Algebras and Representation Theory **7:** 53–57, 2004.

The Derived Picard Group is a Locally Algebraic Group

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(Received: December 2000; accepted: January 2001)

Presented by M. Van der Bergh

Abstract. Let A be a finite-dimensional algebra over an algebraically closed field $\mathbb{K}$. The derived Picard group $\mathrm{DPic}_{\mathbb{K}}(A)$ is the group of two-sided tilting complexes over A modulo isomorphism. We prove that $\mathrm{DPic}_{\mathbb{K}}(A)$ is a locally algebraic group, and its identity component is $\mathrm{Out}^0_{\mathbb{K}}(A)$. If B is a derived Morita equivalent algebra then $\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathrm{DPic}_{\mathbb{K}}(B)$ as locally algebraic groups. Our results extend, and are based on, work of Huisgen-Zimmermann, Saorín and Rouquier.

Mathematics Subject Classifications (2000): Primary: 16D90; secondary: 16G10, 18E30, 20G15.

Key words: Picard group, locally algebraic group.

Let A and B be associative algebras with 1 over a field $\mathbb{K}$. We denote by $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A)$ the bounded derived category of left A-modules. Let B° be the opposite algebra, so an $A \otimes_{\mathbb{K}} B^{\circ}$-module is a $\mathbb{K}$-central A-B-bimodule. A *two-sided tilting complex* over (A, B) is a complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\circ})$ such that there exists a complex $T^{\vee} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\circ})$ and isomorphisms of the derived tensor products $T \otimes^{\mathrm{L}}_{B} T^{\vee} \cong A$ and $T^{\vee} \otimes^{\mathrm{L}}_{A} T \cong B$. Two-sided tilting complexes were introduced by Rickard in [Rd].

When $B = A$ we write $A^{\mathrm{e}} := A \otimes_{\mathbb{K}} A^{\circ}$. The set

$$\mathrm{DPic}_{\mathbb{K}}(A) := \frac{\{\text{two-sided tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})\}}{\text{isomorphism}}$$

is the *derived Picard group of* A (relative to $\mathbb{K}$). The identity element is the class of A, the multiplication is $(T_1, T_2) \mapsto T_1 \otimes^{\mathrm{L}}_{A} T_2$, and the inverse is $T \mapsto T^{\vee} = \mathrm{RHom}_A(T, A)$.

Denote by $\mathrm{Out}_{\mathbb{K}}(A)$ the group of outer $\mathbb{K}$-algebra automorphism of A, and by $\mathrm{Pic}_{\mathbb{K}}(A)$ the Picard group of A (the group of invertible bimodules modulo isomorphism). Then there are inclusions

$$\mathrm{Out}_{\mathbb{K}}(A) \subset \mathrm{Pic}_{\mathbb{K}}(A) \subset \mathrm{DPic}_{\mathbb{K}}(A).$$

* Partially supported by a grant from the US-Israel Binational Science Foundation.

The first inclusion sends the automorphism σ to the invertible bimodule A^{σ} where the right action is twisted by σ. The second inclusion corresponds to the full embedding $\mathsf{Mod}\, A^{\mathrm{e}} \subset \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$. See [Ye] for details.

To simplify notation we use the same symbol to denote an automorphism $\sigma \in \mathrm{Aut}_{\mathbb{K}}(A)$ and its class in $\mathrm{Out}_{\mathbb{K}}(A)$. Likewise for a two-sided tilting complex T and its class in $\mathrm{DPic}_{\mathbb{K}}(A)$. The precise meaning is always clear from the context.

Now assume $\mathbb{K}$ is algebraically closed and A is a finite-dimensional $\mathbb{K}$-algebra. Then the group $\mathrm{Aut}_{\mathbb{K}}(A) = \mathrm{Aut}_{\mathsf{Alg}\,\mathbb{K}}(A)$ of $\mathbb{K}$-algebra automorphisms is a linear algebraic group, being a closed subgroup of $\mathrm{GL}(A) = \mathrm{Aut}_{\mathsf{Mod}\,\mathbb{K}}(A)$. This induces a structure of linear algebraic group on the quotient $\mathrm{Out}_{\mathbb{K}}(A)$. Denote by $\mathrm{Out}^{0}_{\mathbb{K}}(A)$ the identity component.

Examples calculated in [MY] indicated that the whole group $\mathrm{DPic}_{\mathbb{K}}(A)$ should carry a geometric structure (cf. Example 3 below). This is our first main result Theorem 2.

A result of Brauer says that the group $\mathrm{Out}^{0}_{\mathbb{K}}(A)$ is a Morita invariant of A: if A and B are Morita equivalent $\mathbb{K}$-algebras then $\mathrm{Out}^{0}_{\mathbb{K}}(A) \cong \mathrm{Out}^{0}_{\mathbb{K}}(B)$. In [HS] and [Ro] this is extended to derived Morita equivalence. Our Theorem 4 extends these results further.

We shall need the following variant of the result of Huisgen-Zimmermann, Saorín and Rouquier.

THEOREM 1. *Let A and B be finite-dimensional $\mathbb{K}$-algebras. Suppose $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\circ})$ is a two-sided tilting complex over (A, B), with inverse $T^{\vee} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B \otimes_{\mathbb{K}} A^{\circ})$. Then for any element $\sigma \in \mathrm{Out}^{0}_{\mathbb{K}}(A)$ the two-sided tilting complex*

$$\phi^{0}_{T}(\sigma) := T^{\vee} \otimes^{\mathrm{L}}_{A} A^{\sigma} \otimes^{\mathrm{L}}_{A} T \in \mathrm{DPic}_{\mathbb{K}}(B)$$

is in $\mathrm{Out}^{0}_{\mathbb{K}}(B)$. The group homomorphism

$$\phi^{0}_{T}\colon \mathrm{Out}^{0}_{\mathbb{K}}(A) \longrightarrow \mathrm{Out}^{0}_{\mathbb{K}}(B)$$

is an isomorphism of algebraic groups.

Proof. According to [HS, Theorem 17] or [Ro, Théorème 4.2] there is an isomorphism of algebraic groups $\phi^{0}\colon \mathrm{Out}^{0}_{\mathbb{K}}(A) \to \mathrm{Out}^{0}_{\mathbb{K}}(B)$ induced by T. Letting $\tau := \phi^{0}(\sigma) \in \mathrm{Out}^{0}_{\mathbb{K}}(B)$ one has

$$T \otimes_{B} B^{\tau} \cong A^{\sigma} \otimes_{A} T \quad \text{in } \mathsf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\circ}).$$

Applying $T^{\vee} \otimes^{\mathrm{L}}_{A} -$ to this isomorphism we see that $B^{\tau} \cong \phi^{0}_{T}(\sigma)$ in $\mathsf{D}(\mathsf{Mod}\, B^{\mathrm{e}})$, so $\tau = \phi^{0}_{T}(\sigma)$ in $\mathrm{DPic}_{\mathbb{K}}(B)$. We conclude that $\phi^{0}_{T} = \phi^{0}$. □

A *locally algebraic group* over $\mathbb{K}$ is a group G, with a normal subgroup G^{0}, such that G^{0} is a connected algebraic group over $\mathbb{K}$, each coset of G^{0} is a variety, and multiplication and inversion are morphisms of varieties. A morphism $\phi\colon G \to H$ of locally algebraic groups is a group homomorphism such that $\phi(G^{0}) \subset H^{0}$

and the restriction $\phi^0\colon G^0 \to H^0$ is a morphism of varieties. We call ϕ an open immersion if ϕ is injective and ϕ^0 is an isomorphism.

In other words G is the group of rational points $\boldsymbol{G}(\mathbb{K})$ of a reduced group scheme $\boldsymbol{G}$ locally of finite type over $\mathbb{K}$, in the sense of [SGA3, Exposé $\mathrm{VI_A}$]. A morphism $\phi\colon G \to H$ corresponds to a morphism $\boldsymbol{\phi}\colon \boldsymbol{G} \to \boldsymbol{H}$ of group schemes over $\mathbb{K}$.

Here is our first main result.

THEOREM 2. *Let A be a finite-dimensional $\mathbb{K}$-algebra. Then the derived Picard group* $\mathrm{DPic}_{\mathbb{K}}(A)$ *is a locally algebraic group over $\mathbb{K}$. The inclusion* $\mathrm{Out}_{\mathbb{K}}(A) \subset \mathrm{DPic}_{\mathbb{K}}(A)$ *is an open immersion.*

In particular the identity components coincide: $\mathrm{Out}^0_{\mathbb{K}}(A) = \mathrm{DPic}^0_{\mathbb{K}}(A)$.

Proof. Theorem 1 with $A = B$ implies that the subgroup $\mathrm{Out}^0_{\mathbb{K}}(A) \subset \mathrm{DPic}_{\mathbb{K}}(A)$ is normal, and for any two-sided tilting complex T the conjugation $\phi^0_T\colon \mathrm{Out}^0_{\mathbb{K}}(A) \to \mathrm{Out}^0_{\mathbb{K}}(A)$ is an automorphism of algebraic groups.

Let us now switch to the notation $T_1 \cdot T_2$ and T^{-1} for the operations in $\mathrm{DPic}_{\mathbb{K}}(A)$. Define an algebraic variety structure on each coset $C = T \cdot \mathrm{Out}^0_{\mathbb{K}}(A) \subset \mathrm{DPic}_{\mathbb{K}}(A)$ using the multiplication map $P \mapsto T \cdot P$, $P \in \mathrm{Out}^0_{\mathbb{K}}(A)$. Since ϕ^0_T is an automorphism of algebraic groups, the variety structure is independent of the representative $T \in C$.

Let us prove that $\mathrm{DPic}_{\mathbb{K}}(A)$ is a locally algebraic group. For $P_1, P_2 \in \mathrm{Out}^0_{\mathbb{K}}(A)$ and $T_1, T_2 \in \mathrm{DPic}_{\mathbb{K}}(A)$, multiplication is the morphism

$$(T_1 \cdot P_1) \cdot (T_2 \cdot P_2) = (T_1 \cdot T_2) \cdot (\phi^0_{T_2}(P_1) \cdot P_2).$$

Similarly for the inverse:

$$(T \cdot P)^{-1} = T^{-1} \cdot \phi^0_T(P)^{-1}. \qquad \square$$

EXAMPLE 3. Let $\vec{\Omega}_n$ be the quiver with two vertices x, y and n arrows $x \xrightarrow{\alpha_i} y$. Let A be the path algebra $\mathbb{K}\vec{\Omega}_n$. According to [MY, Theorem 5.3], $\mathrm{Out}_{\mathbb{K}}(A) \cong \mathrm{Pic}_{\mathbb{K}}(A) \cong \mathrm{PGL}_n(\mathbb{K})$ and

$$\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathbb{Z} \times (\mathbb{Z} \ltimes \mathrm{PGL}_n(\mathbb{K})).$$

In the semi-direct product a generator T of $\mathbb{Z}$ acts on a matrix $\sigma \in \mathrm{PGL}_n(\mathbb{K})$ by $\phi^0_T(\sigma) = (\sigma^{-1})^{\mathrm{t}}$. This is clearly a morphism of varieties, so $\mathrm{DPic}_{\mathbb{K}}(A)$ is indeed a locally algebraic group.

Our second main result relates two algebras. Recall that the algebras A and B are derived Morita equivalent over $\mathbb{K}$ if there is a $\mathbb{K}$-linear equivalence of triangulated categories $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, B)$.

THEOREM 4. *Suppose A and B are two finite-dimensional $\mathbb{K}$-algebras, and assume they are derived Morita equivalent over $\mathbb{K}$. Then* $\mathrm{DPic}_{\mathbb{K}}(A) \cong \mathrm{DPic}_{\mathbb{K}}(B)$ *as locally algebraic groups.*

Proof. It is known that there exist two-sided tilting complexes $T \in \mathsf{D}(\mathsf{Mod}\, A \otimes_{\mathbb{K}} B^{\circ})$; choose one. We obtain a group isomorphism

$$\phi_T \colon \begin{cases} \mathrm{DPic}_{\mathbb{K}}(A) \longrightarrow \mathrm{DPic}_{\mathbb{K}}(B), \\ S \longmapsto T^{\vee} \otimes^{\mathsf{L}}_A S \otimes^{\mathsf{L}}_A T. \end{cases}$$

By Theorem 1, ϕ_T restricts to an isomorphism of algebraic groups $\phi_T^0\colon \mathrm{Out}^0_{\mathbb{K}}(A) \to \mathrm{Out}^0_{\mathbb{K}}(B)$. So ϕ_T is an isomorphism of locally algebraic groups. □

We end the paper with a corollary and some remarks. Suppose C is a $\mathbb{K}$-linear triangulated category that is equivalent to a small category. Denote by $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{C})$ the group of $\mathbb{K}$-linear triangle auto-equivalences of C modulo natural isomorphism. Let $\mathsf{mod}\, A$ stand for the category of finitely generated A-modules.

COROLLARY 5. *Suppose* C *is a* $\mathbb{K}$*-linear triangulated category that is equivalent to* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)$ *for some hereditary finite-dimensional* $\mathbb{K}$*-algebra* A*. Then* $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{C})$ *is a locally algebraic group.*

Proof. Trivially $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{C}) \cong \mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A))$, and by [MY, Corollary 0.11] we have $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, A)) \cong \mathrm{DPic}_{\mathbb{K}}(A)$. □

EXAMPLE 6. Beilinson [Be] proved that $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, \mathbf{P}^1_{\mathbb{K}}) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, \mathbb{K}\vec{\Omega}_2)$, where $\mathsf{Coh}\, \mathbf{P}^1_{\mathbb{K}}$ is the category of coherent sheaves on the projective line, and $\vec{\Omega}_2$ is the quiver from Example 3. Therefore, $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, \mathbf{P}^1_{\mathbb{K}}))$ is a locally algebraic group. This should be compared to Remark 7 below; see also [MY, Remark 5.4].

Remark 7. Suppose X is a smooth projective variety over $\mathbb{K}$ with ample canonical or anti-canonical bundle. Bondal and Orlov [BO] proved that

$$\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)) \cong (\mathrm{Aut}_{\mathbb{K}}(X) \ltimes \mathrm{Pic}(X)) \times \mathbb{Z}.$$

Here $\mathrm{Pic}(X)$ is the group of line bundles. Thus, $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)) \cong G \times D$, where G is an algebraic group and D is a discrete group and, in particular, this is a locally algebraic group.

Remark 8. In [Or], Orlov gives an example of an Abelian variety over $\mathbb{K}$ such that

$$\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X)) \cong D \ltimes (X \times \widehat{X})(\mathbb{K}),$$

where D is a discrete group (an extension of $\mathrm{SL}_2(\mathbb{Z})$ by $\mathbb{Z}$) and $\widehat{X}$ is the dual Abelian variety. The group D acts (nontrivially) via $\mathrm{Aut}_{\mathbb{K}}(X \times \widehat{X})$ and hence $\mathrm{Out}^{\mathrm{tr}}_{\mathbb{K}}(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\, X))$ is a locally algebraic group.

Acknowledgements

I wish to thank Birge Huisgen-Zimmermann for showing me [HS], and for some very helpful comments on an earlier version of this paper. Thanks also to Raphael Rouquier for calling my attention to [Ro] and for illuminating discussions.

References

[Be] Beilinson, A. A.: Coherent sheaves on $\mathbf{P}^n$ and problems of linear algebra, *Funct. Anal. Appl.* **12** (1978), 214–216.

[BO] Bondal, A. I. and Orlov, D. O.: Reconstruction of a variety from the derived category and groups of autoequivalences, *Compositio Math.* **125**(3) (2001), 327–344.

[HS] Huisgen-Zimmermann, B. and Saorín, M.: Geometry of chain complexes and outer automorphism groups under derived equivalence, *Trans. Amer. Math. Soc.* **353**(12) (2001), 4757–4777.

[MY] Miyachi, J. and Yekutieli, A.: Derived Picard groups of finite dimensional hereditary algebras, *Compositio Math.* **129**(3) (2001), 341–368.

[Or] Orlov, D. O.: On equivalences of derived categories of coherent sheaves on Abelian varieties, *Izv. Math.* **66**(3) (2002), 569–594.

[Rd] Rickard, J.: Derived equivalences as derived functors, *J. London Math. Soc.* **43** (1991), 37–48.

[Ro] Rouquier, R.: Groupes d'automorphismes et équivalences stables ou dérivées, preprint.

[SGA3] *Schémas en groupes*, Lecture Notes in Math. 151, Springer-Verlag, Berlin, 1970.

[Ye] Yekutieli, A.: Dualizing complexes, Morita equivalence and the derived Picard group of a ring, *J. London Math. Soc.* **60** (1999), 723–746.