# RESIDUES AND DIFFERENTIAL OPERATORS ON SCHEMES

AMNON YEKUTIELI

Contents



## 0. Introduction

Suppose $X$ is a finite type scheme over a field $k$, with structural morphism $\pi$. Consider the twisted inverse image functor $\pi^! : \mathsf{D}^+_{\mathrm{c}}(k) \to \mathsf{D}^+_{\mathrm{c}}(X)$ of Grothendieck Duality Theory (see [RD]). The *residue complex* $\mathcal{K}^{\cdot}_X$ is defined to be the Cousin complex of $\pi^! k$. It is a bounded complex of quasi-coherent $\mathcal{O}_X$-modules, possessing remarkable functorial properties. In this paper we provide an explicit construction of $\mathcal{K}^{\cdot}_X$. This construction reveals some new properties of $\mathcal{K}^{\cdot}_X$, and also has applications in other areas of algebraic geometry.

Grothendieck Duality, as developed by Hartshorne in [RD], is an abstract theory, stated in the language of derived categories. Even though this abstraction is suitable for many important applications, often one wants more explicit information. Thus a significant amount of work was directed at finding a presentation of duality in terms of differential forms and residues. Mostly the focus was on the dualizing sheaf $\omega_X$, in various circumstances. The structure of $\omega_X$ as a coherent $\mathcal{O}_X$-module and its variance properties are thoroughly understood by now, thanks to an extended effort including

*Date*: 3.6.97.

1991 *Mathematics Subject Classification.* Primary 14F10; Secondary 14F40, 14F32, 14B10, 13N05.

The author is supported by an Allon Fellowship, and is the incumbent of the Anna and Maurice Boukstein Career Development Chair.

[Kl], [KW], [Li], [HK1], [HK2], [LS] and [HS]. Regarding an explicit presentation of the full duality theory of dualizing complexes, there have been some advances in recent years, notably in the papers [Ye1], [SY], [Hu], [Hg] and [Sa].

In this paper we give a totally new construction of the residue complex $\mathcal{K}^{\cdot}_X$, when $k$ is a perfect field of any characteristic and $X$ is any finite type $k$-scheme. The main idea is the use of *Beilinson Completion Algebras* (BCAs), which were introduced in [Ye2]. These algebras are generalizations of complete local rings, and they carry a mixed algebraic-analytic structure. A review of BCAs and their properties is included in Section 1, for the reader's convenience.

Given a point $x \in X$, the complete local ring $\widehat{\mathcal{O}}_{X,x} = \mathcal{O}_{X,(x)}$ is a BCA, so according to [Ye2] it has a *dual module* $\mathcal{K}(\mathcal{O}_{X,(x)})$. This module is a canonical model for the injective hull of the residue field $k(x)$. If $(x, y)$ is a saturated chain of points (i.e. $y$ is an immediate specialization of $x$) then there is a BCA $\mathcal{O}_{X,(x,y)}$ and homomorphisms $\mathrm{q} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{X,(x,y)})$ and $\mathrm{Tr} : \mathcal{K}(\mathcal{O}_{X,(x,y)}) \to \mathcal{K}(\mathcal{O}_{X,(y)})$. The dual modules $\mathcal{K}(-)$ and the homomorphisms q and Tr have explicit formulas in terms of differential forms and coefficient fields. Set $\delta_{(x,y)} := \mathrm{Tr}\,\mathrm{q} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{X,(y)})$. Define a graded quasi-coherent sheaf $\mathcal{K}^{\cdot}_X$ by

$$\mathcal{K}^q_X := \bigoplus_{\dim \overline{\{x\}} = -q} \mathcal{K}(\mathcal{O}_{X,(x)})$$

and a degree 1 homomorphism

$$\delta := (-1)^{q+1} \sum_{(x,y)} \delta_{(x,y)}.$$

It turns out that $(\mathcal{K}^{\cdot}_X, \delta)$ is a residual complex on $X$, and it is canonically isomorphic to $\pi^! k$ in the derived category $\mathsf{D}(X)$. Hence it is the residue complex of $X$, as defined in the first paragraph. The functorial properties of $\mathcal{K}^{\cdot}_X$ w.r.t. proper and étale morphisms are obtained directly from corresponding properties of BCAs, and therefore are reduced to explicit formulas. All this is worked out in Sections 2 and 3.

An $\mathcal{O}_X$-module $\mathcal{M}$ has a dual complex $\mathrm{Dual}\,\mathcal{M} := \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X)$. Suppose $\mathrm{d} : \mathcal{M} \to \mathcal{N}$ is a differential operator (DO). In Theorem 4.1 we prove there is a dual operator $\mathrm{Dual}(\mathrm{d}) : \mathrm{Dual}\,\mathcal{N} \to \mathrm{Dual}\,\mathcal{M}$, which commutes with $\delta$. The existence of Dual(d) does not follow from formal considerations of duality theory; it is a consequence of our particular construction using BCAs (but cf. Remarks 4.6 and 4.7). The construction also provides explicit formulas for Dual(d) in terms of differential operators and residues, which are used in the applications in Sections 6 and 7.

Suppose $A$ is a finite type $k$-algebra, and let $\mathcal{D}(A)$ be the ring of differential operators of $A$. As an immediate application of Theorem 4.1 we obtain a description of the opposite ring $\mathcal{D}(A)^{\circ}$, as the ring of DOs on $\mathcal{K}^{\cdot}_A$ which commute with $\delta$ (Theorem 4.8). In the case of a Gorenstein algebra it follows that the opposite ring $\mathcal{D}(A)^{\circ}$ is naturally isomorphic to $\omega_A \otimes_A \mathcal{D}(A) \otimes_A \omega_A^{-1}$ (Corollary 4.9).

Applying Theorem 4.1 to the De Rham complex $\Omega^{\cdot}_{X/k}$ we obtain the *De Rham-residue complex* $\mathcal{F}^{\cdot}_X = \operatorname{Dual} \Omega^{\cdot}_{X/k}$. Up to signs this coincides with El-Zein's complex $\mathcal{K}^{\cdot,*}_X$ of [EZ] (Corollary 5.9). The fundamental class $\mathrm{C}_Z \in \mathcal{F}^{\cdot}_X$, for a closed subscheme $Z \subset X$, is easily described in this context (Definition 5.10).

The construction above works also for a formal scheme $\mathfrak{X}$ which is of formally finite type over $k$, in the sense of [Ye3]. An example of such a formal scheme is the completion $\mathfrak{X} = Y_{/X}$, where $X$ is a locally closed subset of the finite type $k$-scheme $Y$. Therefore we get a complex $\mathcal{F}^{\cdot}_{\mathfrak{X}} = \operatorname{Dual} \widehat{\Omega}^{\cdot}_{\mathfrak{X}/k}$. When $X \subset \mathfrak{X}$ is a smooth formal embedding (see Definition 6.5) we prove that the cohomology modules $\mathrm{H}^q(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$ are independent of $\mathfrak{X}$. This is done by analyzing the $E_1$ term of the *niveau spectral sequence* converging to $\mathrm{H}^{\cdot}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$ (Theorem 6.16). Here we assume $\operatorname{char} k = 0$. The upshot is that $\mathrm{H}^q(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) = \mathrm{H}^{\mathrm{DR}}_{-q}(X)$, the De Rham homology. There is an advantage in using smooth formal embeddings. If $U \to X$ is any étale morphism, then there is an étale morphism $\mathfrak{U} \to \mathfrak{X}$ s.t. $U = \mathfrak{U} \times_{\mathfrak{X}} X$, so $U \subset \mathfrak{U}$ is a smooth formal embedding. From this we conclude that $\mathrm{H}^{\mathrm{DR}}_{\cdot}(-)$ is a contravariant functor on $X_{\mathrm{et}}$, the small étale site. Previously it was only known that $\mathrm{H}^{\mathrm{DR}}_{\cdot}(-)$ is contravariant for open immersions (cf. [BlO] Example 2.2).

Suppose $X$ is smooth, and let $\mathcal{H}^p_{\mathrm{DR}}$ be the sheafification of the presheaf $U \mapsto \mathrm{H}^p_{\mathrm{DR}}(U)$ on $X_{\mathrm{Zar}}$. Bloch-Ogus [BlO] give a flasque resolution of $\mathcal{H}^p_{\mathrm{DR}}$, the *arithmetic resolution*. It involves the sheaves $i_{x*}\mathrm{H}^q\Omega^{\cdot}_{k(x)/k}$ where $i_x : \{x\} \to X$ is the inclusion map. Our analysis of the niveau spectral sequence shows that the coboundary operator of this resolution is a sum of Parshin residues (Corollary 6.24).

Our final application of the new construction of the residue complex is to describe the *intersection cohomology $\mathcal{D}$-module* $\mathcal{L}(X,Y)$, when $X$ is an integral curve embedded in a smooth $n$-dimensional variety $Y$ (see [BrK]). Again we assume $k$ has characteristic 0. In fact we are able to describe all coherent $\mathcal{D}_Y$-submodules of $\mathcal{H}^{n-1}_X \mathcal{O}_Y$ in terms of the singularities of $X$ (Corollary 7.6). This description is an algebraic version of Vilonen's work in [Vi], replacing complex analysis with BCAs and algebraic residues. It is our hope that a similar description will be found in the general case, namely $\dim X > 1$. Furthermore, we hope to give in the future an explicit description of the Cousin complex of $\mathrm{DR}\,\mathcal{L}(X,Y) = \Omega^{\cdot}_{Y/k} \otimes \mathcal{L}(X,Y)$. Note that for $X = Y$ one has $\mathcal{L}(X,Y) = \mathcal{O}_X$, so this Cousin complex is nothing but $\mathcal{F}^{\cdot}_X$.

**Acknowledgments.** I wish to thank J. Lipman and S. Kleiman for their continued interest in this work. Thanks also to P. Sastry, R. Hübl, V. Hinich, V. Berkovich, H. Esnault, K. Smith and K. Vilonen for helpful conversations, and thanks to the referee for valuable advice on Sections 4 and 7.

## 1. Review of Beilinson Completion Algebras

Let us begin by reviewing some facts about Topological Local Fields (TLFs) and Beilinson Completion Algebras (BCAs) from the papers [Ye1] and [Ye2].

A semi-topological (ST) ring is a ring $A$, with a linear topology on its underlying additive group, such that for every $a \in A$ the multiplication (on either side) $a : A \to A$ is continuous.

Let $K$ be a field. We say $K$ is an $n$-dimensional local field if there is a sequence of complete discrete valuation rings $\mathcal{O}_1, \ldots, \mathcal{O}_n$, where the fraction field of $\mathcal{O}_1$ is $K$, and the residue field of $\mathcal{O}_i$ is the fraction field of $\mathcal{O}_{i+1}$.

Fix a perfect field $k$. A *topological local field* of dimension $n$ over $k$ is a $k$-algebra $K$ with structures of semi-topological ring and $n$-dimensional local field, satisfying the following parameterization condition: there exists an isomorphism of $k$-algebras $K \cong F((s_1, \ldots, s_n))$ for some field $F$, finitely generated over $k$, which respects the two structures. Here $F((s_1, \ldots, s_n)) = F((s_n)) \cdots ((s_1))$ is the field of iterated Laurent series, with its inherent topology and valuation rings ($F$ is discrete). One should remark that for $n = 1$ we are in the classical situation, whereas for $n \geq 2$, $F((s_1, \ldots, s_n))$ is not a topological ring.

TLFs make up a category $\mathsf{TLF}(k)$, where a morphism $K \to L$ is a continuous $k$-algebra homomorphism which preserves the valuations, and the induced homomorphism of the last residue fields is finite. Write $\Omega^{\mathrm{sep},\cdot}_{K/k}$ for the separated algebra of differentials; with the parameterization above $\Omega^{\mathrm{sep},\cdot}_{K/k} \cong K \otimes_F \Omega^{\cdot}_{F[\underline{s}]/k}$. Then there is a functorial residue map $\mathrm{Res}_{L/K} : \Omega^{\mathrm{sep},\cdot}_{L/k} \to \Omega^{\mathrm{sep},\cdot}_{K/k}$ which is $\Omega^{\mathrm{sep},\cdot}_{K/k}$-linear and lowers degree by $\dim L/K$. For instance if $L = K((t))$ then

$$\mathrm{Res}_{L/K} \left( \sum_i t^i \mathrm{d}t \wedge \alpha_i \right) = \alpha_{-1} \in \Omega^{\mathrm{sep},\cdot}_{K/k}. \tag{1.1}$$

TLFs and residues were initially developed by Parshin and Lomadze, and the theory was enhanced in [Ye1].

The notion of a *Beilinson completion algebra* was introduced in [Ye2]. A BCA is a semi-local, semi-topological $k$-algebra, each of whose residue fields $A/\mathfrak{m}$ is a topological local field. Again there is a parameterization condition: when $A$ is local, there should exist a surjection

$$F((\underline{s}))[[\underline{t}]] = F((\underline{s}))[[t_1, \ldots, t_m]] \twoheadrightarrow A$$

which is strict topologically (i.e. $A$ has the quotient topology) and respects the valuations. Here $F((\underline{s}))$ is as above and $F((\underline{s}))[[\underline{t}]]$ is the ring of formal power series over $F((\underline{s}))$. The notion of BCA is an abstraction of the algebra gotten by Beilinson's completion, cf. Lemma 1.5.

There are two distinguished kinds of homomorphisms between BCAs. The first kind is a *morphism of BCAs* $f : A \to B$ (see [Ye2] Definition 2.5), and the category they constitute is denoted $\mathsf{BCA}(k)$. A morphism is continuous, respects the valuations on the residue fields, but in general is not a local homomorphism. For instance, the homomorphisms $k \to k[[s,t]] \to k((s))[[t]] \to k((s))((t))$ are all morphisms. $\mathsf{TLF}(k)$ is a full subcategory of $\mathsf{BCA}(k)$, consisting of those BCAs which are fields.

The second kind of homomorphism is an *intensification homomorphism* $u : A \to \widehat{A}$ (see [Ye2] Definition 3.6). An intensification is flat, topologically étale (relative to $k$) and unramified (in the appropriate sense). It can be viewed as a sort of localization or completion. Here examples are $k(s)[[t]] \to k((s))[[t]]$ and $k(s,t) \to k(s)((t)) \to k((s))((t))$.

Suppose $f : A \to B$ is a morphism of BCAs and $u : A \to \widehat{A}$ is an intensification. The Intensification Base Change Theorem ([Ye2] Theorem 3.8) says there is a BCA $\widehat{B} = B \otimes_A^{(\wedge)} \widehat{A}$, a morphism $\widehat{f} : \widehat{A} \to \widehat{B}$ and an intensification $v : B \to \widehat{B}$, with $vf = \widehat{f}u$. These are determined up to isomorphism and satisfy certain universal properties. For instance, $k((s))[[t]] = k(s)[[t]] \otimes_{k(s)}^{(\wedge)} k((s))$.

According to [Ye2] Theorem 6.14, every $A \in \mathsf{BCA}(k)$ has a dual module $\mathcal{K}(A)$. The module $\mathcal{K}(A)$ is a ST $A$-module. Algebraically it is an injective hull of $A/\mathfrak{r}$, where $\mathfrak{r}$ is the Jacobson radical. $\mathcal{K}(A)$ is also a right $\mathcal{D}(A)$-module, where $\mathcal{D}(A)$ denotes the ring of continuous differential operators of $A$ (relative to $k$). For a ST $A$-module $M$ let $\operatorname{Dual}_A M := \operatorname{Hom}_A^{\mathrm{cont}}(M, \mathcal{K}(A))$.

The dual modules have variance properties w.r.t. morphisms and intensifications. Given a morphism of BCAs $f : A \to B$, according to [Ye2] Theorem 7.4 there is an $A$-linear map $\operatorname{Tr}_f : \mathcal{K}(B) \to \mathcal{K}(A)$. This induces an isomorphism $\mathcal{K}(B) \cong \operatorname{Hom}_A^{\mathrm{cont}}(B, \mathcal{K}(A))$. Given an intensification homomorphism $u : A \to \widehat{A}$, according to [Ye2] Proposition 7.2 there is an $A$-linear map $\mathrm{q}_u : \mathcal{K}(A) \to \mathcal{K}(\widehat{A})$. It induces an isomorphism $\mathcal{K}(\widehat{A}) \cong \widehat{A} \otimes_A \mathcal{K}(A)$. Furthermore Tr and q commute across intensification base change: $\operatorname{Tr}_{\widehat{B}/\widehat{A}} \mathrm{q}_{B/\widehat{B}} = \mathrm{q}_{\widehat{A}/A} \operatorname{Tr}_{B/A}$.

In case of a TLF $K$, one has $\mathcal{K}(K) = \omega(K) = \Omega^{p,\mathrm{sep}}_{K/k}$, where $p = \operatorname{rank} \Omega^{1,\mathrm{sep}}_{K/k}$. For a morphism of TLFs $f : K \to L$ one has $\operatorname{Tr}_f = \operatorname{Res}_f$, whereas for an intensification $u : K \to \widehat{K}$ the homomorphism $\mathrm{q}_u : \Omega^{p,\mathrm{sep}}_{K/k} \to \Omega^{p,\mathrm{sep}}_{\widehat{K}/k}$ is the canonical inclusion for a topologically étale extension of fields.

**Example 1.2.** Take $L := k(s,t)$, $\widehat{L} := k(s)((t))$, $A := k(s)[[t]]$, $\widehat{A} := k((s))[[t]]$, $K := k(s)$ and $\widehat{K} := k((s))$. The inclusions $L \to \widehat{L}$, $K \to \widehat{K}$ and $A \to \widehat{A}$ are intensifications, whereas $K \to A \to \widehat{L}$ and $\widehat{K} \to \widehat{A}$ are morphisms. Using the isomorphism $\mathcal{K}(A) \cong \operatorname{Hom}_K^{\mathrm{cont}}(A, \Omega^{1,\mathrm{sep}}_{K/k})$ induced by $\operatorname{Tr}_{A/K}$, we see that for $\alpha \in \Omega^{2,\mathrm{sep}}_{\widehat{L}/k}$ the element $\operatorname{Tr}_{\widehat{L}/A}(\alpha) \in \mathcal{K}(A)$ is represented by the functional $a \mapsto \operatorname{Res}_{\widehat{L}/K}(a\alpha)$, $a \in A$. Also for $\phi \in \mathcal{K}(A)$ the element $\widehat{\phi} = \mathrm{q}_{\widehat{A}/A}(\phi) \in \mathcal{K}(\widehat{A})$ is represented by the unique $\widehat{K}$-linear functional $\widehat{\phi} : \widehat{A} \to \Omega^{1,\mathrm{sep}}_{\widehat{K}/k}$ extending $\phi$.

**Remark 1.3.** The proof of existence of dual modules with their variance properties in [Ye2] is straightforward, using Taylor series expansions, differential operators and the residue pairing.

Let $A$ be a noetherian ring and $\mathfrak{p}$ a prime ideal. Consider the exact functor on $A$-modules $M \mapsto M_{(\mathfrak{p})} := \widehat{A}_{\mathfrak{p}} \otimes_A M$. For $M$ finitely generated we have $M_{(\mathfrak{p})} \cong \lim_{\leftarrow i} M_{\mathfrak{p}}/\mathfrak{p}^i M_{\mathfrak{p}}$, and if $M = \lim_{\alpha \to} M_\alpha$, then $M_{(\mathfrak{p})} \cong \lim_{\alpha\to}(M_\alpha)_{(\mathfrak{p})}$. This was generalized by Beilinson (cf. [Be]) as follows.

**Definition 1.4.** Let $M$ be an $A$-module and let $\xi = (\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ be a chain of prime ideals, namely $\mathfrak{p}_i \subset \mathfrak{p}_{i+1}$. Define the *Beilinson completion* $M_\xi$ by recursion on $n$, $n \geq -1$.

1. If $n = -1$ (i.e. $\xi = \emptyset$), let $M_\xi := M$ with the discrete topology.
2. If $n \geq 0$ and $M$ is finitely generated, let
$$M_{(\mathfrak{p}_0,\ldots,\mathfrak{p}_n)} := \lim_{\leftarrow i} (M_{\mathfrak{p}_0}/\mathfrak{p}_0^i M_{\mathfrak{p}_0})_{(\mathfrak{p}_1,\ldots,\mathfrak{p}_n)}.$$
3. For arbitrary $M$, let $\{M_\alpha\}$ be the set of finitely generated submodules of $M$, and let
$$M_{(\mathfrak{p}_0,\ldots,\mathfrak{p}_n)} := \lim_{\alpha\to} (M_\alpha)_{(\mathfrak{p}_0,\ldots,\mathfrak{p}_n)}.$$

A chain $\xi = (\mathfrak{p}_0, \ldots, \mathfrak{p}_n)$ is *saturated* if $\mathfrak{p}_{i+1}$ has height 1 in $A/\mathfrak{p}_i$.

**Lemma 1.5.** *If $\xi = (\mathfrak{p}, \ldots)$ is a saturated chain then $A_\xi$ is a Beilinson completion algebra.*

*Proof.* By [Ye1] Corollary 3.3.5, $A_\xi$ is a complete semi-local noetherian ring with Jacobson radical $\mathfrak{p}_\xi$, and $A_\xi/\mathfrak{p}_\xi = K_\xi$, where $K := A_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$. Choose a coefficient field $\sigma : K \to \widehat{A}_{\mathfrak{p}} = A_{(\mathfrak{p})}$. By [Ye1] Proposition 3.3.6, $K_\xi$ is a finite product of TLFs, and by the proof of [Ye1] Theorem 3.3.8, $\sigma$ extends to a lifting $\sigma_\xi : K_\xi \to A_\xi$. Sending $t_1, \ldots, t_m$ to generators of the ideal $\mathfrak{p}$, we get a topologically strict surjection $K_\xi[[t_1, \ldots, t_m]] \twoheadrightarrow A_\xi$. □

## 2. Construction of the Residue Complex $\mathcal{K}^{\cdot}_X$

Let $k$ be a perfect field, and let $X$ be a scheme of finite type over $k$. By a *chain of points* in $X$ we mean a sequence $\xi = (x_0, \ldots, x_n)$ of points with $x_{i+1} \in \overline{\{x_i\}}$.

**Definition 2.1.** For any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$, define the Beilinson completion $\mathcal{M}_\xi$ by taking an affine open neighborhood $U = \operatorname{Spec} A \subset X$ of $x_n$, and setting $\mathcal{M}_\xi := \Gamma(U, \mathcal{M})_\xi$ as in Definition 1.4.

These completions first appeared as the local factors of Beilinson's adeles in [Be], and were studied in detail in [Ye1].

According to Lemma 1.5, if $\xi = (x_0, \ldots, x_n)$ is saturated, i.e. $\overline{\{x_{i+1}\}} \subset \overline{\{x_i\}}$ has codimension 1, then $\mathcal{O}_{X,\xi}$ is a BCA. We shall be interested in the covertex maps
$$\begin{aligned} \partial^- : \quad & \mathcal{O}_{X,(x_0)} \to \mathcal{O}_{X,\xi} \\ \partial^+ : \quad & \mathcal{O}_{X,(x_n)} \to \mathcal{O}_{X,\xi} \end{aligned}$$

which arise naturally from the completion process (cf. [Ye1] §3.1).

**Lemma 2.2.** *$\partial^+$ is flat, topologically étale relative to $k$, and a morphism in $\mathsf{BCA}(k)$. $\partial^-$ is an intensification homomorphism.*

*Proof.* By definition $\partial^- = \partial^1 \circ \cdots \circ \partial^n$ and $\partial^+ = \partial^0 \circ \cdots \circ \partial^0$, where $\partial^i : \mathcal{O}_{X,\partial_i \xi} \to \mathcal{O}_{X,\xi}$ is the $i$-th coface operator. First let us prove that $\partial^0 : \mathcal{O}_{X,\partial_0 \xi} \to \mathcal{O}_{X,\xi}$ is a morphism of BCAs. This follows from [Ye1] Theorem 3.3.2 (d), since we may assume that $X$ is integral with generic point $x_0$. By part (b) of the same theorem, $\partial^n : \mathcal{O}_{X,\partial_n \xi} \to \mathcal{O}_{X,\xi}$ is finitely ramified and radically unramified (in the sense of [Ye2] Definition 3.1).

Now according to [Ye1] Corollary 3.2.8, $\partial^i : \mathcal{O}_{X,\partial_i \xi} \to \mathcal{O}_{X,\xi}$ is topologically étale relative to $k$, for any $i$. We claim it is also flat. For $i = 0$, $\mathcal{O}_{X,\partial_0 \xi} \to (\mathcal{O}_{X,\partial_0 \xi})_{x_0} = (\mathcal{O}_{X,x_0})_{\partial_0 \xi}$ is a localization, so it's flat. The map from $(\mathcal{O}_{X,x_0})_{\partial_0 \xi}$ to its $\mathfrak{m}_{x_0}$-adic completion $\mathcal{O}_{X,\xi}$ is also flat (these rings are noetherian). For $i > 0$, by induction on the length of chains, $\mathcal{O}_{X,\partial_0 \partial_i \xi} \to \mathcal{O}_{X,\partial_0 \xi}$ is flat, and hence so is $(\mathcal{O}_{X,x_0})_{\partial_0 \partial_i \xi} \to (\mathcal{O}_{X,x_0})_{\partial_0 \xi}$. Now use [CA] Chapter III §5.4 Proposition 4 to conclude that

$$\mathcal{O}_{X,\partial_i \xi} = \lim_{\leftarrow j} (\mathcal{O}_{X,x_0}/\mathfrak{m}_{x_0}^j)_{\partial_0 \partial_i \xi} \to \lim_{\leftarrow j} (\mathcal{O}_{X,x_0}/\mathfrak{m}_{x_0}^j)_{\partial_0 \xi} = \mathcal{O}_{X,\xi}$$

is flat. □

**Example 2.3.** Take $X = \mathbf{A}^2 := \operatorname{Spec} k[s,t]$, $x := (0)$, $y := (t)$ and $z := (s,t)$. Then with $L := k(x) = \mathcal{O}_{X,(x)}$, $\widehat{L} := k(x)_{(y)} = \mathcal{O}_{X,(x,y)}$, $A := \mathcal{O}_{X,(y)}$, $\widehat{A} := \mathcal{O}_{X,(y,z)}$ $K := k(y)$ and $\widehat{K} := k(y)_{(z)}$ we are exactly in the situation of Example 1.2.

**Definition 2.4.** Given a point $x$ in $X$, let $\mathcal{K}_X(x)$ be the skyscraper sheaf with support $\overline{\{x\}}$ and group of sections $\mathcal{K}(\mathcal{O}_{X,(x)})$.

The sheaf $\mathcal{K}_X(x)$ is a quasi-coherent $\mathcal{O}_X$-module, and is an injective hull of $k(x)$ in the category $\mathsf{Mod}(X)$ of $\mathcal{O}_X$-modules.

**Definition 2.5.** Given a saturated chain $\xi = (x, \ldots, y)$ in $X$, define an $\mathcal{O}_X$-linear homomorphism $\delta_\xi : \mathcal{K}_X(x) \to \mathcal{K}_X(y)$, called the coboundary map along $\xi$, by

$$\delta_\xi : \mathcal{K}(\mathcal{O}_{X,(x)}) \xrightarrow{\mathrm{q}_{\partial^-}} \mathcal{K}(\mathcal{O}_{X,\xi}) \xrightarrow{\mathrm{Tr}_{\partial^+}} \mathcal{K}(\mathcal{O}_{X,(y)}).$$

Throughout sections 2 and 3 the following convention shall be used. Let $f : X \to Y$ be a morphism of schemes, and let $x \in X$, $y \in Y$ be points. We will write $x \mid y$ to indicate that $x$ is a closed point in the fiber $X_y := X \times_Y \operatorname{Spec} k(y) \cong f^{-1}(y)$. Similarly for chains: we write $(x_0, \ldots, x_n) \mid (y_0, \ldots, y_n)$ if $x_i \mid y_i$ for all $i$.

**Lemma 2.6.** *Suppose $x \mid y$. Then $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is a morphism of BCAs. If $f$ is quasi-finite then $f^*$ is finite, and if $f$ is étale then $f^*$ is an intensification.*

*Proof.* Immediate from the definitions. □

**Lemma 2.7.** *Suppose $f : X \to Y$ is a quasi-finite morphism. Let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$ and let $x \in X$, $x \mid y_n$. Consider the (finite) set of chains in $X$:*

$$\Xi := \{\xi = (x_0, \ldots, x_n) \mid \xi \mid \eta \text{ and } x_n = x\}.$$

*Then there is a canonical isomorphism of BCAs*

$$\prod_{\xi \in \Xi} \mathcal{O}_{X,\xi} \cong \mathcal{O}_{Y,\eta} \otimes_{\mathcal{O}_{Y,(y_n)}} \mathcal{O}_{X,(x)}.$$

*Proof.* Set $\widehat{X} := \operatorname{Spec} \mathcal{O}_{X,(x)}$ and $\widehat{Y} := \operatorname{Spec} \mathcal{O}_{Y,(y_n)}$, so the induced morphism $\widehat{f} : \widehat{X} \to \widehat{Y}$ is finite. Let us denote by $\xi, \widehat{\xi}, \widehat{\eta}$ variable saturated chains in $X, \widehat{X}, \widehat{Y}$ respectively. For any $\widehat{\eta} \mid \eta$ one has

$$\prod_{\widehat{\xi} \mid \widehat{\eta}} \mathcal{O}_{\widehat{X},\widehat{\xi}} \cong \mathcal{O}_{\widehat{Y},\widehat{\eta}} \otimes_{\mathcal{O}_{Y,(y_n)}} \mathcal{O}_{X,(x)}, \tag{2.8}$$

by [Ye1] Proposition 3.2.3; note that the completion is defined on any noetherian scheme. Now from ibid. Corollary 3.3.13 one has $\mathcal{O}_{X,\xi} \cong \prod_{\widehat{\xi} \mid \xi} \mathcal{O}_{\widehat{X},\widehat{\xi}}$, so taking the product over all $\xi \in \Xi$ and $\widehat{\eta} \mid \eta$ the lemma is proved. $\square$

**Definition 2.9.** Given an étale morphism $g : X \to Y$ and a point $x \in X$, let $y := g(x)$, so $g^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is an intensification. Define

$$\mathrm{q}_g : \mathcal{K}_Y(y) \to g_* \mathcal{K}_X(x)$$

to be the $\mathcal{O}_Y$-linear homomorphism corresponding to $\mathrm{q}_{g^*} : \mathcal{K}(\mathcal{O}_{Y,(y)}) \to \mathcal{K}(\mathcal{O}_{X,(x)})$ of [Ye2] Proposition 7.2.

**Proposition 2.10.** *Let $g : X \to Y$ be an étale morphism.*

(a) *Given a point $y \in Y$, the homomorphism $1 \otimes \mathrm{q}_g : g^* \mathcal{K}_Y(y) \to \bigoplus_{x \mid y} \mathcal{K}_X(x)$ is an isomorphism of $\mathcal{O}_X$-modules.*
(b) *Let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$. Then*

$$(1 \otimes \mathrm{q}_g) \circ g^*(\delta_\eta) = (\sum_{\xi \mid \eta} \delta_\xi) \circ (1 \otimes \mathrm{q}_g)$$

*as homomorphisms $g^* \mathcal{K}_Y(y_0) \to \bigoplus_{x_n \mid y_n} \mathcal{K}_X(x_n)$.*

*Proof.* (a) Because $\mathcal{K}_Y(y)$ is an artinian $\mathcal{O}_{Y,y}$-module, and $g$ is quasi-finite, we find that

$$g^* \mathcal{K}_Y(y) = \bigoplus_{x \mid y} \mathcal{O}_{X,(x)} \otimes_{\mathcal{O}_{Y,(y)}} \mathcal{K}_Y(y) \ .$$

Now use [Ye2] Proposition 7.2 (i).

(b) From Lemma 2.7 and from [Ye2] Theorem 3.8 we see that for every $\xi \mid \eta$, $f^* : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ is both a finite morphism and an intensification. By the definition of the coboundary maps, it suffices to verify that the diagram

$$\begin{array}{ccccc}
\mathcal{K}(\mathcal{O}_{Y,(y_0)}) & \xrightarrow{\mathrm{q}} & \mathcal{K}(\mathcal{O}_{Y,\eta}) & \xrightarrow{\mathrm{Tr}} & \mathcal{K}(\mathcal{O}_{Y,(y_n)}) \\
{\scriptstyle \mathrm{q}}\downarrow & & {\scriptstyle \mathrm{q}}\downarrow & & {\scriptstyle \mathrm{q}}\downarrow \\
\bigoplus_{x_0|y_0} \mathcal{K}(\mathcal{O}_{X,(x_0)}) & \xrightarrow{\mathrm{q}} & \bigoplus_{\xi|\eta} \mathcal{K}(\mathcal{O}_{X,\xi}) & \xrightarrow{\mathrm{Tr}} & \bigoplus_{x_n|y_n} \mathcal{K}(\mathcal{O}_{X,(x_n)})
\end{array}$$

commutes. The left square commutes by [Ye2] Proposition 7.2 (iv), whereas the right square commutes by Lemma 2.7 and [Ye2] Theorem 7.4 (ii). □

**Definition 2.11.** Let $f : X \to Y$ be a morphism between finite type $k$-schemes, let $x \in X$ be a point, and let $y := f(x)$. Define an $\mathcal{O}_Y$-linear homomorphism

$$\mathrm{Tr}_f : f_* \mathcal{K}_X(x) \to \mathcal{K}_Y(y)$$

as follows:

(i) If $x$ is closed in its fiber $X_y$, then $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is a morphism in $\mathsf{BCA}(k)$. Let $\mathrm{Tr}_f$ be the homomorphism corresponding to $\mathrm{Tr}_{f^*} : \mathcal{K}(\mathcal{O}_{X,(x)}) \to \mathcal{K}(\mathcal{O}_{Y,(y)})$ of [Ye2] Theorem 7.4.

(ii) If $x$ is not closed in its fiber, set $\mathrm{Tr}_f := 0$.

**Proposition 2.12.** *Let $f : X \to Y$ be a finite morphism.*

(a) *For any $y \in Y$ the homomorphism of $\mathcal{O}_Y$-modules*

$$\bigoplus_{x|y} f_* \mathcal{K}_X(x) \to \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{K}_Y(y))$$

*induced by $\mathrm{Tr}_f$ is an isomorphism.*

(b) *Let $\eta = (y_0, \dots, y_n)$ be a saturated chain in $Y$. Then*

$$\delta_\eta \, \mathrm{Tr}_f = \mathrm{Tr}_f \sum_{\xi|\eta} f_*(\delta_\xi) : \bigoplus_{x_0|y_0} f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_n).$$

*The sums are over saturated chains $\xi = (x_0, \dots, x_n)$ in $X$.*

*Proof.* (a) By [Ye1] Proposition 3.2.3 we have $\prod_{x|y} \mathcal{O}_{X,(x)} \cong (f_* \mathcal{O}_X)_{(y)}$. Now use [Ye2] Theorem 7.4 (iv).

(b) Use the same diagram which appears in the proof of Proposition 2.10, only reverse the vertical arrows and label them $\mathrm{Tr}_f$. Then the commutativity follows from [Ye2] Theorem 7.4 (i), (ii). □

In [Ye1] §4.3 the notion of a system of residue data on a reduced scheme was introduced.

**Proposition 2.13.** *Suppose $X$ is a reduced scheme. Then $(\{\mathcal{K}_X(x)\}, \{\delta_\xi\}, \{\Psi_\sigma^{-1}\})$, where $x$ runs over the points of $X$, $\xi$ runs over the saturated chains in $X$, and $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ runs over all possible coefficient fields, is a system of residue data on $X$.*

*Proof.* We must check condition (†) of [Ye1] Definition 4.3.10. So let $\xi = (x, \ldots, y)$ be a saturated chain, and let $\sigma : k(x) \to \mathcal{O}_{X,(x)}$ and $\tau : k(y) \to \mathcal{O}_{X,(y)}$ be compatible coefficient fields. Denote also by $\tau$ the composed morphism $\partial^{+}\tau : k(y) \to \mathcal{O}_{X,\xi}$. Then we get a coefficient field $\sigma_\xi : k(\xi) = k(x)_\xi \to \mathcal{O}_{X,\xi}$ extending $\sigma$, which is a $k(y)$-algebra map via $\tau$. Consider the diagram:

$$\begin{array}{ccccccc} \mathcal{K}(\mathcal{O}_{X,(x)}) & \xrightarrow{\mathrm{q}} & \mathcal{K}(\mathcal{O}_{X,\xi}) & \xrightarrow{=} & \mathcal{K}(\mathcal{O}_{X,\xi}) & \xrightarrow{\mathrm{Tr}} & \mathcal{K}(\mathcal{O}_{X,(y)}) \\ \Psi_\sigma \big\downarrow & & \Psi_{\sigma_\xi} \big\downarrow & & \Psi_\tau \big\downarrow & & \Psi_\tau \big\downarrow \\ \mathrm{Dual}_\sigma \, \mathcal{O}_{X,(x)} & \xrightarrow{\mathrm{q}_\sigma} & \mathrm{Dual}_{\sigma_\xi} \, \mathcal{O}_{X,\xi} & \xrightarrow{h} & \mathrm{Dual}_\tau \, \mathcal{O}_{X,\xi} & \xrightarrow{\mathrm{Tr}_\tau} & \mathrm{Dual}_\tau \, \mathcal{O}_{X,(y)} \end{array}$$

where for a $k(\xi)$-linear homomorphism $\phi : \mathcal{O}_{X,\xi} \to \omega(k(\xi))$, $h(\phi) = \mathrm{Res}_{k(\xi)/k(y)} \phi$ (cf. [Ye2] Theorem 6.14). The left square commutes by [Ye2] Proposition 7.2 (iii); the middle square commutes by ibid. Theorem 6.14 (i); and the right square commutes by ibid. Theorem 7.4 (i), (iii). But going along the bottom of the diagram we get $\mathrm{Tr}_\tau \, h \mathrm{q}_\sigma = \delta_{\xi,\sigma/\tau}$, as defined in [Ye1] Lemma 4.3.3. □

**Lemma 2.14.** *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains in the scheme $X$, and let $\xi \vee \partial_0 \eta := (x, \ldots, y, \ldots, z)$ be the concatenated chain. Then there is a canonical isomorphism of BCAs*

$$\mathcal{O}_{X,\xi\vee\partial_0\eta} \cong \mathcal{O}_{X,\xi} \otimes^{(\wedge)}_{\mathcal{O}_{X,(y)}} \mathcal{O}_{X,\eta}$$

(*intensification base change*).

*Proof.* Choose a coefficient field $\sigma : k(y) \to \mathcal{O}_{X,(y)}$. This induces a coefficient ring $\sigma_\eta : k(\eta) \to \mathcal{O}_{X,\eta}$, and using [Ye2] Theorem 3.8 and [Ye1] Theorem 4.1.12 one gets

$$\mathcal{O}_{X,\xi\vee\partial_0\eta} \cong \mathcal{O}_{X,\xi} \otimes^{(\wedge)}_{k(y)} k(\eta) \cong \mathcal{O}_{X,\xi} \otimes^{(\wedge)}_{\mathcal{O}_{X,(y)}} \mathcal{O}_{X,\eta}.$$

□

**Lemma 2.15.** 1. *Let $\xi = (x, \ldots, y)$ and $\eta = (y, \ldots, z)$ be saturated chains and let $\xi \vee \partial_0 \eta = (x, \ldots, y, \ldots, z)$. Then $\delta_\eta \delta_\xi = \delta_{\xi\vee\partial_0\eta}$.*
2. *Given a point $x \in X$ and an element $\alpha \in \mathcal{K}_X(x)$, for all but finitely many saturated chains $\xi = (x, \ldots)$ in $X$ one has $\delta_\xi(\alpha) = 0$.*
3. (*Residue Theorem*) *Let $x, z \in X$ be points s.t. $z \in \overline{\{x\}}$ and* $\mathrm{codim}(\overline{\{z\}}, \overline{\{x\}}) = 2$. *Then $\sum_y \ \delta_{(x,y,z)} = 0$.*

*Proof.* Using Lemma 2.14 we see that part 1 is a consequence of the base change property of traces, cf. [Ye2] Theorem 7.4 (ii). Assertions 2 and 3 are local, by Proposition 2.10, so we may assume there is a closed immersion $f : X \to \mathbf{A}^n_k$. By Proposition 2.12, we can replace $X$ with $\mathbf{A}^n_k$, and so assume that $X$ is reduced. Now according to Proposition 2.13 and [Ye1] Lemma 4.3.19, both 2 and 3 hold. □

**Definition 2.16.** For any integer $q$ define a quasi-coherent sheaf

$$\mathcal{K}_X^q := \bigoplus_{\dim \overline{\{x\}} = -q} \mathcal{K}_X(x).$$

By Lemma 2.15 there an $\mathcal{O}_X$-linear homomorphism

$$\delta := (-1)^{q+1} \sum_{(x,y)} \delta_{(x,y)} : \mathcal{K}_X^q \to \mathcal{K}_X^{q+1},$$

satisfying $\delta^2 = 0$. The complex $(\mathcal{K}_X^{\cdot}, \delta)$ is called the *Grothendieck residue complex* of $X$.

In Corollary 3.8 we will prove that $\mathcal{K}_X^{\cdot}$ is canonically isomorphic (in the derived category $\mathsf{D}(X)$) to $\pi^! k$, where $\pi : X \to \operatorname{Spec} k$ is the structural morphism.

**Remark 2.17.** A heuristic for the negative grading of $\mathcal{K}_X^{\cdot}$ and the sign $(-1)^{q+1}$ is that the residue complex is the "$k$-linear dual" of a hypothetical "complex of localizations" $\cdots \prod \mathcal{O}_{X,y} \to \prod \mathcal{O}_{X,x} \to \cdots$. Actually, there is a naturally defined complex which is built up from all localizations and completions: the Beilinson adeles $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ (cf. [Be] and [HY1]). $\underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\mathcal{O}_X)$ is a DGA, and $\mathcal{K}_X^{\cdot}$ is naturally a right DG-module over it. See [Ye4], and cf. also Remark 6.22.

**Definition 2.18.**
1. Let $f : X \to Y$ be a morphism of schemes. Define a homomorphism of graded $\mathcal{O}_Y$-modules $\operatorname{Tr}_f : f_* \mathcal{K}_X^{\cdot} \to \mathcal{K}_Y^{\cdot}$ by summing the local trace maps of Definition 2.11.
2. Let $g : U \to X$ be an étale morphism. Define $\mathrm{q}_g : \mathcal{K}_X^{\cdot} \to g_* \mathcal{K}_U^{\cdot}$ by summing all local homomorphisms $\mathrm{q}_g$ of Definition 2.9.

**Theorem 2.19.** *Let $X$ be a $k$-scheme of finite type.*

(a) *$(\mathcal{K}_X^{\cdot}, \delta)$ is a residual complex on $X$ (cf.* [RD] *Chapter* VI §1*).*

(b) *If $g : U \to X$ is an étale morphism, then $1 \otimes \mathrm{q}_g : g^* \mathcal{K}_X^{\cdot} \to \mathcal{K}_U^{\cdot}$ is an isomorphism of complexes.*

(c) *If $f : X \to Y$ is a finite morphism, then $\operatorname{Tr}_f : f_* \mathcal{K}_X^{\cdot} \to \mathcal{K}_Y^{\cdot}$ is a homomorphism of complexes, and the induced map*

$$f_* \mathcal{K}_X^{\cdot} \to \mathcal{H}om_{\mathcal{O}_Y}(f_* \mathcal{O}_X, \mathcal{K}_Y^{\cdot})$$

*is an isomorphism of complexes.*

(d) *If $X$ is reduced, then $(\mathcal{K}_X^{\cdot}, \delta)$ is canonically isomorphic to the complex constructed in* [Ye1] §4.3*. In particular, if $X$ is smooth irreducible of dimension $n$, there is a quasi-isomorphism $\mathrm{C}_X : \Omega^n_{X/k}[n] \to \mathcal{K}_X^{\cdot}$.*

*Proof.* Parts (b), (c), (d) are immediate consequences of Propositions 2.10, 2.12 and 2.13 here, and [Ye1] Theorem 4.5.2. (Note that the sign of $\delta$ in [Ye1] is different.) As for part (a), clearly $\mathcal{K}_X^{\cdot}$ is a direct sum of injective hulls of all the residue fields in $X$, with multiplicities 1. It remains to prove that $\mathcal{K}_X^{\cdot}$ has coherent cohomology. Since this is a local question, we can assume using part (b) that $X$ is a closed subscheme of $\mathbf{A}^n_k$. According to parts (c) and (d) of this theorem and [Ye1] Corollary 4.5.6, $\mathcal{K}^{\cdot}_X$ has coherent cohomology. □

From part (b) of the theorem we get:

**Corollary 2.20.** *The presheaf $U \mapsto \Gamma(U, \mathcal{K}^{\cdot}_U)$ is a sheaf on $X_{\mathrm{et}}$, the small étale site over $X$.*

**Definition 2.21.** For an $\mathcal{O}_X$-module $\mathcal{M}$ define dual complex

$$\operatorname{Dual}_X \mathcal{M} := \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X).$$

Observe that since $\mathcal{K}^{\cdot}_X$ is complex of injectives the derived functor $\operatorname{Dual}_X : \mathsf{D}(X)^{\circ} \to \mathsf{D}(X)$ is defined. Moreover, since $\mathcal{K}^{\cdot}_X$ is dualizing, the adjunction morphism $1 \to \operatorname{Dual}_X \operatorname{Dual}_X$ is an isomorphism on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. We shall sometimes write $\operatorname{Dual} \mathcal{M}$ instead of $\operatorname{Dual}_X \mathcal{M}$.

## 3. Duality for Proper Morphisms

In this section we prove that if $f : X \to Y$ is a proper morphism of $k$-schemes, then the trace map $\mathrm{Tr}_f$ of Definition 2.11 is a homomorphism of complexes, and it induces a duality in the derived categories.

**Proposition 3.1.** *Let $f : X \to Y$ be a proper morphism between finite type $k$-schemes, and let $\eta = (y_0, \ldots, y_n)$ be a saturated chain in $Y$. Then there exists a canonical isomorphism of BCAs*

$$\prod_{\xi | \eta} \mathcal{O}_{X,\xi} \cong \prod_{x_0 | y_0} \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,\eta} \ ,$$

*where $\xi = (x_0, \ldots, x_n)$ denotes a variable chain in $X$ lying over $\eta$.*

*Proof.* The proof is by induction on $n$. For $n = 0$ this is trivial. Assume $n = 1$. Let $Z := \overline{\{x_0\}}_{\mathrm{red}}$, so $\mathcal{O}_{Z,x_1}$ is a 1-dimensional local ring inside $k(Z) = k(x_0)$. Considering the integral closure of $\mathcal{O}_{Z,x_1}$ we see that $k((x_0, x_1)) = k(x_0)_{(x_1)} = k(x_0) \otimes \mathcal{O}_{Z,(x_1)}$ is the product of the completions of $k(x_0)$ at all discrete valuations centered on $x_1 \in Z$ (cf. [Ye1] Theorem 3.3.2). So by the valuative criterion for properness we get

$$\prod_{(x_0,x_1)|(y_0,y_1)} k(x_0)_{(x_1)} \cong \prod_{x_0|y_0} k(x_0) \otimes_{k(y_0)} k(y_0)_{(y_1)}. \tag{3.2}$$

For $i \geq 1$ the morphism of BCAs

$$\prod_{x_0|y_0} (\mathcal{O}_{X,x_0}/\mathfrak{m}^i_{x_0}) \otimes_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)} \to \prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)}/\mathfrak{m}^i_{(x_0,x_1)}$$

is bijective, since both sides are flat over $\mathcal{O}_{X,x_0}/\mathfrak{m}^i_{x_0}$, and by equation (3.2) (cf. [Ye2] Proposition 3.5). Passing to the inverse limit in $i$ we get an isomorphism of BCAs

$$\prod_{x_0|y_0} \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)} \cong \prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)} \ . \tag{3.3}$$

Now suppose $n \geq 2$. Then we get

$$\begin{aligned}
&\textstyle\prod_{x_0|y_0} \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,\eta} \\
&\qquad \cong \textstyle\prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0,y_1)}} \mathcal{O}_{Y,\eta} && \text{(i)} \\
&\qquad \cong \textstyle\prod_{(x_0,x_1)|(y_0,y_1)} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_1)}} \mathcal{O}_{Y,\partial_0\eta} && \text{(ii)} \\
&\qquad \cong \textstyle\prod_{\xi|\eta} \mathcal{O}_{X,(x_0,x_1)} \otimes^{(\wedge)}_{\mathcal{O}_{X,(x_1)}} \mathcal{O}_{X,\partial_0\xi} && \text{(iii)} \\
&\qquad \cong \textstyle\prod_{\xi|\eta} \mathcal{O}_{X,\xi} && \text{(iv)}
\end{aligned}$$

where associativity of intensification base change ([Ye2] Proposition 3.10) is used repeatedly; in (i) we use formula (3.3); in (ii) we use Lemma 2.14 applied to $\mathcal{O}_{Y,\eta}$; in (iii) we use the induction hypothesis; and (iv) is another application of Lemma 2.14. □

The next theorem is our version of [RD] Ch. VII Theorem 2.1:

**Theorem 3.4.** (Global Residue Theorem) *Let $f : X \to Y$ be a proper morphism between $k$-schemes of finite type. Then $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ is a homomorphism of complexes.*

*Proof.* Fix a point $x_0 \in X$, and let $y_0 := f(x_0)$. First assume that $x_0$ is closed in its fiber $X_{y_0} = f^{-1}(y_0)$. Let $y_1$ be an immediate specialization of $y_0$. By Proposition 3.1 we have

$$\prod_{x_1|y_1} \mathcal{O}_{X,(x_0,x_1)} \cong \mathcal{O}_{X,(x_0)} \otimes^{(\wedge)}_{\mathcal{O}_{Y,(y_0)}} \mathcal{O}_{Y,(y_0,y_1)},$$

so just as in Proposition 2.12 (b), we get

$$\delta_{(y_0,y_1)} \mathrm{Tr}_f = \sum_{x_1|y_1} \mathrm{Tr}_f\, \delta_{(x_0,x_1)} : f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_1).$$

Next assume $x_0, y_0$ are as above, but $x_0$ is not closed in the fiber $X_{y_0}$. The only possibility to have an immediate specialization $x_1$ of $x_0$ which is closed in its fiber, is if $x_1 \in X_{y_0}$ and $Z := \overline{\{x_0\}}_{\mathrm{red}} \subset X_{y_0}$ is a curve. We have to show that

$$\sum_{x_1|y_0} \mathrm{Tr}_f\, \delta_{(x_0,x_1)} = 0 : f_* \mathcal{K}_X(x_0) \to \mathcal{K}_Y(y_0). \tag{3.5}$$

Since $\mathcal{K}_Z(x_0) \subset \mathcal{K}_X(x_0)$ is an essential submodule over $\mathcal{O}_{Y,y_0}$ it suffices to check (3.5) on $\mathcal{K}_Z(x_0)$. Thus we may assume $X = \overline{\{x_0\}}_{\mathrm{red}}$ and $Y = \overline{\{y_0\}}_{\mathrm{red}}$. After factoring $X \to Y$ through a suitable finite radiciel morphism $X \to \tilde{X}$, and using Proposition 2.12, we may further assume that $K = k(Y) \to k(X)$ is separable. Now

$$\mathrm{Tr}_f\, \delta_{(x_0,x_1)} = \mathrm{Res}_{k((x_0,x_1))/K} : \Omega^{n+1}_{k(X)/k} \to \Omega^n_{K/k}.$$

Since $\Omega^{n+1}_{k(X)/k} = \Omega^1_{k(X)/k} \wedge \Omega^n_{K/k}$, it suffices to check that

$$\sum_{x_1 \in X} \operatorname{Res}_{k((x_0,x_1))/K} = 0 : \Omega^1_{k(X)/k} \to K.$$

Let $K'$ be the maximal purely inseparable extension of $K$ in an algebraic closure, and let $X' := X \times_K K'$. So

$$k((x_0,x_1)) \otimes_K K' \cong \prod_{(x'_0,x'_1)|(x_0,x_1)} k((x'_0,x'_1))$$

where $(x'_0,x'_1)$ are chains in $X'$. According to [Ye1] Lemma 2.4.14 we may assume $k = K = K'$. Since now $K$ is perfect, we are in the position to use the well known Residue Theorem for curves (cf. [Ye1] Theorem 4.2.15). □

**Corollary 3.6.** *Let $f : X \to Y$ be a morphism between $k$-schemes of finite type, and let $Z \subset X$ be a closed subscheme which is proper over $Y$. Then* $\operatorname{Tr}_f : f_* \Gamma_Z \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ *is a homomorphism of complexes.*

*Proof.* Let $\mathcal{I} \subset \mathcal{O}_X$ be the ideal sheaf of $Z$, and define $Z_n := \mathbf{Spec}\, \mathcal{O}_X / \mathcal{I}^{n+1}$, $n \geq 0$. The trace maps $\mathcal{K}^{\cdot}_{Z_0} \to \cdots \to \mathcal{K}^{\cdot}_{Z_n} \to \cdots \to \mathcal{K}^{\cdot}_X$ of Proposition 2.12 induce a filtration by subcomplexes $\Gamma_Z \mathcal{K}^{\cdot}_X = \bigcup_{n=0}^{\infty} \mathcal{K}^{\cdot}_{Z_n}$. Now since each morphism $Z_n \to Y$ is proper, $\operatorname{Tr}_f : f_* \mathcal{K}^{\cdot}_{Z_n} \to \mathcal{K}^{\cdot}_Y$ is a homomorphism of complexes. □

**Theorem 3.7.** (Duality) *Let $f : X \to Y$ be a proper morphism between finite type $k$-schemes. Then for any complex $\mathcal{M}^{\cdot} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$, the homomorphism*

$$\operatorname{Hom}_{\mathsf{D}(X)}(\mathcal{M}^{\cdot}, \mathcal{K}^{\cdot}_X) \to \operatorname{Hom}_{\mathsf{D}(Y)}(\mathrm{R} f_* \mathcal{M}^{\cdot}, \mathcal{K}^{\cdot}_Y)$$

*induced by* $\operatorname{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ *is an isomorphism.*

*Proof.* The proof uses a relative version of Sastry's notion of "residue pairs" and "pointwise residue pairs", cf. [Ye1] Appendix. Define a residue pair relative to $f$ and $\mathcal{K}^{\cdot}_Y$, to be a pair $(\mathcal{R}^{\cdot}, t)$, with $\mathcal{R}^{\cdot}$ a residual complex on $X$, and with $t : f_* \mathcal{R}^{\cdot} \to \mathcal{K}^{\cdot}_Y$ a homomorphism of complexes, which represent the functor $\mathcal{M}^{\cdot} \mapsto \operatorname{Hom}_{\mathsf{D}(Y)}(\mathrm{R} f_* \mathcal{M}^{\cdot}, \mathcal{K}^{\cdot}_Y)$ on $\mathsf{D}^{\mathrm{b}}_{\mathrm{c}}(X)$. Such pairs exist; for instance, we may take $\mathcal{R}^{\cdot}$ to be the Cousin complex $f^{\triangle} \mathcal{K}^{\cdot}_Y$ associated to the dualizing complex $f^{!} \mathcal{K}^{\cdot}_Y$ (cf. [RD] ch. VII §3, or ibid. Appendix no. 4).

A pointwise residue pair relative to $f$ and $\mathcal{K}^{\cdot}_Y$, is by definition a pair $(\mathcal{R}^{\cdot}, t)$ as above, but satisfying the condition: for any closed point $x \in X$, and any coherent $\mathcal{O}_X$-module $\mathcal{M}$ supported on $\{x\}$, the map $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{R}^{\cdot}) \to \operatorname{Hom}_{\mathcal{O}_Y}(f_* \mathcal{M}, \mathcal{K}^{\cdot}_Y)$ induced by $t$ is an isomorphism. By the definition of the trace map $\operatorname{Tr}_f$, the pair $(\mathcal{K}^{\cdot}_X, \operatorname{Tr}_f)$ is a pointwise residue pair. In fact, $k \to \mathcal{O}_{Y,(f(x))} \to \mathcal{O}_{X,(x)}$ are morphisms in $\mathsf{BCA}(k)$, and by [Ye2] Theorem 7.4 (i),(iv) we get

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X) \cong \operatorname{Hom}_{\mathcal{O}_Y}(f_* \mathcal{M}, \mathcal{K}^{\cdot}_Y) \cong \operatorname{Hom}_k(\mathcal{M}_x, k).$$

The proof of [Ye1] Appendix Theorem 2 goes through also in the relative situation: the morphism $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$ in $\mathsf{D}(Y)$ corresponds to a morphism $\zeta : \mathcal{K}^{\cdot}_X \to \mathcal{R}^{\cdot}$ in $\mathsf{D}(X)$. But since both $\mathcal{K}^{\cdot}_X$ and $\mathcal{R}^{\cdot}$ are residual complexes, $\zeta$ is an actual, unique homomorphism of complexes (cf. [RD] Ch. IV Lemma 3.2). By testing on $\mathcal{O}_X$-modules $\mathcal{M}$ as above we see that $\zeta$ is indeed an isomorphism of complexes. So $(\mathcal{K}^{\cdot}_X, \mathrm{Tr}_f)$ is a residue pair. □

Let $\pi : X \to \operatorname{Spec} k$ be the structural morphism. In [RD] §VII.3 we find the twisted inverse image functor $\pi^! : \mathsf{D}^+_{\mathrm{c}}(k) \to \mathsf{D}^+_{\mathrm{c}}(X)$.

**Corollary 3.8.** *There is a canonical isomorphism* $\zeta_X : \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \pi^! k$ *in* $\mathsf{D}(X)$. *It is compatible with proper and étale morphisms. If* $\pi$ *is proper then*

$$\mathrm{Tr}_\pi = \mathrm{Tr}^{\mathrm{RD}}_\pi \, \mathrm{R}\pi_*(\zeta_X) : \pi_* \mathcal{K}^{\cdot}_X \to k$$

*where* $\mathrm{Tr}^{\mathrm{RD}}_\pi : \mathrm{R}\pi_* \pi^! k \to k$ *is the trace map of* [RD] §VII.3.

*Proof.* The uniqueness of $\zeta_X$ follows from considering closed subschemes $i_Z : Z \hookrightarrow X$ finite over $k$. This is because any endomorphism $a$ of $\mathcal{K}^{\cdot}_X$ in $\mathsf{D}(X)$ is a global section of $\mathcal{O}_X$, and $a = 1$ iff $i_Z^*(a) = 1$ for all such $Z$. Existence is proved by covering $X$ with compactifiable (e.g. affine) open sets and using Theorem 3.7, cf. [Ye1] Appendix Theorem 3 and subsequent Exercise. In particular $\zeta_X$ is seen to be compatible with open immersions. Compatibility with proper morphisms follows from the transitivity of traces. As for an étale morphism $g : U \to X$, one has $g^* \mathcal{K}^{\cdot}_X \cong \mathcal{K}^{\cdot}_U$ by Theorem 2.19 (b), and also $g^* \pi^! k = g^! \pi^! k = (\pi g)^! k$. Testing the isomorphisms on subschemes $Z \subset U$ finite over $k$ shows that $g^*(\zeta_X) = \zeta_U$. □

## 4. Duals of Differential Operators

Let $X$ be a $k$-scheme of finite type, where $k$ is a perfect field of any characteristic. Suppose $\mathcal{M}, \mathcal{N}$ are $\mathcal{O}_X$-modules. By a differential operator (DO) $D : \mathcal{M} \to \mathcal{N}$ over $\mathcal{O}_X$, relative to $k$, we mean in the sense of [EGA] IV §16.8. Thus $D$ has order $\leq 0$ if $D$ is $\mathcal{O}_X$-linear, and $D$ has order $\leq d$ if for all $a \in \mathcal{O}_X$, the commutator $[D, a]$ has order $\leq d - 1$.

Recall that the dual of an $\mathcal{O}_X$-module $\mathcal{M}$ is $\operatorname{Dual} \mathcal{M} = \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X)$. In this section we prove the existence of the dual operator $\operatorname{Dual}(D)$, in terms of BCAs and residues. This explicit description of $\operatorname{Dual}(D)$ will be needed for the applications in Sections 5-7. For direct proofs of existence cf. Remarks 4.6 and 4.7.

**Theorem 4.1.** *Let* $\mathcal{M}, \mathcal{N}$ *be two* $\mathcal{O}_X$*-modules, and let* $D : \mathcal{M} \to \mathcal{N}$ *be a differential operator of order* $\leq d$. *Then there is a homomorphism of graded sheaves*

$$\operatorname{Dual}(D) : \operatorname{Dual} \mathcal{N} \to \operatorname{Dual} \mathcal{M}$$

*having the properties below:*

(i) $\operatorname{Dual}(D)$ *is a DO of order* $\leq d$.
(ii) $\operatorname{Dual}(D)$ *is a homomorphism of complexes.*

(iii) *Functoriality: if* $E : \mathcal{N} \to \mathcal{L}$ *is another DO, then* $\operatorname{Dual}(ED) = \operatorname{Dual}(D)\operatorname{Dual}(E)$.
(iv) *If* $d = 0$*, i.e.* $D$ *is* $\mathcal{O}_X$*-linear, then* $\operatorname{Dual}(D)(\phi) = \phi \circ D$ *for any* $\phi \in \mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{K}^{\cdot}_X)$.
(v) *Adjunction: under the homomorphisms* $\mathcal{M} \to \operatorname{Dual}\operatorname{Dual}\mathcal{M}$ *and* $\mathcal{N} \to \operatorname{Dual}\operatorname{Dual}\mathcal{N}$*, one has* $D \mapsto \operatorname{Dual}(\operatorname{Dual}(D))$.

*Proof.* By [RD] Theorem II.7.8, an $\mathcal{O}_X$-module $\mathcal{M}'$ is noetherian iff there is a surjection $\bigoplus_{i=1}^n \mathcal{O}_{U_i} \twoheadrightarrow \mathcal{M}'$, for some open sets $U_1, \ldots, U_n$. Here $\mathcal{O}_{U_i}$ is extended by 0 to a sheaf on $X$. One has $\mathcal{M} = \lim_{\alpha\to} \mathcal{M}_\alpha$, where $\{\mathcal{M}_\alpha\}$ is the set of noetherian submodules of $\mathcal{M}$. (We did not assume $\mathcal{M}, \mathcal{N}$ are quasi-coherent!) So

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^q_X) \cong \lim_{\leftarrow \alpha} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}_\alpha, \mathcal{K}^q_X).$$

Since the sheaf $\mathcal{P}^d_{X/k}$ of principal parts is coherent, and $D : \mathcal{M}_\alpha \to \mathcal{N}$ induces

$$\bigoplus_{i=1}^n (\mathcal{P}^d_{X/k} \otimes \mathcal{O}_{U_i}) \twoheadrightarrow \mathcal{P}^d_{X/k} \otimes \mathcal{M}_\alpha \to \mathcal{N},$$

we conclude that the module $\mathcal{N}_\alpha := \mathcal{O}_X \cdot D(\mathcal{M}_\alpha) \subset \mathcal{N}$ is also noetherian. Therefore we may assume that both $\mathcal{M}, \mathcal{N}$ are noetherian.

We have

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X) = \bigoplus_x \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X(x)),$$

and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}_X(x))$ is a constant sheaf with support $\overline{\{x\}}$ and module

$$\mathcal{H}om_{\mathcal{O}_{X,x}}(\mathcal{M}_x, \mathcal{K}_X(x)) = \operatorname{Hom}_A(M, \mathcal{K}(A)) = \operatorname{Dual}_A M,$$

where $A := \widehat{\mathcal{O}}_{X,x} = \mathcal{O}_{X,(x)}$ and $M := A \otimes \mathcal{M}_x$. Note that $M$ is a finitely generated $A$-module. $D : \mathcal{M}_x \to \mathcal{N}_x$ induces a continuous DO $D : M \to N = A \otimes \mathcal{N}_x$ (for the $\mathfrak{m}$-adic topology). According to [Ye2] Theorem 8.6 there is a continuous DO

$$\operatorname{Dual}_A(D) : \operatorname{Dual}_A N \to \operatorname{Dual}_A M.$$

Properties (i), (iii), (iv) and (v) follows directly from [Ye2] Theorem 8.6 and Corollary 8.8. As for property (ii), consider any saturated chain $\xi = (x, \ldots, y)$. Since $\partial^- : \mathcal{O}_{X,(x)} \to \mathcal{O}_{X,\xi}$ is an intensification homomorphism, and since $\partial^+ : \mathcal{O}_{X,(y)} \to \mathcal{O}_{X,\xi}$ is a morphism in $\mathsf{BCA}(k)$ which is also topologically étale, we see that property (ii) is a consequence of [Ye2] Thm. 8.6 and Cor. 8.12. □

Let $\mathcal{D}_X := \mathcal{D}iff_{\mathcal{O}_X/k}(\mathcal{O}_X, \mathcal{O}_X)$ be the sheaf of differential operators on $X$. By definition $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

**Corollary 4.2.** *If* $\mathcal{M}$ *is a left (resp. right)* $\mathcal{D}_X$*-module, then* $\operatorname{Dual}\mathcal{M}$ *is a complex of right (resp. left)* $\mathcal{D}_X$*-modules. In particular this is true for* $\mathcal{K}^{\cdot}_X = \operatorname{Dual}\mathcal{O}_X$.

**Corollary 4.3.** *Suppose $\mathcal{M}^{\cdot}$ is a complex sheaves, where each $\mathcal{M}^q$ is an $\mathcal{O}_X$-module, and $\mathrm{d} : \mathcal{M}^q \to \mathcal{M}^{q+1}$ is a DO. Then there is a dual complex $\operatorname{Dual}\mathcal{M}^{\cdot}$.*

Specifically, $(\operatorname{Dual}\mathcal{M}^{\cdot}, \mathrm{D})$ is the simple complex associated to the double complex $(\operatorname{Dual}\mathcal{M}^{\cdot})^{p,q} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{M}^{-p}, \mathcal{K}^q_X)$. The operator is $\mathrm{D} = \mathrm{D}' + \mathrm{D}''$, where

$$\begin{aligned} \mathrm{D}' &:= (-1)^{p+q+1} \operatorname{Dual}(\mathrm{d}) : (\operatorname{Dual}\mathcal{M}^{\cdot})^{p,q} \to (\operatorname{Dual}\mathcal{M}^{\cdot})^{p+1,q}, \\ \mathrm{D}'' &:= \delta : (\operatorname{Dual}\mathcal{M}^{\cdot})^{p,q} \to (\operatorname{Dual}\mathcal{M}^{\cdot})^{p,q+1}. \end{aligned}$$

It is well known that if $\operatorname{char} k = 0$ and $X$ is smooth of dimension $n$, then $\omega_X = \Omega^n_{X/k}$ is a right $\mathcal{D}_X$-module.

**Proposition 4.4.** *Suppose $\operatorname{char} k = 0$ and $X$ is smooth of dimension $n$. Then $\mathrm{C}_X : \Omega^n_{X/k} \to \mathcal{K}^{-n}_X$ (the inclusion) is $\mathcal{D}_X$-linear.*

*Proof.* It suffice to prove that any $\partial \in \mathcal{T}_X$ (the tangent sheaf), which we view as a DO $\partial : \mathcal{O}_X \to \mathcal{O}_X$, satisfies $\operatorname{Dual}(\partial)(\alpha) = -\mathrm{L}_\partial(\alpha)$, where $\mathrm{L}_\partial$ is the Lie derivative, and $\alpha \in \Omega^n_{X/k}$. Localizing at the generic point of $X$ we get $\partial \in \mathcal{D}(k(X))$ and $\alpha \in \omega(k(X))$. Now use [Ye2] Definition 8.1 and Proposition 4.2. $\square$

**Remark 4.5.** Proposition 4.4 says that in the case $\operatorname{char} k = 0$ and $X$ smooth, the $\mathcal{D}_X$-module structure on $\mathcal{K}^{\cdot}_X$ coincides with the standard one, which is obtained as follows. The quasi-isomorphism $\mathrm{C}_X : \Omega^n_{X/k} \to \mathcal{K}^{\cdot}_X[-n]$ identifies $\mathcal{K}^{\cdot}_X[-n]$ with the Cousin complex of $\Omega^n_{X/k}$, which is computed in the category $\mathsf{Ab}(X)$ (cf. [Ha] Section I.2). Since any $D \in \mathcal{D}_X$ acts $\mathbb{Z}$-linearly on $\Omega^n_{X/k}$, it also acts on $\mathcal{K}^{\cdot}_X[-n]$.

**Remark 4.6.** According to [Sai] there is a direct way to obtain Theorem 4.1 in characteristic 0. Say $X \subset Y$, with $Y$ smooth. Then $\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X) \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{K}^{\cdot}_Y)$. Now by [Sai] §2.2.3 any DO $D : \mathcal{M} \to \mathcal{N}$ of order $\leq d$ can be viewed as

$$D \in \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N} \otimes_{\mathcal{O}_Y} \mathcal{D}^d_Y) \subset \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{N} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y)$$

(right $\mathcal{D}_Y$-modules). Since $\mathcal{K}^q_Y$ is a $\mathcal{D}_Y$-module (cf. Remark 4.5), we get

$$\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{M}, \mathcal{K}^q_Y) \cong \mathcal{H}om_{\mathcal{D}_Y}(\mathcal{M} \otimes_{\mathcal{O}_Y} \mathcal{D}_Y, \mathcal{K}^q_Y)$$

and so we obtain the dual operator $\operatorname{Dual}(D)$. I thank the referee for pointing out this fact to me.

**Remark 4.7.** Suppose $\operatorname{char} k = p > 0$. Then a $k$-linear map $D : \mathcal{M} \to \mathcal{N}$ is a DO over $\mathcal{O}_X$ iff it is $\mathcal{O}_{X^{(p^n/k)}}$-linear, for $n \gg 0$. Here $X^{(p/k)} \to X$ is the Frobenius morphism relative to $k$, cf. [Ye1] Theorem 1.4.9. Since Tr induces an isomorphism

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{K}^{\cdot}_X) \cong \mathcal{H}om^{\cdot}_{\mathcal{O}_{X^{(p^n/k)}}}(\mathcal{M}, \mathcal{K}^{\cdot}_{X^{(p^n/k)}})$$

we obtain the dual operator $\operatorname{Dual}(D)$.

Let us finish this section with an application to rings of differential operators. Given a finitely generated (commutative) $k$-algebra $A$, denote by $\mathcal{D}(A) := \mathrm{Diff}_{A/k}(A, A)$ the ring of differential operators over $A$. Such rings are of interest for ring theorists (cf. [MR] and [HoSt]). It is well known that if $\operatorname{char} k = 0$ and $A$ is smooth, then the opposite ring $\mathcal{D}(A)^{\circ} \cong \omega_A \otimes_A \mathcal{D}(A) \otimes_A \omega_A^{-1}$, where $\omega_A = \Omega^n_{A/k}$. The next theorem is a vast generalization of this fact.

Given complexes $M^{\cdot}, N^{\cdot}$ of $A$-modules let $\mathrm{Diff}^{\cdot}_{A/k}(M^{\cdot}, N^{\cdot})$ be the complex of $k$-modules which in degree $n$ is $\prod_p \mathrm{Diff}_{A/k}(M^p, N^{p+n})$. Let $\mathcal{K}^{\cdot}_A := \Gamma(X, \mathcal{K}^{\cdot}_X)$ with $X := \operatorname{Spec} A$. By Corollary 4.2, it is a complex of right $\mathcal{D}(A)$-modules.

**Theorem 4.8.** *There is a natural isomorphism of filtered $k$-algebras*

$$\mathcal{D}(A)^{\circ} \cong \mathrm{H}^0 \, \mathrm{Diff}^{\cdot}_{A/k}(\mathcal{K}^{\cdot}_A, \mathcal{K}^{\cdot}_A).$$

*Proof.* First observe that since DOs preserve support, $\mathrm{Diff}_{A/k}(\mathcal{K}^p_A, \mathcal{K}^{p-1}_A) = 0$ for all $p$. This means that every local section $D \in \mathrm{H}^0 \, \mathrm{Diff}^{\cdot}_{A/k}(\mathcal{K}^{\cdot}_A, \mathcal{K}^{\cdot}_A)$ is a well defined DO $D : \mathcal{K}^{\cdot}_A \to \mathcal{K}^{\cdot}_A$ which commutes with the coboundary $\delta$. Applying Dual and taking 0-th cohomology we obtain a DO

$$D^{\vee} = \mathrm{H}^0 \, \mathrm{Dual}(D) : \mathrm{H}^0 \, \mathrm{Dual} \, \mathcal{K}^{\cdot}_A \to \mathrm{H}^0 \, \mathrm{Dual} \, \mathcal{K}^{\cdot}_A.$$

But $\mathrm{H}^0 \, \mathrm{Dual} \, \mathcal{K}^{\cdot}_A = A$, so $D^{\vee} \in \mathcal{D}(A)$. Finally according to Theorem 4.1 (v), $D = D^{\vee\vee}$ for $D \in \mathrm{H}^0 \, \mathrm{Diff}^{\cdot}_{A/k}(\mathcal{K}^{\cdot}_A, \mathcal{K}^{\cdot}_A)$ or $D \in \mathcal{D}(A)$. □

Recall that an $n$-dimensional integral domain $A$ is a Gorenstein algebra iff $\omega_A = \mathrm{H}^{-n} \mathcal{K}^{\cdot}_A \to \mathcal{K}^{\cdot}_A[-n]$ is a quasi-isomorphism, and $\omega_A$ is invertible.

**Corollary 4.9.** *If $A$ is a Gorenstein $k$-algebra, there is a canonical isomorphism of filtered $k$-algebras*

$$\mathcal{D}(A)^{\circ} \cong \mathrm{Diff}_{A/k}(\omega_A, \omega_A) \cong \omega_A \otimes_A \mathcal{D}(A) \otimes_A \omega_A^{-1}.$$

**Remark 4.10.** In [Ho], the right $\mathcal{D}(A)$-module structure on $\omega_A$ was exhibited, when $X = \operatorname{Spec} A$ is a curve. Corollary 4.9 was proved there for complete intersection curves.

## 5. The De Rham-Residue Complex

As before $k$ is a perfect field of any characteristic. Let $X$ be a $k$-scheme of finite type. In this section we define a canonical complex on $X$, the De Rham-residue complex $\mathcal{F}^{\cdot}_X$. As we shall see in Corollary 5.9, $\mathcal{F}^{\cdot}_X$ coincides (up to indices and signs) with the double complex $\mathcal{K}^{\cdot,*}_X$ of [EZ].

According to Theorem 4.1, if $\mathcal{M}^{\cdot}$ is a complex of sheaves, with each $\mathcal{M}^q$ an $\mathcal{O}_X$-module and $\mathrm{d} : \mathcal{M}^q \to \mathcal{M}^{q+1}$ a DO, then $\mathrm{Dual} \, \mathcal{M}^{\cdot}$ is a complex of the same kind.

**Definition 5.1.** The *De Rham-residue complex* on $X$ is the complex

$$\mathcal{F}^{\cdot}_X := \mathrm{Dual} \, \Omega^{\cdot}_{X/k}.$$

of Corollary 4.3.

Note that the double complex $\mathcal{F}_X^{\cdot\cdot}$ is concentrated in the third quadrant of the $(p, q)$-plane.

**Proposition 5.2.** *$\mathcal{F}_X^{\cdot}$ is a right DG module over $\Omega^{\cdot}_{X/k}$.*

*Proof.* The graded module structure is clear. It remains to check that

$$\mathrm{D}(\phi \cdot \alpha) = (\mathrm{D}\phi) \cdot \alpha + (-1)^{p+q} \phi \cdot (\mathrm{d}\alpha)$$

for $\phi \in \mathcal{F}_X^{p,q}$ and $\alpha \in \Omega^{p'}_{X/k}$. But this is a straightforward computation using Theorem 4.1. $\square$

**Proposition 5.3.** *Let $g : U \to X$ be étale. Then there is a homomorphism of complexes $\mathrm{q}_g : \mathcal{F}_X^{\cdot} \to g_* \mathcal{F}_U^{\cdot}$, which induces an isomorphism of graded sheaves $1 \otimes \mathrm{q}_g : g^* \mathcal{F}_X^{\cdot} \xrightarrow{\cong} \mathcal{F}_U^{\cdot}$.*

*Proof.* Consider the isomorphisms $g^* \Omega^{\cdot}_{X/k} \cong \Omega^{\cdot}_{U/k}$ and $1 \otimes \mathrm{q}_g : g^* \mathcal{K}_X^{\cdot} \to \mathcal{K}_U^{\cdot}$ of Theorem 2.19. Clearly $1 \otimes \mathrm{q}_g : g^* \mathcal{F}_X^{p,q} \to \mathcal{F}_U^{p,q}$ is an isomorphism. In light of [Ye2] Theorem 8.6 (iv), $\mathrm{q}_g : \mathcal{F}_X^{\cdot} \to g_* \mathcal{F}_U^{\cdot}$ is a homomorphism of complexes. $\square$

Let $f : X \to Y$ be a morphism of schemes. Define a homomorphism of graded sheaves $\mathrm{Tr}_f : f_* \mathcal{F}_X^{\cdot} \to \mathcal{F}_Y^{\cdot}$ by composing $f^* : \Omega^{\cdot}_{Y/k} \to f_* \Omega^{\cdot}_{X/k}$ with $\mathrm{Tr}_f : f_* \mathcal{K}_X^{\cdot} \to \mathcal{K}_Y^{\cdot}$ of Definition 2.11.

**Proposition 5.4.** *$\mathrm{Tr}_f$ commutes with $\mathrm{D}'$. If $f$ is proper then $\mathrm{Tr}_f$ also commutes with $\mathrm{D}''$.*

*Proof.* Let $y \in Y$ and let $x$ be a closed point in $f^{-1}(y)$. Then $f^* : \mathcal{O}_{Y,(y)} \to \mathcal{O}_{X,(x)}$ is a morphism in $\mathsf{BCA}(k)$. Applying [Ye2] Cor. 8.12 to the DOs

$$\mathrm{d} f^* = f^* \mathrm{d} : \Omega^p_{Y/k,(y)} \to \Omega^{p+1}_{X/k,(x)}$$

we get a dual homomorphism

$$\mathrm{Dual}_{f^*}(\mathrm{d} f^*) = \mathrm{Dual}_{f^*}(f^* \mathrm{d}) : \mathrm{Dual}_{\mathcal{O}_{X,(x)}} \Omega^{p+1}_{X/k,(x)} \to \mathrm{Dual}_{\mathcal{O}_{Y,(y)}} \Omega^{p}_{Y/k,(y)},$$

which equals both $\mathrm{Tr}_f \mathrm{Dual}_X(\mathrm{d})$ and $\mathrm{Dual}_Y(\mathrm{d}) \mathrm{Tr}_f$. The commutation of $\mathrm{D}''$ with $\mathrm{Tr}_f$ in the proper case is immediate from Thm. 3.4. $\square$

Of course if $f : X \to Y$ is a closed immersion, then $\mathrm{Tr}_f$ is injective, and it identifies $\mathcal{F}_X^{\cdot}$ with the subsheaf $\mathcal{H}om_{\Omega^{\cdot}_{Y/k}}(\Omega^{\cdot}_{X/k}, \mathcal{F}_Y^{\cdot})$ of $\mathcal{F}_Y^{\cdot}$. Just as in Corollary 3.6 we get:

**Corollary 5.5.** *Let $f : X \to Y$ be a morphism of schemes, and let $Z \subset X$ be a closed subscheme which is proper over $Y$. Then the trace map $\mathrm{Tr}_f : f_* \Gamma_Z \mathcal{F}_X^{\cdot} \to \mathcal{F}_Y^{\cdot}$ is a homomorphism of complexes.*

Suppose $X$ is an integral scheme of dimension $n$. The canonical homomorphism

$$\mathrm{C}_X : \Omega^n_{X/k} \to \mathcal{K}^{-n}_X = k(X) \otimes_{\mathcal{O}_X} \Omega^n_{X/k} \tag{5.6}$$

can be viewed as a global section of $\mathcal{F}^{-n,-n}_X$.

**Lemma 5.7.** *Suppose $X$ is an integral scheme. Then $\mathrm{D}'(\mathrm{C}_X) = \mathrm{D}''(\mathrm{C}_X) = 0$.*

*Proof.* By [Ye1] Section 4.5, $\mathrm{D}''(\mathrm{C}_X) = \pm\delta(\mathrm{C}_X) = 0$. Next, let $K := k(X)$. Choose $t_1, \ldots t_n \in K$ such that $\Omega^1_{K/k} = \bigoplus K \cdot \mathrm{d}t_i$. Taking products of the $\mathrm{d}t_i$ as bases of $\Omega^{n-1}_{K/k}$ and $\Omega^n_{K/k}$, we see from [Ye2] Theorem 8.6 and Definition 8.1 that $\mathrm{Dual}_K(\mathrm{C}_X) = 0$. □

**Proposition 5.8.** *If $X$ is smooth irreducible of dimension $n$, then the DG homomorphism $\Omega^{\cdot}_{X/k} \to \mathcal{F}^{\cdot}_X[-2n]$, $\alpha \mapsto \mathrm{C}_X \cdot \alpha$, is a quasi-isomorphism.*

*Proof.* First note that $\mathrm{D}(\mathrm{C}_X) = 0$, so this is indeed a DG homomorphism. Filtering these complexes according to the $p$-degree we reduce to looking at $\Omega^p_{X/k}[n] \to \mathcal{F}^{p-n,\cdot}_X$. That is a quasi-isomorphism by Theorem 2.19 part d. □

**Corollary 5.9.** *The complex $\mathcal{F}^{\cdot}_X$ is the same as the complex $\mathcal{K}^{\cdot *}_X$ of* [EZ], *up to signs and indexing.*

*Proof.* If $X$ is smooth of dimension $n$ this is because $\mathcal{F}^{\cdot}_X \cong \Omega^{\cdot}_{X/k}[n] \otimes \mathcal{K}^{\cdot}_X$ is the Cousin complex of $\bigoplus \Omega^p_{X/k}[p]$, and $\mathrm{D}'$ is (up to sign) the Cousin functor applied to d. If $X$ is a general scheme embedded in a smooth scheme $Y$, use Proposition 5.4. □

**Definition 5.10.** Given a scheme $X$, let $X_1, \ldots, X_r$ be its irreducible components, with their induced reduced subscheme structures. For each $i$ let $x_i$ be the generic point of $X_i$, and let $f_i : X_i \to X$ be the inclusion morphism. We define the fundamental class $\mathrm{C}_X$ by:

$$\mathrm{C}_X := \sum_{i=1}^{r} \mathrm{length}(\mathcal{O}_{X,x_i}) \,\mathrm{Tr}_{f_i}(\mathrm{C}_{X_i}) \in \mathcal{F}^{\cdot}_X.$$

The next proposition is easily verified using Propositions 5.4 and 5.3. It should be compared to [EZ] Theorem III.3.1.

**Proposition 5.11.** *For any scheme $X$, the fundamental class $\mathrm{C}_X \in \Gamma(X, \mathcal{F}^{\cdot}_X)$ is annihilated by $\mathrm{D}'$ and $\mathrm{D}''$. If $X$ has pure dimension $n$, then $\mathrm{C}_X$ has bidegree $(-n,-n)$. If $f : X \to Y$ is a proper, surjective, generically finite morphism between integral schemes, then $\mathrm{Tr}_f(\mathrm{C}_X) = \deg(f)\,\mathrm{C}_Y$. If $g : U \to X$ is étale, then $\mathrm{C}_U = \mathrm{q}_g(\mathrm{C}_X)$.*

**Remark 5.12.** In [Ye4] it is shown that $\mathcal{F}^{\cdot}_X$ is a right DG module over the DGA of Beilinson adeles $\mathcal{A}^{\cdot}_X = \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\Omega^{\cdot}_{X/k})$. Now let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank $r$, and let $Z \subset X$ be the zero locus of a regular section of $\mathcal{E}$. According to the adelic Chern-Weil theory of [HY2] there is an adelic connection $\nabla$ on $\mathcal{E}$ such that the Chern form $\mathrm{c}_r(\mathcal{E};\nabla) \in \mathcal{A}_X^{2r}$ satisfies $\mathrm{C}_Z = \pm \mathrm{C}_X \cdot \mathrm{c}_r(\mathcal{E};\nabla) \in \mathcal{F}_X^{\cdot}$.

## 6. De Rham Homology and the Niveau Spectral Sequence

Let $X$ be a finite type scheme over a field $k$ of characteristic 0. In [Ye3] it is shown that if $X \subset \mathfrak{X}$ is a smooth formal embedding (see below) then the De Rham complex $\widehat{\Omega}^{\cdot}_{\mathfrak{X}/k}$ calculates the De Rham cohomology $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$. In this section we will show that the De Rham-residue complex $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ of $\mathfrak{X}$ calculates the De Rham homology $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$. This is done by computing the niveau spectral sequence converging to $\mathrm{H}^{\cdot}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$ (Theorem 6.16). We will draw a few conclusions, including the contravariance of homology w.r.t. étale morphisms (Theorem 6.23). As a reference for algebraic De Rham (co)homology we suggest [Ha].

Given a noetherian adic formal scheme $\mathfrak{X}$ and a defining ideal $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$, let $X_n$ be the (usual) noetherian scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$. Suppose $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism between such formal schemes, and let $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ and $\mathcal{J} \subset \mathcal{O}_{\mathfrak{Y}}$ be defining ideals such that $f^{-1}\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$. Such ideals are always available. We get a morphism of (usual) schemes $f_0 : X_0 \to Y_0$.

**Definition 6.1.** A morphism $f : \mathfrak{X} \to \mathfrak{Y}$ between (noetherian) adic formal schemes is called *formally finite type* (resp. *formally finite* or *formally proper*) if the morphism $f_0 : X_0 \to Y_0$ is finite type (resp. finite or proper).

Obviously these notions are independent of the particular defining ideals chosen.

**Example 6.2.** If $X \to Y$ is a finite type morphism of noetherian schemes, $X_0 \subset X$ is a locally closed subset and $\mathfrak{X} = X_{/X_0}$ is the formal completion, then $\mathfrak{X} \to Y$ is formally finite type. Such a morphism is called *algebraizable*.

**Definition 6.3.** A morphism of formal schemes $\mathfrak{X} \to \mathfrak{Y}$ is said to be *formally smooth* (resp. *formally étale*) if, given a (usual) affine scheme $Z$ and a closed subscheme $Z_0 \subset Z$ defined by a nilpotent ideal, the map $\mathrm{Hom}_{\mathfrak{Y}}(Z, \mathfrak{X}) \to \mathrm{Hom}_{\mathfrak{Y}}(Z_0, \mathfrak{X})$ is surjective (resp. bijective).

This is the definition of formal smoothness used in [EGA] IV Section 17.1. We shall also require the next notion.

**Definition 6.4.** A morphism $g : \mathfrak{X} \to \mathfrak{Y}$ between noetherian formal schemes is called *étale* if it is of finite type (see [EGA] I §10.13) and formally étale.

Note that if $\mathfrak{Y}$ is a usual scheme, then so is $\mathfrak{X}$, and $g$ is an étale morphism of schemes.

**Definition 6.5.** A *smooth formal embedding* of $X$ (over $k$) is a closed immersion of $X$ into a formal scheme $\mathfrak{X}$, which induces a homeomorphism on the underlying topological spaces, and such that $\mathfrak{X}$ is of formally finite type and formally smooth over $k$.

**Example 6.6.** If $X$ is smooth over $Y = \operatorname{Spec} k$ and $X_0, \mathfrak{X}$ are as in Example 6.2, then $X_0 \subset \mathfrak{X}$ is a smooth formal embedding.

Let $\xi = (x_0, \dots, x_q)$ be a saturated chain of points in the formal scheme $\mathfrak{X}$. Choose a defining ideal $\mathcal{I}$, and let $X_n$ be as above. Define the Beilinson completion $\mathcal{O}_{\mathfrak{X},\xi} := \lim_{\leftarrow n} \mathcal{O}_{X_n,\xi}$ (which of course is independent of $\mathcal{I}$).

**Lemma 6.7.** *Let $\mathfrak{X}$ be formally finite type over $k$, and let $\xi$ be a saturated chain in $\mathfrak{X}$. Then $\mathcal{O}_{\mathfrak{X},\xi}$ is a BCA over $k$. If $\mathfrak{X} = X_{/X_0}$, then $\mathcal{O}_{\mathfrak{X},\xi} \cong \mathcal{O}_{X,\xi}$.*

*Proof.* First assume $\mathfrak{X} = X_{/X_0}$. Taking $\mathcal{I}$ to be the ideal of $X_0$ in $X$, we have

$$\begin{aligned}\mathcal{O}_{\mathfrak{X},\xi} = \lim_{\leftarrow n} (\mathcal{O}_X/\mathcal{I}^n)_\xi \cong \lim_{\leftarrow m,n} \mathcal{O}_{X,\xi}/(\mathcal{I}^n \mathcal{O}_{X,\xi} + \mathfrak{m}_\xi^m) \\ \cong \lim_{\leftarrow m} \mathcal{O}_{X,\xi}/\mathfrak{m}_\xi^m = \mathcal{O}_{X,\xi}.\end{aligned}$$

Now by [Ye3] Proposition 1.20 and Lemma 1.1, locally there is a closed immersion $\mathfrak{X} \subset \mathfrak{Y}$, with $\mathfrak{Y}$ algebraizable (i.e. $\mathfrak{Y} = Y_{/Y_0}$). So there is a surjection $\mathcal{O}_{\mathfrak{Y},\xi} \to \mathcal{O}_{\mathfrak{X},\xi}$, and this implies that $\mathcal{O}_{\mathfrak{X},\xi}$ is a BCA. □

One can construct the complexes $\mathcal{K}^{\cdot}_{\mathfrak{X}}$ and $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ for a formally finite type formal scheme $\mathfrak{X}$, as follows. Define $\mathcal{K}_{\mathfrak{X}}(x) := \mathcal{K}(\mathcal{O}_{\mathfrak{X},(x)})$. Now let $(x, y)$ be a saturated chain. Then there is an intensification homomorphism $\partial^- : \mathcal{O}_{\mathfrak{X},(x)} \to \mathcal{O}_{\mathfrak{X},(x,y)}$ and a morphism of BCAs $\partial^+ : \mathcal{O}_{\mathfrak{X},(y)} \to \mathcal{O}_{\mathfrak{X},(x,y)}$. Therefore we get a homomorphism of $\mathcal{O}_{\mathfrak{X}}$-modules $\delta_{(x,y)} : \mathcal{K}_{\mathfrak{X}}(x) \to \mathcal{K}_{\mathfrak{X}}(y)$. Define a graded sheaf $\mathcal{K}^{\cdot}_{\mathfrak{X}} = \bigoplus_{x\in\mathfrak{X}} \mathcal{K}_{\mathfrak{X}}(x)$ on $\mathfrak{X}$, as in §1. Let $\widehat{\Omega}^{\cdot}_{\mathfrak{X}/k}$ be the complete De Rham complex on $\mathfrak{X}$, and set $\mathcal{F}^{p,q}_{\mathfrak{X}} := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\widehat{\Omega}^{-p}_{\mathfrak{X}/k}, \mathcal{K}^q_{\mathfrak{X}})$.

**Proposition 6.8.** *Let $\mathfrak{X}$ be a formally finite type formal scheme over $k$.*

1. *$\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is a complex.*
2. *If $g : \mathfrak{U} \to \mathfrak{X}$ is étale, then there is a homomorphism of complexes $\mathrm{q}_g : \mathcal{F}^{\cdot}_{\mathfrak{X}} \to g_* \mathcal{F}^{\cdot}_{\mathfrak{U}}$, which induces an isomorphism of graded sheaves $1 \otimes \mathrm{q}_g : g^* \mathcal{F}^{\cdot}_{\mathfrak{X}} \xrightarrow{\cong} \mathcal{F}^{\cdot}_{\mathfrak{U}}$.*
3. *If $f : \mathfrak{X} \to \mathfrak{Y}$ is formally proper, then there is a homomorphism of complexes $\operatorname{Tr}_f : f^* \mathcal{F}^{\cdot}_{\mathfrak{X}} \to \mathcal{F}^{\cdot}_{\mathfrak{Y}}$.*

*Proof.* 1. Let $X_n \subset \mathfrak{X}$ be as before. Then one has $\mathcal{F}^{\cdot}_{\mathfrak{X}} = \bigcup \mathcal{F}^{\cdot}_{X_n}$, so this is a complex.
2. Take $U_n := \mathfrak{U} \times_{\mathfrak{X}} X_n$; then each $U_n \to X_n$ is an étale morphism of schemes, and we can use Proposition 5.3.
3. Apply Proposition 5.4 to $X_n \to Y_n$. □

**Proposition 6.9.** *Assume $\mathfrak{X} = Y_{/X}$ for some smooth irreducible scheme $Y$ of dimension $n$ and closed set $X \subset Y$. Then there is a natural isomorphism* *of complexes*

$$\mathcal{F}^{\cdot}_{\mathfrak{X}} \cong \underline{\Gamma}_X \mathcal{F}^{\cdot}_Y. \tag{6.10}$$

*Hence* $\mathcal{F}^{\cdot}_{\mathfrak{X}} \cong \mathrm{R}\underline{\Gamma}_X \Omega^{\cdot}_{Y/k}[2n]$ *in the derived category* $\mathsf{D}(\mathsf{Ab}(Y))$, *and consequently*

$$\mathrm{H}^{-q}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) \cong \mathrm{H}^{2n-q}_X(Y, \Omega^{\cdot}_{Y/k}) = \mathrm{H}^{\mathrm{DR}}_q(X).$$

*Proof.* The isomorphism (6.10) is immediate from Lemma 6.7. But according to Proposition 5.8, $\mathcal{F}^{\cdot}_Y$ is a flasque resolution of $\Omega^{\cdot}_{Y/k}[2n]$ in $\mathsf{Ab}(Y)$. $\square$

We need some algebraic results, phrased in the terminology of [Ye1] §1. Let $K$ be a complete, separated semi-topological (ST) commutative $k$-algebra, and let $\underline{t} = (t_1, \ldots, t_n)$ be a sequence of indeterminates. Let $K[[\underline{t}]]$ and $K((\underline{t}))$ be the rings of formal power series, and of iterated Laurent series, respectively. These are complete, separated ST $k$-algebras. Let $T$ be the free $k$-module with basis $\alpha_1, \ldots, \alpha_n$ and let $\bigwedge^{\cdot}_k T$ be the exterior algebra over $k$.

**Lemma 6.11.** ("Poincaré Lemma") *The DGA homomorphisms*

$$\Omega^{\cdot,\mathrm{sep}}_{K/k} \to \Omega^{\cdot,\mathrm{sep}}_{K[[\underline{t}]]/k}$$

*and*

$$\Omega^{\cdot,\mathrm{sep}}_{K/k} \otimes_k \bigwedge\nolimits^{\cdot}_k T \to \Omega^{\cdot,\mathrm{sep}}_{K((\underline{t}))/k}, \ \alpha_i \mapsto \operatorname{dlog} t_i$$

*are quasi-isomorphisms.*

*Proof.* Since

$$\Omega^{\cdot,\mathrm{sep}}_{K[[\underline{t}]]/k} \cong K[[\underline{t}]] \otimes_{k[\underline{t}]} \Omega^{\cdot}_{k[\underline{t}]/k}$$

the homotopy operator ("integration") of the Poincaré Lemma for the graded polynomial algebra $k[\underline{t}]$ works here also.

For $K((t))$ (i.e. $n = 1$) we have

$$\Omega^{\cdot,\mathrm{sep}}_{K((t))/k} \cong \Omega^{\cdot,\mathrm{sep}}_{K[[t]]/k} \oplus \Omega^{\cdot,\mathrm{sep}}_{K[t^{-1}]/k} \wedge \operatorname{dlog} t$$

so we have a quasi-isomorphism. For $n > 1$ use induction on $n$ and the Künneth formula. $\square$

**Lemma 6.12.** *Suppose $A$ is a local BCA and $\sigma, \sigma' : K \to A$ are two coefficient fields. Then*

$$\mathrm{H}(\sigma) = \mathrm{H}(\sigma') : \mathrm{H}\Omega^{\cdot,\mathrm{sep}}_{K/k} \to \mathrm{H}\Omega^{\cdot,\mathrm{sep}}_{A/k}.$$

*Proof.* Choosing generators for the maximal ideal of $A$, $\sigma$ induces a surjection of BCAs $\tilde{A} = K[[\underline{t}]] \to A$. Denote by $\tilde{\sigma} : K \to \tilde{A}$ the inclusion. The coefficient field $\sigma'$ lifts to some coefficient field $\tilde{\sigma}' : K \to \tilde{A}$. It suffices to show that $\mathrm{H}(\tilde{\sigma}) = \mathrm{H}(\tilde{\sigma}')$. But by Lemma 6.11 both of these are bijective, and using the projection $\tilde{A} \to K$ we see they are in fact equal. $\square$

Given a saturated chain $\xi = (x, \dots, y)$ in $X$ and a coefficient field $\sigma : k(y) \to \mathcal{O}_{X,(y)}$, there is the *Parshin residue map*

$$\operatorname{Res}_{\xi,\sigma} : \Omega^{\cdot}_{k(x)/k} \to \Omega^{\cdot}_{k(y)/k}$$

(cf. [Ye1] Definition 4.1.3). It is a map of DG $k$-modules of degree equal to $-(\text{length of } \xi)$.

**Proposition 6.13.** *Let $\xi = (x, \dots, y)$ be a saturated chain in $X$. Then the map of graded $k$-modules*

$$\operatorname{Res}_{\xi} := \mathrm{H}(\operatorname{Res}_{\xi,\sigma}) : \mathrm{H}\Omega^{\cdot}_{k(x)/k} \to \mathrm{H}\Omega^{\cdot}_{k(y)/k}$$

*is independent of the coefficient field $\sigma$.*

*Proof.* Say $\xi$ has length $n$. Let $L$ be one of the local factors of $k(\xi) = k(x)_{\xi}$, so it is an $n$-dimensional topological local field (TLF). Let $K := \kappa_n(L)$, the last residue field of $L$, which is a finite separable $k(y)$-algebra. Then $\sigma$ extends uniquely to a morphism of TLFs $\sigma : K \to L$, and it is certainly enough to check that

$$\mathrm{H}(\operatorname{Res}_{L/K;\sigma}) : \mathrm{H}\Omega^{\cdot,\mathrm{sep}}_{L/k} \to \mathrm{H}\Omega^{\cdot,\mathrm{sep}}_{K/k} \tag{6.14}$$

is independent of $\sigma$.

After choosing a regular system of parameters $\underline{t} = (t_1, \dots, t_n)$ in $L$ we get $L \cong K((\underline{t}))$. According to Lemma 6.11, $\mathrm{H}(\sigma)$ induces an isomorphism of $k$-algebras

$$\mathrm{H}\Omega^{\cdot}_{K/k} \otimes_k \bigwedge\nolimits^{\cdot}_k T \cong \mathrm{H}\Omega^{\cdot,\mathrm{sep}}_{L/k}. \tag{6.15}$$

But by Lemma 6.12 this isomorphism is independent of $\sigma$. The map (6.14) is $\mathrm{H}\Omega^{\cdot}_{K/k}$-linear, and it sends $\bigwedge^p_k T$ to 0 if $p < n$, and $\operatorname{dlog} t_1 \wedge \cdots \wedge \operatorname{dlog} t_n \mapsto 1$. Hence (6.14) is independent of $\sigma$. $\square$

The topological space $X$ has an increasing filtration by families of supports $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots$, where

$$X_q := \{Z \subset X \mid Z \text{ is closed and } \dim Z \leq q\}.$$

We write $x \in X_q/X_{q-1}$ if $\overline{\{x\}} \in X_q - X_{q-1}$, and the set $X_q/X_{q-1}$ is called the $q$-skeleton of $X$. (This notation is in accordance with [BlO]; in [Ye1] $X_q$ denotes the $q$-skeleton.) The *niveau filtration* on $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is $\mathrm{N}_q\mathcal{F}^{\cdot}_{\mathfrak{X}} := \underline{\Gamma}_{X_q}\mathcal{F}^{\cdot}_{\mathfrak{X}}$. Let us write $X^q/X^{q+1} := X_{-q}/X_{-q-1}$ and $\mathrm{N}^q := \mathrm{N}_{-q}$, so $\{\mathrm{N}^q\mathcal{F}^{\cdot}_{\mathfrak{X}}\}$ is a decreasing filtration.

**Theorem 6.16.** *Suppose* $\operatorname{char} k = 0$ *and $X \subset \mathfrak{X}$ is a smooth formal embedding. Then in the niveau spectral sequence converging to $\mathrm{H}^{\cdot}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$, the $E_1$ term is* (*in the notation of* [ML] *Chapter* XI):

$$E_1^{p,q} = \mathrm{H}^{p+q}_{X^p/X^{p+1}}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) \cong \bigoplus_{x \in X^p/X^{p+1}} \mathrm{H}^{q-p}\Omega^{\cdot}_{k(x)/k},$$

*and the coboundary operator is* $(-1)^{p+1} \sum_{(x,y)} \operatorname{Res}_{(x,y)}$.

*Proof.* We shall substitute indices $(p,q) \mapsto (-q,-p)$; this puts us in the first quadrant. Because $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is a complex of flasque sheaves, one has

$$E_1^{-q,-p} = \mathrm{H}^{-p-q}_{X^{-q}/X^{-q+1}}(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) \cong \bigoplus_{x \in X_q/X_{q-1}} \mathrm{H}^{-p} \mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}} \left( \widehat{\Omega}^{\cdot}_{\mathfrak{X}/k}, \mathcal{K}_{\mathfrak{X}}(x) \right)$$

(the operator $\delta$ is trivial on the $q$-skeleton). Fix a point $x$ of dimension $q$ and let $B := \mathcal{O}_{\mathfrak{X},(x)}$. Then $\widehat{\Omega}^{\cdot}_{\mathfrak{X}/k,(x)} \cong \Omega^{\cdot,\mathrm{sep}}_{B/k}$ and by definition

$$\mathcal{H}om^{\cdot}_{\mathcal{O}_{\mathfrak{X}}} \left( \widehat{\Omega}^{\cdot}_{\mathfrak{X}/k}, \mathcal{K}_{\mathfrak{X}}(x) \right) \cong \mathrm{Dual}_B \, \Omega^{\cdot,\mathrm{sep}}_{B/k}.$$

Choose a coefficient field $\sigma : K = k(x) \to B$. By [Ye2] Theorem 8.6 there is an isomorphism of complexes

$$\Psi_\sigma : \mathrm{Dual}_B \, \Omega^{\cdot,\mathrm{sep}}_{B/k} \xrightarrow{\simeq} \mathrm{Dual}_\sigma \, \Omega^{\cdot,\mathrm{sep}}_{B/k} = \mathrm{Hom}^{\mathrm{cont}}_{K;\sigma}(\Omega^{\cdot,\mathrm{sep}}_{B/k}, \omega(K)).$$

Here $\omega(K) = \Omega^q_{K/k}$ and the operator on the right is $\mathrm{Dual}_\sigma(\mathrm{d})$ of [Ye2] Definition 8.1.

According to [Ye3] §3, $k \to B$ is formally smooth; so $B$ is a regular local ring, and hence $B \cong K[[\underline{t}]]$. Put a grading on $\Omega^{\cdot}_{K[\underline{t}]/k}$ by declaring $\deg t_i = \deg \mathrm{d}t_i = 1$, and let $V_l \subset \Omega^{\cdot}_{K[\underline{t}]/k}$ be the homogeneous component of degree $l$. In particular $V_0 = \Omega^{\cdot}_{K/k}$. Since d preserves each $V_l$, from the definition of $\mathrm{Dual}_\sigma(\mathrm{d})$ we see that

$$\mathrm{Dual}_\sigma \, \Omega^{\cdot,\mathrm{sep}}_{B/k} = \bigoplus_{l=0}^{\infty} \mathrm{Hom}_K(V_l, \omega(K))$$

as complexes. Because the $K$-linear homotopy operator in the proof of Lemma 6.11 also preserves $V_l$ we get $\mathrm{H}\,\mathrm{Hom}_K(V_l, \omega(K)) = 0$ for $l \neq 0$, and hence

$$\mathrm{H}^{-p} \, \mathrm{Dual}_\sigma \, \Omega^{\cdot,\mathrm{sep}}_{B/k} \cong \mathrm{H}^{-p} \, \mathrm{Hom}_K(\Omega^{\cdot}_{K/k}, \omega(K)) \cong \mathrm{H}^{q-p} \Omega^{\cdot}_{K/k} \tag{6.17}$$

(cf. proof of Lemma 5.7).

It remains to check that the coboundary maps match up. Given an immediate specialization $(x,y)$, choose a pair of compatible coefficient fields $\sigma : k(x) \to \mathcal{O}_{\mathfrak{X},(x)} = B$ and $\tau : k(y) \to \mathcal{O}_{\mathfrak{X},(y)} = A$ (cf. [Ye1] Definition 4.1.5). Set $\widehat{B} := \mathcal{O}_{\mathfrak{X},(x,y)}$, so $f : A \to \widehat{B}$ is a morphism of BCAs. A cohomology class $[\phi] \in \mathrm{H}^{-p} \, \mathrm{Dual}_B \, \Omega^{\cdot,\mathrm{sep}}_{B/k}$ is sent under the isomorphism (6.17) to the class $[\beta]$ of some form $\beta \in \Omega^{q-p}_{k(x)/k}$, such that $\mathrm{d}\beta = 0$ and on $\sigma(\Omega^{\cdot}_{k(x)/k}) \subset \Omega^{\cdot,\mathrm{sep}}_{B/k}$, $\phi$ acts like left multiplication by $\beta$. So for $\alpha \in \Omega^p_{k(y)/k}$,

$$\begin{aligned} \mathrm{Tr}_{A/k(y)} \, \mathrm{Tr}_{\widehat{B}/A} \, \phi f \tau(\alpha) &= \mathrm{Res}_{k((x,y))/k(y);\tau}(\beta \wedge \tau(\alpha)) \\ &= \mathrm{Res}_{k((x,y))/k(y);\tau}(\beta) \wedge \alpha. \end{aligned}$$

This says that under the isomorphism

$$\mathrm{H}^{-p} \, \mathrm{Dual}_A \, \Omega^{\cdot,\mathrm{sep}}_{A/k} \cong \mathrm{H}^{q-p-1} \Omega^{\cdot}_{k(y)/k},$$

the class $\delta_{(x,y)}([\phi])$ is sent to $\mathrm{Res}_{(x,y)}([\beta])$. $\square$

**Remark 6.18.** Theorem 6.16, but with $\mathrm{R}\,\underline{\Gamma}_X \Omega^{\cdot}_{Y/k}$ instead of $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ (cf. Proposition 6.9), was discovered by Grothendieck (see [Gr] Footnotes 8,9), and proved by Bloch-Ogus [BlO]. Our proof is completely different from that in [BlO], and in particular we obtain the formula for the coboundary operator as a sum of Parshin residues. On the other hand the proof in [BlO] is valid for a general homology theory (including $l$-adic homology). Bloch-Ogus went on to prove additional important results, such as the degeneration of the sheafified spectral sequence $\mathcal{E}_r^{p,q}$ at $r = 2$, for $X$ smooth.

The next result is a generalization of [Ha] Theorem II.3.2. Suppose $X \subset \mathfrak{Y}$ is another smooth formal embedding. By a morphism of embeddings $f : \mathfrak{X} \to \mathfrak{Y}$ we mean a morphism of formal schemes inducing the identity on $X$. Since $f$ is formally finite, according to Proposition 6.8, $\mathrm{Tr}_f : \mathcal{F}^{\cdot}_{\mathfrak{X}} \to \mathcal{F}^{\cdot}_{\mathfrak{Y}}$ is a map of complexes in $\mathsf{Ab}(X)$.

**Corollary 6.19.** *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of embeddings of $X$. Then $\mathrm{Tr}_f : \Gamma(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) \to \Gamma(X, \mathcal{F}^{\cdot}_{\mathfrak{Y}})$ is a quasi-isomorphism. If $g : \mathfrak{X} \to \mathfrak{Y}$ is another such morphism, then $\mathrm{H}(\mathrm{Tr}_f) = \mathrm{H}(\mathrm{Tr}_g)$.*

*Proof.* $\mathrm{Tr}_f$ induces a map of niveau spectral sequences $E_r^{p,q}(\mathfrak{X}) \to E_r^{p,q}(\mathfrak{Y})$. The theorem and its proof imply that these spectral sequences coincide for $r \geq 1$, hence $\mathrm{H}^{\cdot}(\mathrm{Tr}_f)$ is an isomorphism. The other statement is proved like in [Ye3] Theorem 2.7 (cf. next corollary). $\square$

**Corollary 6.20.** *The $k$-module $\mathrm{H}^q(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$ is independent of the smooth formal embedding $X \subset \mathfrak{X}$.*

*Proof.* As shown in [Ye3], given any two embeddings $X \subset \mathfrak{X}$ and $X \subset \mathfrak{Y}$, the completion of their product along the diagonal $(\mathfrak{X} \times_k \mathfrak{Y})_{/X}$ is also a smooth formal embedding of $X$, and it projects to both $\mathfrak{X}$ and $\mathfrak{Y}$. Therefore by Corollary 6.19, $\mathrm{H}^q(X, \mathcal{F}^{\cdot}_{\mathfrak{X}})$ and $\mathrm{H}^q(X, \mathcal{F}^{\cdot}_{\mathfrak{Y}})$ are isomorphic. Using triple products we see this isomorphism is canonical. $\square$

**Remark 6.21.** We can use Corollary 6.20 to *define* $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$ if some smooth formal embedding exists. For a definition of $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$ in general, using a system of local embeddings, see [Ye3] (cf. [Ha] pp. 28-29).

**Remark 6.22.** In [Ye4] it is shown that $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is naturally a DG module over the adele-De Rham complex $\mathcal{A}^{\cdot}_{\mathfrak{X}} = \underline{\mathbb{A}}^{\cdot}_{\mathrm{red}}(\widehat{\Omega}^{\cdot}_{X/k})$, and this multiplication induces the cap product of $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X)$ on $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$.

The next result is new (cf. [BlO] Example 2.2):

**Theorem 6.23.** *De Rham homology $\mathrm{H}^{\mathrm{DR}}_{\cdot}(-)$ is contravariant w.r.t. étale morphisms.*

*Proof.* The "topological invariance of étale morphisms" (see [Mi] Theorem I.3.23) implies that the smooth formal embedding $X \subset \mathfrak{X}$ induces an "embedding of étale sites" $X_{\mathrm{et}} \subset \mathfrak{X}_{\mathrm{et}}$. By this we mean that for every étale morphism $U \to X$ there is some étale morphism $\mathfrak{U} \to \mathfrak{X}$, unique up to isomorphism, s.t. $U \cong \mathfrak{U} \times_{\mathfrak{X}} X$ (see [Ye3]). Then $U \subset \mathfrak{U}$ is a smooth formal embedding. From Proposition 6.8 we see there is a complex of sheaves $\mathcal{F}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}}$ on $X_{\mathrm{et}}$ with $\mathcal{F}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}}|_U \cong \mathcal{F}^{\cdot}_{\mathfrak{U}} \cong g^* \mathcal{F}^{\cdot}_{\mathfrak{X}}$ for every $g : \mathfrak{U} \to \mathfrak{X}$ étale (cf. [Mi] Corollary II.1.6). But by Corollary 6.20, $\mathrm{H}^{\mathrm{DR}}_{\cdot}(U) = \mathrm{H}^{\cdot}(U, \mathcal{F}^{\cdot}_{\mathfrak{U}})$. □

Say $X$ is smooth irreducible of dimension $n$. Define the sheaf $\mathcal{H}^p_{\mathrm{DR}}$ on $X_{\mathrm{Zar}}$ to be the sheafification of the presheaf $U \mapsto \mathrm{H}^p_{\mathrm{DR}}(U)$. For any point $x \in X$ let $i_x : \{x\} \to X$ be the inclusion. Let $x_0$ be the generic point, so $X_n / X_{n-1} = \{x_0\}$. According to [BlO] there is an exact sequence of sheaves

$$0 \to \mathcal{H}^p_{\mathrm{DR}} \to i_{x_0\,*} \mathrm{H}^p \Omega^{\cdot}_{k(x_0)/k} \to \cdots \to \bigoplus_{x \in X_q / X_{q-1}} i_{x*} \mathrm{H}^{p+q-n} \Omega^{\cdot}_{k(x)/k} \to \cdots$$

called the *arithmetic resolution*. Observe that this is a flasque resolution.

**Corollary 6.24.** *The coboundary operator in the arithmetic resolution of* $\mathcal{H}^p_{\mathrm{DR}}$ *is*

$$(-1)^{q+1} \sum_{(x,y)} \mathrm{Res}_{(x,y)}$$

*where* $\mathrm{Res}_{(x,y)}$ *is the Parshin residue of Proposition* 6.13.

*Proof.* Take $\mathfrak{X} = X$ in Theorem 6.16, and use [BlO] Theorem 4.2. □

## 7. The Intersection Cohomology $\mathcal{D}$-module of a Curve

Suppose $Y$ is an $n$-dimensional smooth algebraic variety over $\mathbb{C}$ and $X$ is a subvariety of codimension $d$. Let $\mathcal{H}^d_X \mathcal{O}_Y$ be the sheaf of $d$-th cohomology of $\mathcal{O}_Y$ with support in $X$. According to [BrK], the holonomic $\mathcal{D}_Y$-module $\mathcal{H}^d_X \mathcal{O}_Y$ has a unique simple coherent submodule $\mathcal{L}(X,Y)$, and the De Rham complex $\mathrm{DR}\,\mathcal{L}(X,Y) = \mathcal{L}(X,Y) \otimes \Omega^{\cdot}_{Y^{\mathrm{an}}}[n]$ is the middle perversity intersection cohomology sheaf $\mathcal{IC}^{\cdot}_{X^{\mathrm{an}}}$. Here $Y^{\mathrm{an}}$ is the associated complex manifold. The module $\mathcal{L}(X,Y)$ was described explicitly using complex-analytic methods by Vilonen [Vi] and Barlet-Kashiwara [BaK]. These descriptions show that the fundamental class $\mathrm{C}_{X/Y}$ lies in $\mathcal{L}(X,Y) \otimes \Omega^d_{Y/k}$, a fact proved earlier by Kashiwara using the Riemann-Hilbert correspondence and the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber (see [Br]).

Now let $k$ be any field of characteristic 0, $Y$ an $n$-dimensional smooth variety over $k$, and $X \subset Y$ an integral curve with arbitrary singularities. In this section we give a description of $\mathcal{L}(X,Y) \subset \mathcal{H}^{n-1}_X \mathcal{O}_Y$ in terms of algebraic residues. As references on $\mathcal{D}$-modules we suggest [Bj] and [Bo] Chapter VI.

Denote by $w$ the generic point of $X$. Pick any coefficient field $\sigma : k(w) \to \widehat{\mathcal{O}}_{Y,w} = \mathcal{O}_{Y,(w)}$. As in [Hu] Section 1 there is a residue map

$$\mathrm{Res}^{\mathrm{lc}}_{w,\sigma} : \mathrm{H}^{n-1}_w \Omega^n_{Y/k} \to \Omega^1_{k(w)/k}$$

("lc" is for local cohomology) defined as follows. Choose a regular system of parameters $f_1, \ldots, f_{n-1}$ in $\mathcal{O}_{Y,w}$, so that $\mathcal{O}_{Y,(w)} \cong k(w)[[f_1, \ldots, f_{n-1}]]$. Then for a generalized fraction, with $\alpha \in \Omega^1_{k(X)/k}$, we have

$$\operatorname{Res}^{\mathrm{lc}}_{w,\sigma} \begin{bmatrix} \sigma(\alpha) \wedge \mathrm{d} f_1 \wedge \cdots \wedge \mathrm{d} f_{n-1} \\ f_1^{i_1} \cdots f_{n-1}^{i_{n-1}} \end{bmatrix} = \begin{cases} \alpha & \text{if } (i_1, \ldots, i_{n-1}) = (1, \ldots, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi : \tilde{X} \to X$ be the normalization, and let $\tilde{w}$ be the generic point of $\tilde{X}$. For any closed point $\tilde{x} \in \tilde{X}$ the residue field $k(\tilde{x})$ is étale over $k$, so it lifts into $\mathcal{O}_{\tilde{X},(\tilde{x})}$. Hence we get canonical morphisms of BCAs $k(\tilde{x}) \to \mathcal{O}_{\tilde{X},(\tilde{x})} \to k(\tilde{w})_{(\tilde{x})}$, and a residue map

$$\operatorname{Res}_{(\tilde{w},\tilde{x})} : \Omega^1_{k(w)/k} \to \Omega^{1,\mathrm{sep}}_{k(\tilde{w})_{(\tilde{x})}/k} \to k(\tilde{w}).$$

Define

$$\operatorname{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x})} : \mathrm{H}^{n-1}_w \Omega^n_{Y/k} \xrightarrow{\operatorname{Res}^{\mathrm{lc}}_{w,\sigma}} \Omega^1_{k(w)/k} \xrightarrow{\operatorname{Res}_{(\tilde{w},\tilde{x})}} k(\tilde{x}).$$

We shall see later that $\operatorname{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x})}$ is independent of $\sigma$. Note that $\mathrm{H}^{n-1}_w \Omega^n_{Y/k} = (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_w$.

**Theorem 7.1.** *Let $x \in X$ be a closed point and let $a \in (\mathcal{H}^{n-1}_X \mathcal{O}_Y)_x$. Then $a \in \mathcal{L}(X,Y)_x$ iff $\operatorname{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x})}(a\alpha) = 0$ for all $\alpha \in \Omega^n_{Y/k,x}$ and $\tilde{x} \in \pi^{-1}(x)$.*

This is our algebraic counterpart of Vilonen's formula in [Vi]. The proof of the theorem appears later in this section.

Fix a closed point $x \in X$. Write $B := \mathcal{O}_{Y,(w,x)}$ and $L := \prod_{\tilde{x} \in \pi^{-1}(x)} k(\tilde{x})$.

**Lemma 7.2.** *There is a canonical morphism of BCAs $L \to B$, and $B \cong L((g))[[f_1, \ldots, f_{n-1}]]$ for indeterminates $g, f_1, \ldots, f_{n-1}$.*

*Proof.* Because $\mathcal{O}_{\tilde{X},(\tilde{x})}$ is a regular local ring we get $\mathcal{O}_{\tilde{X},(\tilde{x})} \cong k(\tilde{x})[[g]]$. It is well known (cf. [Ye1] Theorem 3.3.2) that $k(w)_{(x)} = k(w) \otimes \mathcal{O}_{X,(x)} \cong \prod k(\tilde{w})_{(\tilde{x})}$, hence $k(w)_{(x)} \cong L((g))$.

Choose a coefficient field $\sigma : k(w) \to \mathcal{O}_{Y,(w)}$. It extends to a lifting $\sigma_{(x)} : k(w)_{(x)} \to \mathcal{O}_{Y,(w,x)} = B$ (cf. [Ye1] Lemma 3.3.9), and $L \to B$ is independent of $\sigma$. Taking a system of regular parameters $f_1, \ldots, f_{n-1} \in \mathcal{O}_{Y,w}$ we obtain the desired isomorphism. □

The BCA $A := \mathcal{O}_{Y,(x)}$ is canonically an algebra over $K := k(x)$, so there is a morphism of BCAs $L \otimes_K A \to B$. Define a homomorphism

$$T_x : \mathcal{K}(B) \xrightarrow{\mathrm{Tr}} \mathcal{K}(L \otimes_K A) \cong L \otimes_K \mathcal{K}(A).$$

Since $A \to L \otimes_K A \to B$ are topologically étale (relative to $k$), it follows that $T_x$ is a homomorphism of $\mathcal{D}(A)$-modules.

Define

$$V(x) := \operatorname{Coker}(K \to L).$$

Observe that $V(x) = 0$ iff $x$ is either a smooth point or a geometrically unibranch singularity. We have $V(x)^* \subset L^*$, where $(-)^* := \mathrm{Hom}_k(-, k)$. The isomorphism $L^* \cong L$ induced by $\mathrm{Tr}_{L/k}$ identifies $V(x)^* \cong \mathrm{Ker}(L \xrightarrow{\mathrm{Tr}} K)$.

Since $\Omega^n_{Y/k}[n] \to \mathcal{K}^{\cdot}_Y$ is a quasi-isomorphism we get a short exact sequence

$$0 \to (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x \to \mathcal{K}_Y(w) \xrightarrow{\delta} \mathcal{K}_Y(x) \to 0. \tag{7.3}$$

Also we see that $\mathcal{K}(A) = \mathcal{K}_Y(x) \cong \mathrm{H}^n_x \Omega^n_{Y/k}$. Now $\mathcal{K}_Y(w) = \mathcal{K}(\mathcal{O}_{Y,(w)}) \subset \mathcal{K}(B)$. Because the composed map

$$\mathcal{K}_Y(w) \xrightarrow{T_x} L \otimes_K \mathcal{K}_Y(x) \xrightarrow{\mathrm{Tr}_{L/K} \otimes 1} \mathcal{K}_Y(x)$$

coincides with $\delta$, and by the sequence (7.3), we obtain a homomorphism of $\mathcal{D}_{Y,x}$-modules

$$T_x : (\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x \to V(x)^* \otimes_K \mathrm{H}^n_x \Omega^n_{Y/k}. \tag{7.4}$$

**Theorem 7.5.** *The homomorphism $T_x$ induces a bijection between the lattice of nonzero $\mathcal{D}_{Y,x}$-submodules of $(\mathcal{H}^{n-1}_X \Omega^n_{Y/k})_x$ and the lattice of $k(x)$-submodules of $V(x)^*$.*

The proof of the theorem is given later in this section.

In order to globalize we introduce the following notation. Let $Z$ be the reduced subscheme supported on the singular locus $X_{\mathrm{sing}}$, so $\mathcal{O}_Z = \prod_{x \in X_{\mathrm{sing}}} k(x)$. Then $\mathcal{V} := \bigoplus_{x \in X_{\mathrm{sing}}} V(x)$ and $\mathcal{H}^n_Z \mathcal{O}_Y = \bigoplus_{x \in X_{\mathrm{sing}}} \mathcal{H}^n_{\{x\}} \mathcal{O}_Y$ are $\mathcal{O}_Z$-modules. Using $\Omega^n_{Y/k} \otimes$ to switch between left and right $\mathcal{D}_Y$-modules, and identifying $V(x)^* \cong V(x)$ by the trace pairing, we see that Theorem 7.5 implies

**Corollary 7.6.** *The homomorphism of $\mathcal{D}_Y$-modules*

$$T := \sum_x T_x : \mathcal{H}^{n-1}_X \mathcal{O}_Y \to (\mathcal{H}^n_Z \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} \mathcal{V}$$

*induces a bijection between the lattice of nonzero coherent $\mathcal{D}_Y$-submodules of $\mathcal{H}^{n-1}_X \mathcal{O}_Y$ and the lattice of $\mathcal{O}_Z$-submodules of $\mathcal{V}$.*

Since $\mathcal{H}^n_{\{x\}} \mathcal{O}_Y$ is a simple $\mathcal{D}_Y$-submodule, as immediate corollaries we get:

**Corollary 7.7.** *$\mathcal{H}^{n-1}_X \mathcal{O}_Y$ has a unique simple coherent $\mathcal{D}_Y$-submodule $\mathcal{L}(X, Y)$, and the sequence*

$$0 \to \mathcal{L}(X, Y) \to \mathcal{H}^{n-1}_X \mathcal{O}_Y \xrightarrow{T} (\mathcal{H}^n_Z \mathcal{O}_Y) \otimes_{\mathcal{O}_Z} \mathcal{V} \to 0 \tag{7.8}$$

*is exact.*

**Corollary 7.9.** *$\mathcal{H}^{n-1}_X \mathcal{O}_Y$ is a simple coherent $\mathcal{D}_Y$-module iff the singularities of $X$ are all geometrically unibranch.*

According to Proposition 5.11 the fundamental class $\mathrm{C}_{X/Y}$ is a double cocycle in $\mathcal{H}om(\Omega^1_{Y/k}, \mathcal{K}^{-1}_Y)$, so it determines a class in $(\mathcal{H}^{n-1}_X \mathcal{O}_Y) \otimes_{\mathcal{O}_Y} \Omega^{n-1}_{Y/k}$.

**Theorem 7.10.** $\mathrm{C}_{X/Y} \in \mathcal{L}(X,Y) \otimes_{\mathcal{O}_Y} \Omega^{n-1}_{Y/k}$.

This of course implies that if $\alpha_1, \ldots, \alpha_n$ is a local basis of $\Omega^{n-1}_{Y/k}$ and $\mathrm{C}_{X/Y} = \sum a_i \otimes \alpha_i$, then any nonzero $a_i$ generates $\mathcal{L}(X,Y)$ as a $\mathcal{D}_Y$-module. The proof of the theorem is given later in this section.

**Remark 7.11.** As the referee points out, when $k = \mathbb{C}$, Corollary 7.6 follows easily from the Riemann-Hilbert correspondence. In that case we may consider the sheaf $\mathcal{V}$ on the analytic space $X^{\mathrm{an}}$, given by $\mathcal{V} := \operatorname{Coker}(\mathbb{C}_{X^{\mathrm{an}}} \to \pi^{\mathrm{an}}_* \mathbb{C}_{\tilde{X}^{\mathrm{an}}})$. Now $\mathcal{IC}_{X^{\mathrm{an}}} \cong \pi^{\mathrm{an}}_* \mathbb{C}_{\tilde{X}^{\mathrm{an}}}[1]$. The triangle $\mathcal{V} \to \mathbb{C}_{X^{\mathrm{an}}}[1] \to \mathcal{IC}_{X^{\mathrm{an}}} \xrightarrow{+1}$ is an exact sequence in the category of perverse sheaves, and it is the image of (7.8) under the functor $\mathrm{Sol} = \mathrm{R}\mathcal{H}om_{\mathcal{D}_{Y^{\mathrm{an}}}}((-)^{\mathrm{an}}, \mathcal{O}_{Y^{\mathrm{an}}}[n])$. Nonetheless ours seems to be the first purely algebraic proof Theorem 7.5 and its corollaries (but cf. next remark).

**Remark 7.12.** When $Y = \mathbf{A}^2$ (i.e. $X$ is an affine plane curve) and $k$ is algebraically closed, Corollary 7.9 was partially proved by S.P. Smith [Sm], using the ring structure of $\mathcal{D}(X)$. Specifically, he proved that if $X$ has unibranch singularities, then $\mathcal{H}^1_X \mathcal{O}_Y$ is simple.

**Example 7.13.** Let $X$ be the nodal curve in $Y = \mathbf{A}^2 = \operatorname{Spec} k[s,t]$ defined by $f = s^2(s+1) - t^2$, and let $x$ be the origin. Take $r := t/s \in \mathcal{O}_{Y,w}$, so $s = (r+1)(r-1)$. We see that $\tilde{X} = \operatorname{Spec} k[r]$ and $r+1, r-1$ are regular parameters at $\tilde{x}_1, \tilde{x}_2$ respectively on $\tilde{X}$. For any coefficient field $\sigma$,

$$\operatorname{Res}^{\mathrm{lc}}_{w,\sigma} \begin{bmatrix} \mathrm{d}s \wedge \mathrm{d}t \\ f \end{bmatrix} = \operatorname{Res}^{\mathrm{lc}}_{w,\sigma} \begin{bmatrix} -\mathrm{d}(r+1) \wedge \mathrm{d}f \\ (r+1)(r-1)f \end{bmatrix} = \frac{-\mathrm{d}(r+1)}{(r+1)(r-1)}$$

and hence

$$\operatorname{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x}_1)} \begin{bmatrix} \mathrm{d}s \wedge \mathrm{d}t \\ f \end{bmatrix} = \operatorname{Res}_{(\tilde{w},\tilde{x}_1)} \frac{-\mathrm{d}(r+1)}{(r+1)(r-1)} = 2.$$

Likewise $\operatorname{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x}_2)} \left[\begin{smallmatrix} \mathrm{d}s\wedge\mathrm{d}t \\ f \end{smallmatrix}\right] = -2$. Therefore $\left[\begin{smallmatrix} \mathrm{d}s\wedge\mathrm{d}t \\ f \end{smallmatrix}\right] \notin \mathcal{L}(X,Y) \otimes \Omega^2_{Y/k}$. The fundamental class is $\mathrm{C}_{X/Y} = \left[\begin{smallmatrix} \mathrm{d}f \\ f \end{smallmatrix}\right]$, and as generator of $\mathcal{L}(X,Y) \otimes \Omega^2_{Y/k}$ we may take $\left[\begin{smallmatrix} \mathrm{d}s\wedge\mathrm{d}f \\ f \end{smallmatrix}\right]$.

Before getting to the proofs we need some general results. Let $A$ be a BCA over $k$. The fine topology on an $A$-module $M$ is the quotient topology w.r.t. any surjection $\bigoplus A \twoheadrightarrow M$. The fine topology on $M$ is $k$-linear, making it a topological $k$-module (but only a semi-topological (ST) $A$-module). According to [Ye2] Proposition 2.11.c, $A$ is a Zariski ST ring (cf. ibid. Definition 1.7). This means that any finitely generated $A$-module with the fine topology is separated, and any homomorphism $M \to N$ between such modules is topologically strict. Furthermore if $M$ is finitely generated then it is complete, so it is a complete linearly topologized $k$-vector space in the sense of [Ko].

**Lemma 7.14.** *Let $A$ be a BCA. Suppose $M$ is a countably generated ST $A$-module with the fine topology. Then $M$ is separated, and any submodule $M' \subset M$ is closed.*

*Proof.* Write $M = \bigcup_{i=1}^{\infty} M_i$ with $M_i$ finitely generated. Suppose we put the fine topology on $M_i$. Then each $M_i$ is separated and $M_i \to M_{i+1}$ is strict. By [Ye1] Corollary 1.2.6 we have $M \cong \lim_{i\to} M_i$ topologically, so by ibid. Proposition 1.1.7, $M$ is separated. By the same token $M/M'$ is separated too, so $M'$ is closed. $\square$

**Proposition 7.15.** *Let $A \to B$ be a morphism of BCAs, $N$ a finitely generated $B$-module with the fine topology, and $M \subset N$ a finitely generated $A$-module. Then the topology on $M$ induced by $N$ equals the fine $A$-module topology, and $M$ is closed in $N$.*

*Proof.* Since $A$ is a Zariski ST ring we may replace $M$ by any finitely generated $A$ module $M'$, $M \subset M' \subset N$. Therefore we can assume $N = BM$ and $M = \bigoplus_{\mathfrak{n} \in \operatorname{Max} B} M \cap N_{\mathfrak{n}}$. So in fact we may assume $A, B$ are both local. Like in the proof of [Ye2] Theorem 7.4 we may further assume that $\operatorname{res.dim}(A \to B) \leq 1$.

Put on $M$ the fine $A$-module topology. Let $\bar{N}_i := N/\mathfrak{n}^i N$ and $\bar{M}_i := M/(M \cap \mathfrak{n}^i N)$ with the quotient topologies. We claim $\bar{M}_i \to \bar{N}_i$ is a strict monomorphism. This is so because as $A$-modules both have the fine topology, $\bar{M}_i$ is finitely generated and $\bar{N}_i$ is countably generated (cf. part 1 in the proof of [Ye1] Theorem 3.2.14). Just as in part 2 of loc. cit. we get topological isomorphisms $M \cong \lim_{\leftarrow i} \bar{M}_i$ and $N \cong \lim_{\leftarrow i} \bar{N}_i$, so $M \to N$ is a strict monomorphism. But $M$ is complete and $N$ is separated, so $M$ must be closed. $\square$

Given a topological $k$-module $M$ we set $M^* := \operatorname{Hom}_k^{\mathrm{cont}}(M, k)$ (without a topology).

**Lemma 7.16.** *Suppose $M$ is a separated topological $k$-module. Then:*

1. *For any subset $S \subset M^*$ its perpendicular $S^{\perp} \subset M$ is a closed submodule.*
2. *Given a closed submodule $M_1 \subset M$, one has $M_1^{\perp\perp} = M_1$.*
3. *Suppose $M_1 \subset M_2 \subset M$ are closed submodules. Then there is an exact sequence (of untopologized $k$-modules)*

$$0 \to M_2^{\perp} \to M_1^{\perp} \to (M_2/M_1)^* \to 0.$$

*Proof.* See [Ko] Section 10.4, 10.8. $\square$

Let $M, N$ be complete separated topological $k$-modules, and $\langle -,- \rangle : M \times N \to k$ a continuous pairing. We say $\langle -,- \rangle$ is a *topological perfect pairing* if it induces isomorphisms $N \cong M^*$ and $M \cong N^*$.

**Proposition 7.17.** *Assume $k \to A$ is a morphism of BCAs. Then the residue pairing $\langle -,- \rangle_{A/k} : A \times \mathcal{K}(A) \to k$, $\langle a, \phi \rangle_{A/k} = \operatorname{Tr}_{A/k}(a\phi)$, is a topological perfect pairing.*

*Proof.* We may assume $A$ is local. Then $A = \lim_{\leftarrow i} A/\mathfrak{m}^i$ and $\mathcal{K}(A) = \lim_{i\rightarrow} \mathcal{K}(A/\mathfrak{m}^i)$ topologically. Let $K \to A$ be a coefficient field, so both $A/\mathfrak{m}^i$ and $\mathcal{K}(A/\mathfrak{m}^i) \cong \mathrm{Hom}_K(A/\mathfrak{m}^i, \omega(K))$ are finite $K$-modules with the fine topology. By [Ye1] Theorem 2.4.22 the pairing is perfect. □

From here to the end of this section we consider an integral curve $X$ embedded as a closed subscheme in a smooth irreducible $n$-dimensional variety $Y$. Fix a closed point $x \in X$, and set $A := \mathcal{O}_{Y,(x)}$ and $K := k(x)$. Choosing a regular system of parameters at $x$, say $\underline{t} = (t_1, \ldots, t_n)$, allows us to write $A = K[[\underline{t}]]$. Let $\mathcal{D}(A) := \mathrm{Diff}^{\mathrm{cont}}_{A/k}(A, A)$. Since both $K[\underline{t}] \to A$ and $\mathcal{O}_{Y,x} \to A$ are topologically étale relative to $k$, we have

$$\mathcal{D}(A) \cong A \otimes_K K[\tfrac{\partial}{\partial t_1}, \ldots, \tfrac{\partial}{\partial t_n}] \cong A \otimes_{\mathcal{O}_{Y,x}} \mathcal{D}_{Y,x}$$

(cf. [Ye2] Section 4).

Define $B := \mathcal{O}_{Y,(w,x)}$. Since $A \to B$ is topologically étale relative to $k$, we get a $k$-algebra homomorphism $\mathcal{D}(A) \to \mathcal{D}(B)$. In particular, $B$ and $\mathcal{K}(B)$ are $\mathcal{D}(A)$-modules. Define $L := \prod_{\tilde{x} \in \pi^{-1}(x)} k(\tilde{x})$ as before.

**Lemma 7.18.** *The multiplication map $A \otimes_K L \to B$ is injective. Its image is a $\mathcal{D}(A)$-submodule of $B$. Any $\mathcal{D}(A)$-submodule of $B$ which is finitely generated over $A$ equals $A \otimes_K W$ for some $K$-submodule $W \subset L$.*

*Proof.* By [Kz] Proposition 8.9, if $M$ is any $\mathcal{D}(A)$-module which is finitely generated over $A$, then $M = A \otimes_K W$, where $W \subset M$ is the $K$-submodule consisting of all elements killed by the derivations $\frac{\partial}{\partial t_i}$. Note that $\Omega^{1,\mathrm{sep}}_{B/k}$ is free with basis $\mathrm{d}t_1, \ldots, \mathrm{d}t_n$. Thus it suffices to prove that

$$L = \{b \in B \mid \tfrac{\partial}{\partial t_1} b = \cdots = \tfrac{\partial}{\partial t_n} b = 0\} = \mathrm{H}^0 \Omega^{\cdot,\mathrm{sep}}_{B/k}. \tag{7.19}$$

We know that $B \cong L((g))[[f_1, \ldots, f_{n-1}]]$, so $B$ is topologically étale over the polynomial algebra $k[g, f_1, \ldots, f_{n-1}]$ (relative to $k$), and hence $\mathrm{d}g, \mathrm{d}f_1, \ldots, \mathrm{d}f_{n-1}$ is also a basis of $\Omega^{1,\mathrm{sep}}_{B/k}$. It follows that $\mathrm{H}^0 \Omega^{\cdot,\mathrm{sep}}_{B/k} = L$. □

*Proof.* (of Theorem 7.5) Set $\mathcal{M} := \mathcal{H}^{n-1}_X \Omega^n_{Y/k}$ and define $M := A \otimes_{\mathcal{O}_{Y,x}} \mathcal{M}_x$. Tensoring the exact sequence (7.3) with $A$ we get an exact sequence of $\mathcal{D}(A)$-modules

$$0 \to M \to \mathcal{K}(B) \xrightarrow{\delta} \mathcal{K}(A) \to 0. \tag{7.20}$$

The proof will use repeatedly the residue pairing $\langle -, - \rangle_{B/k} : B \times \mathcal{K}(B) \to k$. By definition of $\delta$ (cf. Definition 2.5 and [Ye2] Section 7) we see that $M = A^{\perp}$. Consider the closed $k$-submodules $A \subset A \otimes_K L \subset B$ (cf. Proposition 7.15). Applying Lemma 7.16 to them, and using $V(x) = L/K$ and $\mathcal{K}(A) \cong A^*$, we get an exact sequence of $\mathcal{D}(A)$-modules

$$0 \to (A \otimes_K L)^{\perp} \to M \xrightarrow{T'} V(x)^* \otimes_K \mathcal{K}(A) \to 0.$$

Keeping track of the operations we see that in fact $T' = T_x|_M$.

Put the fine $A$-module topology on $M$ and $\mathcal{K}(A)$, so $M \to V(x)^* \otimes_K \mathcal{K}(A)$ is continuous. By [Ye1] Proposition 1.1.8, $\mathcal{M}_x \to M$ is dense. Since $\mathcal{K}(A)$ is discrete we conclude that $\mathcal{M}_x \to V(x)^* \otimes_K \mathcal{K}(A)$ is a surjection of $\mathcal{D}_{Y,x}$-modules. Thus any $K$-module $W \subset V(x)^*$ determines a distinct nonzero $\mathcal{D}_{Y,x}$-module $\mathcal{N}_x \subset \mathcal{M}_x$.

Conversely, say $\mathcal{N}_x \subset \mathcal{M}_x$ is a nonzero $\mathcal{D}_{Y,x}$-module. On any open set $U \subset Y$ s.t. $U \cap X$ is smooth the module $\mathcal{M}|_U$ is a simple coherent $\mathcal{D}_U$-module (by Kashiwara's Theorem it corresponds to the $\mathcal{D}_{X\cap U}$-module $\Omega^1_{(X\cap U)/k}$). Therefore the finitely generated $\mathcal{D}_{Y,x}$-module $C$ defined by

$$0 \to \mathcal{N}_x \to \mathcal{M}_x \to C \to 0 \tag{7.21}$$

is supported on $\{x\}$. It follows that $C \cong \mathcal{K}(A)^r$ for some number $r$. Tensoring (7.21) with $A$ we get an exact sequence of $\mathcal{D}(A)$-modules

$$0 \to N \to M \to C \to 0$$

with $N \subset M \subset \mathcal{K}(B)$. By faithful flatness of $\mathcal{O}_{Y,x} \to A$ we see that $\mathcal{N}_x = \mathcal{M}_x \cap N$.

We put on $M, N$ the topology induced from $\mathcal{K}(B)$, and on $C$ the quotient topology from $M$. Now $\mathcal{K}(B)$ has the fine $A$-module topology and it is countably generated over $A$ (cf. proof of Proposition 7.15), so by Lemma 7.14 both $M, N$ are closed in $\mathcal{K}(B)$. Using Lemma 7.16 and the fact that $M^\perp = A$ we obtain the exact sequence

$$0 \to A \to N^\perp \to C^* \to 0,$$

with $N^\perp \subset B$. We do not know what the topology on $C$ is; but it is a ST $A$-module. Hence the identity map $\mathcal{K}(A)^r \to C$ is continuous, and it induces an $A$-linear injection $C^* \to A^r$. Therefore $C^*$, and thus also $N^\perp$, are finitely generated over $A$. According to Lemma 7.18, $N^\perp = A \otimes_K W$ for some $K$-module $W$, $K \subset W \subset L$. But $N$ is closed, so $N = (N^\perp)^\perp$. □

*Proof.* (of Theorem 7.10) For each $\tilde{x} \in \pi^{-1}(x)$ define a homomorphism

$$T_{(\tilde{w},\tilde{x})} : \mathcal{K}(B) \xrightarrow{\mathrm{Tr}} \mathcal{K}(L \otimes_K A) \cong L \otimes_K \mathcal{K}(A) \to k(\tilde{x}) \otimes_K \mathcal{K}(A),$$

so $T_x = \sum T_{(\tilde{w},\tilde{x})}$. From the proof of Theorem 7.5 we see that the theorem amounts to the claim that $T_{\tilde{x}}(\mathrm{C}_{X/Y}(\alpha)) = 0$ for every $\tilde{x}$ and $\alpha \in \Omega^1_{Y/k,x}$. But $\mathrm{C}_{X/Y}$ is the image of $\mathrm{C}_X \in \mathcal{H}om(\Omega^1_{X/k}, \mathcal{K}_X^{-1}(w))$, so we can reduce our residue calculation to the curve $\tilde{X}$. In fact it suffices to show that for every $\alpha \in \Omega^1_{X/k,x}$ one has $\mathrm{Res}_{(\tilde{w},\tilde{x})}\, \alpha = 0$. Since $\alpha \in \Omega^1_{\tilde{X}/k,\tilde{x}}$ this is obvious. □

*Proof.* (of Theorem 7.1) According to [SY] Corollary 0.2.11 (or [Hu] Theorem 2.2) one has

$$\mathrm{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x})} = (1 \otimes \mathrm{Tr}_{A/K}) T_{(\tilde{w},\tilde{x})} : \mathrm{H}^{n-1}_w \Omega^n_{Y/k} \to k(\tilde{x}),$$

which shows that $\mathrm{Res}^{\mathrm{lc}}_{(\tilde{w},\tilde{x})}$ is independent of $\sigma$. Now use Theorem 7.5. □

**Problem 7.22.** What is the generalization to $\dim X > 1$? To be specific, assume $X$ has only an isolated singularity at $x$. Then we know there is an exact sequence

$$0 \to \mathcal{L}(X, Y) \to \mathcal{H}^d_X \mathcal{O}_Y \xrightarrow{T} \mathcal{H}^n_{\{x\}} \mathcal{O}_Y \otimes_{k(x)} V(x) \to 0$$

for some $k(x)$-module $V(x)$. What is the geometric data determining $V(x)$ and $T$? Is it true that $T = \sum T_\xi$, a sum of "residues" along chains $\xi \in \pi^{-1}(x)$, for a suitable resolution of singularities $\pi : \tilde{X} \to X$?

## References


[BaK] D. Barlet and M. Kashiwara, Le réseau $L^2$ d'un système holonome régulier, Invent. Math. **86** (1986), 35-62.

[Bj] J.E. Björk, "Rings of Differential Operators," North Holland, Amsterdam, 1979.

[Be] A.A. Beilinson, Residues and adeles, Funkt. Anal. Pril. **14** (1980) no. 1, 44-45; English trans. in Func. Anal. Appl. **14** (1980) no. 1, 34-35.

[BlO] S. Bloch and A. Ogus, Gersten's conjecture and the homology of schemes, Ann. Scient. ENS, série 4, **7** (1974), 181-202.

[Bo] A. Borel et al. , "Algebraic $\mathcal{D}$-Modules", Academic Press, Boston, 1987.

[Br] J.L. Brylinski, La classe fondamentale d'une variété algébrique engendre le $\mathcal{D}$-module qui calcul sa cohomologie d'intersection (d'après M. Kashiwara). In: Système différentiel et singularités, Astérisque **130** (1985).

[BrK] J.L. Brylinski and M. Kashiwara, Kazhdan-Lusztig conjecture and holonomic systems, Invent. Math. **64** (1981), 387-410.

[CA] N. Bourbaki, "Commutative Algebra", Hermann, Paris, 1972.

[EGA] A. Grothendieck and J. Dieudonné, "Éléments de Géometrie Algébrique"; [EGA] I: Springer-Verlag, Berlin, 1971; [EGA] II-IV: Publ. Math. IHES nos. **8**, **11**, **17**, **20**, **24**, **28**, **32**.

[EZ] F. El Zein, Complexe Dualizant et Applications à la Classe Fondamentale d'un Cycle, Bull. Soc. Math. France, Mémoire **58**, 1978.

[Gr] A. Grothendieck, On the De Rham cohomology of algebraic varieties, Publ. Math. IHES **29** (1966), 351-359.

[Ha] R. Hartshorne, On the De Rham Cohomology of algebraic varieties, Publ. Math. IHES **45** (1975), 5-99.

[Hg] I.C. Huang, Functorial construction of Cousin complexes, preprint.

[HK1] R. Hübl and E. Kunz, Integration of Differential Forms on Schemes, J. reine angew. Math. **410** (1990), 53-83.

[HK2] R. Hübl and E. Kunz, Regular differential forms and duality for projective morphisms, J. reine angew. Math. **410** (1990), 84-108.

[Ho] M. Holland, Duality and $\mathcal{D}$-modules on curves, Bull. Soc. Math. Belg. t. XLIII (1991).

[HoSt] M.P. Holland and J.T. Stafford, Differential operators on rational projective curves, J. Algebra **147** (1992), 176-244.

[HS] R. Hübl and P. Sastry, Regular differential forms and relative duality, Amer. J. Math. **115** no. 4 (1993), 749-787.

[Hu] R. Hübl, Residues of regular and meromorphic differential forms, Math. Annalen **300** (1994), 605-628.

[HY1] R. Hübl and A. Yekutieli, Adeles and differential forms, J. reine angew. Math. **147** (1996), 1-22.

[HY2] R. Hübl and A. Yekutieli, Adelic Chern forms and the Bott residue formula, preprint.

[Kl] S.L. Kleiman, Relative duality for quasi-coherent sheaves, Compositio Math. **41** (1980), 39-60.

[Ko] G. Köthe, "Topological Vector Spaces I", Springer-Verlag, New-York, 1969.

[Kz] N. Katz, Nilpotent connections and the monodromy theorem: applications of a result of Turrittin, Publ. Math. IHES **39** (1970), 175-232.

[KW] E. Kunz and R. Waldi, "Regular Differential Forms," Contemp. Math. **79**, AMS, Providence, 1988.

[Li] J. Lipman, Dualizing sheaves, differentials and residues on algebraic varieties, Astérisque **117** (1984).

[LS] J. Lipman and P. Sastry, Regular differentials and equidimensional scheme-maps, J. Alg. Geom. **1** (1992), 101-130.

[Mi] J.S. Milne, "Étale Cohomology", Princeton University Press, Princeton N.J., 1980.

[ML] S. Mac Lane, "Homology," Springer, Berlin, 1995 (reprint of the 1975 edition).

[MR] J.C. McConnell and J.C. Robson, "Noncommutative Noetherian Rings," Wiley, Chichester, 1988.

[RD] R. Hartshorne, "Residues and Duality", Lecture Notes in Math. 20, Springer-Verlag, Berlin, 1966.

[Sa] P. Sastry, Residues and duality on algebraic schemes, Compositio Math. **101** (1996), 133-178.

[Sm] S.P. Smith, The simple $\mathcal{D}$-module associated to the intersection cohomology complex for a class of plane curves, J. Pure Appl. Algebra **50** (1988), 287-294.

[Sai] M. Saito, Modules de hodge polarisables, Publ. RIMS, Kyoto Univ. **24** (1988), 849-995.

[SY] P. Sastry and A. Yekutieli, On residue complexes, dualizing sheaves and local cohomology modules, Israel J. Math. **90** (1995), 325-348.

[Vi] K. Vilonen, Intersection homology $\mathcal{D}$-module on local complete intersections with isolated singularities, Invent. Math. **81** (1985), 107-114.

[Ye1] A. Yekutieli, "An Explicit Construction of the Grothendieck Residue Complex" (with an appendix by P. Sastry), Astérisque **208** (1992).

[Ye2] A. Yekutieli, Traces and differential operators over Beilinson completion algebras, Compositio Math. **99** (1995), 59-97.

[Ye3] A. Yekutieli, Smooth formal embeddings and the residue complex, preprint.

[Ye4] A. Yekutieli, The action of adeles on the residue complex, preprint.



Department of Theoretical Mathematics, The Weizmann Institute of Science, Rehovot 76100, ISRAEL

*E-mail address*: amnon@wisdom.weizmann.ac.il