$$
$$\{f \in \operatorname{Hom}_{\mathbb{K}}(A^{\otimes q}, A) \mid f \text{ is a differential operator in each factor}\}.$$

Hence this is the same kind of Hochschild cochain complex considered by Kontsevich in [Ko].

Theorem 0.3. *Suppose $\mathbb{K}$ is a noetherian ring and X is a smooth separated $\mathbb{K}$-scheme. Given an $\mathcal{O}_X$-module $\mathcal{M}$ there is an isomorphism*
$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}\big(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}\big) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M})$$
in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$. This isomorphism is functorial in $\mathcal{M}$. In particular for $\mathcal{M} = \mathcal{O}_X$ we get
$$\mathcal{C}^{\cdot}_{\mathrm{cd}}(X) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X).$$

The theorem is proved in Section 2, where it is restated as Corollary 2.9, and is deduced from the more general Theorem 2.8.

Theorem 0.3 says that on a smooth scheme the two definitions of Hochschild cochain complexes coincide. In Section 3 we examine a third definition of Hochschild cohomology, due to Swan [Sw]. We prove (Theorem 3.1) that when X is flat over $\mathbb{K}$ this third definition also agrees with Definition 0.1.

In Section 4 we look at the homomorphism $\pi : \widehat{\mathcal{C}}_q(X) \to \Omega^q_X = \Omega^q_{X/\mathbb{K}}$ given by the formula
$$\pi((1 \otimes a_1 \otimes \cdots \otimes a_q \otimes 1) \otimes 1) = \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_q.$$
Let us denote by $\mathcal{T}_X = \mathcal{T}_{X/\mathbb{K}} := \mathcal{H}om_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ the tangent sheaf, and $\bigwedge^q \mathcal{T}_X := \bigwedge^q_{\mathcal{O}_X} \mathcal{T}_X$. Consider the complexes $\bigoplus_q \Omega^q_{X/\mathbb{K}}[q]$ and $\bigoplus_q (\bigwedge^q \mathcal{T}_X)[-q]$ with trivial coboundaries.

Theorem 0.4 (Decomposition). *Let $\mathbb{K}$ be a noetherian ring, let X be a separated smooth $\mathbb{K}$-scheme of relative dimension n, and assume $n!$ is invertible in $\mathbb{K}$. Then for any $\mathcal{M} \in \mathsf{Mod}\,\mathcal{O}_X$ the homomorphism of complexes*

$$\mathcal{H}om_{\mathcal{O}_X}\big(\bigoplus\nolimits_q \Omega^q_X[q], \mathcal{M}\big) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}\big(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}\big)$$

induced by π is a quasi-isomorphism. In particular for $\mathcal{M} = \mathcal{O}_X$ we get a quasi-isomorphism

$$\pi_{\mathrm{cd}} : \bigoplus\nolimits_q \big(\bigwedge\nolimits^q \mathcal{T}_X\big)[-q] \to \mathcal{C}^{\cdot}_{\mathrm{cd}}(X).$$

Theorem 0.4 is restated (in slightly more general form) in Section 4 as Theorem 4.5 and proved there.

The quasi-isomorphism π_{cd} underlies Kontsevich's Formality Theorem. The fact that π_{cd} is a quasi-isomorphism in the case of a C^{∞} real manifold is [Ko] Theorem 4.6.11; cf. also [Ts] Theorem 2.2.2.

Putting Theorems 0.3 and 0.4 together we obtain a decomposition of the Hochschild cochain complex

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \cong \bigoplus\nolimits_q \big(\bigwedge\nolimits^q \mathcal{T}_X\big)[-q] \otimes_{\mathcal{O}_X} \mathcal{M} \tag{0.5}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_{X^2})$.

Passing to global cohomology in (0.5) we obtain the following corollary. It extends Corollary 2.6 of [Sw] where the assumptions are that $\mathbb{K}$ is a field of characteristic 0 and X is smooth and quasi-projective.

Corollary 0.6. *Let $\mathbb{K}$ be a noetherian ring, let X be a separated smooth $\mathbb{K}$-scheme of relative dimension n, and assume $n!$ is invertible in $\mathbb{K}$. Then for any $\mathcal{M} \in \mathsf{Mod}\,\mathcal{O}_X$ the Hochschild cohomology decomposes:*

$$\mathrm{Ext}^i_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \cong \bigoplus\nolimits_q \mathrm{H}^{i-q}\big(X, \big(\bigwedge\nolimits^q \mathcal{T}_X\big) \otimes_{\mathcal{O}_X} \mathcal{M}\big).$$

Observe that for $\mathcal{M} = \mathcal{O}_X$, X affine and $A := \Gamma(X, \mathcal{O}_X)$ we recover the Hochschild-Kostant-Rosenberg Theorem $\mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A) \cong \bigwedge^i \mathcal{T}_A$.

Remark 0.7. This paper replaces "Decomposition of the Hochschild Complex of a Scheme in Arbitrary Characteristic," which has been withdrawn. The proof of the main result of that paper, which relied on minimal injective resolutions, turned out to have a serious gap in it. The gap was discovered by M. Van den Bergh.

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1. Complete Hochschild Chains

Let $\mathbb{K}$ be a commutative ring and A a commutative $\mathbb{K}$-algebra. As usual we write $A^{\mathrm{e}} := A \otimes A$ where $\otimes := \otimes_{\mathbb{K}}$. For any natural number q let $\mathcal{B}_q(A) :=$

$A^{\otimes(q+2)} = A \otimes \cdots \otimes A$. $\mathcal{B}_q(A)$ is an A^{e}-module via the ring homomorphism $a_1 \otimes a_2 \mapsto a_1 \otimes 1 \otimes \cdots \otimes 1 \otimes a_2$. The (unnormalized) bar resolution is

$$\cdots \to \mathcal{B}_2(A) \xrightarrow{\partial} \mathcal{B}_1(A) \xrightarrow{\partial} \mathcal{B}_0(A) \to A \to 0, \tag{1.1}$$

where ∂ is the A^{e}-linear homomorphism

$$\partial(a_0 \otimes \cdots \otimes a_{q+1}) = \sum_{i=0}^{q} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{q+1}.$$

The coboundary ∂ is A^{e}-linear, and the complex (1.1) is split-exact with splitting homomorphism $s(a_0 \otimes \cdots \otimes a_{q+1}) = a_0 \otimes \cdots \otimes a_{q+1} \otimes 1$. The homomorphism s is A-linear when A acts via $a \mapsto a \otimes 1$. Cf. [Lo] Section 1.1.

For any q let

$$\mathcal{C}_q(A) := \mathcal{B}_q(A) \otimes_{A^{\mathrm{e}}} A.$$

$\mathcal{C}_q(A)$ is the module of degree q Hochschild chains of A.

Since we will be using derived categories, whose objects are cochain complexes, we shall unfortunately have to abandon the conventional notations for Hochschild chains. The first departure will be to write the bar resolution as a cochain complex, with $\mathcal{B}^{-q}(A) := \mathcal{B}_q(A)$. Likewise we write $\mathcal{C}^{-q}(A) := \mathcal{C}_q(A)$.

From now on $\mathbb{K}$ is assumed to be a noetherian ring. Let A be a finitely generated $\mathbb{K}$-algebra. Denote by I_q the kernel of the ring epimorphism $\mathcal{B}_q(A) \to A$, $a_0 \otimes \cdots \otimes a_{q+1} \mapsto a_0 \cdots a_{q+1}$. Let $\widehat{\mathcal{B}}_q(A)$ be the I_q-adic completion of $\mathcal{B}_q(A)$. The homomorphisms ∂ and s are continuous for the I_q-adic topologies, and hence $\widehat{\mathcal{B}}^{\cdot}(A)$ is a complex and $\widehat{\mathcal{B}}^{\cdot}(A) \to A$ is a continuously A-split quasi-isomorphism. We call $\widehat{\mathcal{B}}^{\cdot}(A)$ the complete bar resolution.

Next define

$$\widehat{\mathcal{C}}^{-q}(A) = \widehat{\mathcal{C}}_q(A) := \widehat{\mathcal{B}}_q(A) \otimes_{\widehat{A^{\mathrm{e}}}} A \cong \widehat{\mathcal{B}}_q(A) \otimes_{A^{\mathrm{e}}} A$$

and

$$\mathcal{C}^q_{\mathrm{cd}}(A) := \operatorname{Hom}^{\mathrm{cont}}_{\widehat{A^{\mathrm{e}}}}(\widehat{\mathcal{B}}_q(A), A) \cong \operatorname{Hom}^{\mathrm{cont}}_{A}(\widehat{\mathcal{C}}_q(A), A),$$

where the superscript "cont" refers to continuous homomorphisms with respect to the adic topology, and "cd" stands for "continuous dual." We call $\widehat{\mathcal{C}}_q(A)$ the module of complete Hochschild chains, and $\mathcal{C}^q_{\mathrm{cd}}(A)$ the module of continuous Hochschild cochains.

Lemma 1.2. *Assume A is flat over $\mathbb{K}$. Then $\widehat{\mathcal{B}}^{\cdot}(A)$ is a flat resolution of A as $\widehat{\mathcal{B}}_0(A)$-module.*

Proof. Let's write $\mathcal{B}_q(A) = \mathcal{B}_0(A) \otimes A^{\otimes q}$. Since A is a flat $\mathbb{K}$-algebra, it follows that $\widehat{\mathcal{B}}_0(A) \to \widehat{\mathcal{B}}_0(A) \otimes A^{\otimes q}$ is flat . Now $\widehat{\mathcal{B}}_0(A) \otimes A^{\otimes q}$ is noetherian, and $\widehat{\mathcal{B}}_q(A)$ is an adic completion of it, so $\widehat{\mathcal{B}}_0(A) \otimes A^{\otimes q} \to \widehat{\mathcal{B}}_q(A)$ is flat. $\square$

Suppose Y is a noetherian scheme and $Y_0 \subset Y$ is a closed subset. The formal completion of Y along Y_0 is a noetherian formal scheme $\mathfrak{Y}$ with underlying topological space Y_0. The structure sheaf $\mathcal{O}_{\mathfrak{Y}}$ is a sheaf of topological rings with $\mathcal{I}$-adic topology, where $\mathcal{I} \subset \mathcal{O}_Y$ is any coherent ideal sheaf defining the closed set Y_0. The canonical morphism $\mathfrak{Y} \to Y$ is flat, i.e. $\mathcal{O}_{\mathfrak{Y}}$ is a flat $\mathcal{O}_Y$-algebra. See [EGA I] Section 10.8 for details.

Definition 1.3. Let X be a finite type separated $\mathbb{K}$-scheme. For any $q \geq 2$ let $\mathfrak{X}^q$ be the formal completion of the scheme $X^q := X \times_{\mathbb{K}} \cdots \times_{\mathbb{K}} X$ along the diagonal embedding of X.

(1) For any $q \geq 0$ let $\widehat{\mathcal{B}}_q(X) := \mathcal{O}_{\mathfrak{X}^{q+2}}$.
(2) For any $q \geq 0$ the sheaf of degree q complete Hochschild chains of X is $\widehat{\mathcal{C}}_q(X) := \widehat{\mathcal{B}}_q(X) \otimes_{\mathcal{O}_{\mathfrak{X}^2}} \mathcal{O}_X$.

The benefit of the complete sheaves $\widehat{\mathcal{B}}_q(X)$ and $\widehat{\mathcal{C}}_q(X)$ is they are coherent (although over different ringed spaces). Indeed:

Proposition 1.4. *On any affine open set* $U = \operatorname{Spec} A \subset X$ *one has* $\Gamma(U, \widehat{\mathcal{B}}_q(X)) = \widehat{\mathcal{B}}_q(A)$ *and* $\Gamma(U, \widehat{\mathcal{C}}_q(X)) = \widehat{\mathcal{C}}_q(A)$.

Proof. See [EGA I] Section 10.10. $\square$

The homomorphisms $\partial : \widehat{\mathcal{B}}_q(A) \to \widehat{\mathcal{B}}_{q-1}(A)$ and $s : \widehat{\mathcal{B}}_q(A) \to \widehat{\mathcal{B}}_{q+1}(A)$ sheafify, hence $\widehat{\mathcal{B}}^{\cdot}(X)$ and $\widehat{\mathcal{C}}^{\cdot}(X)$ are complexes with continuous coboundary operators, and $\widehat{\mathcal{B}}^{\cdot}(X) \to \mathcal{O}_X$ is a continuously $\mathcal{O}_X$-split quasi-isomorphism.

Given an $\mathcal{O}_X$-module $\mathcal{M}$ we consider $\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{O}_{X^2}} \mathcal{O}_X \cong \mathrm{L}\Delta^* \mathcal{M}$ as an object of $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.

Proposition 1.5. *Assume* X *is flat over* $\mathbb{K}$. *Then there is an isomorphism*

$$\widehat{\mathcal{C}}^{\cdot}(X) \cong \mathcal{O}_X \otimes^{\mathrm{L}}_{\mathcal{O}_{X^2}} \mathcal{O}_X$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.

Proof. As in any completion of a noetherian scheme, $\widehat{\mathcal{B}}_0(X) = \mathcal{O}_{\mathfrak{X}^2}$ is a flat $\mathcal{O}_{X^2}$-algebra. From Lemma 1.2 we see that $\widehat{\mathcal{B}}_q(X)$ is a flat $\widehat{\mathcal{B}}_0(X)$-module. Hence $\widehat{\mathcal{B}}^{\cdot}(X)$ is a flat resolution of $\mathcal{O}_X$ as $\mathcal{O}_{X^2}$-module. But $\widehat{\mathcal{C}}^{\cdot}(X) \cong \widehat{\mathcal{B}}^{\cdot}(X) \otimes_{\mathcal{O}_{X^2}} \mathcal{O}_X$. $\square$

Given an $\mathcal{O}_X$-module $\mathcal{M}$ we have sheaves $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{M})$, where "$\mathcal{H}om^{\mathrm{cont}}$" refers to continuous homomorphisms for the adic topology on $\widehat{\mathcal{C}}_q(X)$ and the discrete topology on $\mathcal{M}$. The continuous coboundary ∂ makes $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M})$ into a complex. In Definition 0.2 this was called the continuous Hochschild cochain complex with values in $\mathcal{M}$.

Proposition 1.6. (1) *If* $\mathcal{M}$ *is quasi-coherent then* $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{M})$ *is also quasi-coherent.*

(2) *For any affine open set* $U = \operatorname{Spec} A \subset X$, *with* $M := \Gamma(U, \mathcal{M})$, *one has*

$$\Gamma\big(U, \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{M})\big) = \operatorname{Hom}^{\mathrm{cont}}_A(\widehat{\mathcal{C}}_q(A), M).$$

(3) *With* U *as above,*

$$\Gamma(U, \mathcal{C}^q_{\mathrm{cd}}(X)) = \mathcal{C}^q_{\mathrm{cd}}(A) \cong$$
$$\{f \in \operatorname{Hom}_{\mathbb{K}}(A^{\otimes q}, A) \mid f \textit{ is a differential operator in each factor}\}.$$

Proof. (1), (2) We have

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}_q(X), \mathcal{M}) \cong \lim_{m \to} \mathcal{H}om_{\mathcal{O}_{X^2}}(\widehat{\mathcal{B}}_q(X)/\widehat{\mathcal{I}}_q^m, \mathcal{M})$$

where $\widehat{\mathcal{I}}_q = \operatorname{Ker}(\widehat{\mathcal{B}}_q(X) \to \mathcal{O}_X)$. But the sheaf $\widehat{\mathcal{B}}_q(X)/\widehat{\mathcal{I}}_q^m$ is a coherent $\mathcal{O}_{X^2}$-module.

(3) This is immediate from the results in [EGA IV] Section 16.8. $\square$

We see from part (3) of the proposition that this approach to Hochschild cochains is the same as the one used by Kontsevich [Ko].

2. Comparison of Two Definitions

In this section we prove that the two definitions of Hochschild cohomology, Definitions 0.1 and 0.2, coincide when X is smooth over $\mathbb{K}$ (Corollary 2.9). Throughout we assume $\mathbb{K}$ is a noetherian ring and X is a separated finite type scheme over $\mathbb{K}$.

We start by recalling the notion of discrete $\mathcal{O}_{\mathfrak{Y}}$-module on a noetherian formal scheme $\mathfrak{Y}$. An $\mathcal{O}_{\mathfrak{Y}}$-module $\mathcal{M}$ is called discrete if it is discrete for the adic topology of $\mathcal{O}_{\mathfrak{Y}}$; in other words, if any local section of $\mathcal{M}$ is annihilated by some defining ideal of $\mathfrak{Y}$. The subcategory $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}} \subset \mathsf{Mod}\,\mathcal{O}_{\mathfrak{Y}}$ of discrete modules is abelian and closed under direct limits. Moreover $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}}$ is locally noetherian, so every injective object in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}}$ is a direct sum of indecomposable ones. The category $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}}$ has enough injectives, but we do not know if every injective in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}}$ is also injective in the bigger category $\mathsf{Mod}\,\mathcal{O}_{\mathfrak{Y}}$. See [RD] Section II.7 and [Ye2] Sections 3-4 for details.

Given a point $y \in \mathfrak{Y}$ let $\boldsymbol{k}(y)$ be the residue field and $\mathcal{O}_{\mathfrak{Y},y}$ the local ring. Denote by $\mathcal{J}(y)$ an injective hull of $\boldsymbol{k}(y)$ as $\mathcal{O}_{\mathfrak{Y},y}$-module. If y' is a specialization of y define $\mathcal{J}(y,y')$ to be a constant sheaf on the closed set $\overline{\{y'\}}$ with stalk $\mathcal{J}(y)$.

Proposition 2.1. *Let $\mathfrak{Y}$ be a noetherian formal scheme. The indecomposable objects in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{Y}}$ are the sheaves $\mathcal{J}(y,y')$.*

Proof. Exactly as in the proof of [Ye2] Proposition 4.2. □

In particular this applies to $\mathfrak{Y} = \mathfrak{X}^2$, and we shall denote by $\mathcal{J}(x,x')$ the indecomposable objects in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2}$. Therefore any injective $\mathcal{J}$ in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2}$ has a decomposition $\mathcal{J} \cong \bigoplus_{x,x'} \mathcal{J}(x,x')^{\mu(x,x')}$, where $\mu(x,x')$ are cardinal numbers and $\mathcal{J}(x,x')^{\mu(x,x')}$ means a direct sum of $\mu(x,x')$ copies.

If $\mathcal{M} \in \mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2}$ then $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{M})$ makes sense. The formula is

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{M}) = \lim_{m\to} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{M}\big)$$

where $\mathcal{I}_q := \operatorname{Ker}(\widehat{\mathcal{B}}_q(X) \to \mathcal{O}_X)$. Hence given a complex $\mathcal{M}^{\cdot} \in \mathsf{D}(\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2})$ we obtain a total complex $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{M}^{\cdot})$ with the usual indexing and signs.

Recall that $\mathcal{O}_{\mathfrak{X}^2}$ is an $\mathcal{O}_X$-algebra via the first projection $X^2 \to X$, namely $a \mapsto a \otimes 1$.

Lemma 2.2. *Let $\mathcal{J}$ be an injective object in $\mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2}$, and define $\mathcal{J}_X := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, \mathcal{J})$. Then there is a homomorphism of $\mathcal{O}_X$-modules $\tau : \mathcal{J} \to \mathcal{J}_X$, such that for any q the induced homomorphism*

$$\tau_q : \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{J}) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{B}}_q(X), \mathcal{J}_X)$$

is an isomorphism.

Proof. For any pair of points $x, x' \in X$ such that x' is a specialization of x let $\mathcal{J}_X(x,x') \cong \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}\big(\mathcal{O}_X, \mathcal{J}(x,x')\big)$ be the indecomposable injective $\mathcal{O}_X$-module.

Let $\mathcal{I}_q := \operatorname{Ker}(\widehat{\mathcal{B}}_q(X) \to \mathcal{O}_X)$, a defining ideal of the formal scheme $\mathfrak{X}^{q+2}$. For any $m \geq 1$ the sheaf of rings $\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m$ is coherent both as $\mathcal{O}_{\mathfrak{X}^2}$-module and as $\mathcal{O}_X$-module. We see that both $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}(x,x')\big)$ and

$\mathcal{H}om_{\mathcal{O}_X}(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}_X(x,x'))$ are constant sheaves on $\overline{\{x'\}}$ with stalks being injective hulls of $\boldsymbol{k}(x)$ as $(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m)_x$-module. Therefore

$$\mathcal{H}om_{\mathcal{O}_{X^2}}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}(x,x')\big) \cong \mathcal{H}om_{\mathcal{O}_X}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}_X(x,x')\big).$$

This isomorphism is not canonical, yet we can fit it into a compatible direct system as m varies. Thus there is a (noncanonical) isomorphism

$$\begin{aligned} \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{X^2}}\big(\widehat{\mathcal{B}}_q(X), \mathcal{J}(x,x')\big) &\cong \lim_{m\to} \mathcal{H}om_{\mathcal{O}_{X^2}}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}(x,x')\big) \\ &\cong \lim_{m\to} \mathcal{H}om_{\mathcal{O}_X}\big(\widehat{\mathcal{B}}_q(X)/\mathcal{I}_q^m, \mathcal{J}_X(x,x')\big) \\ &\cong \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}\big(\widehat{\mathcal{B}}_q(X), \mathcal{J}_X(x,x')\big). \end{aligned} \tag{2.3}$$

Taking $q = 0$ above, and composing with homomorphism "evaluation at 1," we obtain $\tau_{x,x'} : \mathcal{J}(x,x') \to \mathcal{J}_X(x,x')$.

Now consider the given injective object $\mathcal{J}$. Choosing a decomposition $\mathcal{J} \cong \bigoplus_{x,x'} \mathcal{J}(x,x')^{\mu(x,x')}$, and summing up the homomorphisms $\tau_{x,x'}$, we obtain a homomorphism $\tau : \mathcal{J} \to \mathcal{J}_X$. Because

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{X^2}}(\widehat{\mathcal{B}}_q(X), \mathcal{J}) \cong \bigoplus\nolimits_{x,x'} \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{X^2}}\big(\widehat{\mathcal{B}}_q(X), \mathcal{J}(x,x')\big)^{\mu(x,x')}$$

and

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{B}}_q(X), \mathcal{J}_X) \cong \bigoplus\nolimits_{x,x'} \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}\big(\widehat{\mathcal{B}}_q(X), \mathcal{J}_X(x,x')\big)^{\mu(x,x')}$$

it follows from (2.3) that τ_q is an isomorphism. □

Let A be a $\mathbb{K}$-algebra. For an element $a \in A$, an index q and any $1 \leq j \leq q$, let us define

$$\tilde{\mathrm{d}}_j a := \underbrace{1 \otimes \cdots \otimes 1}_{j} \otimes (a \otimes 1 - 1 \otimes a) \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{B}_q(A). \tag{2.4}$$

Also let

$$\tilde{\mathrm{d}}_0 a := a \otimes 1 \otimes \cdots \otimes 1 - 1 \otimes \cdots \otimes 1 \otimes a \in \mathcal{B}_q(A). \tag{2.5}$$

The ring $\mathcal{B}_q(A)$ is an A-algebra by the map $a \mapsto a \otimes 1 \otimes \cdots \otimes 1$.

Let C be a noetherian commutative ring. A C-algebra A is étale if it is finitely generated and formally étale.

Lemma 2.6. *Denote by $\mathbb{K}[t_1, \ldots, t_n]$ the polynomial algebra in n variables, and let $\mathbb{K}[t_1, \ldots, t_n] \to A$ be an étale ring homomorphism. Then for any $q \geq 0$ the ring $\widehat{\mathcal{B}}_q(A)$ is a formal power series algebra over A in the $n(q+1)$ elements $\tilde{\mathrm{d}}_j t_i$.*

Proof. For any q the homomorphism $A \otimes \mathbb{K}[t_1, \ldots, t_n]^{\otimes(q+1)} \to \mathcal{B}_q(A)$ is étale, which implies that $A \to \mathcal{B}_q(A)$ is formally smooth of relative dimension $n(q+1)$. In particular $\Omega^1_{\mathcal{B}_q(A)/A}$ is a free $\mathcal{B}_q(A)$-module with basis $\{\mathrm{d}t_{i,j}\}$, where $1 \leq j \leq q+1$ and

$$t_{i,j} := \underbrace{1 \otimes \cdots \otimes 1}_{j} \otimes t_i \otimes 1 \otimes \cdots \otimes 1 \in \mathcal{B}_q(A).$$

Now for $1 \leq j \leq q$ we have $\tilde{\mathrm{d}}_j t_i = t_{i,j} - t_{i,j+1}$, whereas $\tilde{\mathrm{d}}_0 t_i - t_{i,q+1} \in A$. We see that the set $\{\mathrm{d}(\tilde{\mathrm{d}}_j t_i)\}_{j=0}^{q}$ is also a basis of $\Omega^1_{\mathcal{B}_q(A)/A}$.

Since the elements $\tilde{\mathrm{d}}_j t_i$ are all in the defining ideal I_q and since $\mathcal{B}_q(A) \to \widehat{\mathcal{B}}_q(A)$ is formally étale, we get a formally étale homomorphism

$$\phi : A[[\tilde{\mathrm{d}}_0 t_1, \ldots, \tilde{\mathrm{d}}_q t_n]] \to \widehat{\mathcal{B}}_q(A).$$

Because ϕ lifts the identity $\phi_0 : A \to A$ it follows that ϕ is bijective. □

Recall that X is said to be smooth over $\mathbb{K}$ if it is formally smooth and finite type (see [EGA IV] Section 17). A smooth scheme is also flat.

Lemma 2.7. *Suppose X is smooth over $\mathbb{K}$. Then for any $q \geq 0$ the functor*

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X), -\big) : \mathsf{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2} \to \mathsf{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2}$$

is exact.

Proof. The statement can be verified locally on X, so let $U = \operatorname{Spec} A \subset X$ be an affine open set that is étale over affine space $\mathbf{A}^n_{\mathbb{K}}$; cf. [EGA IV] Corollary 17.11.3. In other words there is an étale ring homomorphism $\mathbb{K}[t_1, \ldots, t_n] \to A$. According to Lemma 2.6, $\widehat{\mathcal{B}}_q(A)$ is a formal power series algebra over $\widehat{A^{\mathrm{e}}} = \widehat{\mathcal{B}}_0(A)$ in the elements $\tilde{\mathrm{d}}_j t_i$, where $1 \leq j \leq q$.

Denote by $I_{q,\mathrm{e}}$ the kernel of the ring homomorphism $\mathcal{B}_q(A) \to A^{\mathrm{e}}$, $a_0 \otimes a_1 \otimes \cdots \otimes a_{q+1} \mapsto a_0 \otimes a_1 \cdots a_{q+1}$. Let $\widehat{I}_{q,\mathrm{e}}$ be its completion. For any $m \geq 0$ the $\widehat{A^{\mathrm{e}}}$-module $\widehat{\mathcal{B}}_q(A)/\widehat{I}^m_{q,\mathrm{e}}$ is free of finite rank – with basis consisting of monomials in the $\tilde{\mathrm{d}}_j t_i$ – and it has the $\widehat{I}_0$-adic topology.

Passing to sheaves we see that for any m the functor $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X)/\widehat{\mathcal{I}}^m_{q,\mathrm{e}}, -\big)$ is exact. But for any discrete module $\mathcal{M}$,

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X), \mathcal{M}\big) \cong \lim_{m \to} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}^2}}\big(\widehat{\mathcal{B}}_q(X)/\widehat{\mathcal{I}}^m_{q,\mathrm{e}}, \mathcal{M}\big).$$

□

Theorem 2.8. *Suppose $\mathbb{K}$ is a noetherian ring and X is a smooth separated $\mathbb{K}$-scheme. Given a complex $\mathcal{M}^{\cdot} \in \mathsf{D}^+(\mathsf{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2})$ there is an isomorphism*

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{M}^{\cdot}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}^{\cdot})$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$. This isomorphism is functorial in $\mathcal{M}^{\cdot}$.

Proof. Let $\mathcal{M}^{\cdot} \to \mathcal{J}^{\cdot}$ be an injective resolution of $\mathcal{M}^{\cdot}$ in $\mathsf{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2}$. By this we mean that $\mathcal{M}^{\cdot} \to \mathcal{J}^{\cdot}$ is a quasi-isomorphism and $\mathcal{J}^{\cdot}$ is a bounded below complex of injectives objects in $\mathsf{Mod}_{\mathrm{disc}}\, \mathcal{O}_{\mathfrak{X}^2}$. Then each $\mathcal{J}^q$ is an injective $\mathcal{O}_{X^2}$-module supported on X, and

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, \mathcal{J}^{\cdot}) = \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{J}^{\cdot}) = \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}^{\cdot}).$$

Since the homomorphism $\widehat{\mathcal{B}}^{\cdot}(X) \to \mathcal{O}_X$ is split by the continuous $\mathcal{O}_X$-linear homomorphism s, Lemma 2.2 says that for any $q \geq 0$ the homomorphism

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, \mathcal{J}^q) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{J}^q)$$

is a quasi-isomorphism. Because $\widehat{\mathcal{B}}^{\cdot}(X)$ is bounded above and $\mathcal{J}^{\cdot}$ is bounded below, the usual spectral sequence shows that

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\mathcal{O}_X, \mathcal{J}^{\cdot}) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{J}^{\cdot})$$

is a quasi-isomorphism.

Next by Lemma 2.7 for any $q \leq 0$ the homomorphism

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^q(X), \mathcal{M}^{\cdot}) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^q(X), \mathcal{J}^{\cdot})$$

is a quasi-isomorphism. Therefore

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{M}^{\cdot}) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_{\mathfrak{X}^2}}(\widehat{\mathcal{B}}^{\cdot}(X), \mathcal{J}^{\cdot})$$

is a quasi-isomorphism. □

Now we may compare the two definitions of Hochschild cochain complexes.

Corollary 2.9. *Suppose $\mathbb{K}$ is a noetherian ring and X is a smooth separated $\mathbb{K}$-scheme. Given an $\mathcal{O}_X$-module $\mathcal{M}$ there is an isomorphism*

$$\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M})$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_{X^2})$. This isomorphism is functorial in $\mathcal{M}$. In particular for $\mathcal{M} = \mathcal{O}_X$ we get

$$\mathcal{C}^{\cdot}_{\mathrm{cd}}(X) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X).$$

Proof. This is immediate from the theorem, since $\mathsf{Mod}\,\mathcal{O}_X \subset \mathsf{Mod}_{\mathrm{disc}}\,\mathcal{O}_{\mathfrak{X}^2}$, and $\widehat{\mathcal{C}}^{\cdot}(X) = \mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \widehat{\mathcal{B}}^{\cdot}(X)$. □

Remark 2.10. Assume $\mathbb{K}$ is a field, and let $\mathcal{K}^{\cdot}_X$ be the residue complex of X, see [Ye3]. If X is smooth of dimension n over $\mathbb{K}$ then $0 \to \Omega^n_X \to \mathcal{K}^{-n}_X \to \cdots \to \mathcal{K}^0_X \to 0$ is a minimal injective resolution. Hence $\mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{K}^{\cdot}_X)$ is a bounded below complex of flasque sheaves isomorphic to $\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \Omega^n_X)[n]$. Moreover if $f : X \to Y$ is a proper morphism the trace $\mathrm{Tr}_f : f_* \mathcal{K}_X \to \mathcal{K}_Y$ induces a homomorphism of complexes

$$f_* \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{K}^{\cdot}_X) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_Y}(\widehat{\mathcal{C}}^{\cdot}(Y), \mathcal{K}^{\cdot}_Y).$$

This angle ought to be explored.

3. A Third Definition

In the paper [Sw] Swan makes the following definition. Let $\mathbb{K}$ be a commutative ring and X a $\mathbb{K}$-scheme. Let $\mathcal{C}_q(X)$ be the sheaf on X associated to the presheaf $U \mapsto \mathcal{C}_q\big(\Gamma(U, \mathcal{O}_X)\big)$. Then $\mathcal{C}^{\cdot}(X)$ is a complex of $\mathcal{O}_X$-modules. Given an $\mathcal{O}_X$-module $\mathcal{M}$ choose an injective resolution $\mathcal{M} \to \mathcal{J}^0 \to \mathcal{J}^1 \to \cdots$. The qth Hochschild cohomology of X with values in $\mathcal{M}$ is defined to be

$$\mathrm{HH}^q(X, \mathcal{M}) := \mathrm{H}^q\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{J}^{\cdot})).$$

This section is devoted to proving the following theorem.

Theorem 3.1. *Let $\mathbb{K}$ be a noetherian ring and X a flat finite type separated $\mathbb{K}$-scheme. Let $\mathcal{M}^{\cdot} \in \mathsf{D}^+(\mathsf{Mod}\,\mathcal{O}_X)$ be a complex. Assume either of the following:*

(i) *X is embeddable as a closed subscheme of some smooth $\mathbb{K}$-scheme, and $\mathbb{K}$ is a regular ring.*
(ii) *Each $\mathrm{H}^q \mathcal{M}^{\cdot}$ is quasi-coherent.*

Then there is an isomorphism

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{M}^{\cdot}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}^{\cdot})$$

in $\mathsf{D}^+(\mathsf{Mod}\,\mathcal{O}_{X^2})$. This isomorphism is functorial in $\mathcal{M}^{\cdot}$.

Corollary 3.2. *Under the assumptions of the theorem, with $\mathcal{M}^{\cdot} = \mathcal{M}$ a single $\mathcal{O}_X$-module, there is an isomorphism*

$$\mathrm{HH}^q(X, \mathcal{M}) \cong \mathrm{Ext}^q_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}).$$

Corollary 3.2 was proved by Swan in the case of a field $\mathbb{K}$ and a quasi-projective scheme X ([Sw] Theorem 2.1).

The proofs of Theorem 3.1 and Corollary 3.2 are at the end of the section, after some preparation.

The sheaves $\mathcal{C}_q(X)$ are ill behaved; they are not quasi-coherent except in trivial cases. The sheaves $\mathcal{B}_q(X)$, associated to the presheaves $U \mapsto \mathcal{B}_q(\Gamma(U, \mathcal{O}_X))$, are even more troublesome: we do not know if $\mathcal{B}_q(X)$ is an $\mathcal{O}_{X^2}$-module. We get around these problems by using the completions $\widehat{\mathcal{C}}_q(X)$.

Proposition 3.3. *Let $\mathbb{K}$ be a noetherian ring and X a flat finite type separated $\mathbb{K}$-scheme. Then there is an isomorphism*

$$\mathcal{C}^{\cdot}(X) \cong \mathcal{O}_X \otimes^{\mathrm{L}}_{\mathcal{O}_{X^2}} \mathcal{O}_X$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$.

Proof. For any affine open set $U = \operatorname{Spec} A \subset X$ there is a quasi-isomorphism $\mathcal{C}^{\cdot}(A) \to \widehat{\mathcal{C}}^{\cdot}(A)$; see Lemma 1.2. Therefore when we pass to sheaves we obtain a quasi-isomorphism $\mathcal{C}^{\cdot}(X) \to \widehat{\mathcal{C}}^{\cdot}(X)$. Now use Proposition 1.5. □

Definition 3.4. Let Y be a noetherian scheme. An $\mathcal{O}_Y$-module $\mathcal{L}$ is called *finite pseudo locally free* if $\mathcal{L} \cong \bigoplus_{i=1}^n g_{i!}\mathcal{L}_i$, where for each i, $g_i : U_i \to Y$ is the inclusion of an affine open set, $g_{i!}$ is extension by zero, and $\mathcal{L}_i$ is a locally free $\mathcal{O}_{U_i}$-module of finite rank.

According to [RD] Theorem II.7.8, for any noetherian scheme Y the category $\mathsf{Mod}\, \mathcal{O}_Y$ is locally noetherian.

Lemma 3.5. *Suppose Y is a noetherian scheme.*

1. *A finite pseudo locally free $\mathcal{O}_Y$-module $\mathcal{L}$ is a noetherian object in* $\mathsf{Mod}\, \mathcal{O}_Y$.
2. *Given a noetherian object $\mathcal{M} \in \mathsf{Mod}\, \mathcal{O}_Y$ there is a surjection $\mathcal{L} \twoheadrightarrow \mathcal{M}$ with $\mathcal{L}$ a finite pseudo locally free $\mathcal{O}_Y$-module.*
3. *Let $\mathcal{L}$ be a finite pseudo locally free $\mathcal{O}_Y$-module. Then $\mathcal{L}$ is a flat $\mathcal{O}_Y$-module.*
4. *If Y is separated and $\mathcal{L}$ is a finite pseudo locally free $\mathcal{O}_Y$-module then the functor*
$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, -) : \mathsf{QCoh}\, \mathcal{O}_Y \to \mathsf{QCoh}\, \mathcal{O}_Y$$
is exact.

Proof. (1) By the proof of [RD] Theorem II.7.8, for any inclusion $g : U \to X$ of an affine open set, the sheaf $g_!\mathcal{O}_U$ is noetherian in $\mathsf{Mod}\, \mathcal{O}_Y$. This implies that for any coherent $\mathcal{O}_U$-module $\mathcal{M}$, $g_!\mathcal{M}$ is noetherian in $\mathsf{Mod}\, \mathcal{O}_Y$.

(2) For every affine open subset $g : U \to Y$ and every section of $\Gamma(U, \mathcal{M})$ we get a homomorphism $g_!\mathcal{O}_U \to \mathcal{M}$. By the ascending chain condition finitely many of these cover $\mathcal{M}$.

(3) In order to verify flatness we may restrict to a sufficiently small open subset $V \subset Y$. Thus we can assume each $\mathcal{L}_i$ in Definition 3.4 is free; and hence we reduce to the case $\mathcal{L} = g_!\mathcal{O}_U$ for an affine open subset $g : U \to Y$.

For any $\mathcal{O}_Y$-module $\mathcal{M}$ we have

$$g_!\mathcal{O}_U \otimes_{\mathcal{O}_Y} \mathcal{M} \cong g_! g^* \mathcal{M}.$$

Since both functors g^* and $g_!$ are exact it follows that $g_!\mathcal{O}_U$ is flat.

(4) After the same reduction as in 3 we have

$$\mathcal{H}om_{\mathcal{O}_Y}(g_!\mathcal{O}_U, \mathcal{M}) \cong g_* g^* \mathcal{M}.$$

Since g is now an affine morphism the functor

$$g_* : \mathsf{QCoh}\,\mathcal{O}_U \to \mathsf{QCoh}\,\mathcal{O}_Y$$

is exact. □

Proof of Theorem 3.1. If condition (i) is satisfied then X^2 is embeddable in a regular scheme. Hence we can find a resolution $\cdots \to \mathcal{L}^{-1} \to \mathcal{L}^0 \to \mathcal{O}_X$ where all the $\mathcal{O}_{X^2}$-modules $\mathcal{L}^q$ are locally free of finite rank. Otherwise by Lemma 3.5 we can at least find such a resolution where the $\mathcal{L}^q$ are finite pseudo locally free $\mathcal{O}_{X^2}$-modules.

Since $\mathcal{L}^{\cdot}$ is a flat resolution of $\mathcal{O}_X$, by Proposition 3.3 we have

$$\mathcal{C}^{\cdot}(X) \cong \mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \mathcal{L}^{\cdot}$$

in $\mathsf{D}^-(\mathsf{Mod}\,\mathcal{O}_X)$.

Choose a quasi-isomorphism $\mathcal{M}^{\cdot} \to \mathcal{K}^{\cdot}$ where $\mathcal{K}^{\cdot}$ is a bounded below complex of injective $\mathcal{O}_X$-modules. Then choose a quasi-isomorphism $\mathcal{K}^{\cdot} \to \mathcal{J}^{\cdot}$ where $\mathcal{J}^{\cdot}$ is a bounded below complex of injective $\mathcal{O}_{X^2}$-modules. If condition (ii) holds then take $\mathcal{K}^{\cdot}$ and $\mathcal{J}^{\cdot}$ to be complexes of quasi-coherent injective modules over $\mathcal{O}_X$ and $\mathcal{O}_{X^2}$ respectively (cf. [RD] Theorem II.7.18).

We have

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}^{\cdot}) = \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{J}^{\cdot}),$$

and there is a quasi-isomorphism

$$\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{J}^{\cdot}) \to \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{L}^{\cdot}, \mathcal{J}^{\cdot}).$$

Since either all the $\mathcal{L}^q$ are locally free $\mathcal{O}_Y$-modules of finite rank (in case condition (i) holds), or all the $\mathcal{L}^q$ are finite pseudo locally free and all the $\mathcal{K}^q$ and $\mathcal{J}^q$ are quasi-coherent (in case condition (ii) holds), it follows that we have a quasi-isomorphism

$$\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{L}^{\cdot}, \mathcal{K}^{\cdot}) \to \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{L}^{\cdot}, \mathcal{J}^{\cdot}).$$

But

$$\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{L}^{\cdot}, \mathcal{K}^{\cdot}) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \mathcal{L}^{\cdot}, \mathcal{K}^{\cdot}).$$

Finally

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \mathcal{L}^{\cdot}, \mathcal{K}^{\cdot}) &= \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X \otimes_{\mathcal{O}_{X^2}} \mathcal{L}^{\cdot}, \mathcal{K}^{\cdot}) \\ &\cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{K}^{\cdot}) \\ &= \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{M}^{\cdot})\end{aligned}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathcal{O}_X)$. To this isomorphism we apply the functor Δ_*. □

Proof of Corollary 3.2. Choose an injective resolution $\mathcal{M} \to \mathcal{J}^{\cdot}$. Then

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{J}^{\cdot}) = \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{M}).$$

Because each sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^q(X), \mathcal{J}^p)$ is flasque it follows that

$$\mathrm{H}^q\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{J}^{\cdot})) = \mathrm{H}^q\mathrm{R}\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{C}^{\cdot}(X), \mathcal{M})).$$

The left hand side is by definition $\mathrm{HH}^q(X,\mathcal{M})$. The right hand side is, according to the theorem,

$$\mathrm{H}^q\mathrm{R}\Gamma(X, \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X,\mathcal{M})) \cong \mathrm{Ext}^q_{\mathcal{O}_{X^2}}(\mathcal{O}_X,\mathcal{M})).$$

□

4. Decomposition in Characteristic 0

In is section we prove that the Hochschild cochain complex decomposes when X is smooth and $\operatorname{char}\mathbb{K} = 0$. Throughout this section the base ring $\mathbb{K}$ is noetherian and X is a separated finite type scheme over $\mathbb{K}$.

Let A be a finitely generated $\mathbb{K}$-algebra and $\Omega^q_A = \Omega^q_{A/\mathbb{K}}$ the module of relative Kähler differentials of degree q. We declare $\bigoplus_q \Omega^q_A[q]$ to be a complex with trivial coboundary. For any $q \geq 0$ there is an A-linear homomorphism

$$\pi : \mathcal{C}_q(A) = \mathcal{B}_q(A) \otimes_{A^{\mathrm{e}}} A \to \Omega^q_A,$$

$$\pi((1 \otimes a_1 \otimes \cdots \otimes a_q \otimes 1) \otimes 1) = \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_q.$$

Since $\pi\partial = 0$ we obtain a homomorphism of complexes $\pi : \mathcal{C}^{\cdot}(A) \to \bigoplus_q \Omega^q_A[q]$.

Recall that $I_q = \operatorname{Ker}(\mathcal{B}_q(A) \to A)$.

Lemma 4.1. *Let $m > q$. Then $\pi(I_q^m \cdot \mathcal{C}_q(A)) = 0$.*

Proof. Let us consider Ω^q_A as a $\mathcal{B}_q(A)$-module. Then π is a differential operator of order $\leq q$. Now use [Ye1] Proposition 1.4.6. □

The lemma shows that π is continuous, so it extends to a homomorphism of complexes

$$\pi : \widehat{\mathcal{C}}^{\cdot}(A) \to \bigoplus_q \Omega^q_A[q].$$

If we take $A = \mathbb{K}[\boldsymbol{t}] := \mathbb{K}[t_1,\ldots,t_n]$ the polynomial algebra in n variables, then $\mathcal{B}_q(\mathbb{K}[\boldsymbol{t}])$ is a polynomial algebra over $\mathbb{K}[\boldsymbol{t}]$ in the $n(q+1)$ elements $\tilde{\mathrm{d}}_j t_i$, cf. Lemma 2.6. Put a $\mathbb{Z}$-grading on $\mathcal{B}_q(\mathbb{K}[\boldsymbol{t}])$ by declaring $\deg \tilde{\mathrm{d}}_j t_i := 1$, and $\deg a := 0$ for $0 \neq a \in \mathbb{K}[\boldsymbol{t}]$. This induces a grading on $\mathcal{C}_q(\mathbb{K}[\boldsymbol{t}]) = \mathcal{B}_q(\mathbb{K}[\boldsymbol{t}]) \otimes_{\mathcal{B}_0(\mathbb{K}[\boldsymbol{t}])} \mathbb{K}[\boldsymbol{t}]$. Also consider $\Omega^q_{\mathbb{K}[\boldsymbol{t}]}$ to be homogeneous of degree q.

Lemma 4.2. *The homomorphism $\pi : \mathcal{C}_q(\mathbb{K}[\boldsymbol{t}]) \to \Omega^q_{\mathbb{K}[\boldsymbol{t}]}$ has degree 0.*

Proof. Since $\mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$ is a free $\mathbb{K}[\boldsymbol{t}]$-module with basis the monomials $\beta = \tilde{\mathrm{d}}_{j_1} t_{i_1} \cdots \tilde{\mathrm{d}}_{j_m} t_{i_m}$ with $1 \leq j_1,\ldots,j_m \leq q$ and $1 \leq i_1,\ldots,i_m \leq n$ it suffices to look at $\pi(\beta)$. We note that $\deg\beta = m$. Now

$$\pi((1 \otimes a_1 \otimes \cdots \otimes a_q \otimes 1) \otimes 1) = 0$$

if any $a_p \in \mathbb{K}$, $1 \leq p \leq q$. Therefore $\pi(\beta) = 0$ unless $\{j_1,\ldots,j_m\} = \{1,\ldots,q\}$. We conclude that $\pi(\beta) = 0$ if $m < q$. On the other hand, since each $\tilde{\mathrm{d}}_j t_i \in I_q$, Lemma 4.1 tells us that $\pi(\beta) = 0$ if $m > q$. □

The lemma says that π is a morphism in the category $\mathsf{GrMod}\,\mathbb{K}[\boldsymbol{t}]$ of $\mathbb{Z}$-graded $\mathbb{K}[\boldsymbol{t}]$-modules and degree 0 homomorphisms.

Lemma 4.3. *Assume $n!$ is invertible in $\mathbb{K}$. Then $\pi : \mathcal{C}^{\cdot}(\mathbb{K}[\boldsymbol{t}]) \to \bigoplus_q \Omega^q_{\mathbb{K}[\boldsymbol{t}]}[q]$ is a homotopy equivalence of complexes over $\mathsf{GrMod}\,\mathbb{K}[\boldsymbol{t}]$.*

Proof. Write $A := \mathbb{K}[\boldsymbol{t}]$. For $q > n$ we have $\Omega^q_A = 0$, and $q!$ is invertible for all $q \leq n$. So by [Lo] Proposition 1.3.16 the homomorphism of complexes $\pi : \mathcal{C}^{\cdot}(A) \to \bigoplus_q \Omega^q_A[q]$ is a quasi-isomorphism. Now the complexes $\mathcal{C}^{\cdot}(A)$ and $\bigoplus_q \Omega^q_A[q]$ are both bounded above complexes of projective objects in $\mathsf{GrMod}\, A$. So the quasi-isomorphism $\pi : \mathcal{C}^{\cdot}(A) \to \bigoplus_q \Omega^q_A[q]$ has to be a homotopy equivalence. Namely there are homomorphisms $\phi : \Omega^q_A \to \mathcal{C}^{-q}(A)$ and $h : \mathcal{C}^{-q}(A) \to \mathcal{C}^{-q-1}(A)$ in $\mathsf{GrMod}\, A$ satisfying: $\partial\phi = 0$, $1_{\mathcal{C}^{-q}(A)} - \phi\pi = h\partial - \partial h$ and $1_{\Omega^q_A} - \pi\phi = 0$. ☐

Proposition 4.4. *Suppose $\mathbb{K}[\boldsymbol{t}] \to A$ is étale and $n!$ is invertible in $\mathbb{K}$. Then $\pi : \widehat{\mathcal{C}}^{\cdot}(A) \to \bigoplus_q \Omega^q_A[q]$ is a homotopy equivalence of topological A-modules. Namely there are continuous A-linear homomorphisms $\phi : \Omega^q_A \to \widehat{\mathcal{C}}^{-q}(A)$ and $h : \widehat{\mathcal{C}}^{-q}(A) \to \widehat{\mathcal{C}}^{-q-1}(A)$ satisfying: $\partial\phi = 0$, $1_{\widehat{\mathcal{C}}^{-q}(A)} - \phi\pi = h\partial - \partial h$ and $1_{\Omega^q_A} - \pi\phi = 0$. Furthermore the homomorphisms ϕ and h are functorial in A.*

Proof. Declare A to be homogeneous of degree 0. From Lemma 4.3 we get homomorphisms
$$\phi : A \otimes_{\mathbb{K}[\boldsymbol{t}]} \Omega^q_{\mathbb{K}[\boldsymbol{t}]} \to A \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}^{-q}(\mathbb{K}[\boldsymbol{t}])$$
and
$$h : A \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}^{-q}(\mathbb{K}[\boldsymbol{t}]) \to A \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}^{-q-1}(\mathbb{K}[\boldsymbol{t}])$$
in $\mathsf{GrMod}\, A$, satisfying the homotopy equations. Because $\mathbb{K}[\boldsymbol{t}] \to A$ is étale there is an isomorphism $A \otimes_{\mathbb{K}[\boldsymbol{t}]} \Omega^q_{\mathbb{K}[\boldsymbol{t}]} \cong \Omega^q_A$. By Lemma 2.6, $\widehat{\mathcal{C}}_q(A)$ is the completion of $A \otimes_{\mathbb{K}[\boldsymbol{t}]} \mathcal{C}_q(\mathbb{K}[\boldsymbol{t}])$ with respect to the grading. Therefore ϕ and h extend uniquely to continuous homomorphisms as claimed. The functoriality in A follows from the uniqueness. ☐

Theorem 4.5. *Let $\mathbb{K}$ be a noetherian ring, let X be a separated smooth $\mathbb{K}$-scheme of relative dimension n, and assume $n!$ is invertible in $\mathbb{K}$. Then for any complex $\mathcal{M}^{\cdot} \in \mathsf{D}(\mathsf{Mod}\, \mathcal{O}_X)$ the homomorphism of complexes*
$$\mathcal{H}om_{\mathcal{O}_X}\big(\bigoplus\nolimits_q \Omega^q_X[q], \mathcal{M}^{\cdot}\big) \to \mathcal{H}om^{\mathrm{cont}}_{\mathcal{O}_X}\big(\widehat{\mathcal{C}}^{\cdot}(X), \mathcal{M}^{\cdot}\big)$$
induced by π is a quasi-isomorphism.

Proof. The assertion may be checked locally on X, so let $U = \operatorname{Spec} A \subset X$ be an affine open set admitting an étale morphism $U \to \mathbf{A}^n_{\mathbb{K}}$. If $U' = \operatorname{Spec} A' \subset U$ is any affine open subset then the ring homomorphisms $\mathbb{K}[\boldsymbol{t}] \to A \to A'$ are étale. We deduce from Proposition 4.4 that $\pi : \widehat{\mathcal{C}}^{\cdot}(U) \to \bigoplus_q \Omega^q_U[q]$ is a homotopy equivalence of topological $\mathcal{O}_U$-modules, i.e. there are continuous $\mathcal{O}_U$-linear homomorphisms $\phi : \Omega^q_U \to \widehat{\mathcal{C}}^{-q}(U)$ and $h : \widehat{\mathcal{C}}^{-q}(U) \to \widehat{\mathcal{C}}^{-q-1}(U)$ satisfying the homotopy equations. ☐

Corollary 4.6. *Under the assumptions of the theorem, for any complex $\mathcal{M}^{\cdot} \in \mathsf{D}^+(\mathsf{Mod}\, \mathcal{O}_X)$ there is an isomorphism*
$$\bigoplus\nolimits_q \big(\bigwedge\nolimits^q \mathcal{T}_X\big)[-q] \otimes_{\mathcal{O}_X} \mathcal{M}^{\cdot} \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}^{\cdot})$$
in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$. This isomorphism is functorial in $\mathcal{M}^{\cdot}$. In particular for $\mathcal{M}^{\cdot} = \mathcal{O}_X$ we obtain
$$\bigoplus\nolimits_q \big(\bigwedge\nolimits^q \mathcal{T}_X\big)[-q] \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$$
in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$.

Proof. Use Theorem 2.8. ☐

Observe that the isomorphism $\bigwedge^q \mathcal{T}_X \cong \mathcal{E}xt^q_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$ deduced from Corollary 4.6 differs by a factor of $q!$ from the Hochschild-Kostant-Rosenberg isomorphism (cf. [HKR] Theorem 5.2 and [Lo] Theorem 3.4.4).

Taking global cohomology in Corollary 4.6 we deduce the next corollary.

Corollary 4.7. *Under the assumptions of the theorem, for any $\mathcal{O}_X$-module $\mathcal{M}$ there is an isomorphism*

$$\operatorname{Ext}^i_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{M}) \cong \bigoplus\nolimits_q \mathrm{H}^{i-q}\big(X, \big(\bigwedge\nolimits^q \mathcal{T}_X\big) \otimes_{\mathcal{O}_X} \mathcal{M}\big).$$

Corollary 4.7 was proved by Swan ([Sw] Corollary 2.6) in the case X is smooth quasi-projective over the field $\mathbb{K} = \mathbb{C}$.

Let us concentrate on the Hochschild cochain complex with values in $\mathcal{O}_X$. Here we give notation to the homomorphism induced by π; it is

$$\pi_{\mathrm{cd}} : \bigwedge\nolimits^q \mathcal{T}_X \to \mathcal{C}^q_{\mathrm{cd}}(X).$$

The precise formula on an affine open set $U = \operatorname{Spec} A$ is

$$\pi_{\mathrm{cd}}(v_1 \wedge \cdots \wedge v_q)(1 \otimes a_1 \otimes \cdots \otimes a_q \otimes 1) = \sum_{\sigma \in \Sigma_q} \operatorname{sgn}(\sigma) v_{\sigma(1)}(a_1) \cdots v_{\sigma(q)}(a_q)$$

for $v_i \in \mathcal{T}_A = \operatorname{Der}_{\mathbb{K}}(A)$ and $a_i \in A$, where $\operatorname{sgn}(\sigma)$ denotes the sign of the permutation σ.

Theorem 4.5 says that π_{cd} is a quasi-isomorphism if X is smooth of relative dimension n and $n!$ is invertible in $\mathbb{K}$. The next result is a converse.

Theorem 4.8. *Let $\mathbb{K}$ be a Gorenstein noetherian ring of finite Krull dimension and let X be a smooth separated $\mathbb{K}$-scheme of relative dimension n. Then the following three conditions are equivalent.*

(i) $\pi : \widehat{\mathcal{C}}^{\cdot}(X) \to \bigoplus_q \Omega^q_X[q]$ *is a quasi-isomorphism.*
(ii) $\pi_{\mathrm{cd}} : \bigoplus_q (\bigwedge^q \mathcal{T}_X)[-q] \to \mathcal{C}^{\cdot}_{\mathrm{cd}}(X)$ *is a quasi-isomorphism.*
(iii) $n!$ *is invertible in* $\mathcal{O}_X$.

Proof. All three conditions can be checked locally. So take a sufficiently small affine open set $U = \operatorname{Spec} A \subset X$ such that there is an étale homomorphism $\mathbb{K}[t_1, \ldots, t_n] \to A$. We will prove that the three conditions are equivalent on U (cf. Propositions 1.4 and 1.6).

(i) $\Leftrightarrow$ (ii): Denote by D the functor $\operatorname{Hom}_A(-, A)$ and by $\mathrm{R}D : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$ its derived functor. Consider the homomorphism of complexes

$$\pi_{\mathrm{cd}} : \bigoplus\nolimits_q \big(\bigwedge\nolimits^q \mathcal{T}_A\big)[-q] \to \mathcal{C}^{\cdot}_{\mathrm{cd}}(A).$$

By Lemma 2.6, $\widehat{\mathcal{C}}_q(A)$ is a power series algebra over A in nq elements. Hence the adjunction map

$$\widehat{\mathcal{C}}_q(A) \to D\mathcal{C}^q_{\mathrm{cd}}(A) = \operatorname{Hom}_A(\operatorname{Hom}^{\mathrm{cont}}_A(\widehat{\mathcal{C}}_q(A), A), A)$$

is bijective, and we get

$$\pi = D(\pi_{\mathrm{cd}}) : \widehat{\mathcal{C}}_q(A) \to \Omega^q_A.$$

We claim that moreover $\widehat{\mathcal{C}}^{\cdot}(A) = \mathrm{R}D\mathcal{C}^{\cdot}_{\mathrm{cd}}(A)$ and

$$\pi = \mathrm{R}D(\pi_{\mathrm{cd}}) : \widehat{\mathcal{C}}^{\cdot}(A) \to \bigoplus\nolimits_q \Omega^q_A[q]. \tag{4.9}$$

To verify this let us choose a bounded injective resolution $A \to J^{\cdot}$ in $\mathsf{Mod}\, A$, which is possible since A is Gorenstein of finite Krull dimension. Each A-module $\mathcal{C}^q_{\mathrm{cd}}(A)$ is free. Then, even though the complex $\mathcal{C}^{\cdot}_{\mathrm{cd}}(A)$ is unbounded,

$$\mathrm{Hom}_A(\mathcal{C}^{\cdot}_{\mathrm{cd}}(A), A) \to \mathrm{Hom}_A(\mathcal{C}^{\cdot}_{\mathrm{cd}}(A), J^{\cdot})$$

is a quasi-isomorphism. Thus the claim is proved.

The functor $\mathrm{R}D$ is a duality of the subcategory $\mathsf{D}_{\mathrm{c}}(\mathsf{Mod}\, A)$ of complexes with finitely generated cohomologies. By Corollary 2.9 we know that $\mathcal{C}^{\cdot}_{\mathrm{cd}}(A) \in \mathsf{D}_{\mathrm{c}}(\mathsf{Mod}\, A)$, and clearly $\bigoplus_q (\bigwedge^q \mathcal{T}_A)[-q] \in \mathsf{D}_{\mathrm{c}}(\mathsf{Mod}\, A)$. We conclude that π_{cd} is an isomorphism in $\mathsf{D}_{\mathrm{c}}(\mathsf{Mod}\, A)$ iff $\pi = \mathrm{R}D(\pi_{\mathrm{cd}})$ is an isomorphism.

(i) $\Leftrightarrow$ (iii): We know that $\mathcal{C}^{\cdot}(A) \to \widehat{\mathcal{C}}^{\cdot}(A)$ is a quasi-isomorphism. Let $\epsilon : \Omega^q_A \to \mathrm{H}^{-q}\mathcal{C}^{\cdot}(A)$ be the isomorphism of the Hochschild-Kostant-Rosenberg Theorem [Lo] Theorem 3.4.4. Then by [Lo] Proposition 1.3.16, $\pi\epsilon(\alpha) = q!\alpha$ for all $\alpha \in \Omega^q_A$.

If $n!$ is invertible in A then so is $q!$ for all $q \leq n$. For $q > n$ we have $\Omega^q_A = 0$. So π is a quasi-isomorphism.

Conversely, suppose π is a quasi-isomorphism. Let α be a basis of the free A-module Ω^n_A. Then $n!\alpha = \pi\epsilon(\alpha)$ is also a basis, so $n!$ must be invertible in A. □

Oddly, if X is affine there is always a decomposition, regardless of characteristic.

Proposition 4.10. *If $\mathbb{K}$ is noetherian and X is affine and smooth over $\mathbb{K}$ then there is a canonical isomorphism*

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X) \cong \bigoplus\nolimits_q \left(\bigwedge\nolimits^q \mathcal{T}_X\right)[-q]$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$.

Proof. Say $X = \operatorname{Spec} A$. Let $A \to J^{\cdot}$ be an injective resolution in $\mathsf{Mod}\, A^{\mathrm{e}}$, and set $N^{\cdot} := \mathrm{Hom}_{A^{\mathrm{e}}}(A, J^{\cdot})$, which is a complex of A-modules. Denote by $F : \mathsf{Mod}\, A \to \mathsf{Mod}\, A^{\mathrm{e}}$ the restriction of scalars functor for the homomorphism $A^{\mathrm{e}} \to A$ (this is the ring version of Δ_*). Then $FN^{\cdot} = \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, A)$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. Let $G : \mathsf{Mod}\, A^{\mathrm{e}} \to \mathsf{Mod}\, \mathcal{O}_{X^2}$ be the sheafication functor. Since $GJ^{\cdot}$ is an injective resolution of $\mathcal{O}_X$ we see that

$$GFN^{\cdot} \cong \mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, GJ^{\cdot}) \cong \mathrm{R}\mathcal{H}om_{\mathcal{O}_{X^2}}(\mathcal{O}_X, \mathcal{O}_X)$$

in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$.

Now according to the Hochschild-Kostant-Rosenberg Theorem (see [HKR] Theorem 5.2 and [Lo] Theorem 3.4.4) the cohomology $\mathrm{H}^q N^{\cdot} = \mathrm{Ext}^q_{A^{\mathrm{e}}}(A, A) \cong \bigwedge^q \mathcal{T}_A$. Since the A-modules $\bigwedge^q \mathcal{T}_A$ are projective and almost all of them are zero, it is easy to see, by truncation and splitting, that $N^{\cdot} \cong \bigoplus_q (\bigwedge^q \mathcal{T}_A)[-q]$ in $\mathsf{D}(\mathsf{Mod}\, A)$. Therefore $GFN^{\cdot} \cong \bigoplus_q (\bigwedge^q \mathcal{T}_X)[-q]$ in $\mathsf{D}(\mathsf{Mod}\, \mathcal{O}_{X^2})$. □

Question 4.11. We have seen that if X is affine or if $\mathbb{K}$ contains enough denominators then the Hochschild cochain complex $\mathcal{C}^{\cdot}_{\mathrm{cd}}(X)$ decomposes in the derived category. Is there decomposition in other circumstances?

Question 4.12. How is the decomposition of Theorem 4.5 related to the Hodge decomposition of Gerstenhaber-Schack [GS]? Perhaps the comparison to Swan's definition of Hochschild cochains (Section 3) can help.

Remark 4.13. In [Ko], $\mathcal{D}^{\cdot}_{\mathrm{poly}}(X) := \mathcal{C}^{\cdot}_{\mathrm{cd}}(X)[1]$ is called the complex of poly-differential operators. The complex $\mathcal{T}^{\cdot}_{\mathrm{poly}}(X) := \bigoplus_q (\bigwedge^q \mathcal{T}_X)[1-q]$ is called the complex of poly-vector fields. Kontsevich's Formality Theorem [Ko] says that $(\frac{1}{q!}\pi^q_{\mathrm{cd}})_{q \geq 0}$ is the degree 1 component of an L_∞-quasi-isomorphism of the DG Lie algebra structures of $\mathcal{D}^{\cdot}_{\mathrm{poly}}(X)$ and $\mathcal{T}^{\cdot}_{\mathrm{poly}}(X)$ when $\mathbb{K}$ is a field of characteristic 0.

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