DUALIZING COMPLEXES AND PERVERSE MODULES OVER DIFFERENTIAL ALGEBRAS

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Abstract. A differential algebra of finite type over a field $\Bbbk$ is a filtered algebra A, such that the associated graded algebra is finite over its center, and the center is a finitely generated $\Bbbk$-algebra. The prototypical example is the algebra of differential operators on a smooth affine variety, when $\operatorname{char} \Bbbk = 0$. We study homological and geometric properties of differential algebras of finite type. The main results concern the rigid dualizing complex over such an algebra A: its existence, structure and variance properties. We also define and study perverse A-modules, and show how they are related to the Auslander property of the rigid dualizing complex of A.

0. Introduction

The "classical" Grothendieck Duality theory, dealing with dualizing complexes over schemes, was developed in the book *Residues and Duality* by Hartshorne [RD]. A duality theory for noncommutative noetherian algebras over a base field $\Bbbk$ was introduced in [Ye1]. Roughly speaking, a dualizing complex over a $\Bbbk$-algebra A is a complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$, such the functor

$$\mathrm{RHom}_A(-, R) : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$$

is a duality, with inverse $\mathrm{RHom}_{A^{\mathrm{op}}}(-, R)$. Here A^{op} is the opposite algebra, $A^{\mathrm{e}} := A \otimes_{\Bbbk} A^{\mathrm{op}}$, and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is the derived category of bounded complexes of A-modules with finite cohomologies. See Definition 4.1 for details. In the decade since its introduction this noncommutative duality theory has progressed in several directions; cf. the papers [VdB], [Jo], [MY], [WZ] and [Ch2].

One of the biggest problems in noncommutative duality theory is *existence of dualizing complexes*. The most effective existence criterion to date is due to Van den Bergh [VdB]. It says the following: suppose the $\Bbbk$-algebra A has some exhaustive nonnegative filtration $G = \{G_i A\}$ such that the graded algebra $\mathrm{gr}^G A$ is connected, noetherian and commutative. Then A has a rigid dualizing complex R_A (see Definition 4.10). Moreover in [YZ1] we proved that the dualizing complex R_A has the Auslander property (see Definition 4.6), and it is unique up to a unique isomorphism.

Let us remind the reader that a graded $\Bbbk$-algebra $\bar{A}$ is called *connected* if $\bar{A} = \bigoplus_{i \in \mathbb{N}} \bar{A}_i$, $\bar{A}_0 = \Bbbk$ and each $\bar{A}_i$ is a finite $\Bbbk$-module. An exhaustive nonnegative

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filtration G on A, such that the graded algebra $\bar{A} := \operatorname{gr}^G A$ is a noetherian connected graded $\Bbbk$-algebra, can be considered as a "noncommutative compactification of $\operatorname{Spec} A$". Indeed, when A is commutative, let $\widetilde{A} := \operatorname{Rees}^G A \subset A[t]$ be the Rees algebra. Then $\operatorname{Proj} \widetilde{A}$ is a projective scheme over $\Bbbk$, the divisor $\{t = 0\}$ is ample, and its complement is isomorphic to $\operatorname{Spec} A$.

We have observed that often in the literature one encounters algebras A that are equipped with an exhaustive nonnegative filtration $F = \{F_i A\}$, such that $\operatorname{gr}^F A$ is noetherian and finite over its center, yet is not connected. The main goal of this paper is to prove that Van den Bergh's existence criterion applies to such algebras too, and furthermore the rigid dualizing complex R_A has especially good homological and geometric properties. These properties shall be used in our sequel paper [YZ3] to construct rigid dualizing complexes over noncommutative ringed schemes.

Let us introduce some conventions now. Throughout the paper $\Bbbk$ is a field. By default all $\Bbbk$-algebras are associative unital algebras, all modules are left modules, and all bimodules are central over $\Bbbk$. Given a $\Bbbk$-algebra A we denote by $\mathsf{Mod}\, A$ the category of A-modules. The unadorned tensor product $\otimes$ will mean $\otimes_{\Bbbk}$.

Let C be a finitely generated commutative $\Bbbk$-algebra and let A be a C-ring (i.e. there is a ring homomorphism $C \to A$). We call A a *differential C-ring of finite type* if there exists a nonnegative exhaustive filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ of A such that $\operatorname{gr}^F A$ is a finite module over its center $\mathrm{Z}(\operatorname{gr}^F A)$, and $\mathrm{Z}(\operatorname{gr}^F A)$ is a finitely generated C-algebra. We also call A a *differential $\Bbbk$-algebra of finite type*. See Definitions 2.1 and 2.2 for a precise formulation. The prototypical examples are:

(1) A is a finite C-algebra (e.g. an Azumaya algebra);
(2) A is the ring $\mathcal{D}(C)$ of differential operators of C, where C is smooth and $\operatorname{char} \Bbbk = 0$; and
(3) A is the universal enveloping algebra $\mathrm{U}(C; L)$ of a finite Lie algebroid L over C.

In (1) and (3) there are no regularity assumptions on C, A or L. It is not hard to see that any quotient A/I is also a differential $\Bbbk$-algebra of finite type. Also if A_1 and A_2 are differential $\Bbbk$-algebras of finite type then so is the tensor product $A_1 \otimes A_2$.

The key technical result is the following "Theorem on the Two Filtrations":

Theorem 0.1. *Let A be a differential $\Bbbk$-algebra of finite type. Then there is a nonnegative exhaustive filtration $G = \{G_i A\}_{i \in \mathbb{Z}}$ of A such that $\operatorname{gr}^G A$ is a commutative, finitely generated, connected graded $\Bbbk$-algebra.*

Theorem 0.1 is proved in Section 3, where it is restated as Theorem 3.1.

Since Van den Bergh's criterion can now be applied, and using results from [YZ1], we obtain:

Corollary 0.2. *Let A be a differential $\Bbbk$-algebra of finite type. Then A has an Auslander rigid dualizing complex R_A. For any finite A-module M one has $\operatorname{Cdim} M = \operatorname{GKdim} M$.*

The corollary is proved in Section 8, where it is restated as Theorem 8.1. We remind that $\operatorname{GKdim} M$ is the Gelfand-Kirillov dimension of M. The canonical dimension $\operatorname{Cdim} M$ is defined by

$$\operatorname{Cdim} M := -\inf \{q \mid \operatorname{Ext}^q_A(M, R_A) \neq 0\} \in \mathbb{Z} \cup \{-\infty\}$$

for a finite A-module M, and by

$$\operatorname{Cdim} M := \sup \{\operatorname{Cdim} M' \mid M' \subset M \text{ is finite}\}$$

in general. The Auslander property says that $\operatorname{Cdim} M' \leq \operatorname{Cdim} M$ for any finite left or right A-module M and any submodule $M' \subset M$; and it implies that Cdim is an exact dimension function.

The *rigid perverse t-structure* on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is defined as follows:

$${}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{H}^{i} \operatorname{RHom}_{A}(M, R_{A}) = 0 \text{ for all } i < 0\}$$

and

$${}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{H}^{i} \operatorname{RHom}_{A}(M, R_{A}) = 0 \text{ for all } i > 0\}.$$

The heart

$${}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{0} := {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} \cap {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}$$

is called the category of *perverse A-modules*. It is an abelian category, dual to the category $\mathsf{Mod}_{\mathsf{f}}\, A^{\mathrm{op}}$ of finite A^{op}-modules.

Here is an alternative characterization of the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, which resembles the original definition in [BBD]. For a module M and any integer i define $\Gamma_{\mathsf{M}_i} M$ to be the biggest submodule of M with $\operatorname{Cdim} \leq i$. This is a functor $\Gamma_{\mathsf{M}_i} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$, and we denote by $\mathrm{H}^{j}_{\mathsf{M}_i}$ its jth right derived functor. The next result is a special case of Theorem 7.9:

Theorem 0.3. *Let A be a differential $\Bbbk$-algebra of finite type and $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.*

(1) *$M \in {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}$ iff $\operatorname{Cdim} \mathrm{H}^{j} M < i$ for all integers i, j such that $j > -i$.*

(2) *$M \in {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}$ iff $\mathrm{H}^{j}_{\mathsf{M}_i} M = 0$ for all integers i, j such that $j < -i$.*

Interestingly, in a recent preprint [Ka] Kashiwara has proved a similar result; see Remark 7.11.

Suppose C is a finitely generated commutative $\Bbbk$-algebra. A C-bimodule M is called a *differential C-bimodule* if it has some bounded below exhaustive filtration $F = \{F_i M\}_{i \in \mathbb{Z}}$ by C-sub-bimodules, such that $\operatorname{gr}^F M$ is a central C-bimodule. This equivalent to the condition that the support of the C^{e}-module M is in the diagonal $\Delta(U) \subset U^2$, where $U := \operatorname{Spec} C$ (see Proposition 5.21).

Theorem 0.4. *Let C be a finitely generated commutative $\Bbbk$-algebra and A a differential C-ring of finite type. Let R_A be the rigid dualizing complex of A. Then for every i the cohomology bimodule $\mathrm{H}^i R_A$ is a differential C-bimodule.*

This theorem is repeated as Theorem 8.14. One consequence is that the rigid dualizing complex R_A localizes on $\operatorname{Spec} C$ (see Corollary 8.15).

Let $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$. It too is a differential $\Bbbk$-algebra of finite type, so it has a rigid dualizing complex $R_{A^{\mathrm{e}}}$, and the category ${}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{e}})^{0}$ of perverse A^{e}-modules exists. By definition the rigid dualizing complex R_A of A is an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Theorem 0.5. *Let A be a differential $\Bbbk$-algebra of finite type. Then the rigid dualizing complex R_A is a perverse A^{e}-module, i.e. $R_A \in {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{e}})^{0}$.*

This theorem is repeated as Theorem 8.9 in the body of the paper. In addition to being interesting in itself, this result is used in [YZ3] to glue rigid dualizing complexes on noncommutative ringed schemes – as perverse bimodules.

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1. Filtrations of Rings

By a *filtration* of a $\Bbbk$-algebra A we mean an ascending filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ by $\Bbbk$-submodules such that $1 \in F_0 A$ and $F_i A \cdot F_j A \subset F_{i+j} A$. We shall call (A, F) a filtered $\Bbbk$-algebra; but often we shall just say that A is a filtered algebra and leave F implicit.

Suppose (A, F) is a filtered $\Bbbk$-algebra. Given an A-module M, by an (A, F)-filtration of M we mean an ascending filtration $F = \{F_i M\}_{i \in \mathbb{Z}}$ of M by $\Bbbk$-submodules such that $F_i A \cdot F_j M \subset F_{i+j} M$ for all i and j. We call (M, F) a filtered (A, F)-module, and allow ourselves to drop reference to F when no confusion may arise.

We say the filtration F on M is *exhaustive* if $M = \bigcup_i F_i M$, F is *separated* if $0 = \bigcap_i F_i M$, F is *bounded below* if $F_{i_0 - 1} M = 0$ for some integer i_0, and F is *nonnegative* if $F_{-1} M = 0$. The trivial filtration on M is $F_{-1} M := 0$, $F_0 M := M$.

Let us recall some facts about associated graded modules, and establish some notation. It shall be convenient to use the ordered semigroup $\mathbb{Z} \cup \{-\infty\}$ where $-\infty < i$ for every $i \in \mathbb{Z}$, and $i + j := -\infty$ if either $i = -\infty$ or $j = -\infty$.

Let (M, F) be an exhaustive filtered module. The associated graded module is

$$\operatorname{gr}(M, F) = \operatorname{gr}^F M = \bigoplus\nolimits_{i \in \mathbb{Z}} \operatorname{gr}^F_i M := \bigoplus\nolimits_{i \in \mathbb{Z}} \frac{F_i M}{F_{i-1} M}.$$

Given an element $m \in M$ the F-degree of m is

$$\deg^F(m) := \inf \{ i \mid m \in F_i M \} \in \mathbb{Z} \cup \{-\infty\}.$$

The F-symbol of m is

$$\operatorname{symb}^F(m) := m + F_{i-1} M \in \operatorname{gr}^F_i M$$

if $i = \deg^F(m) \in \mathbb{Z}$; and $\operatorname{symb}^F(m) := 0$ if $\deg^F(m) = -\infty$. Thus the homogeneous elements of $\operatorname{gr}^F M$ are the symbols.

Recall that the product on the graded algebra $\operatorname{gr}^F A$ is defined on symbols as follows. Given elements $a_1, a_2 \in A$ let $d_i := \deg^F(a_i)$ and $\bar{a}_i := \operatorname{symb}^F(a_i)$. If both $d_i > -\infty$ then

$$\bar{a}_1 \cdot \bar{a}_2 := a_1 \cdot a_2 + F_{d_1 + d_2 - 1} A \in \operatorname{gr}^F_{d_1 + d_2} A.$$

Otherwise $\bar{a}_1 \cdot \bar{a}_2 := 0$. Similarly one defines a graded $(\operatorname{gr}^F A)$-module structure on a filtered module M.

If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a graded algebra then A is also filtered, where

$$F_i A := \bigoplus\nolimits_{j \le i} A_j.$$

The filtration F is exhaustive and separated. Moreover $A \cong \operatorname{gr}^F A$ as graded algebras. The isomorphism sends $a \in A_i$ to its symbol $\operatorname{symb}^F(a) \in \operatorname{gr}^F_i A$.

Lemma 1.1. *Suppose the $\Bbbk$-algebra A is generated by a sequence of elements $\{a_i\}_{i \in I}$, where I is an indexing set (possibly infinite). Given a sequence $\{d_i\}_{i \in I}$ of nonnegative integers, there is a unique nonnegative exhaustive filtration $F = \{F_d A\}_{d \in \mathbb{Z}}$ such that:*

(i) *For every d, $F_d A$ is the $\Bbbk$-linear span of the products $a_{j_1} \cdots a_{j_m}$ such that $d_{j_1} + \cdots + d_{j_m} \leq d$.*
(ii) *The graded algebra $\mathrm{gr}^F A$ is generated by a sequence of elements $\{\bar{a}_i\}_{i \in I}$, where for every $i \in I$ either $\bar{a}_i = \mathrm{symb}^F(a_i)$ or $\bar{a}_i = 0$.*

Proof. Let $\boldsymbol{x} = \{x_i\}_{i \in I}$ be a sequence of distinct indeterminates, and let $\Bbbk\langle \boldsymbol{x} \rangle$ be the free associative algebra on these generators. Define $\phi : \Bbbk\langle \boldsymbol{x} \rangle \to A$ to be the surjection sending $x_i \mapsto a_i$. Put on $\Bbbk\langle \boldsymbol{x} \rangle$ the grading such that $\deg(x_i) = d_i$. This induces a filtration $F = \{F_d \Bbbk\langle \boldsymbol{x} \rangle\}_{d \in \mathbb{Z}}$ where

$$F_d \Bbbk\langle \boldsymbol{x} \rangle := \bigoplus\nolimits_{e \leq d} \Bbbk\langle \boldsymbol{x} \rangle_e .$$

This filtration can now be transferred to A by setting $F_d A := \phi(F_d \Bbbk\langle \boldsymbol{x} \rangle)$. Clearly (A, F) is exhaustive and nonnegative, and also condition (i) holds. This condition also guarantees uniqueness.

As for condition (ii) consider the surjective graded algebra homomorphism

$$\mathrm{gr}^F(\phi) : \mathrm{gr}^F \Bbbk\langle \boldsymbol{x} \rangle \to \mathrm{gr}^F A.$$

Because of the way the filtration on $\Bbbk\langle \boldsymbol{x} \rangle$ was constructed the graded algebra $\mathrm{gr}^F \Bbbk\langle \boldsymbol{x} \rangle$ is a free algebra on the symbols $\bar{x}_i := \mathrm{symb}^F(x_i)$. Define $\bar{a}_i := \mathrm{gr}^F(\phi)(\bar{x}_i)$. These elements have the required properties. □

Conversely we have the following two lemmas, whose standard proofs we leave out.

Lemma 1.2. *Let $F = \{F_d A\}$ be an exhaustive nonnegative filtration of A, and let $\{a_i\}_{i \in I}$ be a sequence in A. Denote by $\bar{a}_i := \mathrm{symb}^F(a_i)$. Suppose that the sequence $\{\bar{a}_i\}_{i \in I}$ generates $\mathrm{gr}^F A$ as $\Bbbk$-algebra. Then:*
(1) *A is generated by $\{a_i\}_{i \in I}$ as $\Bbbk$-algebra.*
(2) *Let $d_i := \max\{0, \deg^F(a_i)\}$. Then F coincides with the filtration from Lemma* 1.1.

Lemma 1.3. *Let (A, F) be a nonnegative exhaustive filtered $\Bbbk$-algebra and let (M, F) be a bounded below exhaustive filtered (A, F)-module. Suppose $\{a_i\}_{i \in I} \subset A$, $\{b_j\}_{j \in J} \subset A$ and $\{c_k\}_{k \in K} \subset M$ are sequences satisfying:*
(i) *The set of symbols $\{\bar{a}_i\}_{i \in I} \cup \{\bar{b}_j\}_{j \in J}$ generates $\mathrm{gr}^F A$ as $\Bbbk$-algebra.*
(ii) *The set of symbols $\{\bar{c}_k\}_{k \in K}$ generates $\mathrm{gr}^F M$ as $(\mathrm{gr}^F A)$-module.*
(iii) *For every i, j the symbols $\bar{a}_i$ and $\bar{b}_j$ commute.*

Then for every integer d the $\Bbbk$-module $F_d M$ is generated by the set of products

$$\{a_{i_1} \cdots a_{i_p} b_{j_1} \cdots b_{j_q} c_k \mid \deg^F(a_{i_1}) + \cdots + \deg^F(b_{j_1}) + \cdots + \deg^F(c_k) \leq d\}.$$

The base field $\Bbbk$ is of course trivially filtered. The filtered $\Bbbk$-modules (M, F) form an additive category $\mathsf{FiltMod}\,\Bbbk$, in which a morphism $\phi : (M, F) \to (N, F)$ is a $\Bbbk$-linear homomorphism $\phi : M \to N$ such that $\phi(F_i M) \subset F_i N$.

The *Rees module* of (M, F) is

$$\mathrm{Rees}\,(M, F) = \mathrm{Rees}^F M := \bigoplus_{i \in \mathbb{Z}} F_i M \cdot t^i \subset M[t] = M \otimes_{\Bbbk} \Bbbk[t]$$

where t is a central indeterminate of degree 1. We get an additive functor

$$\mathrm{Rees} : \mathsf{FiltMod}\,\Bbbk \to \mathsf{GrMod}\,\Bbbk[t]$$

where $\mathsf{GrMod}\,\Bbbk[t]$ is the abelian category of graded $\Bbbk[t]$-modules and degree 0 homomorphisms.

For a scalar $\lambda \in \Bbbk$ let us denote by sp_λ the specialization to λ of a $\Bbbk[t]$-module $\widetilde{M}$, namely

$$\mathrm{sp}_\lambda \widetilde{M} := \widetilde{M}/(t-\lambda)\widetilde{M}.$$

If $\lambda \neq 0$ then

$$\mathrm{sp}_\lambda : \mathsf{GrMod}\,\Bbbk[t] \to \mathsf{Mod}\,\Bbbk$$

is an exact functor, since $\mathrm{sp}_\lambda \widetilde{M}$ is isomorphic to the degree zero component of the localization $\widetilde{M}_t$. For $\lambda = 0$ we get a functor

$$\mathrm{sp}_0 : \mathsf{GrMod}\,\Bbbk[t] \to \mathsf{GrMod}\,\Bbbk.$$

Given any $(M, F) \in \mathsf{FiltMod}\,\Bbbk$ one has

$$\mathrm{sp}_0 \operatorname{Rees}(M, F) \cong \operatorname{gr}(M, F) = \operatorname{gr}^F M.$$

On the other hand, given a graded $\Bbbk[t]$-module $\widetilde{M}$ there is a filtration F on $M := \mathrm{sp}_1 \widetilde{M}$ defined by

$$F_i M := \operatorname{Im}\big(\bigoplus\nolimits_{j \leq i} \widetilde{M}_j \to M\big).$$

This is a functor

$$\mathrm{sp}_1 : \mathsf{GrMod}\,\Bbbk[t] \to \mathsf{FiltMod}\,\Bbbk.$$

If (M, F) is exhaustive then

$$\mathrm{sp}_1 \operatorname{Rees}(M, F) \cong (M, F).$$

For a graded module $\widetilde{M} \in \mathsf{GrMod}\,\Bbbk[t]$ we have

$$\operatorname{Rees} \mathrm{sp}_1 \widetilde{M} \cong \widetilde{M}/\{t\text{-torsion}\}.$$

If A is a filtered $\Bbbk$-algebra then $\widetilde{A} := \operatorname{Rees} A$ and $\bar{A} := \operatorname{gr} A$ are graded algebras, and we obtain corresponding functors Rees, sp_1 and sp_0 between $\mathsf{FiltMod}\,A$, $\mathsf{GrMod}\,\widetilde{A}$ and $\mathsf{GrMod}\,\bar{A}$.

The next lemma says that a filtration can be lifted to the Rees ring.

Lemma 1.4. *Let F be an exhaustive nonnegative filtration of the $\Bbbk$-algebra A, and let $\widetilde{A} := \operatorname{Rees}^F A \subset A[t]$. For any integer i define*

$$\widetilde{F}_i \widetilde{A} := \bigoplus\nolimits_{j \in \mathbb{N}} (F_{\min(i,j)} A) \cdot t^j \in A[t].$$

Then:

1. $\widetilde{A} = \bigcup_i \widetilde{F}_i \widetilde{A}$, $\widetilde{F}_{-1} \widetilde{A} = 0$ *and* $\widetilde{F}_i \widetilde{A} \cdot \widetilde{F}_j \widetilde{A} \subset \widetilde{F}_{i+j} \widetilde{A}$. *Thus* $\widetilde{F} = \{\widetilde{F}_i \widetilde{A}\}$ *is an exhaustive nonnegative filtration of the algebra* $\widetilde{A}$.
2. *There is an isomorphism of $\Bbbk$-algebras*

$$\operatorname{gr}^{\widetilde{F}} \widetilde{A} \cong (\operatorname{gr}^F A) \otimes \Bbbk[t]$$

(*not respecting degrees*).

Proof. (1) Since $F_{-1} A = 0$ we get $\widetilde{F}_{-1} \widetilde{A} = 0$. Let $\widetilde{A}_j := (F_j A) \cdot t^j$, the jth graded component of $\widetilde{A}$. Then for any i, j there is equality

$$\widetilde{A}_j \cap (\widetilde{F}_i \widetilde{A}) = (F_{\min(i,j)} A) \cdot t^j.$$

Hence $\widetilde{A}_j \subset \widetilde{F}_j \widetilde{A}$. It remains to check the products. For any two pairs of numbers (i, k) and (j, l) one has

$$\min(i, k) + \min(j, l) \leq \min(i + j, k + l).$$

Therefore

$$\big((F_{\min(i,k)} A) \cdot t^k\big) \cdot \big((F_{\min(j,l)} A) \cdot t^l\big) \subset \big((F_{\min(i+j,k+l)} A) \cdot t^{k+l}\big).$$

This says that $\widetilde{F}_i \widetilde{A} \cdot \widetilde{F}_j \widetilde{A} \subset \widetilde{F}_{i+j} \widetilde{A}$.

(2) We have isomorphisms

$$\operatorname{gr}^{\widetilde{F}} \widetilde{A} \cong \bigoplus\nolimits_{0 \leq i} \bigoplus\nolimits_{i \leq j} (\operatorname{gr}^F_i A) \cdot t^j$$

and

$$(\operatorname{gr}^F A) \otimes \Bbbk[t] \cong \bigoplus\nolimits_{0 \leq i} \bigoplus\nolimits_{0 \leq j} (\operatorname{gr}^F_i A) \cdot t^j.$$

The isomorphism $\operatorname{gr}^{\widetilde{F}} \widetilde{A} \xrightarrow{\simeq} (\operatorname{gr}^F A) \otimes \Bbbk[t]$ we want is defined on every summand $(\operatorname{gr}^F_i A) \cdot t^j$ by dividing by t^i. □

2. Differential $\Bbbk$-Algebras of Finite Type

Let C be a ring. Recall that a *C-ring* is a ring A together with a ring homomorphism $C \to A$, called the structural homomorphism. Observe that a C-ring is also a C-bimodule.

Definition 2.1. Suppose C is a commutative $\Bbbk$-algebra and A is a C-ring. A *differential C-filtration* on A is a filtration $F = \{F_i A\}_{i \in \mathbb{Z}}$ with the following properties:

(i) $1 \in F_0 A$ and $F_i A \cdot F_j A \subset F_{i+j} A$.
(ii) $F_{-1} A = 0$ and $A = \bigcup F_i A$.
(iii) Each $F_i A$ is a C-sub-bimodule.
(iv) The graded ring $\operatorname{gr}^F A$ is a C-algebra.

A is called a *differential C-ring* if it admits some differential C-filtration.

The name "differential filtration" signifies the similarity to Grothendieck's definition of differential operators; see [EGA IV].

Note that properties (i) and (iii) imply that the image of the structural homomorphism $C \to A$ lies in $F_0 A$, so that (iv) makes sense.

Definition 2.2. Let C be a commutative noetherian $\Bbbk$-algebra and A a C-ring. A *differential C-filtration of finite type* on A is a differential C-filtration $F = \{F_i A\}$ such that the graded C-algebra $\operatorname{gr}^F A$ is a finite module over its center $\mathrm{Z}(\operatorname{gr}^F A)$, and $\mathrm{Z}(\operatorname{gr}^F A)$ is a finitely generated C-algebra. We say A is a *differential C-ring of finite type* if it admits some differential C-filtration of finite type.

If C is a finitely generated commutative $\Bbbk$-algebra and A is a differential C-ring of finite type then A is also a differential $\Bbbk$-ring of finite type. In this case we also call A a *differential $\Bbbk$-algebra of finite type.*

Example 2.3. Let C be a finitely generated commutative $\Bbbk$-algebra and A a finite C-algebra. Then A is a differential C-ring of finite type. As filtration we can take the trivial filtration $F_{-1} A := 0$ and $F_0 A := A$.

Example 2.4. Suppose $\operatorname{char} \Bbbk = 0$, C is a smooth commutative $\Bbbk$-algebra and $A := \mathcal{D}(C)$ is the ring of $\Bbbk$-linear differential operators. Then A is a differential C-ring of finite type. For filtration we can take the filtration $F = \{F_i A\}$ by order of operator, in which $F_{-1}A := 0$, $F_0 A := C$, $F_1 A := C \oplus \mathcal{T}(C)$ and $F_{i+1}A := F_i A \cdot F_1 A$ for $i \geq 1$. Here $\mathcal{T}(C) := \operatorname{Der}_{\Bbbk}(C)$, the module of derivations.

Example 2.5. A special case of Example 2.4 is when $C := \Bbbk[x_1, \ldots, x_n]$, a polynomial ring. Then A is called the nth Weyl algebra. Writing $y_i := \frac{\partial}{\partial x_i}$ the algebra A is generated by the $2n$ elements $x_1, \ldots, x_n, y_1, \ldots, y_n$, with relations $[x_i, x_j] = [y_i, y_j] = [y_i, x_j] = 0$ for $i \neq j$ and $[y_i, x_i] = 1$. In addition to the filtration F above there is also a differential $\Bbbk$-filtration of finite type $G = \{G_i A\}$ where $G_{-1}A := 0$, $G_0 A := \Bbbk$, $G_1 A := \Bbbk + (\sum_i \Bbbk \cdot x_i) + (\sum_i \Bbbk \cdot y_i)$ and $G_{i+1}A := G_i A \cdot G_1 A$ for $i \geq 1$.

Example 2.6. This example generalizes Example 2.5. Let C be a finitely generated commutative $\Bbbk$-algebra (not necessarily smooth, and $\operatorname{char} \Bbbk$ arbitrary), and let L be a finite C-module (not necessarily projective). Suppose L has a $\Bbbk$-linear Lie bracket $[-,-]$. The module of derivations $\mathcal{T}(C) := \operatorname{Der}_{\Bbbk}(C)$ is also a $\Bbbk$-Lie algebra. Suppose $\alpha : L \to \mathcal{T}(C)$ is a C-linear Lie homomorphism, namely $\alpha(c\xi) = c\alpha(\xi)$ and $\alpha([\xi, \zeta]) = [\alpha(\xi), \alpha(\zeta)]$ for all $c \in C$ and $\xi, \zeta \in L$. L is then called a *Lie algebroid* or a *Lie-Rinehart algebra* (cf. [Ch1] or [Ri]). The ring of generalized differential operators $\mathcal{D}(C; L)$, also called the universal enveloping algebra and denoted $\mathrm{U}(C; L)$, is defined as follows. Choose $\Bbbk$-algebra generators $c_1, \ldots, c_p$ for C and C-module generators $l_1, \ldots, l_q$ for L. Let

$$\Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle := \Bbbk\langle x_1, \ldots, x_p, y_1, \ldots, y_q \rangle$$

be the free associative algebra. We have a ring surjection $\phi_0 : \Bbbk\langle \boldsymbol{x} \rangle \to C$ with $\phi_0(x_i) := c_i$. Let $I_0 := \operatorname{Ker}(\phi_0)$. Next there is a surjection of $\Bbbk\langle \boldsymbol{x} \rangle$-modules

$$\phi_1 : \Bbbk\langle \boldsymbol{x} \rangle^q = \bigoplus\nolimits_{j=1}^{q} \Bbbk\langle \boldsymbol{x} \rangle \cdot y_j \to L$$

with $\phi_1(y_j) := l_j$. Define $I_1 := \operatorname{Ker}(\phi_1) \subset \Bbbk\langle \boldsymbol{x} \rangle^q$. For any i, j choose polynomials $f_{i,j}(\boldsymbol{x})$ and $g_{i,j,k}(\boldsymbol{x})$ such that $[l_i, l_j] = \sum_k g_{i,j,k}(\boldsymbol{c}) l_k \in L$ and $\alpha(l_i)(c_j) = f_{i,j}(\boldsymbol{c}) \in C$. Now define

$$\mathrm{U}(C; L) := \frac{\Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle}{I}$$

where $I \subset \Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ is the two-sided ideal generated by I_0, I_1 and the polynomials $[y_i, y_j] - \sum_k g_{i,j,k}(\boldsymbol{x}) y_k$ and $[y_i, x_j] - f_{i,j}(\boldsymbol{x})$.

The ring $\mathrm{U}(C; L)$ has the following universal property: given any ring D, any ring homomorphism $\eta_0 : C \to D$ and any C-linear Lie homomorphism $\eta_1 : L \to D$ satisfying $[\eta_1(l), \eta_0(c)] = \eta_0(\alpha(l)(c))$, there is a unique ring homomorphism $\eta : \mathrm{U}(C; L) \to D$ through which η_0 and η_1 factor.

Put on $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ the filtration F such that $\deg^F(x_i) = 0$ and $\deg^F(y_j) = 1$. Let F be the filtration induced on $\mathrm{U}(C; L)$ by the surjection $\phi : \Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle \twoheadrightarrow \mathrm{U}(C; L)$. Then $\operatorname{gr}^F \mathrm{U}(C; L)$ is a commutative C-algebra, generated by the elements $\bar{l}_j := \operatorname{gr}^F(\phi)(y_j)$, $j \in \{1, \ldots, q\}$. We see that $\mathrm{U}(C; L)$ is a differential C-ring of finite type. If $C = \Bbbk[x_1, \ldots, x_n]$ and $L = \mathcal{T}(C)$ then we are in the situation of Example 2.5. If $C = \Bbbk$ then $\mathrm{U}(C; L) = \mathrm{U}(L)$ is the usual universal enveloping algebra of the Lie algebra L.

Lemma 2.7 ([ATV, Theorem 8.2]). *Suppose $A = \bigoplus_{i \in \mathbb{N}} A_i$ is a graded $\Bbbk$-algebra and $t \in A$ is a central homogeneous element of positive degree. The following are equivalent:*

(i) *A is left noetherian.*
(ii) *$A/(t)$ is left noetherian.*

The next proposition follows almost directly from the definition and Lemma 2.7.

Proposition 2.8. *If A is a differential $\Bbbk$-algebra of finite type then it is a noetherian finitely generated $\Bbbk$-algebra.*

The class of differential $\Bbbk$-algebras of finite type is closed under tensor products as we now show.

Proposition 2.9. *Let C_1 and C_2 be noetherian commutative $\Bbbk$-algebras, and assume that $C_1 \otimes C_2$ is also noetherian. Let A_i be a differential C_i-ring of finite type for $i = 1, 2$. Then $A_1 \otimes A_2$ is a differential $(C_1 \otimes C_2)$-ring of finite type.*

Proof. Choose differential filtrations of finite type $\{F_n A_1\}$ and $\{F_n A_2\}$ of A_1 and A_2. Define a filtration on $A_1 \otimes A_2$ as follows:

$$F_n(A_1 \otimes A_2) := \sum_{l+m=n} F_l A_1 \otimes F_m A_2.$$

Then

$$\mathrm{gr}^F(A_1 \otimes A_2) \cong (\mathrm{gr}^F A_1) \otimes (\mathrm{gr}^F A_2)$$

as graded rings. Since $\mathrm{gr}^F A_1$ and $\mathrm{gr}^F A_2$ are finite modules over their centers it follows that $(\mathrm{gr}^F A_1) \otimes (\mathrm{gr}^F A_2)$ is a finite module over its center. □

3. The Theorem on the Two Filtrations

The next theorem generalizes the case of the nth Weyl algebra and its two filtrations (see Examples 2.4 and 2.5 above). McConnell and Stafford also considered such filtrations, and our result extends their [MS, Corollary 1.7]. The basic idea is attributed in [MS] to Bernstein.

We recall that a graded $\Bbbk$-algebra A is called *connected* if $A = \bigoplus_{i \in \mathbb{N}} A_i$, $A_0 = \Bbbk$ and each A_i is a finite $\Bbbk$-module.

Theorem 3.1. *Let A be a $\Bbbk$-algebra. Assume A has a differential $\Bbbk$-filtration of finite type $F = \{F_i A\}_{i \in \mathbb{Z}}$. Then there is a nonnegative exhaustive $\Bbbk$-filtration $G = \{G_i A\}_{i \in \mathbb{Z}}$ such that $\mathrm{gr}^G A$ is a commutative, finitely generated, connected graded $\Bbbk$-algebra.*

Observe that G is also a differential filtration of finite type on A. As mentioned in the introduction Theorem 3.1 will be used to prove the existence of an Auslander rigid dualizing complex over A.

The following easy lemma will be used often in the proof of the theorem.

Lemma 3.2. *Let $F = \{F_i A\}$ be a nonnegative exhaustive filtration of A and let $a_1, a_2 \in A$ be two elements. Define $\bar{a}_i := \mathrm{symb}^F(a_i) \in \mathrm{gr}^F A$ and $d_i := \deg^F(a_i) \in \mathbb{N} \cup \{-\infty\}$. Then the commutator $[\bar{a}_1, \bar{a}_2] = 0$ if and only if*

$$\deg^F([a_1, a_2]) \leq d_1 + d_2 - 1.$$

Proof of Theorem 3.1. Step 1. Write $\bar{A} := \operatorname{gr}^F A$. Then the center $\mathrm{Z}(\bar{A})$ is a graded, finitely generated, commutative $\Bbbk$-algebra, and $\bar{A}$ is a finite $\mathrm{Z}(\bar{A})$-module. For any element $a \in A$ we write $\bar{a} := \operatorname{symb}^F(a) \in \operatorname{gr}^F A$.

Let $d_1 \in \mathbb{N}$ be large enough such that $\mathrm{Z}(\bar{A})$ is generated as $\mathrm{Z}(\bar{A})_0$-algebra by finitely many elements of degrees $\leq d_1$, and $\bar{A}$ is generated as $\mathrm{Z}(\bar{A})$-module by finitely many elements of degrees $\leq d_1$.

Choose nonzero elements $a_1, \ldots, a_m \in F_0 A \cong \bar{A}_0$ such that their symbols $\bar{a}_1, \ldots, \bar{a}_m$ are in $\mathrm{Z}(\bar{A})_0$, and they generate $\mathrm{Z}(\bar{A})_0$ as $\Bbbk$-algebra. Next choose elements $b_1, \ldots, b_n \in F_{d_1} A - F_0 A$ such that the symbols $\bar{b}_1, \ldots, \bar{b}_n$ are in $\mathrm{Z}(\bar{A})$, and they generate $\mathrm{Z}(\bar{A})$ as $\mathrm{Z}(\bar{A})_0$-algebra. Finally choose nonzero elements $c_1, \ldots, c_p \in F_{d_1} A$ such that the symbols $\bar{c}_1, \ldots, \bar{c}_p$ generate $\bigoplus_{j=0}^{d_1} \bar{A}_j$ as $\mathrm{Z}(\bar{A})_0$-module. This implies that $\bar{c}_1, \ldots, \bar{c}_p$ generate $\bar{A}$ as a $\mathrm{Z}(\bar{A})$-module.

The symbols $\bar{a}_1, \ldots, \bar{a}_m, \bar{b}_1, \ldots, \bar{b}_n, \bar{c}_1, \ldots, \bar{c}_p$ generate $\bar{A}$ as $\Bbbk$-algebra, so by Lemma 1.2 the elements $a_1, \ldots, c_m, b_1, \ldots, b_n, c_1, \ldots, c_p$ generate A as $\Bbbk$-algebra. Let

$$\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle := \langle x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p \rangle$$

be the free associative algebra, and define a surjective ring homomorphism $\phi : \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle \to A$ by sending $x_i \mapsto a_i$, $y_i \mapsto b_i$ and $z_i \mapsto c_i$. We are now in the situation of Lemma 1.2. The free algebra $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$ also has a filtration F, where $\deg^F(x_i) := \deg^F(a_i)$, $\deg^F(y_i) := \deg^F(b_i)$ etc., and ϕ is a strict surjection, meaning that $F_i(A) = \phi(F_i \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle)$.

Let us denote substitution by $f(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}) := \phi(f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}))$. Consider the subrings

$$\Bbbk\langle \boldsymbol{a} \rangle \subset \Bbbk\langle \boldsymbol{a}, \boldsymbol{b} \rangle \subset A = \Bbbk\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle$$

with filtrations F induced by the inclusions into A. Warning: these filtrations might differ from the filtrations induced by $\phi : \Bbbk\langle \boldsymbol{x} \rangle \twoheadrightarrow \Bbbk\langle \boldsymbol{a} \rangle$ and $\phi : \Bbbk\langle \boldsymbol{x}, \boldsymbol{y} \rangle \twoheadrightarrow \Bbbk\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ respectively.

We observe that the commutators $[a_i, a_j] = 0$ for all i, j, since $a_i = \bar{a}_i \in Z(\bar{A})_0$. This also says that $[\bar{a}_i, \bar{b}_j] = 0$, so according to Lemma 3.2 we get

$$[a_i, b_j] \in F_{\deg^F(b_j) - 1} A$$

for all i, j. Therefore by Lemma 1.3, applied to the filtered $\Bbbk$-algebra $\Bbbk\langle \boldsymbol{a} \rangle$ and the filtered $\Bbbk\langle \boldsymbol{a} \rangle$-module $F_{d_1} A$, we see that there are noncommutative polynomials $f^1_{i,j,k}(\boldsymbol{x}) \in \Bbbk\langle \boldsymbol{x} \rangle$ such that

$$\begin{gathered} \deg^F(f^1_{i,j,k}(\boldsymbol{x})) + \deg^F(c_k) \leq \deg^F(b_j) - 1 \\ \text{and} \\ [a_i, b_j] = \sum_k f^1_{i,j,k}(\boldsymbol{a}) \cdot c_k. \end{gathered} \tag{3.3}$$

Note that either $f^1_{i,j,k}(\boldsymbol{x}) \neq 0$, in which case $\deg^F(f^1_{i,j,k}(\boldsymbol{x})) = 0$; or $f^1_{i,j,k}(\boldsymbol{x}) = 0$ and then $\deg^F(f^1_{i,j,k}(\boldsymbol{x})) = -\infty$. The choice $f^1_{i,j,k}(\boldsymbol{x}) = 0$ is of course required when $\deg^F(c_k) \geq \deg^F(b_j)$.

Likewise $[b_i, b_j] \in F_{\deg^F(b_i) + \deg^F(b_j) - 1} A$, so by Lemma 1.3, applied to the filtered $\Bbbk$-algebra $\Bbbk\langle \boldsymbol{a}, \boldsymbol{b} \rangle$ and the filtered $\Bbbk\langle \boldsymbol{a}, \boldsymbol{b} \rangle$-module A, we see that there are noncommutative polynomials $f^2_{i,j,k}(\boldsymbol{x})$ and $g^2_{i,j,k}(\boldsymbol{y})$ such that

$$\deg^F(f^2_{i,j,k}(\boldsymbol{x})) + \deg^F(g^2_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) \leq \deg^F(b_i) + \deg^F(b_j) - 1$$
$$\text{and} \tag{3.4}$$
$$[b_i, b_j] = \sum_k f^2_{i,j,k}(\boldsymbol{a}) \cdot g^2_{i,j,k}(\boldsymbol{b}) \cdot c_k.$$

Similarly there are polynomials $f^3_{i,j,k}(\boldsymbol{x})$ such that

$$\deg^F(f^3_{i,j,k}(\boldsymbol{x})) + \deg^F(c_k) \leq \deg^F(c_j) - 1$$
$$\text{and}$$
$$[a_i, c_j] = \sum_k f^3_{i,j,k}(\boldsymbol{a}) \cdot c_k,$$

and there are polynomials $f^4_{i,j,k}(\boldsymbol{x})$ and $g^4_{i,j,k}(\boldsymbol{y})$ such that

$$\deg^F(f^4_{i,j,k}(\boldsymbol{x})) + \deg^F(g^4_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) \leq \deg^F(b_i) + \deg^F(c_j) - 1$$
$$\text{and}$$
$$[b_i, c_j] = \sum_k f^4_{i,j,k}(\boldsymbol{a}) \cdot g^4_{i,j,k}(\boldsymbol{b}) \cdot c_k.$$

The same idea applies to $c_i c_j$: there are polynomials $f^5_{i,j,k}(\boldsymbol{x})$ and $g^5_{i,j,k}(\boldsymbol{y})$ such that

$$\deg^F(f^5_{i,j,k}(\boldsymbol{x})) + \deg^F(g^5_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) \leq \deg^F(c_i) + \deg^F(c_j)$$
$$\text{and}$$
$$c_i \cdot c_j = \sum_k f^5_{i,j,k}(\boldsymbol{a}) \cdot g^5_{i,j,k}(\boldsymbol{b}) \cdot c_k.$$

Let G be the standard grading on $\Bbbk\langle \boldsymbol{x} \rangle$, namely $\deg^G(x_i) := 1$. This induces a filtration G. Define

$$e_0 := \max\{0, \deg^G(f^l_{i,j,k}(\boldsymbol{x}))\},$$

$e_1 := e_0 + 1$ and $e_2 := e_0 + e_1 + 1$.

Put on the free algebra $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$ a new grading G by declaring

$$\deg^G(y_i) := e_2 \deg^F(b_i)$$
$$\deg^G(z_i) := e_2 \deg^F(c_i) + e_1,$$

and keeping $\deg^G(x_i) = 1$ as above. We get a new filtration G on $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$. Using this we obtain a new filtration G on A with

$$G_i A := \phi(G_i \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle).$$

Step 2. Now we verify that the filtration G has the required properties. Since the filtration G on $\Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$ is nonnegative exhaustive, and ϕ is a strict surjection, it follows that the filtration G on A is also nonnegative exhaustive. The rest requires some work, and in order to simplify our notation we are going to "recycle" the expressions $\bar{A}$, $\bar{a}_i$ etc. From here on we define $\bar{A} := \operatorname{gr}^G A$. We have a surjective graded $\Bbbk$-algebra homomorphism

$$\bar{\phi} := \operatorname{gr}^G(\phi) : \operatorname{gr}^G \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle \to \operatorname{gr}^G A = \bar{A}.$$

Let $\bar{x}_i := \operatorname{symb}^G(x_i)$, $\bar{y}_i := \operatorname{symb}^G(y_i)$, etc. Then

$$\operatorname{gr}^G \Bbbk\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\rangle = \Bbbk\langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{z}}\rangle := \Bbbk\langle \bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_n, \bar{z}_1, \ldots, \bar{z}_p\rangle$$

which is also a free algebra. Define $\bar{a}_i := \bar{\phi}(\bar{x}_i)$, $\bar{b}_i := \bar{\phi}(\bar{y}_i)$ and $\bar{c}_i := \bar{\phi}(\bar{z}_i)$. Observe that either $\deg^G(a_i) = \deg^G(x_i)$, in which case $\bar{a}_i = \operatorname{symb}^G(a_i)$, and it is a nonzero element of $\bar{A}_{\deg^G(a_i)}$; or $\deg^G(a_i) < \deg^G(x_i)$, and then $\bar{a}_i = 0$. Similar statements hold for $\bar{b}_i$ and $\bar{c}_i$.

Since $\bar{\phi}$ is surjective we see that $\bar{A}$ is generated as $\Bbbk$-algebra by the elements $\bar{a}_1, \ldots, \bar{a}_m, \bar{b}_1, \ldots, \bar{b}_n, \bar{c}_1, \ldots, \bar{c}_p$. These elements are either of positive degree or are 0, and hence $\bar{A}$ is connected graded. We claim that $\bar{A}$ is commutative.

We know already that $[\bar{a}_i, \bar{a}_j] = 0$. Let us check that $[\bar{a}_i, \bar{b}_j] = 0$. If either $\deg^G(a_i) < \deg^G(x_i)$ or $\deg^G(b_j) < \deg^G(y_j)$ then $\bar{a}_i\bar{b}_j = \bar{b}_j\bar{a}_i = 0$. Otherwise $\bar{a}_i = \operatorname{symb}^G(a_i)$ and $\bar{b}_j = \operatorname{symb}^G(b_j)$. By formula (3.3) we have

$$\begin{aligned}\deg^G([a_i, b_j]) &\leq \max\{\deg^G(f^1_{i,j,k}(\boldsymbol{a})) + \deg^G(c_k)\}\\ &\leq \max\{\deg^G(f^1_{i,j,k}(\boldsymbol{x})) + \deg^G(c_k)\}.\end{aligned}$$

For any k such that $f^1_{i,j,k}(\boldsymbol{x}) \neq 0$ we have $\deg^F(c_k) \leq \deg^F(b_j) - 1$, and then

$$\deg^G(c_k) \leq \deg^G(z_k) = e_2 \deg^F(c_k) + e_1 \leq e_2(\deg^F(b_j) - 1) + e_1.$$

Also $\deg^G(f^1_{i,j,k}(\boldsymbol{x})) \leq e_0$. Because

$$\deg^G(b_j) = \deg^G(y_j) = e_2 \deg^F(b_j)$$

and

$$\deg^G(a_i) = \deg^G(x_i) = 1,$$

we get

$$\begin{aligned}\deg^G([a_i, b_j]) &\leq e_0 + (e_2(\deg^F(b_j) - 1) + e_1)\\ &= \deg^G(a_i) + \deg^G(b_j) - 2.\end{aligned}$$

Using Lemma 3.2 we conclude that $[\bar{a}_i, \bar{b}_j] = 0$.

Next let's consider the commutator $[\bar{b}_i, \bar{b}_j]$. If either $\deg^G(b_i) < \deg^G(y_i)$ or $\deg^G(b_j) < \deg^G(y_j)$ then $\bar{b}_i\bar{b}_j = \bar{b}_j\bar{b}_i = 0$. Otherwise $\bar{b}_i = \operatorname{symb}^G(b_i)$ and $\bar{b}_j = \operatorname{symb}^G(b_j)$. By formula (3.4), if $f^2_{i,j,k}(\boldsymbol{x}) \neq 0$ then

$$\deg^F(g^2_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) \leq \deg^F(b_i) + \deg^F(b_j) - 1.$$

Also

$$\begin{aligned}\deg^G(g^2_{i,j,k}(\boldsymbol{y})) &= e_2 \deg^F(g^2_{i,j,k}(\boldsymbol{y}))\\ \deg^G(c_k) &\leq \deg^G(z_k) = e_2 \deg^F(c_k) + e_1.\end{aligned}$$

Therefore, looking only at indices k s.t. $f^2_{i,j,k}(\boldsymbol{x}) \neq 0$, we obtain

$$\begin{aligned}\deg^G([b_i, b_j]) &\leq \max\{\deg^G(f^2_{i,j,k}(\boldsymbol{a})) + \deg^G(g^2_{i,j,k}(\boldsymbol{b})) + \deg^G(c_k)\}\\ &\leq \max\{\deg^G(f^2_{i,j,k}(\boldsymbol{x})) + \deg^G(g^2_{i,j,k}(\boldsymbol{y})) + \deg^G(c_k)\}\\ &\leq e_0 + \max\{e_2 \deg^F(g^2_{i,j,k}(\boldsymbol{y})) + e_2 \deg^F(c_k) + e_1\}\\ &= e_2 \max\{\deg^F(g^2_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) + 1\} + (e_0 + e_1 - e_2)\\ &\leq e_2(\deg^F(b_i) + \deg^F(b_j)) - 1\\ &= \deg^G(b_i) + \deg^G(b_j) - 1.\end{aligned}$$

So according to Lemma 3.2 we conclude that $[\bar{b}_i, \bar{b}_j] = 0$.

The calculation for the other commutators is similar.

Finally we show that, amusingly, $\bar{c}_i \bar{c}_j = 0$. If either $\deg^G(c_i) < \deg^G(z_i)$ or $\deg^G(c_j) < \deg^G(z_j)$ then automatically $\bar{c}_i \bar{c}_j = 0$. Otherwise $\bar{c}_i = \operatorname{symb}^G(c_i)$ and $\bar{c}_j = \operatorname{symb}^G(c_j)$. For any k such that $f^5_{i,j,k}(\boldsymbol{x}) \neq 0$ one has

$$\deg^F(g^5_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k) \leq \deg^F(c_i) + \deg^F(c_j).$$

Therefore, looking only at indices k s.t. $f^5_{i,j,k}(\boldsymbol{x}) \neq 0$, we obtain

$$\begin{aligned}
\deg^G(c_i c_j) &\leq \max\{\deg^G(f^5_{i,j,k}(\boldsymbol{a})) + \deg^G(g^5_{i,j,k}(\boldsymbol{b})) + \deg^G(c_k)\} \\
&\leq \max\{\deg^G(f^5_{i,j,k}(\boldsymbol{x})) + \deg^G(g^5_{i,j,k}(\boldsymbol{y})) + \deg^G(c_k)\} \\
&\leq e_0 + \max\{e_2 \deg^F(g^5_{i,j,k}(\boldsymbol{y})) + e_2 \deg^F(c_k) + e_1\} \\
&= e_2 \max\{\deg^F(g^5_{i,j,k}(\boldsymbol{y})) + \deg^F(c_k)\} + (e_0 + e_1) \\
&\leq e_2(\deg^F(c_i) + \deg^F(c_j)) + (e_0 + e_1) \\
&= \deg^G(c_i) + \deg^G(c_j) - 1.
\end{aligned}$$

So by definition of the product in $\bar{A}$ we get $\bar{c}_i \bar{c}_j = 0$. $\square$

Proposition 3.5. *In the situation of Theorem* 3.1 *assume the* $\Bbbk$*-algebra* A *is graded, and also every* $\Bbbk$*-submodule* $F_i A$ *is graded. Then the filtration* G *can be chosen such that every* $\Bbbk$*-submodule* $G_i A$ *is graded.*

Proof. Simply choose the generators $a_1, \ldots, b_1, \ldots, c_1, \ldots, c_p \in A$ used in the proof to be homogeneous. $\square$

4. Review of Dualizing Complexes

For a $\Bbbk$-algebra A we denote by A^{op} the opposite algebra, and by $A^{\mathrm{e}} := A \otimes A^{\mathrm{op}}$ the enveloping algebra. Recall that an A-module means a left A-module. With this convention a right A-module is an A^{op}-module, and an A-bimodule is an A^{e}-module.

In this section we review the definition of dualizing complexes over rings and related concepts.

Let $\mathsf{Mod}\, A$ be the category of A-modules, and let $\mathsf{Mod}_{\mathrm{f}}\, A$ be the full subcategory of finite (i.e. finitely generated) modules. The latter is abelian when A is left noetherian. Let $\mathsf{D}(\mathsf{Mod}\, A)$ be the derived category of A-modules. The full subcategory of bounded complexes is denoted by $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$, the full subcategory of complexes with finite cohomologies is denoted by $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$, and their intersection is $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Definition 4.1 ([Ye1], [YZ1])**.** Let A be a left noetherian $\Bbbk$-algebra and B a right noetherian $\Bbbk$-algebra. A complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\,(A \otimes B^{\mathrm{op}}))$ is called a *dualizing complex over* (A, B) if it satisfies the following three conditions:

(i) R has finite injective dimension over A and over B^{op}.
(ii) R has finite cohomology modules over A and over B^{op}.
(iii) The canonical morphisms $B \to \operatorname{RHom}_A(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$, and $A \to \operatorname{RHom}_{B^{\mathrm{op}}}(R, R)$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$, are both isomorphisms.

In the case $A = B$ we say R is a *dualizing complex over* A.

Whenever we refer to a dualizing complex over (A, B) in the paper we tacitly assume that A is left noetherian and B is right noetherian.

Remark 4.2. There are many non-isomorphic dualizing complexes over a given $\Bbbk$-algebra A. The isomorphism classes of dualizing complexes are parameterized by the derived Picard group $\operatorname{DPic}(A)$, whose elements are the isomorphism classes of two-sided tilting complexes. See [Ye4] and [MY].

Here are two easy examples.

Example 4.3. Suppose A is a Gorenstein noetherian ring, namely the bimodule $R := A$ has finite injective dimension as left and right module. Then R is a dualizing complex over A.

Example 4.4. If A is a finite $\Bbbk$-algebra then the bimodule $A^* := \operatorname{Hom}_{\Bbbk}(A, \Bbbk)$ is a dualizing complex over A. In fact it is a rigid dualizing complex (see Definition 4.10).

Definition 4.5. Let R be a dualizing complex over (A, B). The *duality functors induced by R* are the contravariant functors

$$\mathrm{D} := \operatorname{RHom}_A(-, R) : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}})$$

and

$$\mathrm{D}^{\mathrm{op}} := \operatorname{RHom}_{B^{\mathrm{op}}}(-, R) : \mathsf{D}(\mathsf{Mod}\, B^{\mathrm{op}}) \to \mathsf{D}(\mathsf{Mod}\, A).$$

By [YZ1, Proposition 1.3] the functors D and D^{op} are a duality (i.e. an anti-equivalence) of triangulated categories between $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})$, restricting to a duality between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})$.

Definition 4.6 ([Ye2], [YZ1])**.** Let R be a dualizing complex over (A, B). We say that R has the *Auslander property*, or that R is an *Auslander dualizing complex*, if the conditions below hold.

(i) For every finite A-module M, every integers $p > q$, and every B^{op}-submodule $N \subset \operatorname{Ext}^p_A(M, R)$, one has $\operatorname{Ext}^q_{B^{\mathrm{op}}}(N, R) = 0$.
(ii) The same holds after exchanging A and B^{op}.

Rings with Auslander dualizing complexes can be viewed as a generalization of Auslander regular rings (cf. [Bj] and [Le]).

Example 4.7. If A is either the nth Weyl algebra or the universal enveloping algebra of a finite dimensional Lie algebra, then A is Auslander regular, and the bimodule $R := A$ is an Auslander dualizing complex.

Definition 4.8. An *exact dimension function* on $\mathsf{Mod}\, A$ is a function

$$\dim : \mathsf{Mod}\, A \to \{-\infty\} \cup \mathbb{R} \cup \{\text{infinite ordinals}\},$$

satisfying the following axioms:

(i) $\dim 0 = -\infty$.
(ii) For every short exact sequence $0 \to M' \to M \to M'' \to 0$ one has $\dim M = \max\{\dim M', \dim M''\}$.
(iii) If $M = \bigcup_\alpha M_\alpha$ then $\dim M = \sup\{\dim M_\alpha\}$.

The basic examples of dimension functions are the Gelfand-Kirillov dimension, denoted by GKdim, and the Krull dimension, denoted by Kdim. See [MR, Section 6.8.4]. Here is another dimension function.

Definition 4.9 ([Ye2], [YZ1])**.** Let R be an Auslander dualizing complex over (A,B). Given a finite A-module M the *canonical dimension of M with respect to R* is

$$\operatorname{Cdim}_{R;A} M := -\inf\{q \mid \operatorname{Ext}^q_A(M,R) \neq 0\} \in \mathbb{Z} \cup \{-\infty\}.$$

For any A-module M we define

$$\operatorname{Cdim}_{R;A} M := \sup\{\dim M' \mid M' \subset M \text{ is finite}\}.$$

Likewise we define $\operatorname{Cdim}_{R;B^{\mathrm{op}}} N$ for a B^{op}-module N.

Often we shall abbreviate $\operatorname{Cdim}_{R;A}$ by dropping subscripts when no confusion can arise. According to [YZ1, Theorem 2.1] Cdim is an exact dimension function on $\mathsf{Mod}\, A$ and $\mathsf{Mod}\, B^{\mathrm{op}}$.

The following concept is due to Van den Bergh. Since R is a complex of A^{e}-modules, the complex $R \otimes R$ consists of modules over $A^{\mathrm{e}} \otimes A^{\mathrm{e}} \cong (A^{\mathrm{e}})^{\mathrm{e}}$. In the definition below $\operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R)$ is computed using the "outside" A^{e}-module structure of $R \otimes R$, and the resulting complex retains the "inside" A^{e}-module structure.

Definition 4.10 ([VdB, Definition 8.1])**.** Let R be a dualizing complex over A. If there is an isomorphism

$$\rho : R \to \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ then we call (R,ρ), or just R, a *rigid* dualizing complex. The isomorphism ρ is called a *rigidifying isomorphism.*

A rigid dualizing complex, if it exists, is unique up to isomorphism, by [VdB, Proposition 8.2].

A ring homomorphism $A \to B$ is said to be *finite* if B is a finite A-module on both sides.

Definition 4.11. [YZ1, Definition 3.7] Let $A \to B$ be a finite homomorphism of $\Bbbk$-algebras. Assume the rigid dualizing complexes (R_A, ρ_A) and (R_B, ρ_B) exist. Let $\operatorname{Tr}_{B/A} : R_B \to R_A$ be a morphism in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. We say $\operatorname{Tr}_{B/A}$ is a *rigid trace* if it satisfies the following two conditions:

(i) $\operatorname{Tr}_{B/A}$ induces isomorphisms

$$R_B \cong \operatorname{RHom}_A(B, R_A) \cong \operatorname{RHom}_{A^{\mathrm{op}}}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

(ii) The diagram

$$\begin{array}{ccc} R_B & \xrightarrow{\rho_B} & \operatorname{RHom}_{B^{\mathrm{e}}}(B, R_B \otimes R_B) \\ {\scriptstyle \operatorname{Tr}}\downarrow & & \downarrow{\scriptstyle \operatorname{Tr} \otimes \operatorname{Tr}} \\ R_A & \xrightarrow{\rho_A} & \operatorname{RHom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ is commutative.

Often we shall say that $\operatorname{Tr}_{B/A} : (R_B, \rho_B) \to (R_A, \rho_A)$ is a rigid trace morphism.

By [YZ1, Theorem 3.2], a rigid trace $\operatorname{Tr}_{B/A}$ is unique (if it exists). In particular taking the identity map $A \to A$ and any two rigid dualizing complexes (R,ρ) and (R',ρ') over A, it follows there is a unique isomorphism $R \xrightarrow{\simeq} R'$ that is a rigid trace; see [YZ1, Corollary 3.4]. Given another finite homomorphism $B \to C$ such that the rigid dualizing complex (R_C, ρ_C) and the rigid trace $\mathrm{Tr}_{C/B}$ exist, the composition $\mathrm{Tr}_{C/A} := \mathrm{Tr}_{C/B} \circ \mathrm{Tr}_{B/A}$ is a rigid trace.

Finally we mention that by [YZ1, Corollary 3.6] the cohomology bimodules $\mathrm{H}^i R_A$ of the rigid dualizing complex are central $\mathrm{Z}(A)$-bimodules, where $\mathrm{Z}(A)$ is the center of A.

Example 4.12. Suppose A is a finitely generated commutative $\Bbbk$-algebra. Choose a finite homomorphism $\Bbbk[\boldsymbol{t}] \to A$ where $\Bbbk[\boldsymbol{t}] = \Bbbk[t_1, \ldots, t_n]$ is the polynomial algebra. Define $R_A := \mathrm{RHom}_{\Bbbk[\boldsymbol{t}]}(A, \Omega^n_{\Bbbk[\boldsymbol{t}]/k}[n])$, and consider this as an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$. By [Ye4, Proposition 5.7] the complex R_A is an Auslander rigid dualizing complex, and in fact it is equipped with a canonical rigidifying isomorphism ρ_A.

5. Quasi-Coherent Ringed Schemes and Localization

In order to study geometric properties of dualizing complexes it is convenient to use the language of schemes and quasi coherent sheaves. See [YZ3, Theorems 0.1 and 0.2].

Let $(X, \mathcal{A})$ be a ringed space over $\Bbbk$. Thus X is a topological space and $\mathcal{A}$ is a sheaf of (possibly noncommutative) $\Bbbk$-algebras on X. By an $\mathcal{A}$-bimodule we mean a sheaf $\mathcal{M}$ of $\Bbbk$-modules on X together with a left $\mathcal{A}$-module structure and a right $\mathcal{A}$-module structure that commute with each other. In other words $\mathcal{M}$ is a module over the sheaf of rings $\mathcal{A} \otimes_{\Bbbk_X} \mathcal{A}^{\mathrm{op}}$, where $\Bbbk_X$ is the constant sheaf $\Bbbk$ on X. An $\mathcal{A}$-ring is a sheaf $\mathcal{B}$ of rings on X together with a ring homomorphism $\mathcal{A} \to \mathcal{B}$. Note that $\mathcal{B}$ is an $\mathcal{A}$-bimodule.

Definition 5.1. Let X be $\Bbbk$-scheme. An $\mathcal{O}_X$-ring $\mathcal{A}$ is called a *quasi-coherent $\mathcal{O}_X$-ring* if $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_X$-module on both sides. The pair $(X, \mathcal{A})$ is then called a *quasi-coherent ringed scheme.*

Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme. An $\mathcal{A}$-module $\mathcal{M}$ is called quasi-coherent if locally, on every sufficiently small open set U, it has a free resolution

$$\mathcal{A}|_U^{(J)} \to \mathcal{A}|_U^{(I)} \to \mathcal{M}|_U \to 0;$$

cf. [EGA I]. Equivalently, $\mathcal{M}$ is quasi-coherent as $\mathcal{O}_X$-module. We shall denote the category of quasi-coherent $\mathcal{A}$-modules by $\mathsf{QCoh}\, \mathcal{A}$.

Proposition 5.2. *Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme, let $U \subset X$ be an affine open set and $A := \Gamma(U, \mathcal{A})$. The functor $\Gamma(U, -)$ is an equivalence of categories $\mathsf{QCoh}\, \mathcal{A}|_U \to \mathsf{Mod}\, A$.*

Proof. This is a slight generalization of [EGA I, Corollary 1.4.2 and Theorem 1.5.1]. See also [Ha, Corollary II.5.5]. □

Given an A-module M we shall usually denote the corresponding quasi-coherent $\mathcal{A}|_U$-module by $\mathcal{A}|_U \otimes_A M$.

The following definition is due to Silver [Si, p. 47].

Definition 5.3. Let A be a ring. An A-ring A' is called a *localization* of A if A' is a flat A-module on both sides, and if the multiplication map $A' \otimes_A A' \to A'$ is bijective.

Example 5.4. Let A be a ring and $S \subset A$ a (left and right) denominator set. The ring of fractions A_S of A with respect to S is the prototypical example of a localization of A. For reference we call such a localization an *Ore localization.*

We remind the reader that a denominator set S is a multiplicatively closed subset of A satisfying the left and right Ore conditions and the left and right torsion conditions (see [MR, Section 2.1]). The left Ore condition is that for all $a \in A$ and $s \in S$ there exist $a' \in A$ and $s' \in S$ such that $as' = a's$. The left torsion condition is

$$\{a \in A \mid as = 0 \text{ for some } s \in S\} \subset \{a \in A \mid sa = 0 \text{ for some } s \in S\}.$$

The right Ore and torsion conditions for A are the respective left conditions for A^{op}.

Not all localizations are Ore, as we see in Example 5.7

Here is a list of some nice descent properties enjoyed by localization, that are proved in [Si, Section 1].

Lemma 5.5. *Let A be a ring and let A' be a localization of A.*

(1) *For any A'-module M' the multiplication $A' \otimes_A M' \to M'$ is bijective.*
(2) *Let M' be an A'-module and $M \subset M'$ an A-submodule. Then the multiplication $A' \otimes_A M \to M'$ is injective.*
(3) *Let M be an A-module and $\phi : M \to A' \otimes_A M$ the homomorphism $\phi(m) := 1 \otimes m$. Then for any A'-submodule $N' \subset A' \otimes_A M$ the multiplication $A' \otimes_A \phi^{-1}(N') \to N'$ is bijective.*
(4) *In the situation of part (3) the A-submodule $\phi(M) \subset A' \otimes_A M$ is essential.*
(5) *Localization of a left noetherian ring is left noetherian.*

Proposition 5.6. *Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme, and let $V \subset U$ be affine open sets in X. Define $C := \Gamma(U, \mathcal{O}_X)$, $C' := \Gamma(V, \mathcal{O}_X)$, $A := \Gamma(U, \mathcal{A})$ and $A' := \Gamma(V, \mathcal{A})$.*

(1) *The multiplication maps $C' \otimes_C A \to A'$ and $A \otimes_C C' \to A'$ are bijective.*
(2) *$A \to A'$ is a localization.*
(3) *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{A}$-module. Then the multiplication map*
$$A' \otimes_A \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$
is bijective.

Proof. Define $C := \Gamma(U, \mathcal{O}_X)$ and $C' := \Gamma(V, \mathcal{O}_X)$. We first show that C' is a localization of C, namely that $\phi : C \to C'$ is flat and $\psi : C' \otimes_C C' \to C'$ is bijective. This can be checked locally on V. Choose an affine open covering $V = \bigcup_i V_i$ with $V_i = \operatorname{Spec} C_{s_i}$ for suitable elements $s_i \in C$. We note that $C_{s_i} \cong C'_{s_i} \cong \Gamma(V_i, \mathcal{O}_X)$ for all i. Hence restricting ϕ and ψ to V_i, namely applying $C_{s_i} \otimes_C -$ to them, we obtain bijections.

Let $\mathcal{M}$ be any quasi-coherent $\mathcal{A}$-module. By [Ha, Proposition 5.1] multiplication

$$C' \otimes_C \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

is bijective.

Because $\mathcal{A}$ is a quasi-coherent left and right $\mathcal{O}_X$-module, the previous formula implies that $C' \otimes_C A \to A'$ and $A \otimes_C C' \to A'$ are both bijective. In addition, since we now know that

$$A' \otimes_A \Gamma(U, \mathcal{M}) \cong C' \otimes_C \Gamma(U, \mathcal{M}),$$

we may conclude that

$$A' \otimes_A \Gamma(U, \mathcal{M}) \to \Gamma(V, \mathcal{M})$$

is bijective.

Finally we have a sequence of isomorphisms, all compatible with the multiplication homomorphisms into A':

$$\begin{aligned} A' \otimes_A A' &\cong A' \otimes_A (A \otimes_C C') \cong A' \otimes_C C' \\ &\cong (A \otimes_C C') \otimes_C C' \cong A \otimes_C (C' \otimes_C C') \\ &\cong A \otimes_C C' \cong A'. \end{aligned}$$

□

Example 5.7. Let X be an elliptic curve over $\mathbb{C}$ and $O \in X$ the zero element for the group structure. Let $P \in X$ be any non-torsion point. Define $U := X - \{O\}$ and $V := X - \{O, P\}$, which are affine open sets, and $C := \Gamma(U, \mathcal{O}_X)$, $C' := \Gamma(V, \mathcal{O}_X)$. By the previous proposition $C \to C'$ is a localization. We claim this is not an Ore localization. If it were then there would be some non-invertible nonzero function $s \in C$ that becomes invertible in C'. Hence the divisor of s on X would be $(s) = n(O - P)$ for some positive integer n. In the group structure this would mean that P is a torsion point, and this is a contradiction.

Definition 5.8. Let A be a ring, let M be an A-bimodule and let A' be a localization of A. If the canonical homomorphisms

$$A' \otimes_A M \to A' \otimes_A M \otimes_A A'$$

and

$$M \otimes_A A' \to A' \otimes_A M \otimes_A A'$$

are bijective then M is said to be *evenly localizable* to A'.

Trivially the bimodule $M := A$ is evenly localizable to A'. The next lemma is also easy and we omit its proof.

Lemma 5.9. *Let M be an A-bimodule. Suppose A' is a localization of A.*

1. *If A is commutative, M is a central A-bimodule and A' is an A-algebra, then M is evenly localizable to A'.*
2. *If there is a short exact sequence of A-bimodules $0 \to L \to M \to N \to 0$ with L and N evenly localizable to A', then M is also evenly localizable to A'.*
3. *Suppose $M \cong \lim_{i\to} M_i$ for some directed system of A-bimodules $\{M_i\}$. If each M_i is evenly localizable to A' then so is M.*

Proposition 5.10. *Let C be a commutative ring and let M be a C-bimodule. Define $U := \operatorname{Spec} C$. The following conditions are equivalent.*

- (i) *For any multiplicatively closed subset $S \subset C$, with localization C_S, M is evenly localizable to C_S.*
- (ii) *M is evenly localizable to $C' := \Gamma(V, \mathcal{O}_U)$ for every affine open set $V \subset U$.*
- (iii) *There is a sheaf of $\mathcal{O}_U$-bimodules $\mathcal{M}$, quasi-coherent on both sides, with $M \cong \Gamma(U, \mathcal{M})$. Such $\mathcal{M}$ is unique up to a unique isomorphism.*

Proof. (i) $\Rightarrow$ (ii): Let us write

$$\phi : C' \otimes_C M \to C' \otimes_C M \otimes_C C'.$$

As in the proof of Proposition 5.6 we choose an affine open covering $V = \bigcup_i V_i$ with $V_i = \operatorname{Spec} C_{s_i}$ and $C'_{s_i} \cong C_{s_i}$. It suffices to show that the homomorphism ϕ_i gotten by applying $C'_{s_i} \otimes_{C'} -$ to ϕ (localizing on the left) is bijective for all i. Using the hypothesis (i) with $S := \{s_i^l\}_{l \in \mathbb{N}}$ and the fact that $C_{s_i} \to C'_{s_i}$ is bijective we get

$$C'_{s_i} \otimes_{C'} (C' \otimes_C M) \cong C_{s_i} \otimes_C M \otimes_C C_{s_i}$$

and

$$\begin{aligned} C'_{s_i} \otimes_{C'} (C' \otimes_C M \otimes_C C') &\cong C_{s_i} \otimes_C M \otimes_C C_{s_i} \otimes_C C' \\ &\cong C_{s_i} \otimes_C M \otimes_C C_{s_i}. \end{aligned}$$

So ϕ_i is bijective.

Similarly one shows that

$$M \otimes_C C' \to C' \otimes_C M \otimes_C C'$$

is bijective.

(ii) $\Rightarrow$ (i): For any element $s \in S$ let $V := \operatorname{Spec} C_s \subset U$. By assumption

$$C_s \otimes_C M \cong C_s \otimes_C M \otimes_C C_s \cong M \otimes_C C_s.$$

Taking direct limit over $s \in S$ we get

$$C_S \otimes_C M \cong C_S \otimes_C M \otimes_C C_S \cong M \otimes_C C_S.$$

(ii) $\Rightarrow$ (iii): Let $\mathcal{M} := \mathcal{O}_U \otimes_C M$ be the sheafification of the (left) C-module M to U. By definition $\mathcal{M}$ is a quasi-coherent left $\mathcal{O}_U$-module.

Given an affine open set $V \subset U$ write $C' := \Gamma(V, \mathcal{O}_U)$. By Proposition 5.6 the multiplication map $C' \otimes_C M \to \Gamma(V, \mathcal{M})$ is a bijection. Therefore $\Gamma(V, \mathcal{M}) \cong C' \otimes_C M \otimes_C C'$. Since M is evenly localizable to C' it follows that $M \otimes_C C' \to \Gamma(V, \mathcal{M})$ is also bijective. We conclude that $\mathcal{M}$ is also a quasi-coherent right $\mathcal{O}_U$-module.

Regarding the uniqueness, suppose $\mathcal{N}$ is another $\mathcal{O}_U$-bimodule quasi-coherent on both sides such that $\Gamma(U, \mathcal{N}) \cong M$ as bimodules. For any affine open set V as above we get an isomorphism of C'-bimodules

$$\Gamma(V, \mathcal{M}) \cong C' \otimes_C M \otimes_C C' \cong \Gamma(V, \mathcal{N})$$

which is functorial in V. Therefore $\mathcal{M} \cong \mathcal{N}$ as $\mathcal{O}_U$-bimodules.

(iii) $\Rightarrow$ (ii): Since $\mathcal{M}$ is quasi-coherent on both sides, for any affine open set $V = \operatorname{Spec} C'$ we have

$$\Gamma(V, \mathcal{M}) = C' \otimes_C M = M \otimes_C C',$$

so M is evenly localizable to C'. $\square$

The relation between even localization and Ore localization of a ring is explained in the next theorem.

Theorem 5.11. *Let C be a commutative ring, let A be a C-ring and $S \subset C$ a multiplicatively closed subset. Denote by C_S the ring of fractions of C with respect to S. Then the following two conditions are equivalent.*

(i) *The image $\bar{S}$ of S in A is a denominator set, with ring of fractions $A_{\bar{S}}$.*
(ii) *The C-bimodule A is evenly localizable to C_S.*

When these conditions hold the multiplication map

$$C_S \otimes_C A \otimes_C C_S \to A_{\bar{S}}$$

is bijective.

Proof. (i) $\Rightarrow$ (ii): Since $A_{\bar{S}}$ is the left ring of fractions of A with respect to $\bar{S}$ (see [MR, Section 2.1.3]), it follows that the homomorphism $C_S \otimes_C A \to A_{\bar{S}}$ is bijective. On the other hand, since $A_{\bar{S}}$ is also the right ring of fractions, $A \otimes_C C_S \to A_{\bar{S}}$ is bijective.

(ii) $\Rightarrow$ (i): Write
$$Q := C_S \otimes_C A \otimes_C C_S$$
and
$$\phi : A \to Q,\ \phi(a) := 1 \otimes a \otimes 1.$$
The assumption that A is evenly localizable to C_S implies that
$$\begin{aligned} \operatorname{Ker}(\phi) &= \{a \in A \mid as = 0 \text{ for some } s \in S\} \\ &= \{a \in A \mid sa = 0 \text{ for some } s \in S\}, \end{aligned}$$
verifying the torsion conditions.

The even localization assumption also implies that given $a_1 \in A$ and $s_1 \in S$ there are $a_2 \in A$ and $s_2 \in S$ such that
$$s_1^{-1} \otimes a_1 \otimes 1 = 1 \otimes a_2 \otimes s_2^{-1} \in Q.$$
Multiplying this equation by s_1 on the left and by s_2 on the right we obtain $1 \otimes a_1 s_2 \otimes 1 = 1 \otimes s_1 a_2 \otimes 1$. Therefore $\phi(s_1 a_2 - a_1 s_2) = 0$. Since $Q \cong A \otimes_C C_S$ there exists some $s_3 \in S$ such that $(s_1 a_2 - a_1 s_2) s_3 = 0$ in A, i.e.
$$s_1(a_2 s_3) = a_1(s_2 s_3) \in A.$$
We have verified the right Ore condition. The left Ore condition is verified the same way. $\square$

Remark 5.12. The theorem applies to any ring A and any commutative multiplicatively closed subset $S \subset A$, since we can take $C := \mathbb{Z}[S] \subset A$.

We will need a geometric interpretation of Theorem 5.11.

Corollary 5.13. *Let C be a commutative ring, let $U := \operatorname{Spec} C$ and let A be a C-ring. The following conditions are equivalent:*

(i) *For every multiplicatively closed set $S \subset C$ the C-bimodule A is evenly localizable to C_S.*
(ii) *For every multiplicatively closed set $S \subset C$ its image $\bar{S} \subset A$ is a denominator set.*
(iii) *There is a quasi-coherent $\mathcal{O}_U$-ring $\mathcal{A}$ such that $\Gamma(U, \mathcal{A}) \cong A$ as C-rings.*

When these conditions hold the quasi-coherent $\mathcal{O}_U$-ring $\mathcal{A}$ is unique up to a unique isomorphism.

Proof. (i) and (ii) are equivalent by Theorem 5.11. The implication (iii) $\Rightarrow$ (i) is a special case of Proposition 5.10. It remains to show that (i) $\Rightarrow$ (iii).

By Proposition 5.10 there is an $\mathcal{O}_U$-bimodule $\mathcal{A}$, quasi-coherent on both sides, such that $A \cong \Gamma(U, \mathcal{A})$ as C-bimodules. The bimodule $\mathcal{A}$ is unique up to a unique isomorphism. Next by Theorem 5.11, for any $s \in C$, letting $S := \{s^i\}_{i \in \mathbb{N}}$, the image $\bar{S} \subset A$ is a denominator set. Therefore on $V := \operatorname{Spec} C_s$ we have canonical isomorphisms
$$\Gamma(V, \mathcal{A}) \cong C_s \otimes_C A \otimes_C C_s \cong A_{\bar{S}},$$
where $A_{\bar{S}}$ is the ring of fractions of A with respect to $\bar{S}$. Hence $\mathcal{A}$ has a unique structure of quasi-coherent $\mathcal{O}_X$-ring. $\square$

Here is a (somewhat artificial) example of a C-ring A satisfying the conditions of Corollary 5.13, but the C^{e}-ring A^{e} fails to satisfy them.

Example 5.14. Let $C := \mathbb{Q}[t]$ with t a variable, and let $U := \operatorname{Spec} C$. Take $A := \mathbb{Q}(t)[a;\sigma]$, an Ore extension of the field $\mathbb{Q}(t)$, where σ is the automorphism $\sigma(t) = -t$. Since every nonzero element $s \in C$ is invertible in A, the C-ring A is evenly localizable to C_S for any multiplicatively closed subset $S \subset C$. Hence there is a quasi-coherent ringed scheme $(U, \mathcal{A})$ with $\Gamma(U, \mathcal{A}) \cong A$ as C-rings. (In fact $\mathcal{A}$ is a constant sheaf on U.) Likewise there's a quasi-coherent ringed scheme $(U, \mathcal{A}^{\mathrm{op}})$.

We claim that there does not exist a quasi-coherent ringed scheme $(U^2, \mathcal{A}^{\mathrm{e}})$ such that $\Gamma(U^2, \mathcal{A}^{\mathrm{e}}) \cong A^{\mathrm{e}}$ as C^{e}-rings. By Corollary 5.13 it suffices to exhibit a multiplicatively closed subset $S \subset C^{\mathrm{e}}$ that is not a denominator set in A^{e}. Consider the element $s := t \otimes 1 - 1 \otimes t \in C^{\mathrm{e}}$ and the set $S := \{s^n\}_{n \in \mathbb{N}}$. Let $\mu : A^{\mathrm{e}} \to A$ be the multiplication map $\mu(a_1 \otimes a_2) := a_1 a_2$, which is a homomorphism of (left) A^{e}-modules, and denote by I the left ideal $\operatorname{Ker}(\mu)$. Then $A^{\mathrm{e}} \cdot s \subset I$. On the other hand $s(a \otimes 1) = ta \otimes 1 - a \otimes t$, so $\mu(s(a \otimes 1)) = ta - at = 2ta$, and by induction $\mu(s^n(a \otimes 1)) = (2t)^n a \neq 0$ for all $n \geq 0$. We conclude that $s^n(a \otimes 1) \notin A^{\mathrm{e}} \cdot s$, so S fails to satisfy the left Ore condition.

Example 5.15. The quasi-coherent ringed scheme $(U, \mathcal{A})$ of the previous example also has the following peculiarity: the C^{e}-module A is not supported on the diagonal $\Delta(U) \subset U^2$. Indeed, for every $n \geq 0$ one has $s^n a = (2t)^n a \neq 0$.

Definition 5.16. Let C be a commutative $\mathbb{k}$-algebra and M a C-bimodule. A *differential C-filtration* on M is an exhaustive, bounded below filtration $F = \{F_i M\}$ where each $F_i M$ is a C-sub-bimodule, and $\operatorname{gr}^F M$ is a central C-bimodule. If M admits some differential C-filtration then we call M a *differential C-bimodule*.

Localization of a ring was defined in Definition 5.3, and even localization of a bimodule was introduced in Definition 5.8.

Proposition 5.17. *Let C be a commutative $\mathbb{k}$-algebra and let M be a differential C-bimodule. If C' is a localization of C, then M is evenly localizable to C'.*

Proof. If M is a central C-bimodule then according to Lemma 5.9(1) M is evenly localizable to C'.

Now let M be a C-bimodule equipped with a differential C-filtration F. Say $F_{i_0 - 1} M = 0$. We prove by induction on $i \geq i_0$ that $F_i M$ is evenly localizable to C'. First $F_{i_0} M$ is central, so the above applies to it. For any i there is an exact sequence

$$0 \to F_{i-1} M \to F_i M \to \operatorname{gr}_i^F M \to 0.$$

By the previous paragraph and by the induction hypothesis $F_{i-1} M$ and $\operatorname{gr}_i^F M$ are evenly localizable to C'. The flatness of $C \to C'$ extends this to $F_i M$, see Lemma 5.9(2). Finally we use Lemma 5.9(3). □

Corollary 5.18. *Let C be a finitely generated commutative $\mathbb{k}$-algebra, A a differential C-ring of finite type, $s \in C$ and $S := \{s^i\}_{i \in \mathbb{N}}$. Then:*

1. *The image $\bar{S}$ of S in A is a denominator set.*
2. *Let C_s and A_s be the Ore localizations w.r.t. S. Then C_s is is a finitely generated $\mathbb{k}$-algebra and A_s is a differential C_s-ring of finite type.*

Proof. (1) Use Proposition 5.17 and Theorem 5.11.

(2) Suppose $F = \{F_i A\}$ is a differential C-filtration of finite type. Then setting

$$F_i A_s := C_s \otimes_C (F_i A) \otimes_C C_s \subset A_s$$

we obtain a filtration F of A_s such that $\operatorname{gr}^F A_s \cong C_s \otimes_C \operatorname{gr}^F A$ as graded C_s-algebras. □

Remark 5.19. The ideas in [KL, Theorem 4.9] can be used to show the following. In the setup of the previous corollary let M be a finite A-module and $\bar{M} := \operatorname{Im}(M \to A_s \otimes_A M)$. Then

$$\operatorname{GKdim}_A \bar{M} = \operatorname{GKdim}_{A_s}(A_s \otimes_A M).$$

Corollary 5.20. *Let C be a commutative $\Bbbk$-algebra, let $U := \operatorname{Spec} C$ and let A be a differential C-ring.*

(1) *There is a quasi-coherent $\mathcal{O}_U$-ring $\mathcal{A}$, unique up to a unique isomorphism, such that $\Gamma(U, \mathcal{A}) \cong A$ as C-rings.*
(2) *For any multiplicatively closed set $S \subset C$ its image $\bar{S} \subset A$ is a denominator set.*
(3) *Given an affine open set $V \subset U$ let $C' := \Gamma(V, \mathcal{O}_U)$. Then $A' := C' \otimes_C A \otimes_C C'$ is a $\Bbbk$-algebra and $A \to A'$ is a localization of rings. If A is noetherian then so is A'.*

Proof. A is a differential C-bimodule, so by Proposition 5.17 A is evenly localizable to C_S for any multiplicatively closed set $S \subset C$. By Corollary 5.13 there is a quasi-coherent $\mathcal{O}_U$-ring $\mathcal{A}$, and by Proposition 5.6 we have $A' \cong \Gamma(V, \mathcal{A})$. The assertion about noetherian rings follows from Lemma 5.5(5). □

Proposition 5.21. *Let C be a commutative $\Bbbk$-algebra, let $U := \operatorname{Spec} C$, let M be a C^{e}-module and let $\mathcal{M} := \mathcal{O}_{U^2} \otimes_{C^{\mathrm{e}}} M$, the quasi-coherent $\mathcal{O}_{U^2}$-module associated to M. Assume C^{e} is noetherian. Then the following conditions are equivalent:*

(i) *M is a differential C-bimodule.*
(ii) *$\mathcal{M}$ is supported on the diagonal $\Delta(U) \subset U^2$.*

Proof. (i) ⇒ (ii): Denote by $I := \operatorname{Ker}(C^{\mathrm{e}} \twoheadrightarrow C)$ and $\mathcal{I} := \mathcal{O}_{U^2} \otimes_{C^{\mathrm{e}}} I$. So $\mathcal{I}$ is an ideal defining the diagonal $\Delta(U)$. Suppose $F = \{F_i M\}$ is a differential C-filtration of M, with $F_{i_0 - 1} M = 0$. Then for all $i \geq i_0$ we have

$$I^{i - i_0 + 1} \cdot F_i M = 0.$$

It follows that the $\mathcal{O}_{U^2}$-module $F_i \mathcal{M} := \mathcal{O}_{U^2} \otimes_{C^{\mathrm{e}}} F_i M$ is supported on $\Delta(U)$. But $\mathcal{M} = \bigcup F_i \mathcal{M}$.

(ii) ⇒ (i): Let $\{\mathcal{M}_\alpha\}$ be the set of coherent $\mathcal{O}_{U^2}$-submodules of $\mathcal{M}$, so $\mathcal{M} = \bigcup \mathcal{M}_\alpha$. Now $\mathcal{M}_\alpha$ is a coherent $\mathcal{O}_{U^2}$-module supported on the diagonal $\Delta(U)$, so there is some integer $i_\alpha \geq 0$ such that $\mathcal{I}^{i_\alpha + 1} \cdot \mathcal{M}_\alpha = 0$. It follows that the C^{e}-module $M_\alpha := \Gamma(U^2, \mathcal{M}_\alpha)$ satisfies $I^{i_\alpha + 1} \cdot M = 0$. And $M = \bigcup M_\alpha$.

Define a filtration F on M by $F_i M := \operatorname{Hom}_{C^{\mathrm{e}}}(C^{\mathrm{e}} / I^{i+1}, M)$ for $i \geq 0$, and $F_{-1} M := 0$. Then $M_\alpha \subset F_{i_\alpha} M$, and this implies that $M = \bigcup F_i M$. Finally $I \cdot F_i M \subset F_{i-1} M$, and hence $\operatorname{gr}^F_i M$ is a central C-bimodule. □

6. Localization of Dualizing Complexes

In this section we study the behavior of rigid dualizing complexes over rings with respect to localization (cf. Definition 5.8).

Definition 6.1. Let $A \to A'$ be a localization homomorphism between two noetherian $\Bbbk$-algebras. Suppose the rigid dualizing complexes (R, ρ) and (R', ρ') of A and A' respectively exist. A *rigid localization morphism* is a morphism

$$\mathrm{q}_{A'/A} : R \to R'$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ satisfying the conditions below.

(i) The morphisms $A' \otimes_A R \to R'$ and $R \otimes_A A' \to R'$ induced by $\mathrm{q}_{A'/A}$ are isomorphisms.

(ii) The diagram

$$\begin{array}{ccc} R & \xrightarrow{\rho} & \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \\ {\scriptstyle \mathrm{q}}\downarrow & & \downarrow{\scriptstyle \mathrm{q} \otimes \mathrm{q}} \\ R' & \xrightarrow{\rho'} & \mathrm{RHom}_{(A')^{\mathrm{e}}}(A', R' \otimes R') \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ is commutative, where $\mathrm{q} := \mathrm{q}_{A'/A}$.

We shall sometimes express this by saying that $\mathrm{q}_{A'/A} : (R, \rho) \to (R', \rho')$ is a rigid localization morphism.

Here is a generalization of [YZ2, Theorem 3.8].

Theorem 6.2. *Let A be a noetherian $\Bbbk$-algebra and let A' be a localization of A. Assume A has a dualizing complex R such that the cohomology bimodules $\mathrm{H}^i R$ are evenly localizable to A'. Then:*

(1) *The complex*

$$R' := A' \otimes_A R \otimes_A A'$$

is a dualizing complex over A'.

(2) *If R is an Auslander dualizing complex over A then R' is an Auslander dualizing complex over A'.*

(3) *Suppose R is a rigid dualizing complex over A with rigidifying isomorphism ρ, and A^{e} is noetherian. Then R' is a rigid dualizing complex over A'. Furthermore R' has a unique rigidifying isomorphism ρ' such that the morphism $\mathrm{q}_{A'/A} : R \to R'$ defined by $r \mapsto 1 \otimes r \otimes 1$ is a rigid localization morphism.*

(4) *In the situation of part* (3) *the rigid localization morphism $\mathrm{q}_{A'/A} : (R, \rho) \to (R', \rho')$ is unique.*

Proof. (1) This follows essentially from the proof of [YZ1, Theorem 1.13]. There A was commutative and A' the localization of A at some prime ideal, but the arguments are valid for an arbitrary localization A'.

(2) To check the Auslander property for R' let M' be any finite A'-module. By Lemma 5.5(2) there is a finite A-module M such that $M' \cong A' \otimes_A M$. For any i, [YZ2, Lemma 3.7(1)] implies that

$$\mathrm{Ext}^i_{A'}(M', R') \cong \mathrm{Ext}^i_A(M, R') \cong \mathrm{Ext}^i_A(M, R) \otimes_A A'$$

as $(A')^{\mathrm{op}}$-modules. Given any $(A')^{\mathrm{op}}$-submodule $N' \subset \operatorname{Ext}^i_{A'}(M', R')$, Lemma 5.5(3) tells us that there is an A^{op}-submodule $N \subset \operatorname{Ext}^i_A(M, R)$ such that $N' \cong N \otimes_A A'$. For such N we have

$$\operatorname{Ext}^j_{(A')^{\mathrm{op}}}(N', R') \cong A' \otimes_A \operatorname{Ext}^j_{A^{\mathrm{op}}}(N, R)$$

which is 0 for all $j < i$. By symmetry we get the other half of the Auslander property for R'.

(3) As in the proof of [YZ2, Theorem 3.8(2)] we have a canonical isomorphism

$$A' \otimes_A \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \otimes_A A' \xrightarrow{\simeq} \operatorname{RHom}_{(A')^{\mathrm{e}}}(A', R' \otimes R')$$

in $\mathsf{D}(\mathsf{Mod}\,(A')^{\mathrm{e}})$. This defines a rigidifying isomorphism ρ' that respects $\mathrm{q}_{A'/A}$ as depicted in the diagram in Definition 6.1. Given any other morphism

$$\rho'' : R' \to \operatorname{RHom}_{(A')^{\mathrm{e}}}(A', R' \otimes R')$$

that renders the diagram commutative, applying the functor $A' \otimes_A - \otimes_A A'$ to the whole diagram we deduce that $\rho^\flat = \rho'$.

(4) Write $\mathrm{q}_1 := \mathrm{q}_{A/A'}$. Suppose $\mathrm{q}_2 : (R, \rho) \to (R', \rho')$ is another rigid localization morphism. Consider the commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{\rho} & \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \\ {\scriptstyle \mathrm{q}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{q}_i \otimes \mathrm{q}_i} \\ R' & \xrightarrow{\rho'} & \operatorname{RHom}_{(A')^{\mathrm{e}}}(A', R' \otimes R') \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\,A^{\mathrm{e}})$. Applying the base change functor $(A')^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} -$ to these diagrams we obtain diagrams

$$\begin{array}{ccc} A' \otimes_A R \otimes_A A' & \xrightarrow{1 \otimes \rho \otimes 1} & A' \otimes_A \operatorname{RHom}_{A^{\mathrm{e}}}(A, R \otimes R) \otimes_A A' \\ {\scriptstyle 1 \otimes \mathrm{q}_i \otimes 1}\downarrow & & \downarrow{\scriptstyle 1 \otimes (\mathrm{q}_i \otimes \mathrm{q}_i) \otimes 1} \\ R' & \xrightarrow{\rho'} & \operatorname{RHom}_{(A')^{\mathrm{e}}}(A', R' \otimes R') \end{array}$$

consisting of isomorphisms in $\mathsf{D}(\mathsf{Mod}\,(A')^{\mathrm{e}})$; cf. proof of [YZ2, Theorem 3.8(2)]. We obtain an isomorphism $\tau : R' \to R'$ such that

$$1 \otimes \mathrm{q}_2 \otimes 1 = \tau \circ (1 \otimes \mathrm{q}_1 \otimes 1) : A' \otimes_A R \otimes_A A' \to R'.$$

But then $\tau : (R', \rho') \to (R', \rho')$ is a rigid trace morphism. By [YZ2, Theorem 3.2] τ has to be the identity. This implies $1 \otimes \mathrm{q}_2 \otimes 1 = 1 \otimes \mathrm{q}_1 \otimes 1$ and therefore $\mathrm{q}_2 = \mathrm{q}_1$. $\square$

The next proposition guarantees that under suitable assumptions the rigid trace localizes.

Proposition 6.3. *Let*

$$\begin{array}{ccc} A & \longrightarrow & A' \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array}$$

be a commutative diagram of $\Bbbk$*-algebras, where the horizontal arrows are localizations, the vertical arrows are finite, and the multiplication maps* $A' \otimes_A B \to B'$ *and* $B \otimes_A A' \to B'$ *are bijective. Assume* $A, A', A^{\mathrm{e}}, B, B'$ *and* B^{e} *are all noetherian. Also assume the rigid dualizing complexes* (R_A, ρ_A) *and* (R_B, ρ_B) *exist, and so* *does the rigid trace morphism* $\mathrm{Tr}_{B/A} : R_B \to R_A$*. By Theorem* 6.2 *the complexes* $R_{A'} := A' \otimes_A R_A \otimes_A A'$ *and* $R_{B'} := B' \otimes_B R_B \otimes_B B'$ *are rigid dualizing complexes over* A' *and* B' *respectively, with induced rigidifying isomorphisms* $\rho_{A'}$ *and* $\rho_{B'}$*. Then the morphism*

$$\mathrm{Tr}_{B'/A'} := 1 \otimes \mathrm{Tr}_{B/A} \otimes 1 : R_{B'} \to R_{A'}$$

is a rigid trace.

Proof. We begin by showing that the morphism $\psi' : R_{B'} \to \mathrm{RHom}_{A'}(B', R_{A'})$ induced by $\mathrm{Tr}_{B'/A'}$ is an isomorphism. Let's recall how ψ' is defined: one chooses a quasi-isomorphism $R_{A'} \to I'$ where I' is a bounded below complex of injective $(A')^{\mathrm{e}}$-modules. Then $\mathrm{Tr}_{B'/A'}$ is represented by an actual homomorphism of complexes $\tau' : R_{B'} \to I'$. The formula for $\psi' : R_{B'} \to \mathrm{Hom}_{A'}(B', I')$ is $\psi'(\beta')(b') = \tau'(b'\beta')$ for $\beta' \in R_{B'}$ and $b' \in B'$.

Let $R_A \to I$ be a quasi-isomorphism where I is a bounded below complex of injective A^{e}-modules, and let $\tau : R_B \to I$ be a homomorphism of complexes representing $\mathrm{Tr}_{B/A}$. We know that the homomorphism $\psi : R_B \to \mathrm{Hom}_A(B, I)$ given by the formula $\psi(\beta)(b) = \tau(b\beta)$ is a quasi-isomorphism.

Since $R_{A'} \cong A' \otimes_A I \otimes_A A'$ there is a quasi-isomorphism $A' \otimes_A I \otimes_A A' \to I'$, and using it we can assume that $\tau' = 1 \otimes \tau \otimes 1$ as morphisms

$$R_{B'} = A' \otimes_A R_B \otimes_A A' \to A' \otimes_A I \otimes_A A' \to I'.$$

Thus we get a commutative diagram

$$\begin{array}{ccc} R_B & \xrightarrow{\psi} & \mathrm{RHom}_A(B, R_A) \\ \downarrow & & \downarrow \\ R_{B'} & \xrightarrow{\psi'} & \mathrm{RHom}_{A'}(B', R_{A'}) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. Applying the base change $- \otimes_{A^{\mathrm{e}}} (A')^{\mathrm{e}}$ to the diagram we conclude that $\psi' = 1 \otimes \psi \otimes 1$. So it is an isomorphism.

By symmetry $R_{B'} \to \mathrm{RHom}_{(A')^{\mathrm{op}}}(B', R_{A'})$ is also an isomorphism.

Next we have to show that the diagram

$$\begin{array}{ccc} R_{B'} & \xrightarrow{\rho_{B'}} & \mathrm{RHom}_{(B')^{\mathrm{e}}}(B', R_{B'} \otimes R_{B'}) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr} \otimes \mathrm{Tr}} \\ R_{A'} & \xrightarrow{\rho_{A'}} & \mathrm{RHom}_{(A')^{\mathrm{e}}}(A', R_{A'} \otimes R_{A'}) \end{array} \tag{6.4}$$

is commutative. This is true since (6.4) gotten by applying $- \otimes_{A^{\mathrm{e}}} (A')^{\mathrm{e}}$ to the commutative diagram

$$\begin{array}{ccc} R_B & \xrightarrow{\rho_B} & \mathrm{RHom}_{B^{\mathrm{e}}}(B, R_B \otimes R_B) \\ {\scriptstyle \mathrm{Tr}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr} \otimes \mathrm{Tr}} \\ R_A & \xrightarrow{\rho_A} & \mathrm{RHom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A). \end{array}$$

□

If a $\Bbbk$-algebra A has an Auslander rigid dualizing complex R then we write $\mathrm{Cdim}_A := \mathrm{Cdim}_{R;A}$ for this preferred dimension function.

We finish this section with a digression from our main theme, to present this corollary to Theorem 6.2.

Corollary 6.5. *Suppose* $\operatorname{char} \Bbbk = 0$*, A is the nth Weyl algebra over $\Bbbk$ and D is its total ring of fractions, i.e. the nth Weyl division ring. Then $D[2n]$ is an Auslander rigid dualizing complex over D, and hence the canonical dimension of D is* $\operatorname{Cdim}_D D = 2n$.

Proof. By [Ye3], $A[2n]$ is a rigid Auslander dualizing complex over A. Now use Theorem 6.2 with $A' := D$. □

We see that unlike the Gelfand-Kirillov dimension GKdim, that cannot distinguish between the various Weyl division rings (since $\operatorname{GKdim} D = \infty$), the canonical dimension is an intrinsic invariant of D that does recover the number n. Moreover this fact can be expressed as a "classical" formula, namely

$$\operatorname{Ext}^i_{D \otimes \mathsf{D}^{\mathrm{op}}}(D, D \otimes D) \cong \begin{cases} D & \text{if } i = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

7. Perverse Modules and the Auslander Condition

In this section we discuss t-structures on the derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. We begin by recalling the definition of a t-structure and its basic properties, following [KS, Chapter X].

Definition 7.1. Suppose D is a triangulated category and $\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0}$ are two full subcategories. Let $\mathsf{D}^{\leq n} := \mathsf{D}^{\leq 0}[-n]$ and $\mathsf{D}^{\geq n} := \mathsf{D}^{\geq 0}[-n]$. We say $(\mathsf{D}^{\leq 0}, \mathsf{D}^{\geq 0})$ is a *t-structure on* D if:

(i) $\mathsf{D}^{\leq -1} \subset \mathsf{D}^{\leq 0}$ and $\mathsf{D}^{\geq 1} \subset \mathsf{D}^{\geq 0}$.
(ii) $\operatorname{Hom}_{\mathsf{D}}(M, N) = 0$ for $M \in \mathsf{D}^{\leq 0}$ and $N \in \mathsf{D}^{\geq 1}$.
(iii) For any $M \in \mathsf{D}$ there is a distinguished triangle
$$M' \to M \to M'' \to M'[1]$$
in D with $M' \in \mathsf{D}^{\leq 0}$ and $M'' \in \mathsf{D}^{\geq 1}$.

When these conditions are satisfied we define the *heart of* D to be the full subcategory $\mathsf{D}^0 := \mathsf{D}^{\leq 0} \cap \mathsf{D}^{\geq 0}$.

Given a t-structure there are truncation functors $\tau^{\leq n} : \mathsf{D} \to \mathsf{D}^{\leq n}$ and $\tau^{\geq n} : \mathsf{D} \to \mathsf{D}^{\geq n}$, and functorial morphisms $\tau^{\leq n} M \to M$, $M \to \tau^{\geq n} M$ and $\tau^{\geq n+1} M \to (\tau^{\leq n} M)[1]$ such that

$$\tau^{\leq n} M \to M \to \tau^{\geq n+1} M \to (\tau^{\leq n} M)[1]$$

is a distinguished triangle in D. One shows that the heart D^0 is an abelian category, and the functor

$$\mathrm{H}^0 := \tau^{\leq 0} \tau^{\geq 0} \cong \tau^{\geq 0} \tau^{\leq 0} : \mathsf{D} \to \mathsf{D}^0$$

is a cohomological functor.

Example 7.2. Let A be a left noetherian ring. The *standard t-structure* on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \{ M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{H}^j M = 0 \text{ for all } j > 0 \}$$

and

$$\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \{ M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{H}^j M = 0 \text{ for all } j < 0 \}.$$

For a complex

$$M = (\cdots \to M^n \xrightarrow{\mathrm{d}^n} M^{n+1} \to \cdots)$$

the truncations are

$$\tau^{\leq n} M = (\cdots \to M^{n-2} \to M^{n-1} \to \mathrm{Ker}(\mathrm{d}^n) \to 0 \to \cdots)$$

and

$$\tau^{\geq n} M = (\cdots \to 0 \to \mathrm{Coker}(\mathrm{d}^{n-1}) \to M^{n+1} \to M^{n+2} \to \cdots).$$

The heart $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$ is equivalent to $\mathsf{Mod}_{\mathrm{f}}\, A$.

Other t-structures on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ shall be referred to as *perverse t-structures*, and the notation $\big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}\big)$ shall be used. The letter "p" stands for "perverse", but often it will also signify a specific perversity function (see below).

Now suppose for $i = 1, 2$ we are given triangulated categories D_i, endowed with t-structures $(\mathsf{D}_i^{\leq 0}, \mathsf{D}_i^{\geq 0})$. An exact functor $F : \mathsf{D}_1 \to \mathsf{D}_2$ is called *t-exact* if $F(\mathsf{D}_1^{\leq 0}) \subset \mathsf{D}_2^{\leq 0}$ and $F(\mathsf{D}_1^{\geq 0}) \subset \mathsf{D}_2^{\geq 0}$. The functor $F : \mathsf{D}_1^0 \to \mathsf{D}_2^0$ between these abelian categories is then exact. To apply this definition to a contravariant functor F we note that $((\mathsf{D}^{\geq 0})^{\mathrm{op}}, (\mathsf{D}^{\leq 0})^{\mathrm{op}})$ is a t-structure on the opposite category D^{op}. A contravariant triangle functor $F : \mathsf{D}_1 \to \mathsf{D}_2$ is called t-exact if $F(\mathsf{D}_1^{\leq 0}) \subset \mathsf{D}_2^{\geq 0}$ and $F(\mathsf{D}_1^{\geq 0}) \subset \mathsf{D}_2^{\leq 0}$.

Example 7.3. Let A be a left noetherian $\Bbbk$-algebra and let B be a right noetherian $\Bbbk$-algebra. Suppose we are given a dualizing complex R over (A, B), and let D and D^{op} be the duality functors R induces; see Definitions 4.1 and 4.5. Put on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})$ the standard t-structure (see Example 7.2). Define subcategories

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{D}M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})^{\geq 0}\}$$

and

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{D}M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})^{\leq 0}\}.$$

Since $\mathrm{D} : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})$ is a duality it follows that

$$\big({}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}, {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}\big)$$

is a t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$, which we call the *perverse t-structure induced by R*. The functors D and D^{op} are t-exact, and

$$\mathrm{D} : {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0 \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})^0 \approx \mathsf{Mod}_{\mathrm{f}}\, B^{\mathrm{op}}$$

is a duality of abelian categories.

Definition 7.4. Suppose A is a noetherian $\Bbbk$-algebra with rigid dualizing complex R_A. The perverse t-structure induced on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ by R_A is called the *rigid perverse t-structure*. An object $M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$ is called a *perverse A-module*.

In the remainder of this section we concentrate on another method of producing t-structures on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. This method is of a geometric nature, and closely resembles the t-structures that originally appeared in [BBD].

A *perversity* is a function $p : \mathbb{Z} \to \mathbb{Z}$ satisfying $p(i) - 1 \leq p(i+1) \leq p(i)$. We call the function $p(i) = 0$ the *trivial perversity*, and the function $p(i) = -i$ is called the *minimal perversity*.

Let A be a ring. Fix an exact dimension function dim on $\mathsf{Mod}\, A$ (see Definition 4.8). For an integer i let $\mathsf{M}_i(\dim)$ be the full subcategory of $\mathsf{Mod}\, A$ consisting of the modules M with $\dim M \leq i$. The subcategory $\mathsf{M}_i(\dim)$ is localizing, and there is a functor

$$\Gamma_{\mathsf{M}_i(\dim)} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$$

defined by

$$\Gamma_{\mathsf{M}_i(\dim)} M := \{m \in M \mid \dim Am \leq i\} \subset M.$$

The functor $\Gamma_{\mathsf{M}_i(\dim)}$ has a derived functor

$$\mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} : \mathsf{D}^+(\mathsf{Mod}\, A) \to \mathsf{D}^+(\mathsf{Mod}\, A)$$

calculated using injective resolutions. For $M \in \mathsf{D}^+(\mathsf{Mod}\, A)$ the *jth cohomology of M with supports in* $\mathsf{M}_i(\dim)$ is defined to be

$$\mathrm{H}^j_{\mathsf{M}_i(\dim)} M := \mathrm{H}^j \mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} M.$$

The definition above was introduced in [Ye2] and [YZ2]. It is based on the following geometric paradigm.

Example 7.5. If A is a commutative noetherian ring of finite Krull dimension and we set $\dim M := \dim \operatorname{Supp} M$ for a finite module M, then

$$\mathrm{H}^j_{\mathsf{M}_i(\dim)} M \cong \lim_{\to} \mathrm{H}^j_Z M$$

where Z runs over the closed sets in $\operatorname{Spec} A$ of dimension $\leq i$.

Definition 7.6. Let A be a left noetherian ring. Given an exact dimension function dim on $\mathsf{Mod}\, A$ and a perversity p, define subcategories

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \dim \mathrm{H}^j M < i \text{ for all } i,j \text{ with } j > p(i)\}$$

and

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} := \{M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \mid \mathrm{H}^j_{\mathsf{M}_i(\dim)} M = 0 \text{ for all } i,j \text{ with } j < p(i)\}$$

of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

Example 7.7. Suppose dim is any exact dimension function such that $\dim M > -\infty$ for all $M \neq 0$. Take the trivial perversity $p(i) = 0$. Then

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0} = \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}$$

and

$${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0} = \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0},$$

namely the standard t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$.

The following lemma is straightforward.

Lemma 7.8. *In the situation of Definition* 7.6 *let p be the minimal perversity, namely $p(i) = -i$, and let $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$. Then:*

(1) $M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\leq 0}$ *if and only if* $\dim \mathrm{H}^{-i} M \leq i$ *for all* i.
(2) $M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq 0}$ *if and only if* $\mathrm{H}^j_{\mathsf{M}_i(\dim)} M = 0$ *for all* $j < -i$, *if and only if* $\mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^{\geq -i}$ *for all* i.

Recall that if R is an Auslander dualizing complex over the rings (A, B) then the canonical dimension Cdim_R (Definition 4.9) is an exact dimension function on $\mathsf{Mod}\, A$.

Theorem 7.9. *Let A be a left noetherian $\Bbbk$-algebra and B a right noetherian $\Bbbk$-algebra. Suppose R is an Auslander dualizing complex over (A, B). Let* dim *be the canonical dimension function* $\operatorname{Cdim}_{R;A}$ *on* $\mathsf{Mod}\, A$, *and let p be the minimal perversity $p(i) = -i$. Then:*

(1) *The pair*

$$\left({}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\leq 0}, {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)^{\geq 0}\right)$$

from Definition 7.6 *is a t-structure on* $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$.

(2) *Put on* $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,B^{\mathrm{op}})$ *the standard t-structure. Then the duality functors* D *and* D^{op} *determined by* R *(see Definition* 4.5*) are t-exact.*

Proof. In the proof we shall use the abbreviations $\mathsf{D}(A) := \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ etc.

If $M \in {}^{p}\mathsf{D}(A)^{\leq -1}$ then $M[-1] \in {}^{p}\mathsf{D}(A)^{\leq 0}$. By Lemma 7.8(1) we get $\dim \mathrm{H}^{-i}(M[-1]) \leq i$. Changing indices we get $\dim \mathrm{H}^{-i}M \leq i-1 \leq i$. Again using Lemma 7.8(1) we see that the first part of condition (i) of Definition 7.1 is verified. The second part of condition (i) is verified similarly using Lemma 7.8(2).

By the Auslander condition $\dim \mathrm{H}^{-j} R = \dim \mathrm{Ext}^{-j}_{B^{\mathrm{op}}}(B, R) \leq j$ for all j. Therefore according to Lemma 7.8(1) we get $R \in {}^{p}\mathsf{D}(A)^{\leq 0}$. On the other hand since

$$\mathrm{H}^{j}_{\mathsf{M}_i(\dim)} R \cong \varinjlim_{\mathfrak{a} \in \mathfrak{F}} \mathrm{Ext}^{j}_{A}(A/\mathfrak{a}, R),$$

where $\mathfrak{F}$ is the Gabriel filter of left ideals corresponding to $\mathsf{M}_i(\dim)$, the Auslander condition and Lemma 7.8(2) imply that $R \in {}^{p}\mathsf{D}(A)^{\geq 0}$.

Let $M' \to M \to M'' \to M'[1]$ be a distinguished triangle in $\mathsf{D}(A)$. Since dim is exact, and using the criterion in Lemma 7.8(1), we see that if M' and M'' are in ${}^{p}\mathsf{D}(A)^{\leq 0}$ then so is M. Likewise applying the functor $\mathrm{H}^{j}_{\mathsf{M}_i(\dim)}$ to this triangle and using Lemma 7.8(2) it follows that if M' and M'' are in ${}^{p}\mathsf{D}(A)^{\geq 0}$ then so is M.

Suppose we are given $M \in {}^{p}\mathsf{D}(A)^{\leq 0}$, $N \in {}^{p}\mathsf{D}(A)^{\geq 1}$ and a morphism $\phi : M \to N$. In order to prove that $\phi = 0$ we first assume M is a single finite module, concentrated in some degree $-l$, with $l \geq 0$ and $\dim M \leq l$. Then ϕ factors through $M \xrightarrow{\phi'} \mathrm{R}\Gamma_{\mathsf{M}_l(\dim)} N \to N$. Now $M \in \mathsf{D}(A)^{\leq -l}$ and, by Lemma 7.8(2), $\mathrm{R}\Gamma_{\mathsf{M}_l(\dim)} N \in \mathsf{D}(A)^{\geq -l+1}$; hence $\phi' = 0$. Next let us consider the general case. Let $\mathrm{H}^{-l}M$ be the lowest nonzero cohomology of M. We have a distinguished triangle

$$T := \left((\mathrm{H}^{-l}M)[l] \to M \to M'' \to (\mathrm{H}^{-l}M)[l+1]\right)$$

where M'' is the standard truncation of M. According to Lemma 7.8(1) we have $\dim \mathrm{H}^{-l}M \leq l$, so by the previous argument the composition $(\mathrm{H}^{-l}M)[l] \to M \xrightarrow{\phi} N$ is zero. So applying $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}\,A)}(-, N)$ to the triangle T we conclude that ϕ comes from some morphism $\phi'' : M'' \to N$. Since $M'' \in {}^{p}\mathsf{D}(A)^{\leq 0}$ and by induction on the number of nonvanishing cohomologies we have $\phi'' = 0$. Therefore condition (ii) is verified.

Next suppose $M \in \mathsf{D}(B^{\mathrm{op}})^{\leq 0}$. In order to prove that $\mathrm{D}^{\mathrm{op}} M \in {}^{p}\mathsf{D}(A)^{\geq 0}$ we can assume M is a single finite B^{op}-module, concentrated in degree $-l$ for some $l \geq 0$. By [YZ2, Proposition 5.2] and its proof we deduce

$$\begin{aligned} \mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} \mathrm{D}^{\mathrm{op}} M &= \mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} \mathrm{RHom}_{B^{\mathrm{op}}}(M, R) \\ &\cong \mathrm{RHom}_{B^{\mathrm{op}}}(M, \mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} R). \end{aligned}$$

As we saw above, the Auslander condition implies that $\mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} R \in \mathsf{D}(B^{\mathrm{op}})^{\geq -i}$. But $M \in \mathsf{D}(B^{\mathrm{op}})^{\leq -l}$, and hence

$$\mathrm{RHom}_{B^{\mathrm{op}}}(M, \mathrm{R}\Gamma_{\mathsf{M}_i(\dim)} R) \in \mathsf{D}(A)^{\geq l-i} \subset \mathsf{D}(A)^{\geq -i}.$$

Now the criterion in Lemma 7.8(2) tells us that $\mathrm{D}^{\mathrm{op}} M \in {}^{p}\mathsf{D}(A)^{\geq 0}$.

Let $M \in \mathsf{D}(B^{\mathrm{op}})^{\geq 0}$. We wish to prove that $\mathrm{D}^{\mathrm{op}} M \in {}^p\mathsf{D}(A)^{\leq 0}$. To do so we can assume M is a single finite module, concentrated in some degree $l \geq 0$. Then for every i, $\mathrm{H}^{-i}\mathrm{D}^{\mathrm{op}} M = \mathrm{Ext}^{-i}_{B^{\mathrm{op}}}(M, R)$ has $\dim \mathrm{H}^{-i}\mathrm{D}^{\mathrm{op}} M \leq i$. Now apply Lemma 7.8(1).

At this point we know that $\mathrm{D}^{\mathrm{op}}(\mathsf{D}(B^{\mathrm{op}})^{\leq 0}) \subset {}^p\mathsf{D}(A)^{\geq 0}$ and $\mathrm{D}^{\mathrm{op}}(\mathsf{D}(B^{\mathrm{op}})^{\geq 0}) \subset {}^p\mathsf{D}(A)^{\leq 0}$. Let $M \in \mathsf{D}(A)$ be an arbitrary complex, and consider the distinguished triangle

$$\tau^{\leq -1}\mathrm{D}M \to \mathrm{D}M \to \tau^{\geq 0}\mathrm{D}M \to (\tau^{\leq -1}\mathrm{D}M)[1]$$

in $\mathsf{D}(B^{\mathrm{op}})$ gotten from the standard t-structure there. Applying D^{op} we obtain a distinguished triangle

$$M' \to M \to M'' \to M'[1]$$

in $\mathsf{D}(A)$, where $M' := \mathrm{D}^{\mathrm{op}}\tau^{\geq 0}\mathrm{D}M$ and $M'' := \mathrm{D}^{\mathrm{op}}\tau^{\leq -1}\mathrm{D}M$. This proves that condition (iii) is fulfilled, so we have a t-structure on $\mathsf{D}(A)$, and also the functor D^{op} is t-exact.

To finish the proof we invoke [KS, Corollary 10.1.18] which tells us that D is also t-exact. $\square$

Question 7.10. Let A be a left noetherian ring. Find necessary and sufficient conditions on a dimension function dim on $\mathsf{Mod}\, A$, and on a perversity function p, such that part (1) of Theorem 7.9 holds.

Remark 7.11. The idea for Definition 7.6 comes from [KS, p. 438 Exercise X.2]. In his recent paper [Ka] Kashiwara considers similar t-structures. In particular his results imply that when $A = B$ is a commutative finitely generated $\Bbbk$-algebra, and R is the rigid dualizing complex of A, then Theorem 7.9 holds for any perversity function p. (Note that here canonical dimension coincides with Krull dimension.) In part (2) of the theorem one has to put on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})$ the perverse t-structure determined by the dual perversity $p^*(i) := -i - p(i)$.

8. The Rigid Dualizing Complex of a Differential $\Bbbk$-Algebra

We begin this section with the following consequence of previous work.

Theorem 8.1. *Let A be a differential $\Bbbk$-algebra of finite type. Then A has an Auslander rigid dualizing complex R_A. For a finite A-module M the canonical dimension* $\mathrm{Cdim}\, M$ *coincides with the Gelfand-Kirillov dimension* $\mathrm{GKdim}\, M$.

Proof. According to Theorem 3.1 A has a nonnegative exhaustive filtration $G = \{G_i A\}$ such that $\mathrm{gr}^G A$ is a commutative, finitely generated, connected graded $\Bbbk$-algebra. Now use [YZ1, Corollary 6.9]. $\square$

Recall that a ring homomorphism $f : A \to B$ is called *finite centralizing* if there exist elements $b_1, \ldots, b_n \in B$ that commute with all elements of A and $B = \sum_i A b_i$.

Proposition 8.2. *Let A be a differential $\Bbbk$-algebra of finite type and $f : A \to B$ a finite centralizing homomorphism. Then B is also a differential $\Bbbk$-algebra of finite type, and the rigid trace $\mathrm{Tr}_{B/A} : R_B \to R_A$ exists.*

Proof. By Theorem 3.1 we can find a differential $\Bbbk$-filtration of finite type $F = \{F_i A\}$ of A such that $\mathrm{gr}^F A$ is connected. By [YZ1, Lemma 6.13] and its proof there is a filtration $F = \{F_i B\}$ of B such that $\mathrm{gr}^F B$ is connected, $f(F_i A) \subset F_i B$ and $\mathrm{gr}^F(f) : \mathrm{gr}^F A \to \mathrm{gr}^F B$ is a finite centralizing homomorphism. It follows that

$\operatorname{gr}^F B$ is finite over its center, so B is a differential $\Bbbk$-algebra of finite type. By [YZ1, Theorem 6.17] the rigid trace $\operatorname{Tr}_{B/A} : R_B \to R_A$ exists. □

Let A be a differential $\Bbbk$-algebra of finite type with rigid dualizing complex R_A. The derived category $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ has on it the rigid perverse t-structure induced by R_A, whose heart is the category of perverse A-modules ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$. See Definition 7.4.

Proposition 8.3. *Let $A \to B$ be a finite centralizing homomorphism between two differential $\Bbbk$-algebras of finite type. Denote by $\operatorname{rest}_{B/A} : \mathsf{D}(\mathsf{Mod}\, B) \to \mathsf{D}(\mathsf{Mod}\, A)$ the restriction of scalars functor.*

(1) *Let $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)$. Then $M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)^0$ if and only if $\operatorname{rest}_{B/A} M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$.*

(2) *If $A \to B$ is surjective then the functor*

$$\operatorname{rest}_{B/A} : {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)^0 \to {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$$

is fully faithful.

Proof. (1) Define the duality functors $\mathrm{D}_A := \operatorname{RHom}_A(-, R_A)$ and $\mathrm{D}_B := \operatorname{RHom}_B(-, R_B)$. According to [YZ1, Proposition 3.9(1)] the trace $\operatorname{Tr}_{B/A} : R_B \to R_A$ gives rise to a commutative diagram

$$\begin{array}{ccc}
\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B) & \xrightarrow{\operatorname{rest}_{B/A}} & \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A) \\
\big\downarrow{\scriptstyle \mathrm{D}_B} & & \big\downarrow{\scriptstyle \mathrm{D}_A} \\
\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B^{\mathrm{op}})^{\mathrm{op}} & \xrightarrow{\operatorname{rest}_{B^{\mathrm{op}}/A^{\mathrm{op}}}} & \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^{\mathrm{op}})^{\mathrm{op}}
\end{array}$$

in which the vertical arrows are equivalences. By definition $M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)^0$ if and only if $\mathrm{H}^i \mathrm{D}_B M = 0$ for all $i \neq 0$. Likewise $\operatorname{rest}_{B/A} M \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0$ if and only if $\mathrm{H}^i \mathrm{D}_A \operatorname{rest}_{B/A} M = 0$ for all $i \neq 0$. But

$$\mathrm{H}^i \mathrm{D}_A \operatorname{rest}_{B/A} M \cong \operatorname{rest}_{B^{\mathrm{op}}/A^{\mathrm{op}}} \mathrm{H}^i \mathrm{D}_B M.$$

(2) In view of (1) we have a commutative diagram

$$\begin{array}{ccc}
{}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, B)^0 & \xrightarrow{\operatorname{rest}_{B/A}} & {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)^0 \\
\big\downarrow{\scriptstyle \mathrm{D}_B} & & \big\downarrow{\scriptstyle \mathrm{D}_A} \\
(\mathsf{Mod}_{\mathrm{f}}\, B^{\mathrm{op}})^{\mathrm{op}} & \xrightarrow{\operatorname{rest}_{B^{\mathrm{op}}/A^{\mathrm{op}}}} & (\mathsf{Mod}_{\mathrm{f}}\, A^{\mathrm{op}})^{\mathrm{op}}
\end{array}$$

where the vertical arrows are equivalences. The lower horizontal arrow is a full embedding, since it identifies $\mathsf{Mod}_{\mathrm{f}}\, B^{\mathrm{op}}$ with the full subcategory of $\mathsf{Mod}_{\mathrm{f}}\, A^{\mathrm{op}}$ consisting of modules annihilated by $\operatorname{Ker}(A^{\mathrm{op}} \to B^{\mathrm{op}})$. Hence the top horizontal arrow is fully faithful. □

Lemma 8.4. *Suppose A and B are $\Bbbk$-algebras, $M, M' \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ and $N, N' \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B^{\mathrm{e}})$. Then there is a functorial morphism*

$$\mu : \operatorname{RHom}_A(M, M') \otimes \operatorname{RHom}_B(N, N') \to \operatorname{RHom}_{A \otimes B}(M \otimes N, M' \otimes N')$$

in $\mathsf{D}(\mathsf{Mod}\, (A \otimes B)^{\mathrm{e}})$. If A and B are left noetherian and all the modules $\mathrm{H}^p M$ and $\mathrm{H}^p N$ are finite then μ is an isomorphism.

Proof. Choose projective resolutions $P \to M$ and $Q \to N$ over A^{e} and B^{e} respectively. So $P \otimes Q \to M \otimes N$ is a projective resolution over $(A \otimes B)^{\mathrm{e}}$, and we get a map of complexes

$$\mu : \operatorname{Hom}_A(P, M') \otimes \operatorname{Hom}_B(Q, N') \to \operatorname{Hom}_{A\otimes B}(P \otimes Q, M' \otimes N')$$

over $(A \otimes B)^{\mathrm{e}}$.

Now assume the finiteness of the cohomologies. To prove μ is a quasi-isomorphism we might as well forget the right module structures. Choose resolutions $P_{\mathrm{f}} \to M$ and $Q_{\mathrm{f}} \to N$ by complexes of finite projective modules over A and B respectively. We obtain a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_A(P, M') \otimes \operatorname{Hom}_B(Q, N') & \xrightarrow{\mu} & \operatorname{Hom}_{A\otimes B}(P \otimes Q, M' \otimes N') \\ \downarrow & & \downarrow \\ \operatorname{Hom}_A(P_{\mathrm{f}}, M') \otimes \operatorname{Hom}_B(Q_{\mathrm{f}}, N') & \xrightarrow{\mu} & \operatorname{Hom}_{A\otimes B}(P_{\mathrm{f}} \otimes Q_{\mathrm{f}}, M' \otimes N') \end{array}$$

in which the vertical arrows are quasi-isomorphism and the bottom arrow is an isomorphism of complexes. □

Recall that by Proposition 2.9 the tensor product of two differential $\Bbbk$-algebras of finite type is also a differential $\Bbbk$-algebra of finite type.

Theorem 8.5. *Suppose A and B are differential $\Bbbk$-algebras of finite type. Then the rigid dualizing complexes satisfy*

$$R_{A\otimes B} \cong R_A \otimes R_B$$

in $\mathsf{D}(\mathsf{Mod}\,(A \otimes B)^{\mathrm{e}})$.

Proof. We will prove that $R_A \otimes R_B$ is a rigid dualizing complex over $A \otimes B$.

Consider the Kunneth spectral sequence

$$(\mathrm{H}^p R_A) \otimes (\mathrm{H}^q R_B) \Rightarrow \mathrm{H}^{p+q}(R_A \otimes R_B).$$

Since $A \otimes B$ is noetherian it follows that $\mathrm{H}^{p+q}(R_A \otimes R_B)$ is a finite $(A \otimes B)$-module on both sides.

From Lemma 8.4 we see that the canonical morphism

$$A \otimes B \to \operatorname{RHom}_{A\otimes B}(R_A \otimes R_B, R_A \otimes R_B)$$

in $\mathsf{D}(\mathsf{Mod}\,(A \otimes B)^{\mathrm{e}})$ is an isomorphism. Likewise for $\operatorname{RHom}_{(A\otimes B)^{\mathrm{e}}}$.

Next using this lemma with A^{e} and B^{e}, and by the rigidity of R_A and R_B, we get isomorphisms

$$\begin{aligned} &\operatorname{RHom}_{(A\otimes B)^{\mathrm{e}}}\big(A \otimes B, (R_A \otimes R_B) \otimes (R_A \otimes R_B)\big) \\ &\quad \cong \operatorname{RHom}_{A^{\mathrm{e}}\otimes B^{\mathrm{e}}}\big(A \otimes B, (R_A \otimes R_A) \otimes (R_B \otimes R_B)\big) \\ &\quad \cong \operatorname{RHom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) \otimes \operatorname{RHom}_{B^{\mathrm{e}}}(B, R_B \otimes R_B) \\ &\quad \cong R_A \otimes R_B \end{aligned}$$

in $\mathsf{D}(\mathsf{Mod}\,(A \otimes B)^{\mathrm{e}})$.

It remains to prove that the complex $R_A \otimes R_B$ has finite injective dimension over $A \otimes B$ and over $(A \otimes B)^{\mathrm{op}}$. This turns out to be quite difficult (cf. Corollary 8.6 below). By Theorem 3.1 there is a filtration F of A such that $\operatorname{gr}^F A$ is connected, finitely generated and commutative. Let $\widetilde{A} := \operatorname{Rees}^F A \subset A[s]$, which is a noetherian connected graded $\Bbbk$-algebra. By [YZ1, Theorem 5.13], $\widetilde{A}$ has a balanced dualizing complex $R_{\widetilde{A}} \in \mathsf{D}^{\mathrm{b}}(\mathsf{GrMod}\,(\widetilde{A})^{\mathrm{e}})$. The same holds for B: there's a filtration G, a Rees ring $\widetilde{B} := \mathrm{Rees}^G B \subset B[t]$ and a balanced dualizing complex $R_{\widetilde{B}}$. According to [VdB, Theorem 7.1] the complex $R_{\widetilde{A}} \otimes R_{\widetilde{B}}$ is a balanced dualizing complex over $\widetilde{A} \otimes \widetilde{B}$. In particular $R_{\widetilde{A}} \otimes R_{\widetilde{B}}$ has finite graded-injective dimension over $\widetilde{A} \otimes \widetilde{B}$.

Now $A \cong \widetilde{A}/(s-1)$, so by [YZ1, Lemma 6.3] the complex

$$Q := (A \otimes \widetilde{B}) \otimes_{\widetilde{A} \otimes \widetilde{B}} (R_{\widetilde{A}} \otimes R_{\widetilde{B}})$$

has finite injective dimension over $A \otimes \widetilde{B}$. But the algebra $A \otimes \widetilde{B}$ is graded (the element $1 \otimes t$ has degree 1), and Q is a complex of graded $(A \otimes \widetilde{B})$-modules. Therefore Q has finite graded-injective dimension over this graded ring. Applying [YZ1, Lemma 6.3] again – it works for any graded ring, connected or not – we see that

$$(A \otimes B) \otimes_{A \otimes \widetilde{B}} Q \cong (A \otimes B) \otimes_{\widetilde{A} \otimes \widetilde{B}} (R_{\widetilde{A}} \otimes R_{\widetilde{B}})$$

has finite injective dimension over $A \otimes B$.

According to [YZ1, Theorem 6.2] there is an isomorphism $R_A \cong A \otimes_{\widetilde{A}} R_{\widetilde{A}}[-1]$ in $\mathsf{D}(\mathsf{Mod}\,A)$. Likewise $R_B \cong B \otimes_{\widetilde{B}} R_{\widetilde{B}}[-1]$. Hence

$$R_A \otimes R_B \cong (A \otimes B) \otimes_{\widetilde{A} \otimes \widetilde{B}} (R_{\widetilde{A}} \otimes R_{\widetilde{B}})[-2]$$

has finite injective dimension over $A \otimes B$.

By symmetry $R_A \otimes R_B$ has finite injective dimension also over $(A \otimes B)^{\mathrm{op}}$. □

Corollary 8.6. *Suppose A and B are differential $\Bbbk$-algebras of finite type, and the complexes $M \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A)$ and $N \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,B)$ have finite injective dimension over A and B respectively. Then $M \otimes N$ has finite injective dimension over $A \otimes B$.*

Proof. Let $M^\vee := \mathrm{RHom}_A(M, R_A)$ and $N^\vee := \mathrm{RHom}_B(N, R_B)$. The complexes $M^\vee \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{op}})$ and $N^\vee \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,B^{\mathrm{op}})$ have finite projective dimension (i.e. they are perfect); see [Ye4, Theorem 4.5]. Since the tensor product of projective modules is projective it follows that $M^\vee \otimes N^\vee \in \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,(A^{\mathrm{op}} \otimes B^{\mathrm{op}}))$ has finite projective dimension. Using Theorem 8.5 and Lemma 8.4 we see that

$$\mathrm{RHom}_{A \otimes B}(M \otimes N, R_{A \otimes B}) \cong \mathrm{RHom}_{A \otimes B}(M \otimes N, R_A \otimes R_B) \cong M^\vee \otimes N^\vee.$$

Applying $\mathrm{RHom}_{A \otimes B}(-, R_{A \otimes B})$ to these isomorphisms we get

$$M \otimes N \cong \mathrm{RHom}_{A \otimes B}(M^\vee \otimes N^\vee, R_{A \otimes B}),$$

so this complex has finite injective dimension. □

Question 8.7. Is there a direct proof of the corollary? Is it true in greater generality, e.g. for any two noetherian $\Bbbk$-algebras A and B?

Remark 8.8. We take this opportunity to correct a slight error in [YZ1]. In [YZ1, Theorem 6.2(1)] the complex R should be defined as $R := (\widetilde{R}_t)_0$, namely the degree 0 component of the localization with respect to the element t. The rest of that theorem (including the proof!) is correct.

If A is a differential $\Bbbk$-algebra of finite type then so is the enveloping algebra A^{e}. Hence the rigid dualizing complex $R_{A^{\mathrm{e}}}$ exists, as does the rigid perverse t-structure on $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{e}})$, whose heart is the category ${}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{e}})^0$ of perverse A^{e}-modules.

Theorem 8.9. *Let A be a differential $\Bbbk$-algebra of finite type with rigid dualizing complex R_A. Then $R_A \in {}^p\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\,A^{\mathrm{e}})^0$.*

Proof. Consider the $\Bbbk$-algebra isomorphism

$$\tau : (A^{\mathrm{op}})^{\mathrm{e}} = A^{\mathrm{op}} \otimes A \xrightarrow{\simeq} A \otimes A^{\mathrm{op}} = A^{\mathrm{e}}$$

with formula $\tau(a_1 \otimes a_2) := a_2 \otimes a_1$. Given an A^{e}-module M let ${}^{\tau}M$ be the $(A^{\mathrm{op}})^{\mathrm{e}}$-module with action via τ, i.e.

$$(a_1 \otimes a_2) \cdot_{\tau} m := \tau(a_1 \otimes a_2) \cdot m = a_2 m a_1$$

for $m \in M$ and $a_1 \otimes a_2 \in (A^{\mathrm{op}})^{\mathrm{e}}$. Doing this operation to the complex $R_A \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ we obtain a complex ${}^{\tau}R_A \in \mathsf{D}(\mathsf{Mod}\,(A^{\mathrm{op}})^{\mathrm{e}})$. Each of the conditions in Definitions 4.1 and 4.10 is automatically verified, and hence $R_{A^{\mathrm{op}}} := {}^{\tau}R_A$ is a rigid dualizing complex over A^{op}.

According to Theorem 8.5 we get an isomorphism

$$R_{A^{\mathrm{e}}} \cong R_A \otimes R_{A^{\mathrm{op}}} = R_A \otimes ({}^{\tau}R_A)$$

in $\mathsf{D}(\mathsf{Mod}\,(A^{\mathrm{e}})^{\mathrm{e}})$. But the left (resp. right) A^{e} action on $R_A \otimes ({}^{\tau}R_A)$ is exactly the outside (resp. inside) action on $R_A \otimes R_A$. By rigidity (cf. Definition 4.10) we have

$$R_A \cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) \cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, R_{A^{\mathrm{e}}})$$

in $\mathsf{D}(\mathsf{Mod}\ A^{\mathrm{e}})$.

Finally since $A^{\mathrm{e}} \cong (A^{\mathrm{e}})^{\mathrm{op}}$, via the involution τ, we may view $\mathrm{RHom}_{A^{\mathrm{e}}}(-, R_{A^{\mathrm{e}}})$ as an auto-duality of $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\ A^{\mathrm{e}})$. By definition of the rigid t-structure this duality exchanges $\mathsf{Mod}_{\mathrm{f}}\, A^{\mathrm{e}}$ and ${}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\ A^{\mathrm{e}})^0$. Since $A \in \mathsf{Mod}_{\mathrm{f}}\, A^{\mathrm{e}}$ it follows that $R_A \in {}^{p}\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\ A^{\mathrm{e}})^0$. □

We know that the cohomology bimodules $\mathrm{H}^i R_A$ are central $\mathrm{Z}(A)$-modules. The next lemma shall be used a few times.

Lemma 8.10. *Let A be a differential $\Bbbk$-algebra of finite type and $a \in A$ a non-invertible central regular element. Define $B := A/(a)$. Let R_A and R_B denote the rigid dualizing complexes of A and B respectively. Then there is a long exact sequence*

$$\cdots \to \mathrm{H}^i R_B \to \mathrm{H}^i R_A \xrightarrow{a} \mathrm{H}^i R_A \to \mathrm{H}^{i+1} R_B \to \cdots$$

of A-bimodules.

Proof. Trivially $A \to B$ is a finite centralizing homomorphism. By Proposition 8.2 the trace morphism $\mathrm{Tr}_{B/A} : R_B \to R_A$ exists. In particular, $R_B \cong \mathrm{RHom}_A(B, R_A)$. There is an exact sequence of bimodules

$$0 \to A \xrightarrow{a} A \to B \to 0.$$

Applying the functor $\mathrm{RHom}_A(-, R_A)$ to it and taking cohomologies we obtain the long exact sequence we want. □

Here are a couple of examples of differential $\Bbbk$-algebras of finite type and their rigid dualizing complexes.

Example 8.11. Let C be a smooth n-dimensional $\Bbbk$-algebra in characteristic 0 and $A := \mathcal{D}(C)$ the ring of differential operators. Then the rigid dualizing complex is $R_A = A[2n]$; see [Ye3].

Example 8.12. Let $\mathfrak{g}$ be an n-dimensional Lie algebra over $\Bbbk$ and $A := \mathrm{U}(\mathfrak{g})$ its universal enveloping algebra. By [Ye3] the rigid dualizing complex is $R_A = A \otimes (\bigwedge^n \mathfrak{g})[n]$, where $\bigwedge^n \mathfrak{g}$ has the adjoint A action on the left and the trivial action on the right.

Suppose A is a ring with nonnegative exhaustive filtration F such that the Rees ring $\widetilde{A} := \operatorname{Rees}^F A$ is left noetherian. We remind that a filtered (A,F)-module (M,F) is called *good* if it is bounded below, exhaustive, and $\operatorname{Rees}^F M$ is a finite $\widetilde{A}$-module.

In the two previous examples the cohomology bimodules $\mathrm{H}^i R_A$ all came equipped with filtrations that were both differential and good on both sides. These properties turn out to hold in general, as Theorems 8.13 and 8.14 show.

Theorem 8.13. *Let A be a differential $\Bbbk$-algebra of finite type, and let R_A be the rigid dualizing complex of A. Let F be some differential $\Bbbk$-filtration of finite type of A. Then for every i there is an induced filtration F of $\mathrm{H}^i R_A$, such that $(\mathrm{H}^i R_A, F)$ is a good filtered (A,F)-module on both sides.*

Proof. Define $\widetilde{A} := \operatorname{Rees}^F A \subset A[t]$. Let $\widetilde{F} = \{\widetilde{F}_i \widetilde{A}\}$ be the filtration from Lemma 1.4. Then

$$\operatorname{gr}^{\widetilde{F}} \widetilde{A} \cong (\operatorname{gr}^F A) \otimes \Bbbk[t]$$

as $\Bbbk$-algebras. The center is

$$\mathrm{Z}(\operatorname{gr}^{\widetilde{F}} \widetilde{A}) \cong \mathrm{Z}(\operatorname{gr}^F A) \otimes \Bbbk[t],$$

which is a finitely generated commutative $\Bbbk$-algebra. Also $\operatorname{gr}^{\widetilde{F}} \widetilde{A}$ is a finite $\mathrm{Z}(\operatorname{gr}^{\widetilde{F}} \widetilde{A})$-module. We conclude that $\widetilde{F}$ is a differential $\Bbbk$-filtration of finite type on $\widetilde{A}$. Moreover each $\Bbbk$-submodule $\widetilde{F}_i \widetilde{A}$ is graded, where $\widetilde{A} \subset A[t]$ has the grading F in which $\deg^F(t) = 1$.

Applying Theorem 3.1 and Proposition 3.5 to the filtered $\Bbbk$-algebra $(\widetilde{A}, \widetilde{F})$ we obtain another filtration $\widetilde{G}$ on $\widetilde{A}$. This new filtration is also differential $\Bbbk$-filtration of finite type, and each $\Bbbk$-submodule $\widetilde{G}_i \widetilde{A} \subset \widetilde{A}$ is graded (for the grading F). Furthermore $\operatorname{gr}^{\widetilde{G}} \widetilde{A}$ is a connected graded $\Bbbk$-algebra (when considered as $\mathbb{Z}$-graded ring with the grading $\widetilde{G}$).

Define

$$B := \operatorname{Rees}^{\widetilde{G}} \widetilde{A} \subset \widetilde{A}[s] \subset A[s,t].$$

This is a $\mathbb{Z}^2$-graded ring with grading $(\widetilde{G}, F)$, in which $\deg^{(\widetilde{G},F)}(s) = (1,0)$ and $\deg^{(\widetilde{G},F)}(t) = (0,1)$.

Consider the $\Bbbk$-algebra B with its grading $\widetilde{G}$. This is a connected graded $\Bbbk$-algebra. The quotient $B/(s) \cong \operatorname{gr}^{\widetilde{G}} \widetilde{A}$ is a finitely generated commutative $\Bbbk$-algebra. Therefore by [YZ1, Theorem 5.13], B has a balanced dualizing complex $R_B \in \mathsf{D}^{\mathrm{b}}(\mathsf{GrMod}\, B^{\mathrm{e}})$. By [VdB, Theorem 6.3] we get

$$R_B \cong (\mathrm{R}\,\Gamma_{\mathfrak{m}} B)^*$$

in $\mathsf{D}(\mathsf{GrMod}\, B^{\mathrm{e}})$. Here $\Gamma_{\mathfrak{m}}$ is the torsion functor with respect to the augmentation ideal $\mathfrak{m}$ of B, and

$$(M)^* := \operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}(M, \Bbbk) = \bigoplus\nolimits_i \operatorname{Hom}_{\Bbbk}(M_{-i}, \Bbbk),$$

the graded dual of the graded $\Bbbk$-module M. In particular, for every p there is an isomorphism of B-bimodules $\mathrm{H}^{-p} R_B \cong \big(\mathrm{H}^p_{\mathfrak{m}} B\big)^*$ where

$$\mathrm{H}^p_{\mathfrak{m}} B := \mathrm{H}^p \mathrm{R}\Gamma_{\mathfrak{m}} B \cong \lim_{k \to} \operatorname{Ext}^p_B(B/\mathfrak{m}^k, B).$$

Now for each k

$$\operatorname{Ext}^p_B(B/\mathfrak{m}^k, B) = \bigoplus\nolimits_{(i,j)\in\mathbb{Z}^2} \operatorname{Ext}^p_B(B/\mathfrak{m}^k, B)_{(i,j)}$$

where (i,j) is the $(\widetilde{G}, F)$ degree. Therefore in the direct limit we get a double grading

$$\mathrm{H}^p_{\mathfrak{m}} B = \bigoplus\nolimits_{(i,j)\in\mathbb{Z}^2} (\mathrm{H}^p_{\mathfrak{m}} B)_{(i,j)}.$$

Since for every i the $\Bbbk$-module $(\mathrm{H}^p_{\mathfrak{m}} B)_i = \bigoplus_{j\in\mathbb{Z}} (\mathrm{H}^p_{\mathfrak{m}} B)_{(i,j)}$ is finite it follows that the graded dual $(\mathrm{H}^p_{\mathfrak{m}} B)^*$, which is computed with respect to the $\widetilde{G}$ grading, is also $\mathbb{Z}^2$-graded. We see that $\mathrm{H}^{-p} R_B$ is in fact a $\mathbb{Z}^2$-graded B-bimodule.

By [YZ1, Theorem 6.2] the complex $R_{\widetilde{A}} := ((R_B[-1])_s)_0$ is a rigid dualizing complex over the ring $\widetilde{A} \cong B/(s-1)$; cf. Remark 8.8. Hence each cohomology

$$\mathrm{H}^p R_{\widetilde{A}} \cong \frac{\mathrm{H}^{p-1} R_B}{(s-1)\cdot \mathrm{H}^{p-1} R_B} = \operatorname{sp}^{\widetilde{G}}_1 \mathrm{H}^{p-1} R_B$$

is a graded $\widetilde{A}$-bimodule (with the $\mathbb{Z}$-grading F in which $\deg^F(t) = 1$), finite on both sides.

Next we have $A \cong \widetilde{A}/(t-1)$. Because $t-1$ is a central regular non-invertible element, Lemma 8.10 says there is an exact sequence

$$\cdots \to \mathrm{H}^p R_A \to \mathrm{H}^p R_{\widetilde{A}} \xrightarrow{t-1} \mathrm{H}^p R_{\widetilde{A}} \to \mathrm{H}^{p+1} R_A \to \mathrm{H}^{p+1} R_{\widetilde{A}} \to \cdots$$

of A-bimodules. Since $\mathrm{H}^p R_{\widetilde{A}}$ is graded the element $t-1$ is a non-zero-divisor on it, and therefore we get an exact sequence

$$0 \to \mathrm{H}^{p-1} R_{\widetilde{A}} \xrightarrow{t-1} \mathrm{H}^{p-1} R_{\widetilde{A}} \to \mathrm{H}^p R_A \to 0.$$

Thus the bimodule

$$\mathrm{H}^p R_A \cong \frac{\mathrm{H}^{p-1} R_{\widetilde{A}}}{(t-1)\cdot \mathrm{H}^{p-1} R_{\widetilde{A}}} = \operatorname{sp}^F_1 \mathrm{H}^{p-1} R_{\widetilde{A}}$$

inherits a bounded below exhaustive filtration F, and $\operatorname{Rees}(\mathrm{H}^p R_A, F)$, being a quotient of $\mathrm{H}^{p-1} R_{\widetilde{A}}$, is a finite $\widetilde{A}$-module on both sides. By definition $(\mathrm{H}^p R_A, F)$ is then is a good filtered (A, F)-module on both sides. □

Theorem 8.14. *Let C be a finitely generated commutative $\Bbbk$-algebra, let A be differential C-ring of finite type, and let R_A be the rigid dualizing complex of A. Then for every i the C-bimodule $\mathrm{H}^i R_A$ is differential.*

Proof. Using the same setup as in the proof of Theorem 8.13, define

$$\bar{A} := \operatorname{gr}^F A = \operatorname{sp}^F_0 \widetilde{A} = \widetilde{A}/(t).$$

So $\bar{A}$ is a C-algebra. Let $R_{\bar{A}}$ be the rigid dualizing complex of $\bar{A}$. By [YZ1, Corollary 3.6] the $\bar{A}$-bimodule $\mathrm{H}^i R_{\bar{A}}$ is $Z(\bar{A})$-central, and hence it is a central C-bimodule. According to Lemma 8.10 there is an exact sequence of $\widetilde{A}$-bimodules

$$\mathrm{H}^{i-1} R_{\widetilde{A}} \xrightarrow{t} \mathrm{H}^{i-1} R_{\widetilde{A}} \to \mathrm{H}^i R_{\bar{A}}.$$

Therefore

$$\operatorname{sp}_0 (\mathrm{H}^{i-1} R_{\widetilde{A}}) = (\mathrm{H}^{i-1} R_{\widetilde{A}})/t \cdot (\mathrm{H}^{i-1} R_{\widetilde{A}}) \hookrightarrow \mathrm{H}^i R_{\bar{A}}$$

is a central C-bimodule.

To conclude the proof, consider the filtration F of $\mathrm{H}^i R_A$ from Theorem 8.13. Because $(\mathrm{H}^i R_A, F) \cong \mathrm{sp}_1\,(\mathrm{H}^{i-1} R_{\widetilde{A}})$ is a good filtered (A,F)-module, say on the left, we see that $(\mathrm{H}^i R_A, F)$ is exhaustive and bounded below. Now

$$\operatorname{Rees}\,(\mathrm{H}^i R_A, F) \cong (\mathrm{H}^{i-1} R_{\widetilde{A}})/\{t\text{-torsion}\},$$

so $\operatorname{gr}\,(\mathrm{H}^i R_A, F)$ is a quotient of $\mathrm{sp}_0\,(\mathrm{H}^{i-1} R_{\widetilde{A}})$. It follows that $\operatorname{gr}\,(\mathrm{H}^i R_A, F)$ is a central C-bimodule. Thus F is a differential C-filtration of $\mathrm{H}^i R_A$. □

Corollary 8.15. *In the situation of Theorem* 8.14 *let* $U := \operatorname{Spec} C$. *Given an affine open set* $V \subset U$ *let* $C' := \Gamma(V, \mathcal{O}_U)$. *Then* $A' := C' \otimes_C A \otimes_C C'$ *is a noetherian* $\Bbbk$*-algebra,* $A \to A'$ *is flat and* $R' := A' \otimes_A R_A \otimes_A A'$ *is an Auslander rigid dualizing complex over* A'.

Proof. According to Corollary 5.20 A' is a noetherian $\Bbbk$-algebra and $A \to A'$ is a localization. We know that A^{e} is noetherian. By Theorem 8.14 each of the cohomology bimodules $\mathrm{H}^i R_A$ are differential as C-bimodules, hence by Proposition 5.17 they are evenly localizable to C'. Thus all the hypothesis of Theorem 6.2(1,2,3) are satisfied. □

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