HOMOLOGICAL TRANSCENDENCE DEGREE

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Abstract. Let D be a division algebra over a base field k. The homological transcendence degree of D, denoted by $\operatorname{Htr} D$, is defined to be the injective dimension of the algebra $D \otimes_k D^\circ$. We show that Htr has several useful properties which the classical transcendence degree has. We extend some results of Resco, Rosenberg, Schofield and Stafford, and compute Htr for several classes of division algebras. The main tool for the computation is Van den Bergh's rigid dualizing complex.

0. Introduction

Throughout k is a commutative base field. By default all algebras and rings are k-algebras, and all homomorphisms are over k. This paper is mainly about division algebras that are infinite dimensional over their centers. Such division algebras appear naturally in noncommutative ring theory, and recently there are many new examples coming from noncommutative projective geometry. One important question in noncommutative algebra/geometry is the classification of division algebras of transcendence degree 2 (see some discussion in [1]). Similarly to the commutative situation, the classification of division algebras of transcendence degree 2 would be equivalent to the birational classification of integral noncommutative projective surfaces. Quantum $\mathbb{P}^3$'s (the classification of which has not yet been achieved), will provide new examples of division algebras of transcendence degree 3. Other division algebras such as the quotient division rings of Artin-Schelter regular algebras will certainly play an important role in noncommutative projective geometry.

Most division algebras arising from noncommutative projective geometry should have finite transcendence degree. But what is the definition of transcendence degree for a division algebra infinite dimensional over its center? The first such definition is due to Gelfand and Kirillov [5]. Let GKdim denote the Gelfand-Kirillov dimension. Then the *Gelfand-Kirillov transcendence degree* of a division algebra D is defined to be

$$\operatorname{GKtr} D = \sup_V \inf_z \ \operatorname{GKdim} k[zV],$$

where V runs over all finite dimensional k-subspaces of D, and z runs over all nonzero elements in D. This is probably the first simple invariant that distinguishes between the Weyl skew fields, since $\operatorname{GKtr} D_n = 2n$, where D_n is the n-th Weyl skew field. Note that $\operatorname{GKdim} D_n = \infty$ for all n, so it does not provide any useful information. Partly due to the complicated definition, GKtr is very mysterious. For

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example, it is not known whether $\operatorname{GKtr} D_1 \leq \operatorname{GKtr} D_2$ when $D_1 \subset D_2$ are division algebras. Also there are only a handful families of division algebras for which GKtr was computed explicitly [5, 9, 36]. Recently the GKtr of the quotient division rings of twisted homogeneous coordinate rings was computed in [16, Corollary 5.8].

The second author gave another definition, called *lower transcendence degree*, denoted by Ltr [37]. In general it is not clear whether Ltr = GKtr. Several basic properties of classical transcendence degree have been established for Ltr. Using the properties of Ltr, one can compute both Ltr and GKtr for several more classes of division algebras. For example, both Ltr and GKtr of the quotient division ring of any n-dimensional Sklyanin algebra are equal to n. It is not hard to see from the definition that both GKtr and Ltr are bounded by GKdim. So for some classes of rings one can obtain upper bounds for these two invariants. However it is not easy to compute the exact value in general. One open question is the following.

Question 0.1. Let Q be the quotient division ring of a noetherian Artin-Schelter regular Ore domain of global dimension $n \geq 4$. Is $\operatorname{Ltr} Q = \operatorname{GKtr} Q = n$?

The answer to Question 0.1 is "yes" for $n \leq 3$ (see [36, Theorem 1.1(10)] and [37, Theorem 0.5(4)]). Also it can be shown that both $\operatorname{Ltr} Q$ and $\operatorname{GKtr} Q$ in Question 0.1 are finite.

It is fundamental to have a transcendence degree that is useful and computable for a large class of division algebras including those arising from noncommutative projective geometry. In this paper we introduce a new definition of transcendence degree which is defined homologically; and show that this transcendence degree is computable for many algebras with good homological properties, including all quotient division rings of Artin-Schelter regular algebras. Let D be a division algebra over k. The *homological transcendence degree* of D is defined to be

$$\operatorname{Htr} D = \operatorname{injdim} D \otimes_k D^\circ$$

where $D \otimes_k D^\circ$ is viewed as a left module over itself. Here D° is the opposite ring of D and injdim denotes the injective dimension of a left module.

The idea of studying homological invariants of division algebras first appeared in Resco's papers [12, 14] for commutative fields. Later this was used by Stafford [24] to study the quotient division ring of the group ring kG of a torsion-free polycyclic-by-finite group G, and the quotient division ring of the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$; and by Rosenberg [17] to study the Weyl skew fields. Schofield extended this idea effectively to stratiform simple artinian rings, and proved several wonderful results in [20]. In addition there are other papers that studied various invariants of the tensor product of divisions rings [15, 22, 27]. Our definition of transcendence degree is motivated by the work of Resco, Rosenberg, Schofield and Stafford. Using the results of Schofield [20] one can show that $\operatorname{Htr} D = n$ where D is a stratiform simple artinian ring of length n [Proposition 1.8]. A similar computation works for the division rings studied by Resco, Rosenberg and Stafford. However we intend to cover a large class of division algebras for which the methods of the above people may not apply.

If A is an Ore domain, let $Q(A)$ denote the quotient division ring of A. We prove the following.

Theorem 0.2. *Let A be an Artin-Schelter regular graded Ore domain. Then $\operatorname{Htr} Q(A) = \operatorname{gldim} A$.*

In Theorem 0.2 above we do not assume that A is noetherian. One of the main tools for this computation is the rigid dualizing complex, which we will review in Section 3. As a consequence, if A and B are two Artin-Schelter regular Ore domains of different global dimensions, then $Q(A)$ is not isomorphic to $Q(B)$. Theorem 0.2 fails without the Artin-Schelter condition. By Propositions 7.6 and 7.8 below there is a connected graded Koszul Ore domain A of GK-dimension 4 and global dimension 4 such that

$$\operatorname{Htr} Q(A) = 3 < 4 = \operatorname{GKtr} Q(A) = \operatorname{GKtr} A.$$

One big project would be to compute the homological transcendence degree for all division algebras that are not constructed from Artin-Schelter regular algebras. We start this task with the quotient division rings of connected graded domains with some mild homological hypotheses.

Theorem 0.3. *Let A be a connected graded noetherian domain. Let $Q = Q(A)$.*

(a) *If A has enough normal elements, then* $\operatorname{Htr} Q = \operatorname{GKdim} A$.
(b) *If A has an Auslander balanced dualizing complex and if $A \otimes_k Q^\circ$ is noetherian, then* $\operatorname{Htr} Q = \operatorname{cd} A$. *Here* cd *denotes the cohomological dimension defined in* Definition 6.1(d).
(c) *If A is an Artin-Schelter Gorenstein ring and if $A \otimes_k Q^\circ$ is noetherian, then* $\operatorname{Htr} Q = \operatorname{injdim} A$.

In addition to Theorems 0.2 and 0.3 above we have Propositions 5.4 and 6.4 and Theorem 6.9 which compute Htr for filtered rings. These seemingly technical results cover several different classes of division rings. For example, Htr is computable for the following classes of division rings:

(i) quotient division rings of affine prime PI algebras,
(ii) the Weyl skew fields,
(iii) quotient division rings of enveloping algebras $U(\mathfrak{g})$ of finite dimensional Lie algebras, and
(iv) quotient division rings of some other quantum algebras studied by Goodearl and Lenagan [7].

Another advantage of this new definition is that it is easy to verify some useful properties similar to those of the classical transcendence degree. Let tr denote the classical transcendence degree of a commutative field (or a PI division algebra).

Proposition 0.4. *Let D be a division algebra and C be a division subalgebra of D.*

(a) $\operatorname{Htr} C \le \operatorname{Htr} D$.
(b) *If D is finite as a left (or a right) C-module, then* $\operatorname{Htr} C = \operatorname{Htr} D$.
(c) *Suppose $C \otimes_k C^\circ$ is noetherian of finite global dimension. If D is the quotient division ring of the skew polynomial ring $C[x; \alpha]$ for some automorphism α of C, then* $\operatorname{Htr} D = \operatorname{Htr} C + 1$.
(d) *If D is a PI division ring and if the center is finitely generated over k as a field, then* $\operatorname{Htr} D = \operatorname{tr} D$.
(e) $\operatorname{Htr} D = \operatorname{Htr} D^\circ$.

As a consequence of Theorem 0.2 and Proposition 0.4(a), if A and B are two Artin-Schelter regular Ore domains and $\operatorname{gldim} A < \operatorname{gldim} B$, then there is no algebra homomorphism from $Q(B)$ to $Q(A)$.

1. Definitions and basic properties

Let A be an algebra over the base field k. Let A° be the opposite ring of A, and let A^{e} be the enveloping algebra $A \otimes A^\circ$, where $\otimes$ denotes $\otimes_k$. Note that the switching operation $a \otimes b \mapsto b \otimes a$ extends to an anti-automorphism of the algebra A^{e}. Usually we work with left modules. A right A-module is viewed as an A°-module, and an A-bimodule is the same as an A^{e}-module. An A-module is called *finite* if it is finitely generated over A.

We have already seen the definition of Htr in the introduction. To compute Htr it is helpful to introduce a few related invariants, which are called modifications of Htr. For a ring B and a B-module N, we denote by $\operatorname{injdim}_B N$ (and $\operatorname{projdim}_B N$) the injective dimension (respectively, the projective dimension) of N as a B-module. If $N = B$, we simplify $\operatorname{injdim}_B B$ to $\operatorname{injdim} B$.

Definition 1.1. Let A be a k-algebra.

(a) The *homological transcendence degree* of A is defined to be
$$\operatorname{Htr} A = \operatorname{injdim} A^{\mathrm{e}}.$$
(b) The *first modification* of Htr is defined to be
$$\mathrm{H}_1\mathrm{tr}\, A = \sup\{i \mid \operatorname{Ext}^i_{A^{\mathrm{e}}}(A, A^{\mathrm{e}}) \neq 0\}.$$
(c) The *second modification* of Htr is defined to be
$$\mathrm{H}_2\mathrm{tr}\, A = \sup\{\operatorname{injdim} A \otimes U\}$$
where U ranges over all division rings.

(d) The *third modification* of Htr is defined to be
$$\mathrm{H}_3\mathrm{tr}\, A = \sup\{\operatorname{injdim} A \otimes U\}$$
where U ranges over all division rings such that $A \otimes U$ is noetherian.

(e) A simple artinian ring S is called *homologically uniform* if
$$\operatorname{Htr} S = \mathrm{H}_1\mathrm{tr}\, S = \mathrm{H}_2\mathrm{tr}\, S < \infty.$$
(f) A simple artinian ring S is called *smooth* if $\operatorname{projdim}_{S^{\mathrm{e}}} S < \infty$.

If A is a graded ring, then the graded version of (a), (b), (c) and (d) can be defined and are denoted by $\mathrm{Htr}_{\mathrm{gr}}$, $\mathrm{H}_1\mathrm{tr}_{\mathrm{gr}}$, $\mathrm{H}_2\mathrm{tr}_{\mathrm{gr}}$ and $\mathrm{H}_3\mathrm{tr}_{\mathrm{gr}}$ respectively.

These definitions were implicitly suggested by the work of Resco [12, 13, 14], Rosenberg [17], Schofield [20] and Stafford [24]. The idea in Definition 1.1(e,f) of working in the class of simple artinian algebras instead of division algebras is due to Schofield [20]. Smooth simple artinian rings are called *regular* by Schofield [20] (see also Lemma 1.3).

We are mainly interested in Htr, but the modifications $\mathrm{H}_1\mathrm{tr}$, $\mathrm{H}_2\mathrm{tr}$ and $\mathrm{H}_3\mathrm{tr}$ are closely related to Htr. In fact we are wondering whether division rings of finite Htr are always homologically uniform [Question 7.9] . For several classes of division algebras, $\mathrm{H}_1\mathrm{tr}$ and $\mathrm{H}_2\mathrm{tr}$ are relatively easy to compute; and then Htr is computable by the following easy lemma.

An algebra A is called *doubly noetherian* if A^{e} is noetherian, and it is called *rationally noetherian* if $A \otimes U$ is noetherian for every division ring U.

Lemma 1.2. *Let S be a simple artinian algebra.*

(a) $\mathrm{H}_1\mathrm{tr}\, S \leq \operatorname{Htr} S \leq \mathrm{H}_2\mathrm{tr}\, S$. *If* $\mathrm{H}_1\mathrm{tr}\, S \geq \mathrm{H}_2\mathrm{tr}\, S$ *and* $\mathrm{H}_2\mathrm{tr}\, S < \infty$*, then S is homologically uniform.*

(b) $\mathrm{H}_3\mathrm{tr}\, S \le \mathrm{H}_2\mathrm{tr}\, S$. *If S is rationally noetherian, then equality holds.*
(c) *If S is doubly noetherian, then* $\mathrm{Htr}\, S \le \mathrm{H}_3\mathrm{tr}\, S$.

By Proposition 7.1 below there is a smooth, homologically uniform, but not doubly noetherian, commutative field F over k such that $\mathrm{H}_3\mathrm{tr}\, F < \mathrm{Htr}\, F$.

Lemma 1.3. *Let S be a simple artinian ring. In parts(a,c) suppose S is smooth and* $\mathrm{projdim}_{S^e} S = n < \infty$.

(a) [20, Lemma 2, p. 269] $\mathrm{gldim}\, S^{\mathrm{e}} = n \ge \mathrm{gldim}\, S \otimes U$ *for every simple artinian ring U. As a consequence,* $\mathrm{H}_2\mathrm{tr}\, S \le n$.
(b) *S is smooth if and only if* $\mathrm{gldim}\, S^e < \infty$. *In this case* $\mathrm{projdim}_{S^e} S = \mathrm{gldim}\, S^e$.
(c) *If* $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, \bigoplus_I S^{\mathrm{e}}) \cong \bigoplus_I \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$ *for any index set I (e.g., if S is doubly noetherian), then S is homologically uniform and* $\mathrm{Htr}\, S = n$.

Proof. (b) Follows from (a).

(c) We claim that $\mathrm{H}_1\mathrm{tr}\, S = n$, which is equivalent to $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \neq 0$. Since $\mathrm{projdim}_{S^{\mathrm{e}}} S = n$, there is an S^{e}-module M such that $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, M) \neq 0$ and $\mathrm{Ext}^{n+1}_{S^{\mathrm{e}}}(S, -) = 0$. There is an index set I and a short exact sequence of S^{e}-modules

$$0 \to N \to \textstyle\bigoplus_I S^{\mathrm{e}} \to M \to 0.$$

Applying $\mathrm{Ext}^i_{S^{\mathrm{e}}}(S, -)$ to the above exact sequence we have a long exact sequence

$$\to \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, N) \to \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, \textstyle\bigoplus_I S^{\mathrm{e}}) \to \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, M) \to \mathrm{Ext}^{n+1}_{S^{\mathrm{e}}}(S, N) \to .$$

Since $\mathrm{Ext}^{n+1}_{S^{\mathrm{e}}}(S, N) = 0$ and $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, M) \neq 0$, the above exact sequence implies that $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, \bigoplus_I S^{\mathrm{e}}) \neq 0$. By hypothesis, $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, \bigoplus_I S^{\mathrm{e}}) \cong \bigoplus_I \mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$, hence we have $\mathrm{Ext}^n_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) \neq 0$. Thus our claim is proved. The assertion follows from part (a) and Lemma 1.2(a). ☐

Lemma 1.3 says that every doubly noetherian smooth division algebra D is homologically uniform and $\mathrm{Htr}\, D = \mathrm{gldim}\, D^{\mathrm{e}}$. We can use Lemma 1.3 and results of Resco and Stafford to compute the Htr of some division rings. For example, the commutative field $D = k(x_1, \cdots, x_n)$ is homologically uniform with $\mathrm{Htr} = n$, since $\mathrm{gldim}\, D^{\mathrm{e}} = n$ [14, Theorem p. 215]. A similar statement holds for the quotient division rings of $U(\mathfrak{g})$ and kG (see details in [24, Theorem, p. 33]). Proposition 1.8 below is also useful for such a computation.

Our main result Theorem 0.2 deals with the case when $Q(A)$ may fail to be doubly noetherian, and Theorem 0.3 deals with the case when $Q(A)$ may fail to be smooth.

Let us now review the basic properties of the classical transcendence degree of commutative fields over k. Let $F \subset G$ be commutative fields over k.

(TD1) $\mathrm{tr}\, k(x_1, \cdots, x_n) = n$ for every $n \ge 0$.
(TD2) $\mathrm{tr}\, F \le \mathrm{tr}\, G$.
(TD3) If $\dim_F G$ is finite, then $\mathrm{tr}\, F = \mathrm{tr}\, G$.
(TD4) If $G = F(x)$, then $\mathrm{tr}\, G = \mathrm{tr}\, F + 1$.
(TD5) If $\{F_i\}$ is a directed set of subfields of G such that $G = \bigcup F_i$, then $\mathrm{tr}\, G = \sup\{\mathrm{tr}\, F_i\}$.
(TD6) If G is finitely generated as a field and $\mathrm{tr}\, F = \mathrm{tr}\, G$, then $\dim_F G$ is finite.

We will try to prove some versions of (TD1-TD4) for Htr. However, Proposition 7.1(b) below shows that (TD5) fails for Htr, which is an unfortunate deficiency of homological transcendence degree. And we have not proven any generalization of (TD6). The following lemma is a collection of some well-known facts.

Lemma 1.4. *Let $A \subset B$ be k-algebras.*

(a) *Assume that $B = P \bigoplus N$ as left A-modules with P a projective generator of the category of left A-modules and that a similar decomposition holds for the right A-module B. Suppose the right A-module B is flat. Then* $\operatorname{injdim} A \leq \operatorname{injdim} B$.

(b) *If A is an A-bimodule direct summand of B, then* $\operatorname{gldim} A \leq \operatorname{gldim} B + \operatorname{projdim}_A B$.

(c) *If $B = A[x; \alpha, \delta]$ where α is an automorphism and δ is an α-derivation, then* $\operatorname{injdim} A \leq \operatorname{injdim} B \leq \operatorname{injdim} A + 1$.

(d) *If B is a localization of A, then* $\operatorname{gldim} B \leq \operatorname{gldim} A$.

(e) *If A is noetherian and B is a localization of A, then* $\operatorname{injdim} B \leq \operatorname{injdim} A$.

Proof. (a) Since B is a flat A°-module, the Hom-$\otimes$ adjunction gives the isomorphism

$$\operatorname{Ext}^i_B(B \otimes_A M, B) \cong \operatorname{Ext}^i_A(M, B) \tag{E1.4.1}$$

for all A-modules M and all i. The isomorphism (E1.4.1) is also given in Lemma 2.2(b) below. Since the right A-module B contains a projective generator as a direct summand, $B \otimes_A M \neq 0$ for every $M \neq 0$. So (E1.4.1) implies that $\operatorname{injdim}_B B \geq \operatorname{injdim}_A B$. Since the left A-module B contains a projective generator as a direct summand, $\operatorname{injdim}_A B \geq \operatorname{injdim}_A A$. The assertion follows.

(b) This is [10, Theorem 7.2.8]

(c) By part (a), $\operatorname{injdim} A \leq \operatorname{injdim} B$.

By [10, Proposition 7.5.2] (or [20, Lemma 1, p. 268]), for any B-module M, there is an exact sequence

$$0 \to B \otimes_A ({}^\alpha M) \to B \otimes_A M \to M \to 0$$

of B-modules. This short exact sequence induces a long exact sequence

$$\cdots \to \operatorname{Ext}^i_B(B \otimes_A ({}^\alpha M), B) \to \operatorname{Ext}^{i+1}_B(M, B) \to \operatorname{Ext}^{i+1}_B(B \otimes_A M, B) \to \cdots .$$

Since B is a flat A°-module, the Hom-$\otimes$ adjunction gives

$$\operatorname{Ext}^i_B(B \otimes_A M, B) \cong \operatorname{Ext}^i_A(M, B) = 0$$

for all $i > \operatorname{injdim} A$ and all M. Thus the two ends of the above long exact sequence are zero, which implies that the middle term $\operatorname{Ext}^{i+1}_B(M, B) = 0$ for all $i > \operatorname{injdim} A$. This shows that $\operatorname{injdim} B \leq \operatorname{injdim} A + 1$.

(d) This is [10, Corollary 7.4.3].

(e) This is also well-known, and is a special case of Lemma 2.3 below. □

Proposition 1.5. *Let $D \subset Q$ be simple artinian algebras.*

(a) $\operatorname{Htr} D \leq \operatorname{Htr} Q$.

(b) *If A is Morita equivalent to D, then* $\operatorname{Htr} D = \operatorname{Htr} A$.

(c) *If Q is finite over D on the left, or on the right, then* $\operatorname{Htr} D = \operatorname{Htr} Q$.

(d) *If D is PI and its center C is finitely generated over k as a field, then* $\operatorname{Htr} D = \operatorname{tr} C = \operatorname{tr} D$.

(e) $\operatorname{Htr} D = \operatorname{Htr} D^\circ$.

Remark 1.6. (a) The hypothesis "C is finitely generated over k as a field" in part (d) of Proposition 1.5 is necessary as Proposition 7.1 shows.

(b) There are division algebras D such that $D \not\cong D^{\circ}$. For example, let F be a field extension of k such that the Brauer group of F has an element $[D]$ of order larger than 2. Then the central division ring D corresponding to $[D]$ has the property $D \not\cong D^{\circ}$. If we want such a division algebra that is infinite over its center, then let Q be the Goldie quotient ring of $D \otimes k_q[x, y]$ where q is not a root of 1. It is eay to check that the center of Q is $F \otimes k \cong F$ and Q is infinite over F. Since D is the division subring of Q consisting of all elements integral over the center F, then $Q \not\cong Q^{\circ}$.

Proof of Proposition 1.5. (a) Since D is simple artinian, every nonzero (left or right) D-module is a projective generator. So the D-module Q and the D°-module Q° are projective generators. Hence $Q \otimes Q^{\circ}$ is a projective generator over $D \otimes D^{\circ}$. Similarly, $Q \otimes Q^{\circ}$ is a projective generator over $D \otimes D^{\circ}$ on the right. The assertion follows from Lemma 1.4(a).

(b) Since A and D are Morita equivalent and both are simple artinian, $A \cong \mathrm{M}_n(B)$ and $D \cong \mathrm{M}_s(B)$ for some division algebra B and some n, s. So we may assume that A is a division ring and $D = \mathrm{M}_n(A)$. Now $D^{\mathrm{e}} = (\mathrm{M}_n(A))^{\mathrm{e}} \cong \mathrm{M}_{n^2}(A^{\mathrm{e}})$. Hence we have $\operatorname{injdim} D^{\mathrm{e}} = \operatorname{injdim} \mathrm{M}_{n^2}(A^{\mathrm{e}}) = \operatorname{injdim} A^{\mathrm{e}}$.

(c) By part (a) it suffices to show that $\operatorname{Htr} Q \leq \operatorname{Htr} D$. Assume the right D-module Q is finite. Then $B := \operatorname{End}_{D^{\circ}}(Q)$ is Morita equivalent to D. Also by the definition of B there is a natural injection $Q \to B$. By (a,b), $\operatorname{Htr} Q \leq \operatorname{Htr} B = \operatorname{Htr} D$. By symmetry the assertion holds when Q is finite over D on the left.

(d) Since D is PI, D is finite over its center C [10, Theorem 13.3.8]. By part (c), $\operatorname{Htr} D = \operatorname{Htr} C$. It suffices to show that $\operatorname{Htr} C = \operatorname{tr} C$. Let F be a subfield of C such that $F \cong k(x_1, \cdots, x_n)$ for some integer n and that C is algebraic over F. Then C is finite over F and $\operatorname{tr} C = \operatorname{tr} F = n$. By part (c) it suffices to show that $\operatorname{Htr} F = n$. By Lemma 1.3 and [14, Theorem, p. 215], $\operatorname{Htr} F = n$. Hence the assertion follows.

(e) This follows from the fact that there is an anti-automorphism $D^{\mathrm{e}} \to D^{\mathrm{e}}$. □

Similarly one can prove the following version of Proposition 1.5 for $\mathrm{H}_2\mathrm{tr}$.

Proposition 1.7. *Let $D \subset Q$ be simple artinian algebras.*

(a) $\mathrm{H}_2\mathrm{tr}\, D \leq \mathrm{H}_2\mathrm{tr}\, Q$.

(b) *If Q is finite over D on the left (or on the right), then* $\mathrm{H}_2\mathrm{tr}\, D = \mathrm{H}_2\mathrm{tr}\, Q$*. As a consequence, if a simple artinian ring A is Morita equivalent to D, then* $\mathrm{H}_2\mathrm{tr}\, D = \mathrm{H}_2\mathrm{tr}\, A$.

(c) *If D is PI and its center, denoted by C, is finitely generated as a field, then* $\mathrm{H}_2\mathrm{tr}\, D = \operatorname{tr} C$.

Unlike Proposition 1.5(e), we don't know whether $\mathrm{H}_2\mathrm{tr}\, D = \mathrm{H}_2\mathrm{tr}\, D^{\circ}$ or not.

Recall from [20] that a simple artinian ring is *stratiform over k* if there is a chain of simple artinian rings

$$S = S_n \supset S_{n-1} \supset \cdots \supset S_1 \supset S_0 = k$$

where, for every i, either (i) S_{i+1} is finite over S_i on both sides; or (ii) S_{i+1} is isomorphic to $S_i(x_i; \alpha_i, \delta_i)$ for an automorphism α_i of S_i and and α_i-derivation δ_i of S_i. Such a chain of simple artinian rings is called a *stratification* of S. The *stratiform length* of S is the number of steps in the chain that are of type (ii). One basic property proved in [20] is that the stratiform length is an invariant of S.

Proposition 1.8. *If S is a stratiform simple artinian ring of stratiform length m, then S is rationally noetherian, homologically uniform and* $\operatorname{Htr} S = m$.

Proof. It follows from induction on the steps of the stratification that S and S° are rationally noetherian (and hence doubly noetherian). But S might not be smooth.

Next we show $\mathrm{H}_1\mathrm{tr}\, S = m$. Applying [20, Lemma 20, p. 277] to the S-bimodule S, we have the following statement: there is a simple artinian ring $S'' \supset S$ such that S'', as $(S'' \otimes S^\circ)$-module, has projective dimension m. Since S° is rationally noetherian, $S'' \otimes S^\circ$ is noetherian. Hence we have

$$\operatorname{Ext}^i_{S''\otimes S^\circ}(S'', S'' \otimes S^\circ) = \begin{cases} \text{nonzero} & \text{if} \quad i = m, \\ 0 & \text{if} \quad i > m. \end{cases}$$

Since $S'' \otimes S^\circ$ is projective over S^{e}, we have

$$\operatorname{Ext}^i_{S''\otimes S^\circ}(S'', S'' \otimes S^\circ) \cong \operatorname{Ext}^i_{S^{\mathrm{e}}}(S, S'' \otimes S^\circ)$$

which is a direct sum of copies of $\operatorname{Ext}^i_{S^{\mathrm{e}}}(S, S^{\mathrm{e}})$. Thus

$$\operatorname{Ext}^i_{S^{\mathrm{e}}}(S, S^{\mathrm{e}}) = \begin{cases} \text{nonzero} & \text{if} \quad i = m, \\ 0 & \text{if} \quad i > m. \end{cases}$$

Therefore $\mathrm{H}_1\mathrm{tr}\, S = m$.

By Lemma 1.2(a) it remains to show that $\mathrm{H}_2\mathrm{tr}\, S \leq m$. We use induction on the steps of the stratification. Suppose $\mathrm{H}_2\mathrm{tr}\, S_{n-1}$ is no more than the stratiform length of S_{n-1}. We want to show that this statement holds for S_n.

Case (i): S_n is finite over S_{n-1} on both sides. By Proposition 1.7(b), $\mathrm{H}_2\mathrm{tr}\, S_n = \mathrm{H}_2\mathrm{tr}\, S_{n-1}$. The claim follows.

Case (ii): $S_n = S_{n-1}(x; \alpha, \delta)$. Let U be any simple artinian ring and let $A = S_{n-1} \otimes U$ and $B = S_{n-1}[x; \alpha, \delta] \otimes U = A[x; \alpha, \delta]$. By Lemma 1.4(c), $\operatorname{injdim} B \leq \operatorname{injdim} A + 1$. Since $S_n \otimes U$ is a localization of the noetherian ring B, we have $\operatorname{injdim} S_n \otimes U \leq \operatorname{injdim} B$ by Lemma 1.4(e). Combining these two inequalities, the claim follows. ☐

Next we give a list of known examples, and a few more examples will be given in Section 7.

Example 1.9. (a) Let F be a separable field extension of k that is finitely generated as a field. Then F is rationally noetherian and $\operatorname{gldim} F^{\mathrm{e}} = \operatorname{tr} F < \infty$. Hence F is smooth, homologically uniform and $\operatorname{Htr} F = \operatorname{tr} F$.

(b) Let F be the commutative field $k(x_1, x_2, \ldots)$, which is an infinite pure transcendental extension of k. The ring F is not doubly noetherian. For each integer m let F_m be the subfield $k(x_1, \ldots, x_m) \subset F$. Then F_m is rationally noetherian, smooth, homologically uniform with Htr m. Since $F_m \subset F$ for all m, one sees that

$$\operatorname{Htr} F = \mathrm{H}_2\mathrm{tr}\, F = \mathrm{H}_3\mathrm{tr}\, F = \infty.$$

But $\mathrm{H}_1\mathrm{tr}\, F = -\infty$ since $\operatorname{Ext}^i_{F^{\mathrm{e}}}(F, F^{\mathrm{e}}) = 0$ for all i [31, Example 3.13].

(c) Let D be a simple artinian ring finite dimensional over k. By Proposition 1.5(c), $\operatorname{Htr} D = \operatorname{Htr} k = 0$.

(d) Let F be a finite dimensional purely inseparable field extension of k. Since $\operatorname{gldim} F^{\mathrm{e}} = \infty$, F is not smooth over k. By part (c) $\operatorname{injdim} F^{\mathrm{e}} = \operatorname{Htr} F = 0$.

(e) Let D_n be the n-th Weyl skew field. Since D_n is rationally noetherian and $\operatorname{gldim} D_n^{\mathrm{e}} = 2n$ [17, 24], by Lemma 1.3, D_n is smooth and homologically uniform and $\operatorname{Htr} D_n = 2n$.

(f) Let $D(\mathfrak{g})$ be the quotient division ring of the universal enveloping algebra $U(\mathfrak{g})$ of a finite dimensional Lie algebra $\mathfrak{g}$. Then $D(\mathfrak{g})$ is rationally noetherian and $\operatorname{gldim} D(\mathfrak{g})^{\mathrm{e}} = \dim_k \mathfrak{g}$ [24]. By Lemma 1.3, $D(\mathfrak{g})$ is smooth and homologically uniform and $\operatorname{Htr} D = \dim_k \mathfrak{g}$. A similar statement holds for quotient division rings of group rings kG studied in [24].

(g) Let $\{p_{ij} \mid i < j\}$ be a set of nonzero scalars in k. Let A be the skew polynomial ring $k_{\{p_{ij}\}}[x_1, \ldots, x_n]$ that is generated by elements $x_1, \ldots, x_n$ and subject to the relations $x_j x_i = p_{ij} x_i x_j$ for all $i < j$. Let Q be the quotient division ring of A. Then Q is a stratiform division ring of stratiform length n. Hence Q is rationally noetherian, homologically uniform, and $\operatorname{Htr} Q = n$. This is a generalization of (TD1). Since Q^{e} is a localization of another skew polynomial ring of finite global dimension, Q is smooth.

2. Polynomial extension

In this section we discuss the property (TD4) for Htr. We have not yet proved a satisfactory generalization of (TD4). Let S be a simple artinian ring with automorphism α and α-derivation δ of S. The Goldie quotient ring of $S[t; \alpha, \delta]$ is denoted by $S(t; \alpha, \delta)$. We don't know if

$$\operatorname{Htr} S(t; \alpha, \delta) = \operatorname{Htr} S + 1$$

holds in general, but we present some partial results in Proposition 2.7 below.

Recall that the third modification of Htr is

$$\mathrm{H}_3\mathrm{tr}\, S = \sup\{\operatorname{injdim} S \otimes U\}$$

where U ranges over all division algebras such that $S \otimes U$ is noetherian. If S is a doubly noetherian simple artinian ring, then

$$\mathrm{H}_1\mathrm{tr}\, S \leq \operatorname{Htr} S \leq \mathrm{H}_3\mathrm{tr}\, S \leq \mathrm{H}_2\mathrm{tr}\, S.$$

For doubly noetherian simple artinian rings S, $\mathrm{H}_3\mathrm{tr}$ is a good replacement for $\mathrm{H}_2\mathrm{tr}$. In this case we call S *weakly uniform* if

$$\mathrm{H}_1\mathrm{tr}\, S = \operatorname{Htr} S = \mathrm{H}_3\mathrm{tr}\, S < \infty.$$

If S is rationally noetherian, then "weakly uniform" is equivalent to "homologically uniform".

Lemma 2.1. *Let S be a simple artinian ring and let $Q = S(t; \alpha, \delta)$.*

(a) *S is doubly noetherian if and only if Q is.*
(b) *S is rationally noetherian if and only if Q is.*
(c) *If S is smooth, so is Q. The converse holds when $\delta = 0$.*

Proof. Note that $Q^\circ \cong S^\circ(t; \alpha^{-1}, -\delta\alpha^{-1})$.

(a) If $S^{\mathrm{e}} = S \otimes S^\circ$ is noetherian, so is $S[t; \alpha, \delta] \otimes S^\circ[t; \alpha^{-1}, -\delta\alpha^{-1}]$. Therefore its localization $Q \otimes Q^\circ$ is noetherian.

In the other direction, we suppose Q^{e} is noetherian. Since Q is faithfully flat (and projective) as left and right S-module, Q^{e} is a faithfully flat left module over S^{e}. Hence S^{e} is left (and hence right) noetherian.

(b) Similar to part (a).

(c) If S is smooth, an argument similar to the proof of part (a) shows that Q is smooth.

To show the converse we assume that $\delta = 0$. Decompose Q into $Q = S \bigoplus C$ where

$$C = \{f(t)(g(t))^{-1} \mid \deg_t f(t) < \deg_t g(t)\} \oplus (\textstyle\bigoplus_{n=1}^{\infty} t^n S).$$

Hence S is a S-bimodule direct summand of Q. Thus S^{e} is a S^{e}-bimodule direct summand of Q^{e}. The assertion follows from Lemma 1.4(b). □

It is not clear to us if the "converse" part of Lemma 2.1(c) holds when $\delta \neq 0$.

The following lemma is basically [32, Lemma 3.7]. Note that in [32, Lemma 3.7], an extra hypothesis "M being bounded below" was forgotten. Various versions of the following lemma exist in the literature, especially for modules instead of complexes.

Let $\mathsf{Mod}\, A$ denote the category of A-modules and let $\mathsf{D}(\mathsf{Mod}\, A)$ denote the derived category of $\mathsf{Mod}\, A$. If A is graded, $\mathsf{GrMod}\, A$ is the category of graded A-modules and $\mathsf{D}(\mathsf{GrMod}\, A)$ is the derived category of $\mathsf{GrMod}\, A$. We refer to [29] for basic material about complexes and derived categories.

A complex $L \in \mathsf{D}^{-}(\mathsf{Mod}\, A)$ is called *pseudo-coherent* if L has a bounded above free resolution $P = (\cdots \to P^i \to P^{i+1} \to \cdots)$ such that each component P^i is a finite free A-module [8, Exposé 1 (L. Illusie), Section 2]. If L is pseudo-coherent, then

$$\mathrm{RHom}_A(L, \textstyle\bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{RHom}_A(L, M_i)$$

and

$$\mathrm{Ext}^n_A(L, \textstyle\bigoplus_{i\in I} M_i) \cong \bigoplus_{i\in I} \mathrm{Ext}^n_A(L, M_i) \quad \text{for all } n$$

where $\{M_i\}_{i\in I}$ is a set of uniformly bounded below complexes.

There are two different definitions of injective dimension of a complex existing in the literature, one of which is given as follows. Let X be a bounded below complex of A-modules. Then the *injective dimension* of X is defined to be

$$\operatorname{injdim}_A X = \sup\{i \mid Y^i \neq 0\}$$

where Y is a minimal injective resolution of X. If A is $\mathbb{Z}$-graded, the graded injdim can be defined. If $\operatorname{injdim}_A X = n$, then $\mathrm{Ext}^i_A(M, X) = 0$ for all A-modules M and for all $i > n$; and there is an A-module M such that $\mathrm{Ext}^n_A(M, X) \neq 0$.

Lemma 2.2. [32, Lemma 3.7] *Let A, B be algebras. Let L be a complex in $\mathsf{D}^{-}(\mathsf{Mod}\, A)$.*

(a) *Let N be a B-module of finite flat dimension and let $M \in \mathsf{D}^{+}(\mathsf{Mod}\, A \otimes B^{\circ})$. Suppose L is pseudo-coherent. Then the functorial morphism*

$$\mathrm{RHom}_A(L, M) \otimes^{\mathrm{L}}_B N \to \mathrm{RHom}_A(L, M \otimes^{\mathrm{L}}_B N)$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, k)$.

(b) *Suppose $A \to B$ is a ring homomorphism such that B is a flat A°-module. Let $M \in \mathsf{D}^{+}(\mathsf{Mod}\, B)$. Then the functorial morphism*

$$\mathrm{RHom}_A(L, M) \to \mathrm{RHom}_B(B \otimes_A L, M)$$

is an isomorphism in $\mathsf{D}(\mathsf{Mod}\, k)$.

The following lemma is well-known; and follows easily from the above lemma.

Lemma 2.3. *Let A be a noetherian ring and let B be any ring. Suppose R is a bounded complex of $A \otimes B^\circ$-modules. Suppose that A' and B' are Ore localizations of A and B respectively and that $A' \otimes_A R \cong R \otimes_B B'$ in $\mathsf{D}(\mathsf{Mod}\, A \otimes B^\circ)$. Then* $\operatorname{injdim}_{A'}(A' \otimes_A R) \leq \operatorname{injdim}_A R$.

The following lemma is similar to [20, Theorem 8, p. 272] and is known to many researchers. Note that there is a typographical error in the statement of [20, Theorem 8, p. 272]: "$i \neq 1$" should be "$i \geq 1$".

Lemma 2.4. *Let A be a left noetherian ring with an automorphism α. Let $T = A[t^{\pm 1}; \alpha]$. If M is a T-module that is finitely generated as A-module, then*

$$\operatorname{Ext}^i_T(M, T) \cong \operatorname{Ext}^{i-1}_A({}^\alpha M, A)$$

as A°-modules, for all $i \geq 1$.

Definition 2.5. A simple artinian algebra S is called *rigid* if $\operatorname{RHom}_{S^e}(S, S^e) \cong S^\sigma[-n]$ for some integer n and some automorphism σ of S; or equivalently

$$\operatorname{Ext}^i_{S^e}(S, S^e) = \begin{cases} S^\sigma & \text{if } \; i = n, \\ 0 & \text{if } \; i \neq n. \end{cases}$$

In the above definition $[n]$ denotes the n-th complex shift and the bimodule S^σ is defined by

$$a * s * b = as\sigma(b)$$

for all $a, s, b \in S$. Clearly $\mathrm{H}_1\mathrm{tr}\, S = n$.

For example the n-th Weyl skew field D_n in Example 1.9(e) is rigid. This follows from the computation given at the end of [33, Section 6]:

$$\operatorname{Ext}^i_{D_n^e}(D_n, D_n^e) = \begin{cases} D_n & \text{if } \; i = 2n, \\ 0 & \text{if } \; i \neq 2n. \end{cases}$$

Other doubly noetherian division rings in Example 1.9 are also rigid [Corollary 6.11]. If S is not doubly noetherian, then S might not be rigid even if it is homologically uniform [Proposition 7.1(d)].

Proposition 2.6. *Let D be a doubly noetherian simple artinian ring and B be the skew Laurent polynomial ring $D[t^{\pm 1}; \alpha]$ where α is an automorphism of D. Let $Q = D(t; \alpha)$. Then* $\mathrm{H}_1\mathrm{tr}\, B = \mathrm{H}_1\mathrm{tr}\, Q = \mathrm{H}_1\mathrm{tr}\, D + 1$. *Furthermore D is rigid if and only if Q is rigid.*

Proof. It is easy to reduce to the case when D is a division ring, so we assume that D is a division ring in the proof below. Since D^e is noetherian, so are B^e and Q^e.

We can view B as a $\mathbb{Z}$-graded ring with $\deg t = 1$ and $\deg D = 0$. Hence B^e is also $\mathbb{Z}$-graded and $B^e \cong (D^e[w^{\pm 1}; \alpha \otimes \alpha])[(t \otimes 1)^{\pm 1}, \sigma]$ where $w = t \otimes t^{-1}$ and $\sigma : w \mapsto w, d_1 \otimes d_2 \mapsto \alpha(d_1) \otimes d_2$ for all $d_1 \in D$ and $d_2 \in D^\circ$. Clearly B^e is strongly $\mathbb{Z}$-graded.

Since B^e is noetherian and B is a finite graded B^e-module, $\operatorname{Ext}^i_{B^e}(B, B^e)$ can be computed in the category of the $\mathbb{Z}$-graded B^e-modules. Since B^e is strong $\mathbb{Z}$-graded, $\mathsf{GrMod}\, B^e \cong \mathsf{Mod}\, C$ where $C = D^e[w^{\pm 1}; \alpha \otimes \alpha]$. Since the degree zero parts of B^e and B are equal to C and D respectively, we have $\operatorname{Ext}^i_B(B, B^e) \neq 0$ if and only if $\operatorname{Ext}^i_C(D, C) \neq 0$ where D is a left C-module. Furthermore the degree zero part of $\operatorname{Ext}^i_B(B, B^e)$ is isomorphic to $\operatorname{Ext}^i_C(D, C)$. By Lemma 2.4, for all $i \geq 1$,

$$\operatorname{Ext}^i_C(D, C) \cong \operatorname{Ext}^{i-1}_{D^e}({}^{(\alpha \otimes \alpha)}D, D^e) \cong \operatorname{Ext}^{i-1}_{D^e}(D, D^e)^{\alpha^{-1} \otimes \alpha^{-1}}. \tag{E2.6.1}$$

This shows that $\mathrm{H}_1\mathrm{tr}\, B = \mathrm{H}_1\mathrm{tr}\, D + 1$.

Let n be any integer such that $V := \mathrm{Ext}^n_{B^e}(B, B^e) \neq 0$; and let $W = \mathrm{Ext}^n_C(D, C)$. There is a natural D-bimodule structure on W. Let's think about the left D-action on W. Since D is a division ring, W is a faithful D-module. Since V is basically equal to $W[t^{\pm 1}]$, it is a faithful $B(= D[t^{\pm 1}; \alpha])$-module. Therefore $Q \otimes_B V \neq 0$. Similarly V is a faithful B°-module. Hence $Q \otimes_B V$ is a faithful B°-module and $Q \otimes_B V \otimes_B Q \neq 0$. Since B^e is noetherian, by Lemma 2.2,

$$\mathrm{Ext}^n_{Q^e}(Q, Q^e) \cong \mathrm{Ext}^n_{B^e}(B, Q^e) \cong V \otimes_{B^e} Q^e \cong Q \otimes_B V \otimes_B Q \neq 0.$$

This implies that $\mathrm{H}_1\mathrm{tr}\, B = \mathrm{H}_1\mathrm{tr}\, Q$.

If $\mathrm{Ext}^{n-1}_{D^e}(D, D^e) \cong D^\sigma$ for some automorphism σ of D, then by (E2.6.1) we have $\mathrm{Ext}^n_C(D, C) \cong D^{\sigma'}$ for another automorphism σ' of D. The above argument shows that $\mathrm{Ext}^n_{Q^e}(Q, Q^e) \cong Q^{\sigma''}$ for some automorphism σ'' of Q. Therefore if D is rigid so it Q. The converse can be proved similarly. □

Proposition 2.7. *Suppose S is a doubly noetherian simple artinian ring. Let $Q = S(t; \alpha)$.*

(a) *If S (or Q) is smooth, then* $\mathrm{Htr}\, Q = \mathrm{Htr}\, S + 1$.

(b) *If S is weakly uniform, then so is Q, and* $\mathrm{Htr}\, Q = \mathrm{Htr}\, S + 1$.

(c) *If $\alpha = id_S$ and if* $\mathrm{H}_1\mathrm{tr}\, Q = \mathrm{Htr}\, Q$, *then* $\mathrm{H}_1\mathrm{tr}\, S = \mathrm{Htr}\, S = \mathrm{Htr}\, Q - 1$.

Proof. (a) The assertion follows from Lemma 1.3 and Proposition 2.6.

(b) Note that $S \otimes U$ is noetherian if and only if $Q \otimes U$ is. By Lemma 1.4(c,e), we have $\mathrm{H}_3\mathrm{tr}\, Q \leq \mathrm{H}_3\mathrm{tr}\, S + 1$. If S is weakly uniform, we have

$$\mathrm{H}_3\mathrm{tr}\, S + 1 = \mathrm{H}_1\mathrm{tr}\, S + 1 = \mathrm{H}_1\mathrm{tr}\, Q$$

where the last equality is Proposition 2.6. Combining these facts with Lemma 1.2(a,c), we obtain that Q is weakly uniform and that $\mathrm{Htr}\, Q = \mathrm{Htr}\, S + 1$.

(c) By hypothesis and Proposition 2.6 we have

$$\mathrm{Htr}\, Q = \mathrm{H}_1\mathrm{tr}\, Q = \mathrm{H}_1\mathrm{tr}\, S + 1 \leq \mathrm{Htr}\, S + 1.$$

Let $A = S(t) \otimes_{k(t)} S^\circ(t)$. Then $A \cong Q^e/(t \otimes 1 - 1 \otimes t)$ where $(t \otimes 1 - 1 \otimes t)$ is a central regular element of Q^e. By Rees' lemma, $\mathrm{injdim}\, Q^e \geq \mathrm{injdim}\, A + 1$. We can decompose A into a direct sum of S^e-modules

$$A = S^e \bigoplus (\textstyle\bigoplus_{n \geq 1} S^e t^n) \bigoplus \{f_1 g_1^{-1} \otimes f_2 g_2^{-1} \mid \deg_t f_i < \deg_t g_i\}$$

$$\bigoplus \{S \otimes f_2 g_2^{-1} \mid \deg_t f_2 < \deg_t g_2\} \bigoplus \{f_1 g_1^{-1} \otimes S^\circ \mid \deg_t f_1 < \deg_t g_1\}.$$

Also, A is flat over S^e on the right. By Lemma 1.4(a), $\mathrm{injdim}\, A \geq \mathrm{injdim}\, S^e$. Hence we have

$$\mathrm{Htr}\, Q = \mathrm{injdim}\, Q^e \geq \mathrm{injdim}\, A + 1 \geq \mathrm{injdim}\, S^e + 1 = \mathrm{Htr}\, S + 1.$$

Combining these inequalities with $\mathrm{H}_1\mathrm{tr}\, Q = \mathrm{Htr}\, Q$, $\mathrm{Htr}\, S + 1 \geq \mathrm{H}_1\mathrm{tr}\, S + 1$ (follows from Lemma 1.2(a)) and $\mathrm{H}_1\mathrm{tr}\, Q = \mathrm{H}_1\mathrm{tr}\, S + 1$ (follows from Proposition 2.6), one sees that all inequalities are equalities; and hence $\mathrm{Htr}\, S + 1 = \mathrm{H}_1\mathrm{tr}\, S + 1 = \mathrm{Htr}\, Q$. □

Now we are ready to prove Proposition 0.4.

Proof of Proposition 0.4. Parts (a,b,d,e) are proved in Proposition 1.5 and part (c) in Proposition 2.7(a). □

In the rest of the paper we will compute Htr for various classes of division algebras that are not in Example 1.9.

3. Review of Dualizing complexes

The dualizing complex is one of the main tools in the computation of homological transcendence degree. In this section we review several basic definitions related to dualizing complexes. We refer to [28, 29, 31] for other details. Some material about local duality will be reviewed in Section 6.

Definition 3.1. Let A be an algebra. A complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$ is called a *dualizing complex* over A if it satisfies the following conditions:

(a) R has finite injective dimension over A and over A° respectively.
(b) R is pseudo-coherent over A and over A° respectively.
(c) The canonical morphisms $A \to \mathrm{RHom}_A(R, R)$ and $A \to \mathrm{RHom}_{A^{\circ}}(R, R)$ are isomorphisms in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

If A is $\mathbb{Z}$-graded, a graded dualizing complex is defined similarly.

If A is noetherian (or more generally, coherent) then the definition agrees with [29, Definition 3.3] (or [31, Definition 1.1] for $A = B$).

Let R be a dualizing complex over a noetherian ring A and let M be a finite A-module. The *grade* of M with respect to R is defined to be

$$j_R(M) = \inf\{q \mid \mathrm{Ext}_A^q(M, R) \neq 0\}.$$

The grade of an A°-module is defined similarly.

Definition 3.2. [30, 31] A dualizing complex R over a noetherian ring A is called *Auslander* if

(a) For every finite A-module M, every integer q and every finite A°-submodule $N \subset \mathrm{Ext}_A^q(M, R)$ one has $j_R(N) \geq q$.
(b) The same holds after exchanging A and A°.

The *canonical dimension* of a finite A-module M with respect to R is defined to be

$$\mathrm{Cdim}\, M = -j_R(M).$$

Let R be a complex of A^{e}-modules, viewed as a complex of A-bimodules. Let R° denote the "opposite complex" of R which is defined as follows: as a complex of k-modules $R = R^{\circ}$ and the left and right A°-module actions on R° is given by

$$a * r * b = bra$$

for all $a, b \in A^{\circ}$ and $r \in R^{\circ}(= R)$. If $R \in \mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ then $R^{\circ} \in \mathsf{D}(\mathsf{Mod}(A^{\circ})^{\mathrm{e}})$. Since $(A^{\circ})^{\mathrm{e}}$ is isomorphic to A^{e}, there is a natural isomorphism $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}}) \cong \mathsf{D}(\mathsf{Mod}(A^{\circ})^{\mathrm{e}})$. The following definition is due to Van den Bergh [28, Definition 8.1].

Definition 3.3. [28] A dualizing complex R over A is called *rigid* if there is an isomorphism

$$\rho : R \to \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R^{\circ})$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$. Here the left A^{e}-module structure of $R \otimes R^{\circ}$ comes from the left A-module structure of R and the left A°-module structure of R°. To be precise (R, ρ) is called a *rigid* dualizing complex and the isomorphism ρ is called a *rigidifying isomorphism*.

A simple artinian ring S is rigid (see Definition 2.5) if and only if S has a rigid dualizing complex. In fact an easy computation shows that $\mathrm{RHom}_{S^e}(S, S^e) \cong S^\sigma[-n]$ if and only if $R := S^{\sigma^{-1}}[n]$ is a rigid dualizing complex over S.

When A is connected graded, there is a notion of balanced dualizing complex introduced in [29], which is related to the rigid dualizing complex. Let A be a connected graded algebra and let $\mathfrak{m} = A_{>0}$. Let $\Gamma_{\mathfrak{m}}$ denote the $\mathfrak{m}$-torsion functor $\lim_{n\to\infty} \mathrm{Hom}_A(A/\mathfrak{m}^n, -)$ (also see Section 6). If M is a graded A-module, let M' denote the graded vector space dual of M.

Definition 3.4. [29] A graded dualizing complex $R \in \mathsf{D}^{\mathrm{b}}(\mathsf{GrMod}\, A^e)$ over a connected graded ring A is called *balanced* if there are isomorphisms

$$\mathrm{R}\Gamma_{\mathfrak{m}}(R) \cong A' \cong \mathrm{R}\Gamma_{\mathfrak{m}^\circ}(R)$$

in $\mathsf{D}^b(\mathsf{GrMod}\, A^e)$.

By [28, Proposition 8.2(2)] a balanced dualizing complex over a noetherian connected graded ring is rigid after forgetting the grading.

Definition 3.5. A connected graded ring A is called *Artin-Schelter Gorenstein* (or AS Gorenstein) if

(a) A has graded injective dimension $n < \infty$ on the left and on the right,
(b) $\mathrm{Ext}^i_A(k, A) = \mathrm{Ext}^i_{A^\circ}(k, A) = 0$ for all $i \neq n$, and
(c) $\mathrm{Ext}^n_A(k, A) \cong \mathrm{Ext}^n_{A^\circ}(k, A) \cong k(l)$ for some l.

If moreover A has finite graded global dimension, then A is called *Artin-Schelter regular* (or AS regular).

In the above definition (l) denotes the lth degree shift of a graded module. If A is AS regular, then $\mathrm{gldim}\, A = n = \mathrm{injdim}\, A$. By [29] if A is noetherian and AS Gorenstein (or regular), then A has a balanced dualizing complex $A^\sigma(-l)[-n]$ for some automorphism σ. Note that in Definition 3.5 neither A is noetherian nor is the GK-dimension of A finite.

4. Computation of $\mathrm{H}_1 \mathrm{tr}$

In this section we use Van den Bergh's rigidity formula to compute $\mathrm{H}_1 \mathrm{tr}$ of some division algebras.

Let A be an algebra and let S be a left and right Ore set of regular elements of A. Let $B = S^{-1}A = AS^{-1}$. An A-bimodule complex R is called *evenly localizable* to B if

$$B \otimes_A R \to B \otimes_A R \otimes_A B \quad \text{and} \quad R \otimes_A B \to B \otimes_A R \otimes_A B$$

are quasi-isomorphisms [33, Definition 5.8]. If B is $Q(A)$, the total Goldie quotient ring of A, then we simply say R is evenly localizable without reference to B. It is easy to see that R is evenly localizable to B if and only if $\mathrm{H}^i(R)$ is evenly localizable to B for all i. The following lemma was proved a few times in slightly different versions (e.g., [33, Theorem 6.2]).

Lemma 4.1. *Let A be an algebra and let B be a localization of A with respect to an Ore set S. Let R be a dualizing complex over A. Assume that*

(i) *R is evenly localizable to B, and*
(ii) *either A is noetherian or B has finite global dimension.*

Then $R_B := B \otimes_A R \otimes_A B$ *is a dualizing complex over* B.

The graded version of the assertion also holds.

Proof. We only sketch a proof in the case when B has finite (ungraded) global dimension. By the definition of even localizibility we have $R \otimes_A B \cong R_B \cong B \otimes_A R$.

To prove R_B is a dualizing complex over B we need to show (a,b,c) in Definition 3.1. Part (a) is clear since B has finite global dimension and R_B is bounded. Part (b) follows from the fact that the pseudo-coherence is preserved under flat change of rings. Part (c) follows from Lemma 2.2 and the fact R is pseudo-coherent.

The noetherian case is similar; in fact it was proved in [33, Theorem 6.2(a)]. $\square$

Let A be a Goldie prime ring and let $Q(A)$ denote the Goldie quotient ring of A. Then $Q(A)$ is simple artinian; and in particular, it has global dimension 0. If A is graded Goldie prime, let $Q_{\mathrm{gr}}(A)$ denote the graded Goldie quotient ring of A. Then $Q_{\mathrm{gr}}(A)$ is graded simple artinian of graded global dimension 0. As an ungraded ring, $Q_{\mathrm{gr}}(A)$ is noetherian and has global dimension at most 1.

Suppose R is a dualizing complex over A that is evenly localizable to $Q := Q(A)$. Since Q is simple artinian of global dimension 0, Lemma 4.1 applies and $R_Q (\cong R \otimes_A Q \cong Q \otimes_A R)$ is a dualizing complex over Q. Let B be any simple artinian ring. Then a dualizing complex over B is isomorphic to $P[n]$ where P is an invertible B-bimodule [35, Theorem 0.2]; and every invertible B-bimodule is isomorphic to B^σ for some automorphism σ of B. Hence every dualizing complex over Q is isomorphic to $Q^\sigma[n]$ for some n and some automorphism σ of Q. Therefore one has

$$\{i \mid Q \otimes_A \mathrm{H}^i(R) \neq 0\} = \{i \mid \mathrm{H}^i(R) \otimes_A Q \neq 0\} = \{i \mid \mathrm{H}^i(R_Q) \neq 0\} = \{-n\}.$$

Use this equation we define *hammerhead* of R to be n and write $\xi(R) = n$. In the graded setting, every graded dualizing complex over a graded simple artinian ring Q_{gr} is of the form $Q_{\mathrm{gr}}^\sigma(l)[n]$. So one can define a graded version of this nation, called the *hammerhead* of the graded dualizing complex R, and denoted by $\xi_{\mathrm{gr}}(R)$.

Proposition 4.2. *Let* A *be a Goldie prime ring and let* $Q = Q(A)$. *Let* R *be a rigid dualizing complex over* A *that is evenly localizable to* Q. *If the* A^{e}*-module* A *is pseudo-coherent, then* Q *is rigid and* $\mathrm{H}_1 \operatorname{tr} Q = \xi(R)$.

If A *is graded, then the graded version of the assertion also holds.*

In the graded case, we have further $\mathrm{H}_1 \operatorname{tr} Q = \mathrm{H}_1 \operatorname{tr}_{\mathrm{gr}} Q_{\mathrm{gr}}(A) = \xi(R) = \xi_{\mathrm{gr}}(R)$.

Proof. It is easy to show that the ring Q^{e} is an Ore localization of A^{e}. So Q^{e} is a flat A^{e}-module. By Lemma 2.2(a) and pseudo-coherence of A,

$$\mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R^\circ) \otimes_{A^{\mathrm{e}}} Q^{\mathrm{e}} \cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, (R \otimes R^\circ) \otimes_{A^{\mathrm{e}}} Q^{\mathrm{e}}) =: (*).$$

Since $R_Q := Q \otimes_A R \otimes_A Q$ is a dualizing complex over Q, $R \otimes_A Q \cong X[n]$ and $R^\circ \otimes_A Q^\circ = X^\circ[n]$ where X is is isomorphic to a Q-bimodule Q^τ for some automorphism τ and $n = \xi(R)$. Hence

$$(*) = \mathrm{RHom}_{A^{\mathrm{e}}}(A, X \otimes X^\circ[2n]) \cong \mathrm{RHom}_{Q^{\mathrm{e}}}(Q^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A, X \otimes X^\circ[2n])$$

where the last isomorphism is Lemma 2.2(b). Note that $Q^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A \cong Q$ as Q^{e}-module. By the rigidity of R,

$$\mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R^\circ) \otimes_{A^{\mathrm{e}}} Q^{\mathrm{e}} \cong R \otimes_{A^{\mathrm{e}}} Q^{\mathrm{e}} \cong X[n].$$

Combining these we have

$$\mathrm{RHom}_{Q^{\mathrm{e}}}(Q, X \otimes X^\circ[2n]) \cong \mathrm{RHom}_{A^{\mathrm{e}}}(A, R \otimes R^\circ) \otimes_{A^{\mathrm{e}}} Q^{\mathrm{e}} \cong X[n].$$

After a complex shift we have

$$\mathrm{RHom}_{Q^{\mathrm{e}}}(Q, X \otimes X^{\circ}) \cong X[-n].$$

Since $X \cong Q^{\tau}$ and $X^{\circ} \cong (Q^{\tau})^{\circ}$ as Q-bimodules, we have $\mathrm{RHom}_{Q^{\mathrm{e}}}(Q, Q^{\mathrm{e}}) \cong Q^{\sigma}[-n]$ where $\sigma = \tau^{-1}$. Hence Q is rigid and $\mathrm{H}_1 \operatorname{tr} Q = n$. The first assertion follows.

The proof of the graded case is similar.

In the graded case let $\tilde{R} = Q_{\mathrm{gr}} \otimes_A R \otimes_A Q_{\mathrm{gr}}$ where $Q_{\mathrm{gr}} = Q_{\mathrm{gr}}(A)$. By Lemma 4.1, $\tilde{R}$ is a graded dualizing complex over Q_{gr}. Since Q_{gr} is noetherian and has global dimension at most 1, $\tilde{R}$ is also an ungraded dualizing complex over Q_{gr}. As said before, $\tilde{R} \cong Q_{\mathrm{gr}}^{\sigma}(l)[n]$; so $\tilde{R}$ is evenly localizable to Q. By hypothesis the A^{e}-module A is pseudo-coherent. Since $Q_{\mathrm{gr}} \cong Q_{\mathrm{gr}}^{\mathrm{e}} \otimes_{A^{\mathrm{e}}} A$, Q_{gr} is pseudo-coherent over $Q_{\mathrm{gr}}^{\mathrm{e}}$. Hence we can apply the first assertion to $\tilde{R}$. The last assertion follows by the fact $\xi(\tilde{R}) = \xi_{\mathrm{gr}}(\tilde{R}) = \xi_{\mathrm{gr}}(R) = \xi(R)$. $\square$

To use Proposition 4.2 we need to check the following:

(C1) the A^{e}-module A is pseudo-coherent;
(C2) there exists a rigid dualizing complex R over A;
(C3) R is evenly localizable to Q;
(C4) $\xi(R)$ is computable.

In the rest of this section we discuss (C1), (C2) and (C3). First we consider condition (C1).

If A^{e} is noetherian, then A has a free resolution over A^{e} with each term being a finite free A^{e}-module. So A is pseudo-coherent over A^{e}.

Let A be a connected graded ring. Following [28], A is called Ext-*finite* if $\mathrm{Ext}^i_A(k,k)$ is finite dimensional over k for all i. If A is noetherian, then it is Ext-finite. There are many non-noetherian graded rings which are Ext-finite. For example, if A is AS regular (not necessarily noetherian), then A is Ext-finite [26, Proposition 3.1(3)].

Let $F \to k$ be the minimal free resolution of k as A-module. Then $F^{-i} \cong A \otimes V_i$ where V_i is the graded vector space $\mathrm{Tor}^A_i(k,k)$. Hence A is Ext-finite if and only if $\mathrm{Tor}^A_i(k,k)$ is finite dimensional for all i, if and only if k is pseudo-coherent over A.

Lemma 4.3. *Let A be a connected graded algebra. Let $V_i = \mathrm{Tor}^A_i(k,k)$.*

(a) *The graded A^{e}-module A has a minimal graded free resolution P such that $P^{-i} \cong A^{\mathrm{e}} \otimes V_i$.*
(b) *The projective dimension* $\operatorname{projdim}_{A^{\mathrm{e}}} A$ *is equal to the projective dimension* $\operatorname{projdim}_A k (= \operatorname{gldim} A)$.
(c) *If A is* Ext-*finite, then the A^{e}-module A is pseudo-coherent.*
(d) *Let t be a homogeneous regular normal element of A. Let M be a graded pseudo-coherent $A/(t)$-module. Then M is pseudo-coherent viewed as an A-module.*
(e) *Let t be a homogeneous regular normal element of A. Then* $\operatorname{projdim}_A k = \operatorname{projdim}_{A/(t)} k + 1$ *if* $\operatorname{projdim}_{A/(t)} k$ *is finite.*

We would like to remark that Lemma 4.3(b) is similar to a result of Rouquier [19, Lemma 7.2] which says that $\operatorname{projdim}_{A^{\mathrm{e}}} A = \operatorname{gldim} A$ for finite dimensional algebras A or commutative algebras A essentially of finite type.

Proof of Lemma 4.3. (a) It suffices to show that $\operatorname{Tor}_i^A(k,k) \cong \operatorname{Tor}_i^{A^e}(k,A)$ as graded k-modules. Let P be a minimal graded free resolution of the graded A^e-module A. We think of P as an A-bimodule free resolution of A. Restricted to the left (and to the right), the exact complex $P \to A \to 0$ is a split sequence since every term of it is a free A-module (respectively, free A°-module). This implies that $k \otimes_{A^\circ} (P \to A \to 0)$ is exact and hence $P' := k \otimes_{A^\circ} P$ is a free resolution of the A-module k. Hence

$$\operatorname{Tor}_i^A(k,k) = \mathrm{H}^{-i}(k \otimes_A P') \cong \mathrm{H}^{-i}(k \otimes_A (k \otimes_{A^\circ} P)) =: (*).$$

Since $k \otimes_{A^\circ} P = (A \otimes k) \otimes_{A^e} P$ and $k \cong k \otimes_A (A \otimes k)$, we have

$$(*) \cong \mathrm{H}^{-i}(k \otimes_A (A \otimes k) \otimes_{A^e} P) \cong \mathrm{H}^{-i}(k \otimes_{A^e} P) = \operatorname{Tor}^{A^e}(k,A).$$

(b,c) are immediate consequences of part (a).

(d,e) We use the double Tor spectral sequence:

$$\operatorname{Tor}_p^B(\operatorname{Tor}_q^A(k,B),M) \Rightarrow_p \operatorname{Tor}_n^A(k,M) \tag{E4.3.1}$$

where $B = A/(t)$. Note that $\operatorname{Tor}_i^A(k,B) \cong k$ for $i = 0,1$ and $\operatorname{Tor}_i^A(k,B) = 0$ for all $i > 1$. If M is pseudo-coherent as B-module, then $\operatorname{Tor}_i^B(k,M)$ is finite for all i. By the above spectral sequence $\operatorname{Tor}_i^A(k,M)$ is finite for all i. Thus we proved part (d).

For part (e) we assume that $\operatorname{projdim}_B k < \infty$ and let $M = k$ in the spectral sequence. Then we see that $\operatorname{projdim}_A k \leq \operatorname{projdim}_B k + 1$. Also the spectral sequence (E4.3.1) implies that

$$\operatorname{Tor}_{n+1}^A(k,k) = \operatorname{Tor}_n^B(\operatorname{Tor}_1^A(k,B),k) \neq 0$$

for $n = \operatorname{projdim}_B k$. The assertion follows. $\square$

Similarly, $k \otimes_A P$ is a minimal free resolution of k as A°-module. Combining Lemma 4.3 with the comments before Lemma 4.3 we have proved the following.

Proposition 4.4. *Condition (C1) holds if A satisfies one of the following conditions:*

(a) *A^e is noetherian.*
(b) *A is connected graded and noetherian.*
(c) *A is connected graded and AS regular.*

Secondly we consider condition (C2).

There are already many results about the existence of rigid dualizing complexes in [28, 29, 31]. For example, if A has a filtration such that $\operatorname{gr} A$ is connected graded, noetherian, and AS Gorenstein, then $R = A^\sigma[n]$ is a rigid dualizing complex over A where n is the injective dimension of $\operatorname{gr} A$ [31, Proposition 6.18(2)]. For non-noetherian AS regular algebras rigid dualizing complexes also exist.

Proposition 4.5. *Let A be an AS regular algebra of (graded) global dimension n.*

(a) *A^e is AS regular of global dimension $2n$.*
(b) $\mathrm{RHom}_{A^e}(A, A^e) \cong A^\tau(-l)[-n]$ *for some automorphism τ, where l is the number given in* Definition 3.5(c).
(c) *Let $R = A^\sigma(l)[n]$ where $\tau = \sigma^{-1}$. Then R is a rigid dualizing complex over A.*
(d) *If A is (graded) Goldie prime, then $Q(A)$ is rigid and smooth and*

$$\mathrm{H}_1 \operatorname{tr} A = \mathrm{H}_1 \operatorname{tr} Q(A) = \mathrm{H}_1 \operatorname{tr}_{\mathrm{gr}} Q_{\mathrm{gr}}(A) = n.$$

Proof. (a) For any connected graded ring B, $\operatorname{gldim} B = \operatorname{projdim}_B k$. It is clear that

$$\operatorname{projdim}_{A^e} k = \operatorname{projdim}_{A\otimes A^\circ} {}_Ak \otimes {}_{A^\circ}k \leq \operatorname{projdim}_A k + \operatorname{projdim}_{A^e} k = 2n.$$

Hence $\operatorname{gldim} A^e \leq 2n$. Note that ${}_Ak$ and ${}_{A^\circ}k$ have finite minimal free resolutions over A and A° respectively [26, Proposition 3.1(3)]. The Künneth formula and the AS Gorenstein property for A and A^e imply that

$$\mathrm{RHom}_{A^e}(k, A^e) = \mathrm{RHom}_A(k, A) \otimes \mathrm{RHom}_{A^\circ}(k, A^\circ) = k(-2l)[2n].$$

Therefore A^e is an AS regular of global dimension $2n$.

(b) Since A is AS regular, the A^e-module A is pseudo-coherent [Proposition 4.4(c)]. Since A has finite global dimension, k has finite projective dimension. By Lemma 2.2(a),

$$\mathrm{RHom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_A k \cong \mathrm{RHom}_{A^e}(A, A^e \otimes^{\mathrm{L}}_A k) \cong \mathrm{RHom}_{A^e}(A, k \otimes A^\circ) =: (*).$$

Since A is AS regular, we have

$$\mathrm{RHom}_A(k, A) = k(-l)[-n] \qquad \text{or} \qquad \mathrm{RHom}_A(k, A(l)[n]) = k,$$

where (l) is the degree shift. Hence

$$k \otimes A^\circ = \mathrm{RHom}_{A^e}(k \otimes A^\circ, A \otimes A^\circ(l)[n]).$$

The computation continues

$$(*) \cong \mathrm{RHom}_{A^e}(A, \mathrm{RHom}_{A^e}(k \otimes A^\circ, A^e(l)[n]) \cong \mathrm{RHom}_{A^e}((k \otimes A^\circ) \otimes^{\mathrm{L}}_{A^e} A, A^e(l)[n])$$

The computation in the proof of Lemma 4.3(a) shows that

$$(k \otimes A^\circ) \otimes^{\mathrm{L}}_{A^e} A \cong (k \otimes A^\circ) \otimes_{A^e} P \cong k \otimes_A P \cong k.$$

Therefore

$$(*) \cong \mathrm{RHom}_{A^e}((k \otimes A^\circ) \otimes^{\mathrm{L}}_{A^e} A, A^e(l)[n]) \cong \mathrm{RHom}_{A^e}(k, A^e(l)[n]) =: (**).$$

By the proof of part (a), A^e is AS regular and $\mathrm{RHom}_{A^e}(k, A^e) = k(-2l)[-2n]$. Therefore

$$(**) \cong k(-l)[-n].$$

Thus we have proved that

$$\mathrm{RHom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_A k \cong k(-l)[-n].$$

Since the free resolution P of the A^e-module A is bounded with each term being finite, $\mathrm{RHom}_{A^e}(A, A^e) \cong \mathrm{Hom}_{A^e}(P, A^e) := P^\vee$, which is a bounded complex of finite free right A^e-modules. Let V be the minimal graded free resolution of $P^\vee$ viewed as A°-module complex. Note that the existence of V follows from the facts that each term of $P^\vee$ is locally finite and that $(P^\vee)_{\ll 0} = 0$. Then

$$V \otimes^{\mathrm{L}}_A k \cong \mathrm{RHom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_A k \cong k(-l)[-n]$$

which implies that $V \cong A^\circ(-l)[-n]$; or equivalently, $\mathrm{RHom}_{A^e}(A, A^e) \cong A^\circ(-l)[-n]$ as A°-modules. Similarly, $\mathrm{RHom}_{A^e}(A, A^e) \cong A(-l)[-n]$ as A-modules. Thus

$$\mathrm{Ext}^n_{A^e}(A, A^e) \cong A^\sigma(-l)$$

for some automorphism σ; and

$$\mathrm{Ext}^i_{A^e}(A, A^e) = 0$$

for all $i \neq n$.

(c) The rigidifying isomorphism follows from part (b). It is easy to see that R is a dualizing complex.

(d) Follows from part (c) and Proposition 4.2. □

Thirdly we look at condition (C3).

If $R = A^\sigma[n]$, then R is clearly evenly localizable to $Q(A)$ and $\xi(R) = n$. So we can now use Proposition 4.2 to compute the $\mathrm{H_1 tr}$ (similar to Proposition 4.5(d)).

Condition (C3) may hold even for $R \neq A^\sigma[n]$. In fact we don't have any examples of rigid dualizing complexes such that (C3) fails. Let's consider two cases: (i) R is an Auslander dualizing complexes; and (ii) A is a graded (or filtered) ring. We have already seen that Auslander dualizing complexes are evenly localizable to $Q(A)$ [34, Proposition 3.3].

Proposition 4.6. *Let A be a noetherian prime ring and let $Q = Q(A)$. Let R be an Auslander dualizing complex over A.*

(a) [34, Proposition 3.3] *R is evenly localizable to Q and* $\xi(R) = \operatorname{Cdim} A$.
(b) *If R is rigid and if A^{e}-module A is pseudo-coherent, then Q is rigid and* $\mathrm{H_1 tr}\, Q = \operatorname{Cdim} A$. *If A is also graded, then* $\mathrm{H_1 tr}\, Q_{\mathrm{gr}}(A) = \operatorname{Cdim} A$.

Proof. (a) is [34, Proposition 3.3]. To prove part (b) we use Proposition 4.2. □

A useful consequence is the following. Recall from [31] that a noetherian connected graded ring A is said to *have enough normal elements* if every non-simple prime graded factor ring A/I contains a nonzero normal element of positive degree.

Corollary 4.7. *Let A be a prime ring with a noetherian connected filtration such that* $\operatorname{gr} A$ *has enough normal elements. Then Q is rigid and* $\mathrm{H_1 tr}\, Q = \operatorname{GKdim} A$.

Proof. By [2, Section 4], $A \otimes B$ is noetherian for any noetherian algebra B. So A^{e} is noetherian and hence A is pseudo-coherent. By [31, Corollary 6.9], A has an Auslander rigid dualizing complex R such that $\operatorname{Cdim} = \operatorname{GKdim}$. The assertion follows from Proposition 4.6(b). □

Finally we mention that (C3) holds for graded noetherian rings.

Lemma 4.8. *Let A be a graded noetherian prime ring with a balanced dualizing complex R. Then R is evenly localizable to $Q(A)$ and to $Q_{\mathrm{gr}}(A)$.*

The proof follows from Lemma 6.3(b,c) below where the filtered case is considered.

5. Proof of Theorem 0.2

In this section we prove Theorem 0.2. Let U denote some division k-algebra. We sometimes use A_U to denote the ring $A \otimes U$.

Lemma 5.1. *Let A be a connected graded ring and let U be a division ring. Then the ungraded global dimension* $\operatorname{gldim} A_U$ *is equal to the graded projective dimension* $\operatorname{projdim}_A k$.

Proof. It is clear that

$$\operatorname{projdim}_A k \le \operatorname{gldim} A \le \operatorname{gldim} A_U.$$

So it suffices to show that $\operatorname{projdim}_A k \ge \operatorname{gldim} A_U$. Hence we may assume that $\operatorname{projdim}_A k < \infty$. Let P be a minimal free A^{e}-resolution of A. By Lemma 4.3(b), the length of P is equal to $\operatorname{projdim}_A k$. Let $P_U := P \otimes U$ and consider U as a

U^{e}-module. Then P_U is a complex of A_U^{e}-module. Let M be any ungraded A_U-module, we claim that $M \otimes_{A_U^\circ} P_U$ is a free A_U-module resolution of M. By the reasoning given in the proof of Lemma 4.3(a), $P \to A \to 0$ is a split exact sequence of A°-modules. Hence $P_U \to A_U \to 0$ is a split exact sequence of A_U°-modules. Thus $M \otimes_{A_U^\circ} P_U \to M \to 0$ is exact, or $M \otimes_{A_U^\circ} P_U$ is a resolution of M. Now every term in $M \otimes_{A_U^\circ} P_U$ is a direct sum of copies of

$$M \otimes_{A_U^\circ} (A^{\mathrm{e}} \otimes U) \cong M \otimes A$$

where M is viewed as a U-module. Since U is a division ring, M is free over U, and whence $M \otimes A$ is free over A_U. Therefore the $M \otimes_{A_U^\circ} P_U$ is a projective resolution of M. Consequently, the projective dimension of M is bounded by the length of P. Thus $\operatorname{gldim} A_U \leq \operatorname{projdim}_{A^{\mathrm{e}}} A = \operatorname{projdim}_A k$. □

Proposition 5.2. *If A is a connected graded Goldie prime ring of finite global dimension, then* $\mathrm{H}_2\mathrm{tr}\, Q(A) \leq \operatorname{gldim} A$.

Proof. The assertion follows from the (in)equalities

$$\operatorname{injdim} Q(A) \otimes U \leq \operatorname{gldim} Q(A) \otimes U \leq \operatorname{gldim} A \otimes U = \operatorname{gldim} A$$

where the last equality is Lemma 5.1. □

We now prove a restatement of Theorem 0.2.

Theorem 5.3. *Let A be an AS regular graded ring that is Goldie prime. Then $Q(A)$ is rigid, smooth and homologically uniform and* $\mathrm{Htr}\, Q(A) = \operatorname{gldim} A$.

Proof. Since A is AS regular, so is $A \otimes A^\circ$ [Proposition 4.5(a)]. By Lemma 1.4(d), $Q \otimes Q^\circ$ has finite global dimension where $Q := Q(A)$. Hence Q is is smooth [Lemma 1.3(b)]. By Proposition 4.5(d), Q is rigid and $\mathrm{H}_1\mathrm{tr}\, Q = \operatorname{gldim} A$. By Proposition 5.2, $\mathrm{H}_2\mathrm{tr}\, Q \leq \operatorname{gldim} A$. The assertion follows from Lemma 1.2(a). □

We do not assume that A is noetherian in the above statement. There is no reason to expect that an AS regular Goldie prime ring (or even Ore domain) should be noetherian or have finite GK-dimension. There are many connected graded Ore domains have exponential growth (hence not noetherian). For example any Rees ring of an affine Ore domain of exponential growth is connected graded with exponential growth.

Next we consider the degree zero part of the graded quotient ring. Let A be a graded prime ring and let $Q_{\mathrm{gr}}(A)$ be the graded Goldie quotient ring. Let $Q_0(A)$ (or Q_0 for short) be the degree zero part of $Q_{\mathrm{gr}}(A)$. In general, Q_0 is semisimple artinian and it is a finite direct sum of simple artinian rings which are isomorphic to each other. We define $\mathrm{Htr}\, Q_0$ to be the Htr of one copy of its simple artinian summands. If A is an Ore domain, then Q_0 is a division algebra. We now prove a version of Theorem 5.3 for Q_0.

Proposition 5.4. *Let A be an AS regular graded ring that is Goldie prime. Suppose that $Q_{\mathrm{gr}}(A) = Q_0[t^{\pm 1}; \alpha]$ for some $t \in A_{\geq 1}$ and some automorphism α. If Q_0 is doubly noetherian, then Q_0 is rigid, smooth and homologically uniform and* $\mathrm{Htr}\, Q_0 = \operatorname{gldim} A - 1$.

Proof. Let Q be the Goldie quotient ring of A. Then $Q = Q_0(t; \alpha)$. By Theorem 5.3, Q is rigid, smooth and homologically uniform. By Lemma 2.1(c), Q_0 is smooth; and by hypothesis, Q_0 is doubly noetherian. Lemma 1.3 implies that Q_0 is homologically uniform. To show $\operatorname{Htr} Q_0 = \operatorname{gldim} A - 1$, it suffices to show that $\mathrm{H}_1 \operatorname{tr} Q_0 = \mathrm{H}_1 \operatorname{tr} Q - 1$, which follows from Proposition 2.6. Also by Proposition 2.6, Q_0 is rigid. $\square$

6. Proof of Theorem 0.3

At the end of this section we prove Theorem 0.3. In order to prove Theorem 0.3 we need to review (and extend) some work of Van den Bergh [28] and of the first author [29] on local duality for graded modules, and to study filtered rings.

When applied to graded modules Hom and $\otimes$ and their derived functors will be in the graded sense. If M is a graded k-vector space, then M' is the graded k-linear dual of M.

Recall that a connected graded ring A is Ext-finite if $\operatorname{Ext}^i_A(k,k)$ is finite for all i. A consequence of this condition is that $A/A_{\geq n}$ is pseudo-coherent over A for every $n \geq 1$. Let U be any division algebra over k. By tensoring with U we see that $A_U/(A_U)_{\geq n}$ pseudo-coherent over A_U for all n. The trivial graded A_U-module $A_U/(A_U)_{\geq 1}$ is also denoted by U.

Definition 6.1. Let B be any $\mathbb{N}$-graded ring (not necessarily connected graded) and let $\mathfrak{m} = B_{\geq 1}$.

(a) For any graded B-module M, the $\mathfrak{m}$-*torsion functor* $\Gamma_{\mathfrak{m}}$ is defined to be

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n x = 0, \text{ for } n \gg 0\}.$$

The right derived functor of $\Gamma_{\mathfrak{m}}$, denoted by $\mathrm{R}\,\Gamma_{\mathfrak{m}}$, is defined on the derived category $\mathsf{D}^+(\mathsf{GrMod}\, B)$.

(b) The *ith local cohomology* of $X \in \mathsf{D}^+(\mathsf{GrMod}\, B)$ is defined to be

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \mathrm{R}^i\,\Gamma_{\mathfrak{m}}(X).$$

(c) The *local cohomological dimension* of a graded B-module M is defined to be

$$\operatorname{lcd} M = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(M) \neq 0\}.$$

(d) The *cohomological dimension* of $\Gamma_{\mathfrak{m}}$, also called the *cohomological dimension* of B, is defined to be

$$\operatorname{cd} \Gamma_{\mathfrak{m}} = \operatorname{cd} B = \sup\{\operatorname{lcd} M \mid \text{for all graded } B\text{-modules } M\}.$$

Obviously, $\Gamma_{\mathfrak{m}}(M) = \varinjlim \operatorname{Hom}_B(B/\mathfrak{m}^n, M)$, which implies that

$$\mathrm{H}^i_{\mathfrak{m}}(X) = \varinjlim \operatorname{Ext}^i_B(B/\mathfrak{m}^n, X)$$

for all $X \in \mathsf{D}^+(\mathsf{GrMod}\, B)$. If B is left noetherian and $\operatorname{cd} \Gamma_{\mathfrak{m}} < \infty$, then

$$\operatorname{cd} \Gamma_{\mathfrak{m}} = \operatorname{lcd} B = \sup\{i \mid \mathrm{H}^i_{\mathfrak{m}}(B) \neq 0\}.$$

Lemma 6.2. *Assume A is* Ext-*finite. Let $R = \mathrm{R}\Gamma_{\mathfrak{m}}(A)'$ and suppose it is locally finite. Let $R_U = R \otimes U$.*

(a) *Let $E = A' \otimes U$. Then E is the graded injective hull of the graded trivial A_U-module U and $\operatorname{Hom}_U(M, U) \cong \operatorname{Hom}_{A_U}(M, E)$ for all graded A_U-modules M.*

(b) *Let M be a graded A_U-module. Then $\mathrm{R}\Gamma_{\mathfrak{m}_U}(M) \cong \mathrm{R}\Gamma_{\mathfrak{m}}(M)$. As a consequence, $\operatorname{cd} A = \operatorname{cd} A_U$.*

(c) *Suppose* $\operatorname{cd} A < \infty$. *Then*

$$\operatorname{Hom}_U(\mathrm{R}\Gamma_{\mathfrak{m}_U}(M), U) \cong \mathrm{RHom}_{A_U}(M, R_U)$$

for $M \in \mathsf{D}^b(\mathsf{GrMod}\, A)$.

(d) *Suppose* $\operatorname{cd} A < \infty$. *Then* R_U *has graded injective dimension 0.*

(e) *If* A *is noetherian with balanced dualizing complex* R, *then* R_U *is a graded dualizing complex over* A_U.

(f) *If* A *is AS Gorenstein, then the graded injective dimension of* A_U *is equal to the graded injective dimension of* A.

Proof. (a) Consider the exact functor $F : M \mapsto \operatorname{Hom}_U(M, U)$ from $\mathsf{GrMod}\, A_U$ to $\mathsf{GrMod}\, A_U^{\circ}$. This functor sends coproducts to products. By Watts' theorem [18, Theorem 3.36], F is equivalent to $M \mapsto \operatorname{Hom}_{A_U}(M, E_U)$ where

$$E_U = \operatorname{Hom}_U(A_U, U) \cong A' \otimes U = E.$$

Since the functor is exact, E_U is injective. The socle of E_U is the trivial module U, so E_U is the injective hull of U.

(b) Let M be a graded A_U-module. Then the Hom-$\otimes$ adjunction implies that

$$\operatorname{Ext}^i_{A_U}(A_U/(A_U)_{\geq n}, M) = \operatorname{Ext}^i_A(A/A_{\geq n}, M).$$

Therefore

$$\mathrm{R}\Gamma_{\mathfrak{m}_U}(M) = \mathrm{R}\Gamma_{\mathfrak{m}}(M).$$

The assertion follows.

(c) Since A is Ext-finite, $\mathrm{R}\Gamma_{\mathfrak{m}}(-)$ commutes with coproducts [28, Lemma 4.3]. By part (b) $\mathrm{R}\Gamma_{\mathfrak{m}_U}(-)$ commutes with coproducts.

Using part (b) and the fact that R is locally finite, we have

$$\operatorname{Hom}_U(\mathrm{R}\Gamma_{\mathfrak{m}_U}(A_U), U) \cong \operatorname{Hom}_U(\mathrm{R}\Gamma_{\mathfrak{m}}(A_U), U) \cong \operatorname{Hom}_U(R' \otimes U, U) \cong R \otimes U = R_U.$$

Let F be a bounded resolution of A_U as graded A_U^{e}-module whose restriction consists of $\Gamma_{\mathfrak{m}_U}$-acyclic A-modules. Then we have $\mathrm{R}\Gamma_{\mathfrak{m}_U}(A) = \Gamma_{\mathfrak{m}_U}(F)$. Let K be a projective resolution of M. Since $\mathrm{R}\Gamma_{\mathfrak{m}_U}(-)$ commutes with coproducts, $F \otimes_{A_U} K$ is $\Gamma_{\mathfrak{m}_U}$-acyclic. This implies that

$$\mathrm{R}\Gamma_{\mathfrak{m}_U}(M) = \Gamma_{\mathfrak{m}_U}(F \otimes_{A_U} K) = \Gamma_{\mathfrak{m}_U}(F) \otimes_{A_U} K.$$

Let E be the injective resolution of U as a graded A_U-module. By part (a) we have

$$\mathrm{RHom}_U(\mathrm{R}\Gamma_{\mathfrak{m}_U}(M), U) \cong \operatorname{Hom}_{A_U}(\Gamma_{\mathfrak{m}_U}(F) \otimes_{A_U} K, E) =: (*).$$

By the Hom-$\otimes$ adjunction we have

$$(*) \cong \operatorname{Hom}_{A_U}(K, \operatorname{Hom}_{A_U}(\Gamma_{\mathfrak{m}_U}(F), E)) \cong \mathrm{RHom}_{A_U}(K, R_U) \cong \mathrm{RHom}_{A_U}(M, R_U)$$

where the middle isomorphism follows from

$$\operatorname{Hom}_{A_U}(\Gamma_{\mathfrak{m}_U}(F), E) \cong \operatorname{Hom}_{A_U}(\mathrm{R}\Gamma_{\mathfrak{m}_U}(A_U), E) \cong R_U.$$

(d) Since the complex $\mathrm{R}\Gamma_{\mathfrak{m}_U}(M)$ lives in non-negative positions, the "dual" complex $\operatorname{Hom}_U(\mathrm{R}\Gamma_{\mathfrak{m}_U}(M), U)$ lives in the non-positive positions. Hence by part (c), R_U has injective dimension less than or equal to 0. If M is $\mathfrak{m}$-torsion, then, by part (c), $\operatorname{Ext}^0_{A_U}(M, R_U) \neq 0$. The assertion follows.

Note that if M is $\mathfrak{m}$-torsion-free then $\operatorname{Ext}^0_{A_U}(M, R_U) = 0$.

(e) By definition R is pseudo-coherent over A on both sides. So R_U is pseudo-coherent over A_U on both sides. By part (d) R_U has finite injective dimension on the left; by symmetry also on the right. Finally $\mathrm{RHom}_{A_U}(R_U, R_U) \cong A_U$ follows from the fact R is pseudo-coherent and R is a dualizing complex over A. By symmetry, $\mathrm{RHom}_{A_U^\circ}(R_U, R_U) \cong A_U$. Therefore R_U is a dualizing complex over A_U.

(f) If A is AS Gorenstein (and Ext-finite), then the balanced dualizing complex over A is $R = A^\sigma(l)[n]$ where n is the graded injective dimension of A. The assertion follows from part (d). $\square$

An ascending $\mathbb{N}$-filtration $F = \{F_i A\}_{i\geq 0}$ on a ring A is called *connected filtration* (respectively, *noetherian connected filtration*) if

(i) $1 \in F_0 A$,
(ii) $F_i A\, F_j A \subset F_{i+j} A$,
(iii) $A = \bigcup_{i\geq 0} F_i A$, and
(iv) the associated graded ring
$$\operatorname{gr} A := \bigoplus_{i=0}^{\infty} F_i A / F_{i-1} A$$
is connected graded (respectively, connected graded and noetherian).

The Rees ring of A with a given filtration F defined to be
$$L := \bigoplus_{i=0}^{\infty} (F_i A) t^i.$$
So L is a subring of $A[t]$ such that $L/(t) = \operatorname{gr} A$ and $L/(t-1) = A$. By [3, Theorem 8.2], $\operatorname{gr} A$ is noetherian if and only if L is.

An A-bimodule is called *filtered finite* if there is a filtration on M compatible with the filtration on A such that $\operatorname{gr} M$ is a finite left and a finite right graded $\operatorname{gr} A$-module. If A is connected graded, then A has an obvious filtration such that $\operatorname{gr} A = A$. Sometimes we view a graded ring A as a filtered ring so that we can pass some graded properties to the ungraded setting.

Lemma 6.3. *Let A be a ring with a connected filtration and let L be the Rees ring. In parts (c-f) assume that A has a noetherian connected filtration and that A is prime. Let R be a rigid dualizing complex over A (if it exists).*

(a) *If $\operatorname{gr} A$ is Ext-finite, then so is L.*
(b) *If A is Goldie prime, then L is both graded Goldie prime and ungraded Goldie prime. Furthermore $Q(L) = Q(A)(t)$.*
(c) *Every filtered finite bimodule is evenly localizable.*
(d) *If $\operatorname{gr} A$ has a balanced dualizing complex, then any rigid dualizing complex R over A is evenly localizable.*
(e) *If $\operatorname{gr} A$ has a balanced dualizing complex, then*
$$0 \leq \xi(R) \leq \operatorname{cd} \operatorname{gr} A = -\inf\{i \mid \mathrm{H}^i(R) \neq 0\}.$$
(f) *If $\operatorname{gr} A$ is noetherian and AS Gorenstein, then $\xi(R) = \operatorname{injdim} \operatorname{gr} A$.*

Proof. (a) We know that t is a central regular element in L such that $L/(t) \cong \operatorname{gr} A$. Applying Lemma 4.3(d) to the trivial module k the assertion follows.

(b) For every regular element $x \in A$, xt^i for some $i \geq 0$ is a homogeneous regular element in L. The set of homogeneous regular elements in L form an Ore set. By inverting all homogeneous elements we obtain that
$$Q_{\mathrm{gr}}(L) = Q(A)[t^{\pm 1}],$$

which is prime and graded simple artinian. So L is both graded Goldie prime and ungraded Goldie prime, and $Q(L) = Q(A)(t)$.

(c) Let M be a filtered finite A-bimodule. By [25, Lemma 3.1], M is left Goldie torsion if and only if M is a right Goldie torsion. By passing to the factor module of M modulo the largest Goldie torsion A-submodule we may assume that M is two-sided Goldie torsionfree. Since $Q \otimes_A M$ is an artinian left Q-module and Goldie torsionfree right A-module, any regular element in A° acts on $Q \otimes_A M$ bijectively. Hence $Q \otimes_A M \cong Q \otimes_A M \otimes_A Q$. Similarly, $M \otimes_A Q \cong Q \otimes_A M \otimes_A Q$. Thus M is evenly localizable.

(d) It follows from the construction of the rigid dualizing complex R in [31, Theorem 6.2] that $\mathrm{H}^i(R)$ is filtered finite. By part (c) each $\mathrm{H}^i(R)$ is evenly localizable. So R is evenly localizable.

(e) Let $R_{\operatorname{gr} A}$ be a balanced dualizing complex over $\operatorname{gr} A$. By [28, Theorem 6.3] $R_{\operatorname{gr} A} = \mathrm{R}\Gamma_{\mathfrak{m}}(\operatorname{gr} A)'$. Since $\Gamma_{\mathfrak{m}}$ has finite cohomological dimension, $\operatorname{cd} \operatorname{gr} A = \operatorname{cd} \Gamma_{\mathfrak{m}} = -\inf\{i \mid \mathrm{H}^i(R_{\operatorname{gr} A}) \neq 0\}$. By the construction of R in [31, Theorem 6.2] one sees that

$$\inf\{i \mid \mathrm{H}^i(R) \neq 0\} = \inf\{i \mid \mathrm{H}^i(R_{\operatorname{gr} A}) \neq 0\} = -\operatorname{cd} \operatorname{gr} A.$$

Since $R_{\operatorname{gr} A}$ has injective dimension 0, R has injective dimension at most 0 by the construction. If $\mathrm{H}^i(R) \neq 0$ then i lies in between $-(\operatorname{cd} \operatorname{gr} A)$ and 0. The assertion follows.

(f) When A is filtered AS Gorenstein, then $R = A^\sigma[n]$ where $n = \operatorname{injdim} \operatorname{gr} A$ [31, Proposition 6.18]. Hence $\xi(R) = \operatorname{injdim} \operatorname{gr} A$. □

Proposition 6.4. *Let A be a Goldie prime ring. Suppose A has a connected filtration such that* $\operatorname{gr} A$ *is AS regular. If $Q(A)$ is doubly noetherian, then $Q(A)$ is smooth, rigid and homologically uniform, and* $\operatorname{Htr} Q(A) = \operatorname{gldim} \operatorname{gr} A$.

Proof. Let L be the Rees ring of A. By Lemma 4.3(e) and the Rees lemma, L is AS regular and $\operatorname{gldim} L = \operatorname{gldim} \operatorname{gr} A + 1$. By Lemma 6.3(b), L is Goldie prime. Since $Q_{\mathrm{gr}}(L) = Q(A)[t^{\pm 1}]$ and $Q(A)$ is doubly noetherian, the assertion follows from Proposition 5.4. □

Proposition 6.5. *Let A be a prime ring with a noetherian connected filtration such that* $\operatorname{gr} A$ *has a balanced dualizing complex. Let R be a rigid dualizing complex over A. If Q is doubly noetherian, then* $\mathrm{H}_1 \operatorname{tr} Q = \xi(R)$.

Proof. Let L be the Rees ring and let R_L be a balanced (and rigid) dualizing complex over L. Since L is noetherian, it is pseudo-coherent as L^{e}-module (see Proposition 4.4(b)). All conditions in Proposition 4.2 holds for L in both the graded and the ungraded settings. Hence $\mathrm{H}_1 \operatorname{tr}_{\mathrm{gr}} Q_{\mathrm{gr}}(L) = \xi_{\mathrm{gr}}(R_L) = \mathrm{H}_1 \operatorname{tr} Q(L) = \xi(R_L)$.

By [31, Theorem 6.2], a rigid dualizing complex R_A over A is given by the degree zero part of $R_L[t^{-1}][-1]$. Thus $\mathrm{H}^i(R_A)$ is not Goldie A-torsion if and only if $\mathrm{H}^{i-1}(R_L)$ is not Goldie L-torsion. This implies that $\xi(R_A) = \xi_{\mathrm{gr}}(R_L) - 1$. It remains to show that $\mathrm{H}_1 \operatorname{tr} Q(A) = \mathrm{H}_1 \operatorname{tr} Q_{\mathrm{gr}}(L) - 1$. But this follows from Proposition 2.6. □

Next we study the injective dimension of $Q(A) \otimes U$. The following lemma is [6, Theorem 1.3].

Lemma 6.6. [6, Theorem 1.3]. *Let C be a noetherian ring and t a central element of C. If I is an injective C-module, then $I[t^{-1}]$ is an injective $C[t^{-1}]$-module. A graded version of the assertion also holds.*

Proposition 6.7. *Let A be a filtered ring and let L be the Rees ring. Assume L is noetherian and has a balanced dualizing complex R_L. Let U be a division ring such that L_U is noetherian. Let R_A be a rigid dualizing complex over A. Then $R_A \otimes U$ is a dualizing complex over A of injective dimension at most 0.*

Proof. By [31, Theorem 6.2], $R_A = (R_L[t^{-1}])_0[-1]$. Hence, for all U,

$$R_A \otimes U = ((R_L \otimes U)[t^{-1}])_0[-1].$$

Since L_U is noetherian, A_U is noetherian. This implies that $R_A \otimes U$ is pseudo-coherent on both sides. Since $\mathrm{RHom}_A(R_A, R_A) \cong A$, Lemma 2.2 implies that

$$\mathrm{RHom}_{A_U}(R_A \otimes U, R_A \otimes U) \cong \mathrm{RHom}_A(R_A, R_A) \otimes U \cong A_U.$$

It remains to show that the injective dimension of $R_A \otimes U$ is at most 0 on both sides. By Lemma 6.2(d), $R_L \otimes U$ has graded injective dimension 0. Let I be the minimal graded injective dimension of $R_L \otimes U$.

We claim that $I^0 \cong L' \otimes U$. Since $L \otimes U$ is noetherian, I^0 is a direct sum of indecomposable injectives, say $\bigoplus J_i$. By Lemma 6.2(c),

$$\mathrm{Ext}^0_{L_U}(U, R_L \otimes U) = \mathrm{Hom}_U(U, U) = U.$$

This shows that I^0 contains only one copy of $L' \otimes U$. If $I^0 \neq L' \otimes U$, then there is an $\mathfrak{m}$-torsionfree graded A_U-module M such that $\mathrm{Ext}^0_{L_U}(M, R_L \otimes U) \neq 0$. But this contradicts Lemma 6.2(c) since $\Gamma_{\mathfrak{m}}(M) = 0$. So we proved our claim.

Since I^0 is t-torsion, $I^0[t^{-1}] = 0$; and by Lemma 6.6 the complex $(R_L \otimes U)[t^{-1}]$ of $\mathbb{Z}$-graded $L[t^{-1}]$-modules has injective dimension at most -1. Since $L[t^{-1}]$ is strongly graded and $(L[t^{-1}])_0 = A$, the complex $((R_L \otimes U)[t^{-1}])_0[-1]$ of A-modules has injective dimension at most 0. □

Proposition 6.8. *Let A be a prime ring with a noetherian connected filtration such that* $\operatorname{gr} A$ *has a balanced dualizing complex. Let U be a division ring such that* $\operatorname{gr} A \otimes U$ *is noetherian. Let R be a rigid dualizing complex over A. Then* $\operatorname{injdim} Q(A) \otimes U \leq \xi(R)$.

Proof. It is clear that $\operatorname{gr} A \otimes U$ is noetherian if and only if $L \otimes U$ is noetherian where L is the Rees ring of A. In this case $A \otimes U$ is also noetherian. By Proposition 6.7, $R_A \otimes U$ is a dualizing complex over A_U with injective dimension ≤ 0. By Lemma 4.1, $(Q \otimes_A R_A \otimes_A Q) \otimes U$ is a dualizing complex over $Q \otimes U$ where $Q = Q(A)$. Since $Q \otimes_A R_A \otimes_A Q \cong Q^\sigma[-d]$ where $d = \xi(R_A)$, $(Q \otimes_A R_A \otimes_A Q) \otimes U \cong Q^\sigma[-d] \otimes U$. The injective dimension will not increase under localization by Lemma 2.3. So $\operatorname{injdim}_{Q\otimes U}(Q \otimes U[-d]) \leq 0$. The assertion follows by a complex shift. □

We are now ready to prove a generalization of Theorem 0.3.

Theorem 6.9. *Let A be a filtered Goldie prime ring with noetherian connected filtration. Let $Q = Q(A)$.*

(a) *If* $\operatorname{gr} A$ *has a enough normal elements, then Q is rationally noetherian, rigid and homologically uniform, and* $\operatorname{Htr} Q = \operatorname{GKdim} A$.

(b) *If* $\operatorname{gr} A$ *has an Auslander balanced dualizing complex and* $\operatorname{gr} A \otimes Q^\circ$ *is noetherian, then Q is rigid and* $\mathrm{H_1tr}\, Q = \operatorname{Htr} Q = \operatorname{cd} \operatorname{gr} A$.

(c) *If* $\operatorname{gr} A$ *is AS Gorenstein and* $\operatorname{gr} A \otimes Q^{\circ}$ *is noetherian, then* Q *is rigid and* $\mathrm{H}_1\operatorname{tr} Q = \operatorname{Htr} Q = \operatorname{injdim} \operatorname{gr} A$.

Proof. (a) If $\operatorname{gr} A$ has enough normal elements, then $\operatorname{gr} A \otimes U$ is noetherian for all k-algebras U [2, Proposition 4.9]. Consequently, A and Q are rationally noetherian. Also by [31, Corollary 6.9], A has an Auslander, GKdim-Macaulay, rigid dualizing complex R_A. Hence

$$\xi(R_A) = \operatorname{cd} \operatorname{gr} A = \operatorname{GKdim} \operatorname{gr} A = \operatorname{GKdim} A.$$

By Proposition 6.8, $\operatorname{injdim} Q \otimes U \leq \xi(R_A) = \operatorname{GKdim} A$ for all division rings U. This says that $\mathrm{H}_2\operatorname{tr} Q \leq \operatorname{GKdim} A$. The assertion follows from Lemma 1.2 and Corollary 4.7.

(b) By [31, Corollary 6.8] and the proof of [31, Theorem 6.2], A has an Auslander dualizing complex R_A, and $\operatorname{cd} \operatorname{gr} A = \operatorname{Cdim} \operatorname{gr} A = \operatorname{Cdim} A$. By Proposition 4.6(a), $\xi(R_A) = \operatorname{Cdim} A$. By Proposition 6.8, $\operatorname{injdim} Q \otimes Q^{\circ} \leq \xi(R_A)$ since $\operatorname{gr} A \otimes Q^{\circ}$ is noetherian. By Proposition 6.5, $\mathrm{H}_1\operatorname{tr} Q = \xi(R_A)$. Hence $\mathrm{H}_1\operatorname{tr} Q = \operatorname{Htr} Q = \xi(R) = \operatorname{cd} \operatorname{gr} A$. By Proposition 4.2 Q is rigid.

(c) Similar to the proof of part (b). Using Proposition 4.2, Lemmas 6.3(f) and 1.2(a) and Proposition 6.8, we have

$$\xi(R) = \mathrm{H}_1\operatorname{tr} Q \leq \operatorname{Htr} Q = \operatorname{injdim} Q \otimes Q^{\circ} \leq \xi(R) = \operatorname{injdim} \operatorname{gr} A.$$

Hence $\mathrm{H}_1\operatorname{tr} Q = \operatorname{Htr} Q = \operatorname{injdim} \operatorname{gr} A$. By Proposition 4.2 Q is rigid. □

Theorem 0.3 is an immediate consequence of Theorem 6.9. Theorem 6.9 also applies to affine prime PI rings and various quantum algebras which have noetherian filtrations such that $\operatorname{gr} A$ has Auslander dualizing complexes. Here we give an example of this kind.

Example 6.10. Let Q be a division algebra finite over its center C, and assume C is finitely generated as a field. It is easy to pick an affine prime PI subalgebra A of Q such that Q is the quotient ring of A, and A has a filtration such that $\operatorname{gr} A$ is connected graded noetherian and affine PI. By Theorem 6.9(a), Q is a rationally noetherian, rigid and homologically uniform; and $\operatorname{Htr} Q = \operatorname{GKdim} Q = \operatorname{tr} Q$.

The following corollary follows from Theorem 6.9(a); the proof is omitted.

Corollary 6.11. *The simple artinian rings given in* Example 1.9(a,c,d,e,f,g) *are rigid.*

7. Examples

In this section we will give some examples to show that Htr can be different from other versions of transcendence degrees for certain division algebras.

Proposition 7.1. *Let* F *be a countably infinite dimensional separable algebraic field extension of* k.

(a) F *is smooth, homologically uniform and* $\operatorname{Htr} F = \operatorname{gldim} F^{\mathrm{e}} = 1$.
(b) F *is not doubly noetherian and*

$$\operatorname{Htr} F = 1 > 0 = \sup\{\operatorname{Htr} G_i \mid \textit{for all } G_i \subset F \textit{ with } \dim_k G_i < \infty\}.$$

(c) $\mathrm{H}_3\operatorname{tr} F = \operatorname{tr} F = 0$.
(d) F *is not rigid.*

The proof of the above proposition follows from several lemmas below.

Lemma 7.2. *Let F be as in Proposition 7.1. Then F is smooth of* $\mathrm{H_2tr}\, F \leq 1$ *and* $\operatorname{gldim} F^{\mathrm{e}} = \operatorname{projdim}_{F^{\mathrm{e}}} F = 1$.

Proof. By [11, Theorem 10], $\operatorname{projdim}_{F^{\mathrm{e}}} F = 1$. The assertion follows from Lemma 1.3(a). □

Consider the short exact sequence

$$0 \to J \to F^{\mathrm{e}} \to F \to 0 \tag{E7.2.1}$$

where the map $F^{\mathrm{e}} \to F$ is the multiplication and J is the kernel of this map.

The following Lemma 7.3(a) is due to Ken Goodearl. The authors thank him for providing the result.

Lemma 7.3. *Let F be as in Proposition 7.1 and let J be as in* (E7.2.1). *Then*

(a) *There is an infinite sequence of nonzero orthogonal idempotents $\{e_i\}_{i=1}^{\infty} \subset F^{\mathrm{e}}$ such that $J = \bigoplus_{i=1}^{\infty} e_i F^{\mathrm{e}}$.*
(b) $\operatorname{Hom}_{F^{\mathrm{e}}}(F, J) = \operatorname{Hom}_{F^{\mathrm{e}}}(J, F) = 0$.
(c) $\operatorname{Ext}^1_{F^{\mathrm{e}}}(F, F) = \operatorname{Hom}_{F^{\mathrm{e}}}(F, F^{\mathrm{e}}) = 0$.
(d) $\operatorname{Ext}^1_{F^{\mathrm{e}}}(F, F^{\mathrm{e}})$ *is infinite dimensional over F.*

Proof. (a) Write F as the union of countable sequence of finite dimensional subfields

$$k \subset F_1 \subset F_2 \subset \cdots \subset F.$$

Let J_i be the kernel of the map $F_i^{\mathrm{e}} \to F_i$. Since F is separable, F_i^{e} is a finite direct sum of field extensions of k. This implies that there are idempotents u_i and $v_i = 1 - u_i$ in F_i^{e} such that $F_i^{\mathrm{e}} = u_i F_i^{\mathrm{e}} \bigoplus v_i F_i^{\mathrm{e}}$ where $J_i = u_i F_i^{\mathrm{e}}$ and $F_i \cong v_i F_i^{\mathrm{e}}$ as F_i^{e}-module. Since $u_i \in J_j$ for all $j > i$, one sees that $u_i u_j = u_i$ for all $i < j$. This implies that $v_j v_i = v_j$ for all $j > i$. There are only two possibilities: either Case 1: $v_i = v_{i+1}$ for all $i \gg 0$, or Case 2: $v_i \neq v_{i+1}$ for infinitely many i. In Case 1 we may assume that $v_i = v_{i+1} := v$ for all i by passing to a subsequence. Then $F_i^{\mathrm{e}} = (1-v) F_i^{\mathrm{e}} \bigoplus v F_i^{\mathrm{e}}$ for all i. Thus $F^{\mathrm{e}} = (1-v)F^{\mathrm{e}} \bigoplus v F^{\mathrm{e}}$. Since

$$(1-v)F^{\mathrm{e}} = \textstyle\bigcup_i (1-v) F_i^{\mathrm{e}} = \bigcup_i J_i = J,$$

$vF^{\mathrm{e}} \cong F$. This implies that F is projective, a contradiction to Lemma 7.2. So Case 1 is impossible and that leaves us Case 2. By choosing a subsequence of $\{F_i\}$ we may assume that $v_i \neq v_{i+1}$ for all i. Let $e_1 = u_1 = 1 - v_1$ and $e_i = v_i - v_{i+1}$. Then $\{e_i\}$ is a set of nonzero orthogonal idempotents of F^{e} and $J_n = u_n F_n^{\mathrm{e}} = \bigoplus_{n=1}^{i} e_i F_i^{\mathrm{e}}$ for all n. Since $J = \bigcup_n J_n$, the assertion follows.

(b) If $f : J \to F$ is a nonzero F^{e}-homomorphism, then f is surjective since F is a field. Pick $b \in J_i \subset J$ such that $f(b) = 1 \in F$. Thus f induces a nonzero F_i^{e}-homomorphism from $bF_i^{\mathrm{e}} \to F_i$. But F_i^{e} is a direct sum $F_i \bigoplus J_i$ as rings, any homomorphism from a submodule of J_i to F_i is zero. This is a contradiction, hence $\operatorname{Hom}_{F_i^{\mathrm{e}}}(J, F) = 0$. A similar argument shows that $\operatorname{Hom}_{F_i^{\mathrm{e}}}(F, J) = 0$.

(c) Applying $\operatorname{Hom}_{F^{\mathrm{e}}}(-, F)$ to the short exact sequence (E7.2.1) we obtain an exact sequence

$$\to \operatorname{Hom}_{F^{\mathrm{e}}}(J, F) \to \operatorname{Ext}^1_{F^{\mathrm{e}}}(F, F) \to \operatorname{Ext}^1_{F^{\mathrm{e}}}(F^e, F) \to .$$

By part (b) the left end of the above sequence is zero and the right end is zero since F^{e} is a free F^{e}-module. Hence $\operatorname{Ext}^1_{F^{\mathrm{e}}}(F, F) = 0$.

If $\operatorname{Hom}_{F^{\mathrm{e}}}(F, F^e) \neq 0$, then let F be the image of some nonzero map $F \to F^{\mathrm{e}}$. By part (b) $F \cap J = 0$. Hence $F^{\mathrm{e}} = F \bigoplus J$ because $F^{\mathrm{e}}/J \cong F$. This contradicts the fact $\operatorname{projdim}_{F^{\mathrm{e}}} F = 1$ in Lemma 7.2. Therefore $\operatorname{Hom}_{F^{\mathrm{e}}}(F, F^e) = 0$.

(d) Applying $\mathrm{Hom}_{F^{\mathrm{e}}}(F,-)$ to the short exact sequence (E7.2.1) we obtain an exact sequence

$$\to \mathrm{Hom}_{F^{\mathrm{e}}}(F,F) \to \mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J) \to \mathrm{Ext}^1_{F^{\mathrm{e}}}(F,F^{\mathrm{e}}) \to .$$

Since $\mathrm{Hom}_{F^{\mathrm{e}}}(F,F)$ is 1-dimensional over F, it suffices to show that $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J)$ is infinite dimensional over F. Recall that the short exact sequence (E7.2.1) is non-split since $\operatorname{projdim}_{F^{\mathrm{e}}} F = 1$. Hence (E7.2.1) represents a nonzero element in $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J)$, which we denote by ψ.

By part (a), $J = \bigoplus_{i=1}^{\infty} e_i F^{\mathrm{e}}$ where $\{e_i\}$ is an infinite set of nonzero orthogonal idempotents in F^{e}. Now let Φ be any infinite subset of $\mathbb{N}$ and let $\Lambda = \bigoplus_{i\in\Phi} e_i F^{\mathrm{e}}$. We claim that $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,\Lambda) \neq 0$. Otherwise if $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,\Lambda) = 0$, then

$$\psi \in \mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J) = \mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J')$$

where $J' = \bigoplus_{i\notin\Phi} e_i F^{\mathrm{e}}$. Hence ψ represents a non-split short sequence

$$0 \to J' \to E \to F \to 0$$

such that

$$0 \to J' \bigoplus \Lambda \to E \bigoplus \Lambda \to F \to 0$$

is equivalent to (E7.2.1). Therefore $F^e \cong E \bigoplus \Lambda$; but this is impossible because Λ is an infinite direct sum. So we proved our claim that $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,\Lambda) \neq 0$.

Next we decompose $\mathbb{N}$ into a disjoint union of infinitely many infinite subsets $\{\Phi_n\}_{n\in\mathbb{N}}$ and define $\Lambda_n = \bigoplus_{i\in\Phi_n} e_i F^{\mathrm{e}}$. By the last paragraph, $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,\Lambda_n) \neq 0$ for all n. Hence the F-vector space dimension of $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,\bigoplus_{n=1}^{p}\Lambda_n)$ is at least p. Finally note that $\bigoplus_{n=1}^{p}\Lambda_n$ is a direct summand of J, therefore $\mathrm{Ext}^1_{F^{\mathrm{e}}}(F,J)$ is infinite dimensional over F, as desired. □

Proof of Proposition 7.1. (a) By Lemma 7.2, F is smooth and $\mathrm{H}_2\mathrm{tr}\, F \leq 1$. By Lemma 7.3(d), $\operatorname{injdim} F^e > 0$ and $\mathrm{H}_1\mathrm{tr}\, F > 0$. The assertion follows from Lemma 1.2(a).

(b) Since F is not finitely generated as a field, F^e is not noetherian [15, Proposition 1]. We have seen that $F = \bigcup_i G_i$ where G_i ranges over all finite dimensional subfields of F. For each G_i, we know that $\mathrm{Htr}\, G_i = 0$ [Example 1.9(c)]. The assertion follows.

(c) Clearly $\mathrm{tr}\, F = 0$.

Let U be a division algebra such that $F \otimes U$ is noetherian. We claim that $F\otimes U$ is semisimple artinian. If the claim is proved, then $\operatorname{injdim} F\otimes U = \operatorname{gldim} F\otimes U = 0$. This implies that $\mathrm{H}_3\mathrm{tr}\, F = 0$.

Now we prove the claim. Let $F = \bigcup_i F_i$ where $\{F_i\}$ is an ascending chain of finite dimensional subfields of F. Then $F\otimes U = \bigcup_i F_i \otimes U$. For each i, $F_i\otimes U$ is artinian since F_i is finite dimensional. Let Z be the center of U. Then $F_i \otimes Z$ is a direct sum of fields, say $\bigoplus_t G_t$, since F_i is separable. Then $F_i\otimes U = \bigoplus_t G_t \otimes_Z U$. Since Z is the center of U, each $G_t\otimes_Z U$ is simple. Hence $F_i \otimes U$ is semisimple artinian. Write

$$F_i \otimes U = \bigoplus_{t=1}^{p} \mathrm{M}_{n_t}(D^i_t) \qquad \text{(E7.3.1)}$$

where D^i_t are division rings. The Goldie rank of $F_i\otimes U$ is $\sum_{t=1}^{p} n_t$. Since $F\otimes U$ is faithfully flat over $F_i\otimes U$, then Goldie rank of $F_i \otimes U$ is bounded by the Goldie rank of $F\otimes U$; the latter is finite because $F\otimes U$ is noetherian. Therefore for $i \gg 0$,

$F_i \otimes U$ has the same form of the decomposition (E7.3.1) with $D_t^i \subset D_t^{i+1}$ for all i. Thus

$$F \otimes U = \bigoplus_{t=1}^{p} \mathrm{M}_{n_t}(D_t)$$

where $D_t = \bigcup_i D_t^i$. Since each D_t^i is a division ring, so it D_t. Therefore the claim is proved.

(d) This follows from Lemma 7.3(d). $\square$

Example 7.4. Let F be the separable algebraic field extension of k as in Proposition 7.1. Let F' be another separable algebraic field extension of k such that F is a subfield of F' and F' is countably infinite dimensional over F. By Proposition 7.1(a) $\operatorname{Htr} F' = 1$. So $\operatorname{Htr} F' = \operatorname{Htr} F$ but $\dim_F F' = \infty$.

Example 7.5. Let the base field k be $\mathbb{C}(\{x_i\}_{i\in I})$ where $|I| > \aleph_m$ for all integers m. Let F be the field $\mathbb{C}(\{x_i^{1/2}\}_{i\in I})$. Then F is a separable algebraic field extension of k such that $\dim_k F > \aleph_m$ for all integers m. For each m there is a subfield $F_m \subset F$ such that $\dim_k F_m = \aleph_m$. By [11, Theorem 10], $\operatorname{projdim}_{F_m^{\mathrm{e}}} F_m = \operatorname{gldim} F_m^{\mathrm{e}} = m$. We claim that there is no smooth simple artinian ring S such that $F \subset S$. If on contrary such S exists, then S^{e} has finite global dimension and S contains F_m for all m. By [10, Theorem 7.2.5], $\operatorname{gldim} F_m^{\mathrm{e}} \leq \operatorname{gldim} S^{\mathrm{e}}$. Since m is arbitrary, $\operatorname{gldim} S^{\mathrm{e}} = \infty$, a contradiction.

It is unclear to us if $\operatorname{Htr} F_m = m$ and if F_m is homologically uniform for all m.

Next we are going to construct an algebra A which is regular, but not AS regular.

Let G be the nilpotent group generated by a, b, c with relations $ab = ba, ac = ca, bc = cba$. Then the group algebra kG is isomorphic to $k[a^{\pm 1}, b^{\pm 1}][c^{\pm 1}, \sigma^{-1}]$ where $\sigma : a \mapsto a, b \mapsto ab$. The group algebra kG is obviously G-graded. Let A be the subalgebra of kG generated by $x := c$, $y := ac$, $z := bc$, $t := abc$. Since kG is a domain, so is A. If we set $\deg(c) = 1$ and $\deg(a) = \deg(b) = 0$, then A is a connected $\mathbb{N}$-graded domain generated in degree 1.

The following proposition are due to a joint work of Paul Smith and the second author [23]. The authors thank Paul Smith for allowing them to use this unpublished result.

Proposition 7.6. [23] *Let A be the connected graded algebra constructed above.*

(a) *It is Koszul of global dimension* 4.
(b) *It is a domain with $H_A(t) = (1-t)^{-4}$.*
(c) *It is not AS regular.*
(d) *It is neither left nor right noetherian.*
(e) *It has no non-trivial normal elements.*

Remark 7.7. A part of the proof of Proposition 7.6 was based on a long and tedious computation about the minimal free resolution of the trivial graded A-module k. It seems sensible to omit the proof here.

We show now that Theorem 0.2 is false without the Artin-Schelter condition. Note that a domain of finite GK-dimension is an Ore domain. Hence the ring A in Proposition 7.6 has a Goldie quotient ring.

Proposition 7.8. *Let A be the algebra given in Proposition 7.6. Let $Q(A)$ be the Goldie quotient ring of A. Then*

$$\operatorname{Htr} Q(A) = 3 < 4 = \operatorname{GKtr} Q(A) = \operatorname{GKtr} A = \operatorname{GKdim} A.$$

Proof. The division ring $Q(A)$ is isomorphic to $k(a,b)(c;\sigma)$ where σ maps $a \mapsto a$, $b \mapsto ba$. So it is stratiform of length 3. By Proposition 1.8, $Q(A)$ is homologically uniform of Htr 3. The division algebra $Q(A)$ is also the quotient division ring of the nilpotent group $G = \langle a,b\rangle/(ab=ba, ac=ca, bc=cba)$. By a result of Lorenz [9, Theorem 2.2], $\operatorname{GKtr} Q(A) = 4$. By [36, Propositions 2.1 and 3.1(3)], we have $\operatorname{GKtr} Q(A) \le \operatorname{GKtr} A \le \operatorname{GKdim} A = 4$. Therefore $\operatorname{GKtr} A = \operatorname{GKdim} A = 4$. □

Finally we post a question and make two remarks.

Question 7.9. Let S be a doubly noetherian simple artinian algebra. Is then S rigid and homologically uniform?

Remark 7.10. The work of Resco and Stafford [12, 13, 14, 24] suggests that one can define *Krull transcendence degree* of a simple artinian ring S to be

$$\operatorname{Ktr} S = \operatorname{Kdim} S^{\mathrm{e}}$$

where Kdim denotes Krull dimension. Krull transcendence degree has nice properties similar to those listed in Proposition 1.5 (the proofs are also similar), we believe that this invariant deserves further study.

Remark 7.11. The division algebras in this paper are different from the free skew fields constructed by Cohen [4] and Schofield [21]. We expect that the free skew fields have infinite homological transcendence degree.

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