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Dualizing Complexes over Noncommutative Graded Algebras

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INTRODUCTION

The motivation for the work presented in this paper was two questions posed by M. Artin. Let A be a d dimensional regular graded algebra over the field k, with augmentation ideal m (see Def. 4.13). Denote by $\mathrm{H}^d_m(A)$ (resp. $\mathrm{H}^d_{m^\circ}(A)$) the dth left (resp. right) local cohomology of A at the ideal m; i.e., $\mathrm{H}^d_m(A) = \lim_{n\to} \mathrm{Ext}^d_A(A/m^n, A)$ and similarly for $\mathrm{H}^d_{m^\circ}(A)$. As left modules and as right modules, $\mathrm{H}^d_m(A)$ and $\mathrm{H}^d_{m^\circ}(A)$ are both isomorphic to $A' := \mathrm{Hom}_k(A, k)$ (up to a twist in degrees; all operations are taking place in the graded category).

Question 1. Can the bimodule structure of $\mathrm{H}^d_m(A)$ be described in terms of other invariants of A?

Question 2. Are the bimodules $\mathrm{H}^d_m(A)$ and $\mathrm{H}^d_{m^\circ}(A)$ isomorphic?

In order to answer these questions it became necessary to extend Grothendieck's duality formalism of [RD] to deal with noncommutative graded k-algebras. Let A be such an algebra. Denote by A° its opposite algebra and by A^e the algebra $A \otimes_k A^\circ$. Given a complex $R^\cdot \in \mathbf{D}^+(A^e)$ we have derived functors $\mathrm{RHom}^\cdot_A(-, R^\cdot)\colon \mathbf{D}(A)^\circ \to \mathbf{D}(A^\circ)$ and $\mathrm{RHom}^\cdot_{A^\circ}(-, R^\cdot)\colon \mathbf{D}(A^\circ)^\circ \to \mathbf{D}(A)$ (we are assuming the reader is familiar with the language of derived categories). Let $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ be the derived category of bounded complexes of A-modules with coherent (Def. 1.1) cohomologies. A dualizing complex over A is, loosely speaking, a complex $R^\cdot \in \mathbf{D}^+(A^e)$ s.t. the functors $\mathrm{RHom}^\cdot_A(-, R^\cdot)$ and $\mathrm{RHom}^\cdot_{A^\circ}(-, R^\cdot)$ interchange the categories $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^\circ)$ and are inverses to each other (see Def. 3.3 and Prop. 3.5). We prove the following uniqueness theorem (Thm. 3.9):

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THEOREM. *Let $R^\cdot$ be a dualizing complex over A, and let $\tilde{R}^\cdot$ be any complex in $\mathbf{D}^+(A^e)$. Then $\tilde{R}^\cdot$ is dualizing iff $\tilde{R}^\cdot \cong R^\cdot \otimes_A L[n]$ in $\mathbf{D}(A^e)$ for some invertible bimodule L and some integer n. Moreover, n is unique and L is unique up to isomorphism.*

Unlike the commutative theory, where one is free to localize and pass to subschemes, these methods are not available to us, and we cannot prove a local duality theorem [RD, Chap. V Thm. 6.2]. In fact, such a theorem does not hold in general, as is demonstrated by Examples 4.15 and 4.16. Therefore we make it a feature of the dualizing complex (Def. 4.1):

DEFINITION. A dualizing complex $R^\cdot$ over A is called balanced if $\mathrm{H}^0_m(R^\cdot) \cong \mathrm{H}^0_{m^\circ}(R^\cdot) \cong A'$ as bimodules, and $\mathrm{H}^q_m(R^\cdot) = \mathrm{H}^q_{m^\circ}(R^\cdot) = 0$ for $q \neq 0$.

When A is noetherian and has a balanced dualizing complex, local duality holds—see Theorem 4.18. Such a complex is unique (up to isomorphism). We prove the existence of a balanced dualizing complex for the following classes of algebras:

(i) Noetherian regular algebras (Cor. 4.14).

(ii) Finitely generated k-algebras finite over their centers (Cor. 5.6).

(iii) Skew homogeneous coordinate rings (Thm. 7.3).

For the first two classes of algebras the proofs are formal and make use of "infinitesimal calculus"—computing duality on finite length modules (e.g., Thm. 4.8). The proof of the existence of a balanced dualizing complex over a skew homogeneous coordinate ring is constructive. Let $B = B(X, \sigma, \mathscr{L})$ be such a ring (Def. 6.6). We show how Γ_*, initially defined as a functor from $\mathcal{O}_X$-modules to B-modules (see [AV, Sect. 3]), extends to a functor from σ-equivariant $\mathcal{O}_X$-modules, to B–B-modules. Let $\pi: X \to \operatorname{Spec} k$ be the structural morphism. The residue complex $\mathscr{K}^\cdot_X$ on X, which is the Cousin complex associated to $\pi^! k$ (see [RD, Chap. VI, Sect. 1), has a canonical equivariance which is denoted by ε. We introduce a complex of bimodules $\mathscr{K}^\cdot_B$ by adding a term in dimension 0 to the complex $\Gamma_*(\mathscr{K}^\cdot_X, \varepsilon)[1]$ (Def. 7.2). We then use Grothendieck's duality theorem fo proper morphisms to prove (Thm. 7.3)

THEOREM. *The complex $\mathscr{K}^\cdot_B$ is a balanced dualizing complex over B.*

Let us return to the two original questions. Since over a regular algebra A the bimodule A is a dualizing complex, it is easily seen that a positive answer to the second question is equivalent to the existence of a balanced dualizing complex over A. Hence when A is noetherian, we are through. It is known that regular algebras of dimensions $\leqslant 3$ generated in degree 1 are noetherian, and Artin conjectures that this is true in general (see [AS, Sect. 0]).

As far as the first question is concerned, we have $\mathrm{H}^d_m(A) \cong \mathrm{Hom}_k(\mathrm{H}^{-d}R^{\cdot}_A, k)$, where $R^{\cdot}_A$ is a balanced dualizing complex over A (by local duality, Thm. 4.18). Thus a description of the complex $R^{\cdot}_A$ will answer this question positively. If A happens to be the skew homogeneous coordinate ring associated to a triple $(X, \sigma, \mathscr{L})$, then $\Gamma_*(\omega_X, \varepsilon)[d]$ is a balanced dualizing complex over A (see Cor. 7.13).

The other case we are able to handle is that of an elliptic 3 dimensional algebra. There is a normalizing non-zero-divisor $g \in A_{s+1}$ such that $B = A/(g)$ is the skew homogeneous coordinate ring associated to an elliptic triple $(E, \sigma, \mathscr{L})$. Let λ be the eigenvalue of σ^{-1} acting on $\Gamma(E, (\omega_E, \varepsilon))$, and let ϕ_λ be the automorphism of A defined by $\phi_\lambda(a) = \lambda^n a$, $a \in A_n$. Let ϕ_g be the automorphism of A s.t. $ga = \phi_g(a) g$. Finally, let $A(\phi_g \phi_\lambda, -s-1)$ be the invertible bimodule with generator γ in degree $s+1$ s.t. $\gamma a = \phi_g \phi_\lambda(a)\gamma$. Then we have (Thm. 7.18):

THEOREM. *The complex $A(\phi_g \phi_\lambda, -s-1)[3]$ is a balanced dualizing complex over A.*

The duality formalism should prove useful for various applications. We sketch a theory of traces for finite algebra homomorphisms (Remark 5.7). We also define Cohen–Macaulay and Gorenstein algebras in terms of their dualizing complexes. In Proposition 5.8 these definitions are compared to the familiar ones for algebras finite over their centers.

All of our algebras are connected positively graded, that is to say of the form $A = \bigoplus_{n=0}^{\infty} A_n$ and $A_0 \cong k$. This is quite restrictive—for instance, a matrix ring $M_n(A)$ is ruled out. However, it seems that many of the results in the paper will remain valid if A_0 is allowed to be any finite separable k-algebra. It is less certain whether letting A be filtered, instead of graded, and working with filtered modules, will yield a useful theory. With enveloping algebras in mind, this is very tempting.

The paper is organized as follows:

Section 1. In this section we introduce notation and conventions regarding graded algebras and modules and recall some well-known results.

Section 2. We define four derived categories related to a graded algebra A and consider various functors between them: Res_A, $\mathrm{RHom}^{\cdot}_A(-,-)$, etc.

Section 3. Dualizing complexes are defined. The key result is the uniqueness Theorem 3.9. We give some examples of dualizing complexes.

Section 4. This section deals with balanced dualizing complexes. The key result is Theorem 4.8, which gives an "infinitesimal" criterion for a dualizing complex over a noetherian algebra to be balanced.

Section 5. Given a finite algebra homomorphism $f\colon A \to B$ we show that it is sometimes possible to relate dualizing complexes over the two algebras. This is the case, for example, when A is commutative and f is centralizing.

Section 6. Here we define the skew homogeneous coordinate ring B associated to a triple $(X, \sigma, \mathcal{L})$. We discuss the passage from $\mathcal{O}_X$-modules (resp. σ-equivariant $\mathcal{O}_X$-modules) to B-modules (resp. B–B-bimodules). A formula for computing multiplication in bimodules is given. We demonstrate its use in Example 6.21.

Section 7. In this section dualizing complexes over skew homogeneous coordinate rings are discussed. The main result of this section is Theorem 7.3, which was mentioned earlier. We work out examples for triples $(X, \sigma, \mathcal{L})$, where X is $\mathbf{P}^n_k$ (Examples 7.14 and 7.15) and where X is an abelian variety (Examples 7.16 and 7.17). The section concludes with the description of a balanced dualizing complex over an elliptic 3 dimensional regular algebra.

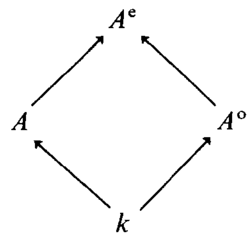
1. Graded Algebras and Modules

Let k be a field, and let $A = \bigoplus_{n=0}^{\infty} A_n$ be an associative $\mathbf{Z}$-graded k-algebra. Unless otherwise stated, the term "A-module" will mean "graded left A-module"; similarly, "k-module" will mean "graded k-vector space." Let $M = \bigoplus_{i=-\infty}^{\infty} M_i$ be an A-module. The nth *twist* of M is the module $M(n)$ defined by $M(n)_i = M_{n+i}$. A module M is called *left* (resp. *right*) *limited* if $M_i = 0$ for $i \ll 0$ (resp. $i \gg 0$). The augmentation ideal $\bigoplus_{n>0} A_n$ is denoted by m. An unadorned tensor symbol $\otimes$ will mean tensor over k.

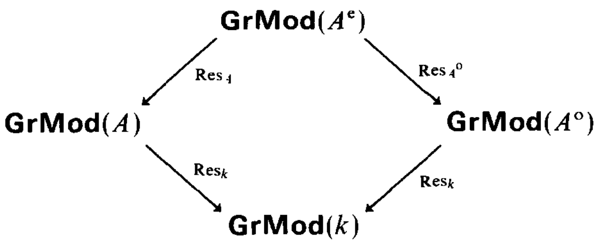
Denote by $\mathsf{GrMod}(A)$ the category whose objects are A-modules and whose morphisms are A-linear homomorphisms of degree 0. This is an abelian category with enough projectives and injectives, and it has direct sums and products (see [NV, Sect. I.3.2]). It should be pointed out that given a collection $\{M_\alpha\}$ of A-modules, their product in $\mathsf{GrMod}(A)$ is the module $\bigoplus_n [\prod_\alpha (M_\alpha)_n]$. A projective (resp. injective) object in $\mathsf{GrMod}(A)$ will be referred to simply as an A-projective (resp. A-injective).

Caution. All objects and operations related to a graded algebra will be tacitly considered as graded, unless otherwise stated.

Let A° be the opposite of A, and let A^e be the algebra $A \otimes A^\circ$. We will identify graded right A-modules (resp. graded A–A-bimodules) with A°-modules (resp. A^{e}-modules). The diagram of inclusions



induces a diagram of restriction functors



all of which are exact. For a module M over any of these algebras, denote by $\mathrm{Ungr}(M)$ the ungraded module over the same algebra. The functors Res and Ungr will occasionally remain implicit, for the sake of legibility.

Given A-modules M and N, let $\mathrm{Hom}_A(M, N)_n$ be the set of A-linear homomorphisms of degree n from M to N. It is a k-module which we may identify with $\mathrm{Hom}_{\mathbf{GrMod}(A)}(M, N(n))$. Define

$$\mathrm{Hom}_A(M, N) := \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_A(M, N)_n.$$

Given an A°-module M and an A-module N, we introduce a grading on the tensor product of the ungraded modules. For any integer n, let $(M \otimes_A N)_n$ be the additive subgroup of $M \otimes_A N$ generated by the tensors $x \otimes y$ with $x \in M_i$, $y \in N_j$, $i + j = n$.

If we let B and C denote either k or A, then $\otimes_A$ is a functor

$$\mathbf{GrMod}(B \otimes A^{\circ}) \times \mathbf{GrMod}(A \otimes C^{\circ}) \to \mathbf{GrMod}(B \otimes C^{\circ}).$$

Similarly, $\mathrm{Hom}_A(-, -)$ is a functor

$$\mathbf{GrMod}(A \otimes B^{\circ})^{\circ} \times \mathbf{GrMod}(A \otimes C^{\circ}) \to \mathbf{GrMod}(B \otimes C^{\circ})$$

and $\text{Hom}_{A^\circ}(-, -)$ is a functor

$$\mathbf{GrMod}(B \otimes A^\circ)^\circ \times \mathbf{GrMod}(C \otimes A^\circ) \to \mathbf{GrMod}(C \otimes B^\circ),$$

where $\mathbf{GrMod}(-)^\circ$ is the opposite category. Observe that $\otimes_A$, $\text{Hom}_A(-, -)$, and $\text{Hom}_{A^\circ}(-, -)$ commute with the relevant restriction functors; as for the ungrading, $\otimes_A$ commutes with it, and the Hom functors commute with it when the first argument is finitely generated.

An A-module F is *free* if it is isomorphic to $\bigoplus A(n_i)$ for some $n_i \in \mathbf{Z}$. An A-module M is *finitely presented* if there exists an exact sequence $F_1 \to F_0 \to M \to 0$ with F_0 and F_1 finitely generated free A-modules.

DEFINITION 1.1. Let M be an A-module. If M is finitely generated, and if every finitely generated submodule $N \subset M$ is finitely presented, then M is called a (graded) coherent A-module. The algebra A is called a (graded, left and right) coherent ring if A is coherent both as an A-module and as an A°-module.

Remark 1.2. It is a standard fact that the graded algebra A is noetherian iff the ungraded algebra $\text{Ungr}(A)$ is noetherian. The author does not know whether the same holds for "coherent."

A subcategory $\mathbf{B}$ of an abelian category $\mathbf{A}$ is called *thick* if, given any exact sequence $M_1 \to M_2 \to N \to M_3 \to M_4$ in $\mathbf{A}$ with all $M_i \in \mathbf{B}$, then also $N \in \mathbf{B}$.

PROPOSITION 1.3. *The full subcategory of* $\mathbf{GrMod}(A)$ *consisting of coherent modules is a thick abelian subcategory.*

Proof. The proof of [Bo2, Chap. II, Prop. 1.7] works for graded modules. ∎

Of course any noetherian ring is coherent. We shall not require A to be noetherian but instead shall impose the following conditions on it:

$$\begin{aligned}&\text{(i)} \quad A \text{ is a (graded) coherent ring.}\\ &\text{(ii)} \quad A/m \text{ is a coherent } A\text{-module.}\\ &\text{(iii)} \quad A_0 \cong k.\end{aligned} \tag{1.4}$$

We shall identify k with either A_0 or A/m, according to context. The apparent lack of symmetry in condition (ii) will be removed in Corollary 1.8. It is not assume that A is generated in degree 1. However, conditions (1.4) imply that:

PROPOSITION 1.5. *A is a finitely generated k-algebra.*

Proof. Since A and A/m are coherent, so is m. Say $m = \sum_{i=1}^{r} Aa_i$ for some homogeneous a_i. Then by induction on degree we have $A = k[a_1, ..., a_r]$. ∎

Let F be a free A-module. Then $\mathrm{Ungr}(F)$ is also free and hence the functors $\mathrm{Tor}_q^A(-, -)$ commute with ungrading. Observe that any free A-module is of the form $F \cong A \otimes V$, where $V = \bigoplus k(n_i)$ is a k-module. The next proposition shows that any left limited projective in **GrMod**(A) is free.

PROPOSITION 1.6. *Let M be a left limited A-module.*

(a) *If $mM = M$ then $M = 0$ (Nakayama's lemma).*

(b) *There exist left limited k-modules V_q, $q \geq 0$, and an exact sequence*

$$\cdots \to A \otimes V_2 \xrightarrow{d_1} A \otimes V_1 \xrightarrow{d_0} A \otimes V_0 \xrightarrow{\varepsilon} M \to 0$$

such that $\mathrm{im}(d_q) \subset m \otimes V_q$ *for all q. Hence $V_q \cong \mathrm{Tor}_q^A(k, M)$ as k-modules.*

(c) *M is projective in* **GrMod**(A) *iff the homomorphism $\varepsilon: A \otimes V_0 \to M$ above is bijective, iff* $\mathrm{Ungr}(M)$ *is projective in* **Mod**($\mathrm{Ungr}(A)$).

Proof. (a) Trivial.

(b) Take $V_0 := k \otimes_A M$ and lift it to get a projective cover $\varepsilon: A \otimes V_0 \to M$ (cf. [Ro, Thm. 2.8.40]). Since $\ker(\varepsilon)$ is left limited we may continue recursively.

(c) Follows immediately from (b). ∎

Remark 1.7. One can easily show that for *any* projective A-module M the $\mathrm{Ungr}(A)$-module $\mathrm{Ungr}(M)$ is projective [NV, Chap. I, Cor. 3.3.7]. This is false for injective modules.

COROLLARY 1.8. *$A/m \cong k$ is a coherent A°-module.*

Proof. Consider the ranks of the k-modules $\mathrm{Tor}_q^A(k, k)$. ∎

PROPOSITION 1.9. (a) *Any coherent A-module M has a resolution $\cdots \to F_1 \to F_0 \to M \to 0$, with the F_q fintely generated free A-modules.*

(b) *Finitely presented A-modules are coherent.*

(c) *Finite length A-modules are coherent.*

Proof. (a, b) Immediate from condition (1.4)(i) and Proposition 1.3.

(c) Since any finite length A-module is an extension of k, we may use induction on length, condition (1.4)(ii), and Proposition 1.3. ∎

Let A' be the A^e-module $\mathrm{Hom}_k(A, k)$. For any A-module M there is a natural isomorphism

$$\mathrm{Hom}_A(M, A')_0 \xrightarrow{\cong} \mathrm{Hom}_k(M, k)_0 = \mathrm{Hom}_k(M_0, k)$$

(sending $f: M \to A'$ to $f(-)(1): M \to k$). Since the latter is an exact functor of M, this shows that A' is injective over A. Moreover, the homomorphism $k \to A'$ corresponding to the identity $k \to k$ is essential. Thus A' is an injective envelope of k in $\mathbf{GrMod}(A)$. By symmetry, the same is true over A°. If M is an A-module such that $\dim_k M_n < \infty$ for all n, then the natural homomorphism

$$M \to \mathrm{Hom}_{A^\circ}(\mathrm{Hom}_A(M, A'), A')$$

is an isomorphism (Matlis duality).

An A^e-module L is called an invertible bimodule if there exists some A^e-module $L^\vee$ such that

$$L \otimes_A L^\vee \cong L^\vee \otimes_A L \cong A$$

over A^e. By Morita theory (see [Ja, p. 167]) the ungraded modules L and $L^\vee$ are finitely generated projective modules over A and over A°. Moreover, there are isomorphisms $L^\vee \cong \mathrm{Hom}_A(L, A)$ and $L^\vee \cong \mathrm{Hom}_{A^\circ}(L, A)$. As remarked previously, these statements hold also for the graded modules.

Given a k-algebra automorphism ϕ of A and an integer n, an invertible bimodule $A(\phi, n)$ can be constructed as follows. As a left module, $A(\phi, n)$ is the A-module $A(n)$ with generator $e = 1$ in degree $-n$. The right multiplication is given by the rule $ea = \phi(a)e$, $a \in A$. It is easily seen that $A(\phi, n) \otimes_A A(\phi', n') \cong A(\phi\phi', n + n')$.

PROPOSITION 1.10. *Any invertible bimodule is isomorphic to some* $A(\phi, n)$.

Proof. Let L be an invertible bimodule with inverse $L^\vee$. Since L and $L^\vee$ are finitely generated over A, they are left limited. Consider the surjective A^e-module homomorphism

$$k \otimes_A L^\vee \otimes_A L \twoheadrightarrow (k \otimes_A L^\vee \otimes_A k) \otimes (k \otimes_A L).$$

Because $L^\vee \otimes_A L \cong A$, it follows that $k \otimes_A L^\vee \otimes_A L \cong k$. On the other hand, by Nakayama's lemma, $k \otimes_A L^\vee \otimes_A k \neq 0$. Therefore $k \otimes_A L \cong k(n)$ for some integer n and because L is A-projective, we have $L \cong A(n)$ as A-module. By symmetry, $L \cong A(n)$ also as A^0-modules. Choose a generator $e \in L_{-n}$ for L. The map $\phi: A \to A$ defined implicitly by $ea = \phi(a)e$ is a k-algebra automorphism of A. Thus $L \cong A(\phi, n)$. ∎

2. The Derived Categories Related to A

In this section we will be generalizing to the noncommutative situation some of the definitions and results contained in Chapter I of [RD]. The original notation of that text will be retained as much as possible.

Let B denote any of the algebras k, A, A°, or A^{e}. Define $\mathbf{K}(B) = \mathbf{K}(\mathbf{GrMod}(B))$ to be the category of complexes of B-modules. A morphism $\phi: M^{\cdot} \to N^{\cdot}$ in $\mathbf{K}(B)$ is a homotopy class of complex homomorphisms. A quasi-isomorphism is a morphism which induces isomorphisms in cohomology. The derived category $\mathbf{D}(B)$ is obtained by inverting the quasi-isomorphisms in $\mathbf{K}(B)$. We denote by $\mathbf{D}^{+}(B)$, $\mathbf{D}^{-}(B)$, and $\mathbf{D}^{b}(B)$ the full subcategories of $\mathbf{D}(B)$ consisting of bounded below, bounded above, and bounded complexes, respectively. The exact restriction functors Res extend to functors on the various derived categories, e.g., $\mathrm{Res}_A: \mathbf{D}(A^{e}) \to \mathbf{D}(A)$.

Lemma 2.1. *The functors* Res_A *and* $\mathrm{Res}_{A^{\circ}}$ *map projectives to projectives and injectives to injectives.*

Proof. Since A^{e} itself is A-projective, it follows that any A^{e}-projective is an A-projective too. Next, consider the functor $M \mapsto A^{e} \otimes_A M$ from $\mathbf{GrMod}(A)$ to $\mathbf{GrMod}(A^{e})$. It is an exact left adjoint to Res_A. By standard arguments this implies that Res_A maps injectives to injectives (cf. [Bo1, Chap. V, Sect. 1.6]). ∎

Let B and C denote either A or k. Given complexes $M^{\cdot} \in \mathbf{K}(A \otimes B^{\circ})$ and $N^{\cdot} \in \mathbf{K}(A \otimes C^{\circ})$, set

$$\mathrm{Hom}^{\cdot}_A(M^{\cdot}, N^{\cdot}) := \bigoplus_{n \in \mathbf{Z}} \left[\prod_{p \in \mathbf{Z}} \mathrm{Hom}_A(M^{p}, N^{p+n}) \right]$$

(where $\prod_p$ is taken in the category $\mathbf{GrMod}(B \otimes C^{\circ})$). This becomes a complex with differential $d^{n} := \prod_p (d_M^{p-1} + (-1)^{n+1} d_N^{p+n})$. One gets a bi-$\partial$-functor

$$\mathrm{Hom}^{\cdot}_A: \mathbf{K}(A \otimes B^{\circ})^{\circ} \times \mathbf{K}(A \otimes C^{\circ}) \to \mathbf{K}(B \otimes C^{\circ}).$$

Theorem 2.2. (a) *The functor* $\mathrm{Hom}^{\cdot}_A$ *has a derived functor*

$$\mathrm{RHom}^{\cdot}_A: \mathbf{D}(A \otimes B^{\circ})^{\circ} \times \mathbf{D}^{+}(A \otimes C^{\circ}) \to \mathbf{D}(B \otimes C^{\circ}).$$

When $N^{\cdot} \in \mathbf{D}^{+}(A \otimes C^{\circ})$ *is a complex of* A*-injectives, then* $\mathrm{RHom}^{\cdot}_A(M^{\cdot}, N^{\cdot}) = \mathrm{Hom}^{\cdot}_A(M^{\cdot}, N^{\cdot})$ *for any* $M^{\cdot} \in \mathbf{D}(A \otimes B^{\circ})^{\circ}$.

(b) *The functor* $\mathrm{Hom}^{\cdot}_A$ *also has a derived functor*

$$\mathrm{RHom}^{\cdot}_A: \mathbf{D}^{-}(A \otimes B^{\circ})^{\circ} \times \mathbf{D}(A \otimes C^{\circ}) \to \mathbf{D}(B \otimes C^{\circ}).$$

When $M^\cdot \in \mathbf{D}^-(A \otimes B^\circ)$ *is a complex of A-projectives, then* $\mathrm{RHom}^\cdot_A(M^\cdot, N^\cdot) = \mathrm{Hom}^\cdot_A(M^\cdot, N^\cdot)$ *for any* $N^\cdot \in \mathbf{D}(A \otimes C^\circ)$.

(c) *The two derived functors coincide on* $\mathbf{D}^-(A \otimes B^\circ)^\circ \times \mathbf{D}^+(A \otimes C^\circ)$.

Proof. (a) The proof is essentially the same as that in [RD, p. 65]; we indicate the necessary modifications. Let $\mathbf{L}$ be the full subcategory of $\mathbf{K}^+(A \otimes C^\circ)$, consisting of complexes which are isomorphic (in $\mathbf{K}^+(A \otimes C^\circ)$) to complexes of A-injectives. Since a direct sum of two A-injectives is also A-injective, $\mathbf{L}$ is a triangulated subcategory (cf. [RD, Chap. I, Cor. 5.4.β]). If $M^\cdot \in \mathbf{K}(A \otimes B^\circ)^\circ$ and $N^\cdot \in \mathbf{L}$, then

$$\mathrm{Res}_k(\mathrm{Hom}^\cdot_A(M^\cdot, N^\cdot)) = \mathrm{Hom}^\cdot_A(\mathrm{Res}_A(M^\cdot), \mathrm{Res}_A(N^\cdot)).$$

Therefore Lemma 6.2 of [RD, Chap. I] holds for objects of $\mathbf{L}$. By Lemma 2.1, any $A \otimes C^\circ$-injective is also an A-injective, so any $A \otimes C^\circ$-module embeds in a A-injective. Having established these properties of $\mathbf{L}$, the proof in [RD, p. 65] applies to the present situation.

(b) Again we quote [RD, p. 65], "reversing arrows," and note that any $A \otimes B^\circ$-module is a quotient of an A-projective.

(c) This is Lemma 6.3 of [RD, Chap. I]. ∎

Corollary 2.3. *The functor* $\mathrm{RHom}^\cdot_A$ *commutes with the various restriction functors.*

By symmetry we also have a functor $\mathrm{RHom}^\cdot_{A^\circ}$ with the corresponding properties.

Given complexes $M^\cdot \in \mathbf{D}(A)$ and $N^\cdot \in \mathbf{D}^+(A)$, define

$$\mathrm{Ext}^n_A(M^\cdot, N^\cdot) := \mathrm{H}^n\mathrm{RHom}^\cdot_A(M^\cdot, N^\cdot).$$

A complex $N^\cdot \in \mathbf{D}^+(A)$ is said to have finite injective dimension over A if there exists an integer n_0 such that $\mathrm{Ext}^n_A(M, N^\cdot) = 0$ for all $n > n_0$ and all $M \in \mathbf{GrMod}(A)$. Here M is considered to be a complex concentrated in dimension 0. The next proposition is a variant of [RD, Chap. I, Prop. 7.6].

Proposition 2.4. *The following are equivalent for any complex* $N^\cdot \in \mathbf{D}^+(A^e)$:

(i) $N^\cdot$ *is isomorphic in* $\mathbf{D}^+(A^e)$ *to a bounded complex of* A^e*-modules which are both A-injective and* A°*-injective.*

(ii) $N^\cdot$ *has finite injective dimension over both A and* A°.

Proof. Condition (i) implies condition (ii) because of Theorem 2.2(a). Assume now that condition (ii) holds, and choose n_0 such that $\mathrm{Ext}^n_A(M, N^\cdot) = 0$ and $\mathrm{Ext}^n_{A^\circ}(L, N^\cdot) = 0$ for all $n > n_0$, all $M \in \mathbf{GrMod}(A)$, and all $L \in \mathbf{GrMod}(A^{\circ})$. Let $N^{\cdot} \to I^{\cdot}$ be a quasi-isomorphism in $\mathbf{K}^{+}(A^{e})$, where $I^{\cdot}$ is a complex of A^{e}-injectives. Define $\sigma_{\leqslant n_0+1}(I^{\cdot})$ to be the truncated complex $\cdots \to I^{n_0} \to \ker(d^{n_0+1}) \to 0 \to \cdots$. As in the proof of [RD, Chap. I, Prop. 7.6], the homomorphism $\sigma_{\leqslant n_0+1}(I^{\cdot}) \to I^{\cdot}$ is a quasi-isomorphism. The proof also shows that $\ker d^{n_0+1}$ is both A-injective and A°-injective. ∎

Let $M^{\cdot} \in \mathbf{K}(B \otimes A^{\circ})$ and let $N^{\cdot} \subset \mathbf{K}(A \otimes C^{\circ})$, where B and C are either A or k. Define a complex $(M^{\cdot} \otimes_A N^{\cdot}, d)$ as

$$M^{\cdot} \otimes_A N^{\cdot} := \bigoplus_{n \in \mathbf{Z}} \left[\bigoplus_{p+q=n} M^{p} \otimes_A N^{q} \right]$$

and $d := d_M + (-1)^{n} d_N$ on $(M^{\cdot} \otimes_A N^{\cdot})^{n}$. This gives rise to a bi-∂-functor

$$\otimes_A : \mathbf{K}(B \otimes A^{\circ}) \times \mathbf{K}(A \otimes C^{\circ}) \to \mathbf{K}(B \otimes C^{\circ}).$$

THEOREM 2.5. *The functor* $\otimes_A$ *has a derived functor*

$$\otimes_A^{\mathsf{L}} : \mathbf{D}^{-}(B \otimes A^{\circ}) \times \mathbf{D}^{-}(A \otimes C^{\circ}) \to \mathbf{D}^{-}(B \otimes C^{\circ}).$$

If either $M^{\cdot}$ *is a complex of* A°*-projectives or* $N^{\cdot}$ *is a complex of* A*-projectives, then* $M^{\cdot} \otimes_A^{\mathsf{L}} N^{\cdot} = M^{\cdot} \otimes_A N^{\cdot}$.

Proof. The full subcategory $\mathbf{L}$ of $\mathbf{K}^{-}(B \otimes A^{\circ})$ (resp. $\mathbf{K}^{-}(A \otimes C^{\circ})$) consisting of complexes which are isomorphic to complexes of A°-projectives (resp. A-projectives) is a triangulated subcategory. Since Lemma 4.1 of [RD, Chap. II] holds for complexes in $\mathbf{L}$, we can apply the arguments on pages 94–95 of [RD] to our situation. ∎

3. DUALIZING COMPLEXES

In this section we define dualizing complexes over the noncommutative graded algebra A. Since we take care to state the definition of a dualizing complex in terms of coherent modules, it applies to algebras which are not noetherian (see Example 3.7). We prove a uniqueness theorem for these complexes. Our treatment is suggested by [RD, Chap. V].

Let B denote either of the algebras k or A. Given a complex $R^{\cdot} \in \mathbf{D}^{+}(A^{e})$, we can define functors

$$D := \mathrm{Hom}^{\cdot}_{A}(-, R^{\cdot}) : \mathbf{D}(A \otimes B^{\circ})^{\circ} \to \mathbf{D}(B \otimes A^{\circ}) \tag{3.1}$$

and

$$D^{\circ} := \mathrm{RHom}^{\cdot}_{A^{\circ}}(-, R^{\cdot}) : \mathbf{D}(B \otimes A^{\circ})^{\circ} \to \mathbf{D}(A \otimes B^{\circ}), \tag{3.2}$$

called the duality functors associated to $R^{\cdot}$. For $M^{\cdot} \in \mathbf{D}(A \otimes B^{\circ})$ there is a natural morphism $M^{\cdot} \to D^{\circ}D(M^{\cdot})$. Indeed, we may assume that $R^{\cdot}$ is a bounded below complex of A^{e}-modules which are both A-injective and A°-injective. Then $D^{\circ}D(M^{\cdot}) = \mathrm{Hom}^{\cdot}_{A^{\circ}}(\mathrm{Hom}^{\cdot}_{A}(M^{\cdot}, R^{\cdot}), R^{\cdot})$ and the morphism $M^{\cdot} \to D^{\circ}D(M^{\cdot})$ arises from the canonical module homomorphisms $M^{n} \to \prod_{p} [\mathrm{Hom}_{A^{\circ}}(\mathrm{Hom}_{A}(M^{n}, R^{p}), R^{p})]$. Symmetrically, there is a natural morphism $M^{\cdot} \to DD^{\circ}(M^{\cdot})$ in $\mathbf{D}(B \otimes A^{\circ})$. Note that $D^{\circ}D(A) = \mathrm{RHom}^{\cdot}_{A^{\circ}}(R^{\cdot}, R^{\cdot})$ and $DD^{\circ}(A) = \mathrm{RHom}^{\cdot}_{A}(R^{\cdot}, R^{\cdot})$.

Now let B denote either of the algebras A or A°. Define $\mathbf{D}_{\mathrm{c}}(B)$ to be the full subcategory of $\mathbf{D}(B)$ consisting of the complexes $M^{\cdot}$ such that for all q, the module $\mathrm{H}^{q}M^{\cdot}$ is coherent over B. Let $\mathbf{D}^{+}_{\mathrm{c}}(B)$, $\mathbf{D}^{-}_{\mathrm{c}}(B)$, and $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(B)$ be the intersections of $\mathbf{D}_{\mathrm{c}}(B)$ with $\mathbf{D}^{+}(B)$, $\mathbf{D}^{-}(B)$, and $\mathbf{D}^{\mathrm{b}}(B)$, respectively.

DEFINITION 3.3. A complex $R^{\cdot} \in \mathbf{D}^{+}(A^{e})$ is called dualizing if it satisfies the following conditions:

(i) $R^{\cdot}$ has finite injective dimension over A and over A°.

(ii) $\mathrm{Res}_{A}(R^{\cdot}) \in \mathbf{D}^{+}_{\mathrm{c}}(A)$ and $\mathrm{Res}_{A^{\circ}}(R^{\cdot}) \in \mathbf{D}^{+}_{\mathrm{c}}(A^{\circ})$.

(iii) The natural morphisms $\Phi: A \to \mathrm{RHom}^{\cdot}_{A}(R^{\cdot}, R^{\cdot})$ and $\Phi^{\circ}: A \to \mathrm{RHom}^{\cdot}_{A^{\circ}}(R^{\cdot}, R^{\cdot})$ are isomorphisms in $\mathbf{D}(A^{e})$.

PROPOSITION 3.4. *Let $R^{\cdot} \in \mathbf{D}^{+}(A^{e})$ satisfy conditions* (i) *and* (ii) *of the previous definition. Then the duality functors D and D° interchange the categories $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^{\circ})$.*

Proof. According to Proposition 2.4 we may assume that $R^{\cdot}$ is a bounded complex of A^{e}-modules which are injective over A and over A°. Thus the functor D sends $\mathbf{D}^{\mathrm{b}}(A)^{\circ}$ into $\mathbf{D}^{\mathrm{b}}(A^{\circ})$, and is way out on both sides (see [RD, p. 68]). By the reversed form of [RD, Chap. I, Prop. 7.3], in order to show that D sends $\mathbf{D}^{-}_{\mathrm{c}}(A)^{\circ}$ into $\mathbf{D}_{\mathrm{c}}(A^{\circ})$, it suffices to check this for finitely generated projective A-modules. In view of Proposition 1.6(c) and the fact that $\mathrm{H}^{\cdot}D$ commutes with direct sums and twisting, we reduce the problem to showing that $D(\mathrm{Res}_{A}(A)) = \mathrm{Res}_{A^{\circ}}(R^{\cdot})$ is in $\mathbf{D}_{\mathrm{c}}(A^{\circ})$; but this is just condition (ii). By symmetry, the statements regarding D° also hold. ∎

The following proposition justifies the name “dualizing complex.”

PROPOSITION 3.5. *Let $R^{\cdot} \in \mathbf{D}^{+}(A^{e})$ satisfy conditions* (i) *and* (ii) *of Definition* 3.3. *Then $R^{\cdot}$ satisfies condition* (iii) *of the same definition iff it satisfies condition*

(iii′) *Let D and D° be the dualizing functors associated to $R^{\cdot}$. Then for every $M^{\cdot} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ (resp. $M^{\cdot} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^{\circ})$) the natural morphism $M^{\cdot} \to D^{\circ}D(M^{\cdot})$ (resp. $M^{\cdot} \to DD^{\circ}(M^{\cdot})$) is an isomorphism.*

Proof. The morphism $\Phi^\circ: A \to D^\circ D(A)$ is an isomorphism in $\mathbf{D}(A^e)$ iff the morphism $\text{Res}_A(A) \to D^\circ D\,\text{Res}_A(A)$ is an isomorphism in $\mathbf{D}(A)$. Similarly for Φ. Therefore (iii′) implies (iii).

Conversely, since the functor $D^\circ D$ is way out on both sides, we can use the reversed form of [RD, Chap. I, Prop. 7.1]; so we must show that $M \to \text{H}^\cdot D^\circ D(M)$ is an isomorphism for finitely generated projective A-modules M. Since the functor $\text{H}^\cdot D^\circ D$ commutes with direct sums and twisting, it suffices to show this for $M = \text{Res}_A(A)$—which is done by applying cohomology to condition (iii). By symmetry we verify that $M^\cdot \to DD^\circ(M^\cdot)$ is an isomorphism on $\mathbf{D}_c^b(A^\circ)$. ∎

Here are a few easy examples of algebras which have dualizing complexes. In later sections more complicated examples are discussed.

EXAMPLE 3.6. Let $A = k[X_1, ..., X_n]$ be a commutative polynomial ring, with X_i homogeneous of degree 1. As an A-module, A has finite injective dimension (since the ring has finite global dimension n). Clearly $A = \text{Hom}_A(A, A) = \text{Hom}_{A^\circ}(A, A)$, so the bimodule A is a dualizing complex. The algebra A is noetherian, so all finitely generated modules are coherent.

In this example things can be simplified by observing that any complex $M^\cdot \in \mathbf{D}_c^b(A)$ is isomorphic to a bounded complex $P^\cdot$ of finitely generated projectives. Then $\text{Hom}_A(P^\cdot, A)$ is a bounded complex of finitely generated A°-projectives and the duality is the isomorphism

$$P^\cdot \xrightarrow{\simeq} \text{Hom}_{A^\circ}(\text{Hom}_A(P^\cdot, A), A).$$

EXAMPLE 3.7. Let $A = k\langle X_1, ..., X_n\rangle$ be a free associative k-algebra, i.e., a polynomial ring in the noncommuting indeterminates X_i, which are homogeneous of degree 1. The algebra A has global dimension 1, so it is a coherent ring (cf. [Ro, p. 266]); if $n > 1$ it is not noetherian. As in the first example, the bimodule A is dualizing, and duality takes on a simple form using projectives.

EXAMPLE 3.8. Let A be a finite length k-algebra. Then the bimodule $A' = \text{Hom}_k(A, k)$ is a dualizing complex over A.

The following uniqueness theorem is a generalization of Theorem 3.1 of [RD, Chap. VI]. Given a complex $M^\cdot$, let $M^\cdot[n]$ be the shifted complex s.t. $M^q[n] = M^{q+n}$.

THEOREM 3.9. *Let $R^\cdot$ be a dualizing complex over A, and let $\tilde{R}^\cdot$ be any complex in $\mathbf{D}^+(A^e)$. Then $\tilde{R}^\cdot$ is dualizing iff $\tilde{R}^\cdot \cong R^\cdot \otimes_A L[n]$ in $\mathbf{D}(A^e)$ for* *some invertible bimodule L and some integer n. Moreover, n is unique and L is unique up to isomorphism.*

Before proving the theorem, we state and prove two lemmas.

LEMMA 3.10. *Let $R^{\cdot}$ and $\tilde{R}^{\cdot}$ be two dualizing complexes on A. Then there is a natural isomorphism*

$$M^{\cdot} \otimes^{\mathrm{L}}_{A} \mathrm{RHom}^{\cdot}_{A}(R^{\cdot}, \tilde{R}^{\cdot}) \xrightarrow{\cong} \mathrm{RHom}^{\cdot}_{A}(\mathrm{RHom}^{\cdot}_{A^{\circ}}(M^{\cdot}, R^{\cdot}), \tilde{R}^{\cdot})$$

for all $M^{\cdot} \in \mathbf{D}^{-}(A^{\mathrm{e}})$ such that $\mathrm{Res}_{A^{\circ}}(M^{\cdot}) \in \mathbf{D}^{-}_{\mathrm{c}}(A^{\circ})$.

Note that $M^{\cdot} \otimes^{\mathrm{L}}_{A} \mathrm{RHom}^{\cdot}_{A}(R^{\cdot}, \tilde{R}^{\cdot})$ is defined, because $\mathrm{RHom}^{\cdot}_{A}(R^{\cdot}, \tilde{R}^{\cdot})$ is in $\mathbf{D}^{\mathrm{b}}(A^{\mathrm{e}})$.

Proof. We may assume that $M^{\cdot}$ is a complex of A°-projectives, and that $R^{\cdot}$ (resp. $\tilde{R}^{\cdot}$) is a bounded complex of A-injectives (resp. A°-injectives) (see Thm. 2.2). Then the morphism

$$M^{\cdot} \otimes_{A} \mathrm{Hom}^{\cdot}_{A}(R^{\cdot}, \tilde{R}^{\cdot}) \to \mathrm{Hom}^{\cdot}_{A}(\mathrm{Hom}^{\cdot}_{A^{\circ}}(M^{\cdot}, R^{\cdot}), \tilde{R}^{\cdot})$$

is the obvious one. To show that it is an isomorphism we may restrict the complex $M^{\cdot}$ to A°, and so we may assume that $M^{\cdot} \in \mathbf{D}^{-}_{\mathrm{c}}(A^{\circ})$. The two functors in question are way out left, hence it suffices to check for an isomorphism when $M^{\cdot} = \mathrm{Res}_{A^{\circ}}(A)$ (cf. Prop. 3.5). But in this case the morphism is clearly an isomorphism. ∎

LEMMA 3.11. *Let $M^{\cdot}$ and $\tilde{M}^{\cdot}$ be complexes in $\mathbf{D}^{-}(A^{\mathrm{e}})$ satisfying*

(i) *For all q, $\mathrm{H}^{q}M^{\cdot}$ and $\mathrm{H}^{q}\tilde{M}^{\cdot}$ are left limited.*

(ii) $M^{\cdot} \otimes^{\mathrm{L}}_{A} \tilde{M}^{\cdot} \cong \tilde{M}^{\cdot} \otimes^{\mathrm{L}}_{A} M^{\cdot} \cong A$ *in* $\mathbf{D}^{-}(A^{\mathrm{e}})$.

Then $M^{\cdot} \cong L[-n]$ and $\tilde{M}^{\cdot} L^{\vee}[n]$ for some invertible bimodule L and some integer n.

Proof. Let n be the maximal integer such that $\mathrm{H}^{n}(M^{\cdot}) \neq 0$. After replacing $M^{\cdot}$ with $\sigma_{\leqslant n}(M^{\cdot})$ as in the proof of Proposition 2.4, we may assume that $M^{q} = 0$ for $q > n$. Similarly, we may assume that for some integer m, $\mathrm{H}^{m}(\tilde{M}^{\cdot}) \neq 0$ and $\tilde{M}^{q} = 0$ for $q > m$.

By the spectral sequence for a double complex one has isomorphisms of A^{e}-modules

$$\mathrm{H}^{m}(\tilde{M}^{\cdot}) \otimes_{A} \mathrm{H}^{n}(M^{\cdot}) \cong \mathrm{H}^{m+n}(\tilde{M}^{\cdot} \otimes^{\mathrm{L}}_{A} M^{\cdot})$$

and

$$\mathrm{H}^{n}(M^{\cdot}) \otimes_{A} \mathrm{H}^{m}(\tilde{M}^{\cdot}) \cong \mathrm{H}^{n+m}(M^{\cdot} \otimes^{\mathrm{L}}_{A} \tilde{M}^{\cdot}).$$

Since both $\mathrm{H}^m(\tilde{M}^{\cdot})$ and $\mathrm{H}^n(M^{\cdot})$ are left limited, their tensor product is non-zero. Therefore we must have $n+m=0$ and the bimodule $L := \mathrm{H}^n(M^{\cdot})$ is invertible with inverse $L^{\vee} = \mathrm{H}^{-n}(\tilde{M}^{\cdot})$.

Because L is projective over A, there is a decomposition of complexes of A-modules

$$\mathrm{Res}_A(M^{\cdot}) \cong N^{\cdot} \oplus \mathrm{Res}_A(L)[-n]$$

for some $N^{\cdot} \in \mathbf{D}^-(A)$. Similarly, we have

$$\mathrm{Res}_{A^\circ}(\tilde{M}^{\cdot}) \cong \tilde{N}^{\cdot} \otimes \mathrm{Res}_{A^\circ}(L^{\vee})[n]$$

for some $\tilde{N}^{\cdot} \in \mathbf{D}^-(A^\circ)$. Thus one gets an isomorphism of graded k-modules (i.e., bigraded k-vector spaces)

$$\begin{aligned}\mathrm{Res}_k(A) &\cong \mathrm{H}^{\cdot}(\mathrm{Res}_{A^\circ}(\tilde{M}^{\cdot}) \otimes^{\mathrm{L}}_A \mathrm{Res}_A(M^{\cdot}))\\ &\cong (\mathrm{Res}_{A^\circ}(L^{\vee}) \otimes_A \mathrm{Res}_A(L)) \oplus \mathrm{H}^{\cdot}(\mathrm{Res}_{A^\circ}(L^{\vee})[n] \otimes_A N^{\cdot})\\ &\quad \oplus \mathrm{H}^{\cdot}(\tilde{N}^{\cdot} \otimes_A \mathrm{Res}_A(L)[-n]) \oplus \mathrm{H}^{\cdot}(\tilde{N}^{\cdot} \otimes^{\mathrm{L}}_A N^{\cdot}).\end{aligned}$$

Comparing these modules we see that the last three summands are zero. The complexes $N^{\cdot}$ and $\tilde{N}^{\cdot}$ are therefore acyclic, and we have isomorphisms $M^{\cdot} \xrightarrow{\cong} L[-n]$ and $\tilde{M}^{\cdot} \xrightarrow{\cong} L^{\vee}[n]$ in $\mathbf{D}(A^{\mathrm{e}})$. ∎

Proof (*of Theorem* 3.9). Given an invertible bimodule L and an integer n, the complex $R^{\cdot} \otimes_A L[n]$ is a dualizing complex. The isomorphism

$$L[n] \cong \mathrm{RHom}_A(R^{\cdot}, R^{\cdot} \otimes_A L[n])$$

determines L and n. Conversely, suppose that $\tilde{R}^{\cdot}$ is a dualizing complex. Define the functors

$$D := \mathrm{RHom}^{\cdot}_A(-, R^{\cdot}), \qquad D^\circ := \mathrm{RHom}^{\cdot}_{A^\circ}(-, R^{\cdot}),$$
$$\tilde{D} := \mathrm{RHom}^{\cdot}_A(-, \tilde{R}^{\cdot}), \qquad \tilde{D}^\circ := \mathrm{RHom}^{\cdot}_{A^\circ}(-, \tilde{R}^{\cdot}),$$

and set $M^{\cdot} = := \tilde{D}D^\circ(A)$, $\tilde{M} := D\tilde{D}^\circ(A)$. Thus $M^{\cdot} = \mathrm{RHom}^{\cdot}_A(R^{\cdot}, \tilde{R}^{\cdot})$ and $\tilde{M} = \mathrm{RHom}^{\cdot}_A(\tilde{R}^{\cdot}, R^{\cdot})$.

The objects $M^{\cdot}$ and $\tilde{M}^{\cdot}$ of $\mathbf{D}^{\mathrm{b}}(A^{\mathrm{e}})$ satisfy the conditions of Lemma 3.10, so according to the conclusion of this lemma we get isomorphisms

$$\tilde{M}^{\cdot} \otimes^{\mathrm{L}}_A M^{\cdot} \cong \tilde{D}D^\circ(\tilde{M}^{\cdot}) \cong \tilde{D}D^\circ D\tilde{D}^\circ(A) \cong A.$$

By symmetry, we also have $M^{\cdot} \otimes^{\mathrm{L}}_A \tilde{M}^{\cdot} \cong A$. Applying Lemma 3.11 we conclude that $M^{\cdot} \cong L[n]$ for some invertible bimodule L and some integer n. Using Lemma 3.10 again we get the isomorphisms

$$R^{\cdot} \otimes_A^{\mathrm{L}} M^{\cdot} \cong \tilde{D}D^{\circ}(R^{\cdot}) = \tilde{D}D^{\circ}D(A) = \tilde{R}^{\cdot},$$

yielding $R^{\cdot} \otimes_A L[n] \cong \tilde{R}^{\cdot}$. ∎

COROLLARY 3.12. *Let $R^{\cdot}$ be a dualizing complex over A and let $\tilde{L}$ be an invertible bimodule. Then $\tilde{L} \otimes_A R^{\cdot} \cong R^{\cdot} \otimes_A L$ for some invertible bimodule L. Moreover, L and $\tilde{L}$ are generated in the same degree.*

Proof. According to the theorem $\tilde{L} \otimes_A R^{\cdot} \cong R^{\cdot} \otimes_A L[n]$ for some L and n. Now over A the modules $\tilde{L} \otimes_A R^{\cdot}$ and $R^{\cdot} \otimes_A \tilde{L}$ are isomorphic. Therefore

$$\mathrm{Res}_A(\mathrm{H}^{\cdot}(R^{\cdot}) \otimes_A \tilde{L}) \cong \mathrm{Res}_A(\mathrm{H}^{\cdot}(R^{\cdot}) \otimes_A L[n])$$

so $n = 0$ and the bimodules L and $\tilde{L}$ are generated in the same degree. ∎

4. BALANCED DUALIZING COMPLEXES

A dualizing complex over the algebra A, if it exists, is unique only up to shifting in dimension and tensoring with an invertible bimodule (Thm. 3.9). In order to single our a particular isomorphism class of dualizing complexes, we examine torsion at the augmentation ideal m.

For $n \geq 0$ define m^n to be the ideal $\bigoplus_{i \geq n} A_i$. Thus $m^1 = m$, but unless $A = k[A_1]$, m^n need not be the nth power of m. Observe that A/m^n has finite length for all n. Let B denote either of the algebras k or A. Given an $(A \otimes B^{\circ})$-module M, let $\Gamma_m(M)$ be its left m-torsion submodule $\lim_{n \to} \mathrm{Hom}_A(A/m^n, M)$. The functor Γ_m: $\mathbf{GrMod}(A \otimes B^{\circ}) \to \mathbf{GrMod}(A \otimes B^{\circ})$ has a derived functor $\mathrm{R}\Gamma_m$: $\mathbf{D}^+(A \otimes B^{\circ}) \to \mathbf{D}^+(A \otimes B^{\circ})$ with the property that $\mathrm{R}\Gamma_m(M^{\cdot}) = \Gamma_m(M^{\cdot})$ when $M^{\cdot}$ is a complex of A-injectives. The proof is the same as that of Theorem 2.2, since A-injectives are acyclic for Γ_m. Symmetrically, define $\Gamma_{m^{\circ}}(M) := \lim_{n \to} \mathrm{Hom}_{A^{\circ}}(A/m^n, M)$, and let $\mathrm{R}\Gamma_{m^{\circ}}$ be its derived functor. Denote the local cohomology at m, $\mathrm{H}^{\cdot}\mathrm{R}\Gamma_m$ (resp. $\mathrm{H}^{\cdot}\mathrm{R}\Gamma_{m^{\circ}}$), by $\mathrm{H}^{\cdot}_m$ (resp. $\mathrm{H}^{\cdot}_{m^{\circ}}$). Thus for any $A \otimes B^{\circ}$-module M we have

$$\mathrm{H}^q_m(M) = \lim_{n \to} \mathrm{Ext}^q_A(A/m^n, M)$$

and a corresponding equality for $\mathrm{H}^q_{m^{\circ}}$.

DEFINITION 4.1. A dualizing complex $R^{\cdot}$ over A is called balanced if there are isomorphisms $\mathrm{R}\Gamma_m(R^{\cdot}) \cong A'$ and $\mathrm{R}\Gamma_{m^{\circ}}(R^{\cdot}) \cong A'$ in $\mathbf{D}(A^{\mathrm{e}})$.

We will show that if $R^\cdot$ is balanced then it satisfies the "noncommutative version" of the local duality theorem [RD, Chap. V, Thm. 6.2]. Any two balanced dualizing complexes are isomorphic; to see this, assume both $R^\cdot$ and $R^\cdot \otimes_A L[n]$ are balanced dualizing complexes. Then

$$A' \cong \mathrm{R}\Gamma_m(R^\cdot \otimes_A L[n]) \cong \mathrm{R}\Gamma_m(R^\cdot) \otimes_A L[n] \cong A' \otimes_A L[n]$$

so $n = 0$ and $L \cong A$ over A^e.

Before continuing with balanced dualizing complexes, we need more information about injective modules and torsion. A complex $I^\cdot \in \mathbf{K}(A)$ is called a minimal injective complex if for all q the A-module I^q is injective, and if the inclusion $\ker(d^q) \subsetneq I^q$ is essential.

LEMMA 4.2. *Any complex $M^\cdot \in \mathbf{K}^+(A)$ admits a quasi-isomorphism into a minimal injective complex.*

Proof. Start with a quasi-isomorphism $M^\cdot \to J^\cdot$ with $J^\cdot \in \mathbf{K}^+(A)$ a complex of A-injectives. Let $Z^q := \ker(d^q)$ and let $E_A(Z^q)$ be an injective envelope of Z^q. For every q there is an isomorphism $J^q \cong E_A(Z^q) \oplus K^q$. Define $I^q := E_A(Z^q)/d^{q-1}(K^{q-1})$. The induced homomorphism $J^\cdot \to I^\cdot$ is a quasi-isomorphism and $I^\cdot$ is then a minimal injective complex. ∎

LEMMA 4.3. *Let $I^\cdot \in \mathbf{K}^+(A)$ be a minimal injective complex. Then the complex* $\mathrm{Hom}_A(k, I^\cdot)$ *has a zero differential.*

Proof. Suppose that d^q is nonzero on $\mathrm{Hom}_A(k, I^q)$. Since this is a semi-simple A-module (A acts via k) we can find a nonzero A-submodule $M \subset \mathrm{Hom}_A(k, I^q)$ such that $d^q|_M$ is an injection. But then $M \cap Z^q = 0$, contradicting the minimality of $I^\cdot$. ∎

PROPOSITION 4.4. *Let $R^\cdot$ be a dualizing complex. Then the following conditions are equivalent:*

(i) $\mathrm{Ext}^\cdot_A(k, R^\cdot) \cong k$.

(i′) $\mathrm{Ext}^\cdot_{A^\circ}(k, R^\cdot) \cong k$.

(ii) $\mathrm{R}\Gamma_m(R^\cdot) \cong A' \otimes_A L$ *in* $\mathbf{D}(A^e)$ *for some invertible bimodule* L.

(ii′) $\mathrm{R}\Gamma_{m^\circ}(R^\cdot) \cong \tilde{L} \otimes_A A'$ *in* $\mathbf{D}(A^e)$ *for some invertible bimodule* $\tilde{L}$.

Proof. (i) $\Leftrightarrow$ (i′) We first observe that (i) implies that $D(k) = \mathrm{RHom}^\cdot_A(k, R^\cdot)$ is isomorphic in $\mathbf{D}(A^e)$ to k, by the standard truncation trick (see proof of Prop. 2.4). From duality we have $D^\circ D(k) = k$ and hence $D^\circ(k) \cong k$. Taking cohomology we obtain (i′). The converse follows by symmetry.

(i) and (i′) ⇒ (ii) By the truncation trick, (ii) is equivalent to $H^{\cdot}_m(R^{\cdot}) \cong A' \otimes_A L$ over A^e, which in turn is equivalent to $H^{\cdot}_m(R^{\cdot}) \cong A'$ over A and over A° (separately). First replace $\mathrm{Res}_A(R^{\cdot})$ with a minimal injective complex $I^{\cdot}$ isomorphic to it in $\mathbf{D}^+(A)$. According to Lemma 4.3 and condition (i), there is an isomorphism of complexes $\mathrm{Hom}^{\cdot}_A(k, I^{\cdot}) \cong k$. Considering the torsion in $I^{\cdot}$, we see that $\Gamma_m(I^{\cdot}) \cong A'$. Therefore $\mathrm{Res}_A H^{\cdot}_m(R^{\cdot}) \cong \mathrm{Res}_A(A')$.

Next, we note that the functor $M \mapsto \mathrm{Ext}^0_A(M, R^{\cdot}) = H^0 D(M)$ is a k-linear exact functor from the category of finite length A-modules to the category of finite length A°-modules. This follows from condition (i), the long exact sequence of cohomology (see [RD, Chap. I, Prop. 6.1]) and induction on length. Using (i′) and the identities $D^{\circ}D = 1$ and $DD^{\circ} = 1$ we see that the functor $M \mapsto H^0 D^{\circ}(M)$ is adjoint to the first functor, and there is an isomorphism of k-modules

$$\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A^{\circ}}(H^0 D(N), H^0 D(M)).$$

For every integer n, we have

$$\mathrm{Hom}_{A^{\circ}}(k, H^0 D(A/m^n)) \cong \mathrm{Hom}_A(A/m^n, k) = k.$$

This means that $\mathrm{Hom}_{A^{\circ}}(k, \mathrm{Ext}^0_A(A/m^n, R^{\cdot})) \cong k$ and passing to the direct limit in n we get $\mathrm{Hom}_{A^{\circ}}(k, H^0_m(R^{\cdot})) \cong k$. This gives rise to an essential monomorphism of A°-modules $H^0_m(R^{\cdot}) \subsetneq A'$. We already know that $H^0_m(R^{\cdot}) \cong A'$ over A, so by comparing dimensions of homogeneous components we conclude that $\mathrm{Res}_{A^{\circ}} H^0_m(R^{\cdot}) \cong \mathrm{Res}_{A^{\circ}}(A')$ too.

(i) and (i′) ⇒ (ii′) By symmetry.

(ii) ⇒ (i) Again replace $\mathrm{Res}_A(R^{\cdot})$ with a minimal injective complex $I^{\cdot}$. Then $M^{\cdot} := \mathrm{Hom}_A(k, I^{\cdot})$ is a subcomplex of $\Gamma_m(I^{\cdot})$ with zero differential. Consider the minimal q such that $M^q \neq 0$, which is also the minimal q such that $M^q \neq 0$, which is also the minimal q such that $\Gamma_m(I^q) \neq 0$. Then $M^q = H^q(M^{\cdot}) \subset H^q \Gamma_m(I^{\cdot})$, hence from (ii) we conclude that $q = 0$, $H^0 \Gamma_m(I^{\cdot}) \cong A'$ and $M^0 \cong k$. This implies that $\Gamma_m(I^0) \cong A'$ and that $d^0(\Gamma_m(I^0)) = 0$. A repetition of this argument shows that $\Gamma_m(I^q) = 0$ for $q > 0$. Thus $\Gamma_m(I^{\cdot}) \cong A'$, $M^{\cdot} \cong k$, and condition (i) holds.

(ii′) ⇒ (i′) By symmetry. ∎

DEFINITION 4.5. A dualizing complex $R^{\cdot}$ satisfying any of the equivalent conditions of Proposition 4.4 is called a pre-balanced dualizing complex.

A pre-balanced dualizing complex is determined up to tensoring with an invertible bimodule generated in degree 0. A balanced dualizing complex is clearly pre-balanced; however, unless A is noetherian, it is not known whether the existence of a pre-balanced dualizing complex implies the existence of a balanced one. The significance of the noetherian property is in:

PROPOSITION 4.6. *If the algebra A is left noetherian then for every injective A-module I, the m-torsion submodule $\Gamma_m(I)$ is injective too.*

(Compare [Ha, Chap. III, Lemma 3.2].)

Proof. If A is left noetherian, then an arbitrary direct sum of injective A-modules is injective. Let $M := \mathrm{Hom}_A(k, I)$, so $M \cong \bigoplus_\alpha k(n_\alpha)$ as an A-module. Then $\bigoplus_\alpha A'(n_\alpha)$ is an injective envelope of M. Now let $E \subset I$ be an injective A-module with $M \subset E$ an essential submodule. Therefore $E \cong \bigoplus_\alpha A'(n_\alpha)$ and so $E \subset \Gamma_m(I)$. There exists some splitting $I \cong E \oplus I'$; because $M \cap I' = 0$, it follows that $\Gamma_m(I') = 0$ and hence $\Gamma_m(I) \subset E$. We conclude that $E = \Gamma_m(I)$, and in particular that $\Gamma_m(I)$ is an injective A-module. ▮

If M is any finite length A-module, then the functors $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_A(M, \Gamma_m(-))$ are naturally isomorphic. According to [RD, Chap. I, Prop. 5.4(b)] and to the proposition just proved, there is an isomorphism of functors

$$\mathrm{RHom}^{\cdot}_A(M, -) \cong \mathrm{RHom}^{\cdot}_A(M, \mathrm{R}\Gamma_m(-)) \tag{4.7}$$

when A is left noetherian.

THEOREM 4.8. *Assume the algebra A is noetherian. The following are equivalent for any dualizing complex $R^{\cdot}$ over A:*

(i) *$R^{\cdot}$ is balanced.*

(ii) $\mathrm{R}\Gamma_m(R^{\cdot}) \cong A'$ *in* $\mathbf{D}(A^e)$.

(iii) *For every $n \geq 1$ there is an isomorphism* $\mathrm{RHom}^{\cdot}_A(A/m^n, R^{\cdot}) \cong \mathrm{Hom}_k(A/m^n, k)$ *in* $\mathbf{D}(A^e)$.

If A is generated over k by elements of degrees $\leq n_0$, then the above are equivalent to

(iii′) *For $n = 1, n_0 + 1$ there is an isomorphism* $\mathrm{RHom}^{\cdot}_A(A/m^n, R^{\cdot}) \cong \mathrm{Hom}_k(A/m^n, k)$ *in* $\mathbf{D}(A^e)$.

Proof. (i) ⇒ (ii) Trivial.

(ii) ⇒ (iii) Consider the isomorphisms in $\mathbf{D}(A^e)$:

$$\begin{aligned} D(A/m^n) &= \mathrm{RHom}^{\cdot}_A(A/m^n, R^{\cdot}) \\ &\cong \mathrm{RHom}^{\cdot}_A(A/m^n, \mathrm{R}\Gamma_m(R^{\cdot})) && \text{[Eq. (4.7)]} \\ &\cong \mathrm{RHom}_A(A/m^n, A') && \text{[condition (ii)]} \\ &\cong \mathrm{Hom}_k(A/m^n, k). \end{aligned}$$

(iii) ⇒ (i) Let $R^{\cdot}$ be a dualizing complex satisfying condition (iii). Taking $n = 1$ we see that $R^{\cdot}$ is pre-balanced, so by Proposition 4.4, $\mathrm{R}\Gamma_m(R^{\cdot}) \cong A' \otimes_A L$ for some invertible bimodule L. For every $n \geqslant 1$ there are isomorphisms

$$\begin{aligned} \mathrm{Hom}_k(A/m^n, k) &\cong \mathrm{RHom}^{\cdot}_A(A/m^n, R^{\cdot}) && \text{[condition (iii)]} \\ &\cong \mathrm{RHom}^{\cdot}_A(A/m^n, \mathrm{R}\Gamma_m(R^{\cdot})) && \text{[Eq. (4.7)]} \\ &\cong \mathrm{RHom}^{\cdot}_A(A/m^n, A' \otimes_A L) \\ &\cong \mathrm{RHom}^{\cdot}_A(A/m^n, A') \otimes_A L \\ &\cong \mathrm{Hom}_k(A/m^n, k) \otimes_A L. \end{aligned}$$

After applying $\mathrm{Hom}_k(-, k)$ we obtain an isomorphism $L^{\vee} \otimes_A A/m^n \cong A/m^n$, where $L^{\vee} := \mathrm{Hom}_A(L, A)$. Thus if $L \cong A(\phi, 0)$, the induced automorphism $\bar{\phi}$ on A/m^n is the identity. Letting $n \to \infty$ we conclude that $L \cong A$ over A^e and so $\mathrm{R}\Gamma_m(R^{\cdot}) \cong A'$. We observe that this implies

$$D(M) \cong \mathrm{Hom}_k(M, k) \tag{4.9}$$

for any finite length A^e-module M.

On the other hand, $\mathrm{R}\Gamma_{m^{\circ}}(R^{\cdot}) \cong \tilde{L} \otimes_A A'$ for some invertible bimodule $\tilde{L}$, so

$$\begin{aligned} D^{\circ}(A/m^n) &= \mathrm{RHom}^{\cdot}_{A^{\circ}}(A/m^n, R^{\cdot}) \\ &\cong \mathrm{RHom}^{\cdot}_{A^{\circ}}(A/m^n, \mathrm{R}\Gamma_{m^{\circ}}(R^{\cdot})) && \text{[Eq. (4.7)]} \\ &\cong \mathrm{RHom}^{\cdot}_{A^{\circ}}(A/m^n, \tilde{L} \otimes_A A') \\ &\cong \tilde{L} \otimes_A \mathrm{RHom}_{A^{\circ}}(A/m^n, A') \\ &\cong \tilde{L} \otimes_A \mathrm{Hom}_k(A/m^n, k). \end{aligned}$$

Upon applying D to these objects one gets

$$\begin{aligned} A/m^n &\cong DD^{\circ}(A/m^n) \\ &\cong D(\tilde{L} \otimes_A \operatorname{Hom}_k(A/m^n, k)) \\ &\cong \operatorname{Hom}_k(\tilde{L} \otimes_A \operatorname{Hom}_k(A/m^n, k), k) \qquad [\text{Eq. } (4.9)] \\ &\cong A/m^n \otimes_A \tilde{L}^{\vee}. \end{aligned}$$

Therefore $\tilde{L} \cong A$ over A^e, finishing the proof that $R^{\cdot}$ is balanced.

(iii) ⇒ (iii′) Trivial.

(iii′) ⇒ (i) Since A is generated by $\bigoplus_{n=0}^{n_0} A_n$, an automorphism ϕ of A is completely determined by its action on this set, and hence by the action of $\bar{\phi}$ on A/m^{n_0+1}. The proof of (iii) ⇒ (i) can now be used. ▌

Corollary 4.10. *If A is noetherian and if it has a pre-balanced dualizing complex then it has a balanced dualizing complex.*

Proof. Let $R^{\cdot}$ be a pre-balanced dualizing complex and let L be an invertible bimodule s.t. $R\Gamma_m(R^{\cdot}) \cong A' \otimes_A L$. Then $R\Gamma_m(R^{\cdot} \otimes_A L^{\vee}) \cong A'$ and by the theorem the dualizing complex $R^{\cdot} \otimes_A L^{\vee}$ is balanced. ▌

Corollary 4.11. *Assume A is noetherian. Let $R^{\cdot}$ be a balanced dualizing complex over A, and let L be an invertible bimodule. Then $L \otimes_A R^{\cdot} \cong R^{\cdot} \otimes_A L$ in $\mathbf{D}(A^e)$.*

Proof. We have $R\Gamma_m(L^{\vee} \otimes_A R^{\cdot} \otimes_A L) \cong L^{\vee} \otimes_A A' \otimes_A L \cong A'$ so by the theorem $L^{\vee} \otimes_A R^{\cdot} \otimes_A L \cong R^{\cdot}$. Tensoring on the left with L gives the desired result. ▌

Remark 4.12. The last corollary shows that if $A(\phi, n)[m]$ is a balanced dualizing complex, then ϕ must be in the center of $\operatorname{Aut}(A)$.

The following definition is taken from [ATV1]:

Definition 4.13. A graded algebra A, which in addition to the assumptions (1.4) also

(i) has finite global dimension d,

(ii) is Gorenstein: $\operatorname{Ext}_A^{\cdot}(k, A) \cong k(e)[-d]$ for some integer e, and

(iii) has polynomial growth,

is called a d-dimensional regular graded algebra.

Since the bimodule $A(-e)[d]$ is a pre-balanced dualizing complex over A, we have

COROLLARY 4.14. *A noetherian regular graded algebra has a balanced dualizing complex.*

A regular algebra of dimension $d \leqslant 3$ and generated in degree 1 is known to be noetherian (see [ATV1, Thm. 8.1] for $d=3$, and [AS, p. 172] for $d=2$). An analysis of balanced dualizing complexes over such algebras is done in the last section of the paper. As the next examples demonstrate, not all algebras which have a dualizing complex also have a pre-balanced one.

EXAMPLE 4.15. Let $A = k\langle X, Y\rangle$ be a free k-algebra. As shown in Example 3.7, the bimodule A is a dualizing complex over A. The A-module k has a projective resolution

$$0 \to A(-1)^2 \to A \to k \to 0.$$

Therefore $\operatorname{Ext}^1_A(k, A) \cong A(1)^2/A$ and so $\dim_k \operatorname{Ext}^1_A(k, A) = \infty$. Any dualizing complex is isomorphic to $A(m)[n]$ in $\mathbf{D}(A)$ for some m, n; hence A does not have a pre-balanced dualizing complex.

EXAMPLE 4.16. Let $A = k\langle X, Y\rangle/(YX)$. This algebra is not noetherian, but one can show it is (graded) coherent. The global dimension of A is 2 (see [AS, p. 172]), and it is a Koszul quadratic algebra, as can be seen from the resolution of k as an A-module:

$$0 \to A(-2) \xrightarrow{(Y,0)} A(-1)^2 \xrightarrow{(X,Y)^t} A \to k \to 0.$$

From the resolution we get $\operatorname{Ext}^2_A(k, A) \cong (A/YA)(2)$, which has infinite rank over k. Thus A has no pre-balanced dualizing complex. (This example was suggested by the referee.)

We conclude this section by proving a version of the local duality theorem for a balanced dualizing complex (cf. [RD, Chap. V, Thm. 6.2]). For any $M^\cdot \in \mathbf{D}^+(A)$ there is a natural morphism

$$\mathrm{R}\Gamma_m(M^\cdot) \to \mathrm{RHom}^\cdot_{A^\circ}(\mathrm{RHom}^\cdot_A(M^\cdot, R^\cdot), \mathrm{R}\Gamma_m(R^\cdot))$$

(see beginning of Section 3). By choosing an isomorphism $\mathrm{R}\Gamma_m(R^\cdot) \cong A'$ in $\mathbf{D}^+(A^e)$ we get a morphism in $\mathbf{D}^+(A)$:

$$\theta\colon \mathrm{R}\Gamma_m(M^\cdot) \to \mathrm{RHom}^\cdot_{A^\circ}(\mathrm{RHom}^\cdot_A(M^\cdot, R^\cdot), A'). \tag{4.17}$$

THEOREM 4.18. *Assume A is noetherian. Then for any $M^\cdot \in \mathbf{D}^b_c(A)$, θ is an isomorphism.*

There are variants of this theorem with $M^{\cdot} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^{\circ})$ or $M^{\cdot} \in \mathbf{D}^{\mathrm{b}}(A^{\mathrm{e}})$. The following lemma is needed for the proof of the theorem. As usual, we set $D(-) := \mathrm{RHom}^{\cdot}_{A}(-, R^{\cdot})$.

LEMMA 4.19. *For any $M^{\cdot}, N^{\cdot} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ and any integer q, there is a natural isomorphism of k-modules*:

$$\mathrm{Ext}^{q}_{A}(M^{\cdot}, N^{\cdot}) \cong \mathrm{Ext}^{q}_{A^{\circ}}(D(N^{\cdot}), D(M^{\cdot})).$$

Proof. According to [RD, Chap. I, Thm. 6.4] one has

$$\mathrm{Ext}^{q}_{A}(M^{\cdot}, N^{\cdot})_{i} \cong \mathrm{Hom}_{\mathbf{D}(A)}(M^{\cdot}, N^{\cdot}(i)[q])$$

in each degree i. Since D is an equivalence of categories between $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A)$ and $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^{\circ})$, which are full subcategories of $\mathbf{D}(A)$ and $\mathbf{D}(A^{\circ})$, respectively, there is a k-linear isomorphism

$$\mathrm{Hom}_{\mathbf{D}(A)}(M^{\cdot}, N^{\cdot}(i)[q]) \cong \mathrm{Hom}_{\mathbf{D}(A^{\circ})}(D(N^{\cdot}), D(M^{\cdot})(i)[q]).$$

But the latter is isomorphic to $\mathrm{Ext}^{q}_{A^{\circ}}(D(N^{\cdot}), D(M^{\cdot}))_{i}$. ∎

Proof (*of theorem*). Given $q \in \mathbf{Z}$ we will show that $\mathrm{H}^{q}(\theta)$ is bijective. For every $n \geqslant 0$ there is a commutative diagram in $\mathbf{D}(A)$,

$$\begin{array}{ccc} \mathrm{RHom}^{\cdot}_{A}(A/m^{n}, M^{\cdot}) & \xrightarrow{\theta_n} & \mathrm{RHom}^{\cdot}_{A^{\circ}}(D(M^{\cdot}), D(A/m^{n})) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma_{m}(M^{\cdot}) & \xrightarrow{\theta} & \mathrm{RHom}^{\cdot}_{A^{\circ}}(D(M^{\cdot}), A') \end{array}$$

where the left vertical arrow comes from the transformation of functors $\mathrm{Hom}_{A}(A/m^{n}, -) \to \Gamma_{m}$. The right vertical arow comes from the isomorphism $D(A/m^{n}) \cong \mathrm{Hom}_{k}(A/m^{n}, k)$ of Theorem 4.8, and so it depends on our choice of an isomorphism $\mathrm{R}\Gamma_{m}(M^{\cdot}) \cong A'$. Set $(A/m^{n})' := \mathrm{Hom}_{k}(A/m^{n}, k)$. Passing to the qth cohomology and then taking direct limits with respect to n we obtain the diagram

$$\begin{array}{ccc} \lim_{n\to} \mathrm{Ext}^{q}_{A}(A/m^{n}, M^{\cdot}) & \xrightarrow{\lim_{n\to} \mathrm{H}^{q}(\theta_n)} & \lim_{n\to} \mathrm{Ext}^{q}_{A^{\circ}}(D(M^{\cdot}), (A/m^{n})') \\ \downarrow \cong & & \downarrow \phi \\ \mathrm{H}^{q}_{m}(M^{\cdot}) & \xrightarrow{\mathrm{H}^{q}(\theta)} & \mathrm{Ext}^{q}_{A^{\circ}}(D(M^{\cdot}), A') \end{array}$$

Lemma 4.19 guarantees that each $\mathrm{H}^{q}(\theta_{n})$ is bijective. Thus it remains to show that ϕ is bijective.

The complex $D(M^{\cdot})$ is in $\mathbf{D}^{\mathrm{b}}_{\mathrm{c}}(A^{\circ})$, so it can be replaced by a complex

$P^{\cdot} \in \mathbf{D}_c^-(A^\circ)$ with each P^q a finitely generated projective A°-module. Then ϕ looks like

$$\begin{aligned}
&\lim_{n\to} \mathrm{Ext}^q_{A^\circ}(D(M^{\cdot}), (A/m^n)') \\
&\quad = \lim_{n\to} \mathrm{H}^q\mathrm{Hom}^{\cdot}_{A^\circ}(P^{\cdot}, (A/m^n)') \\
&\quad \cong \mathrm{H}^q \lim_{n\to} \mathrm{Hom}^{\cdot}_{A^\circ}(P^{\cdot}, (A/m^n)') \quad [\lim_{n\to} \text{ and } \mathrm{H}^q \text{ commute}] \\
&\quad \cong \mathrm{H}^q\, \mathrm{Hom}^{\cdot}_{A^\circ}(P^{\cdot}, \lim_{n\to} (A/m^n)') \quad [P^{-q} \text{ and } P^{-q+1} \text{ are finitely generated}] \\
&\quad \cong \mathrm{H}^q\, \mathrm{Hom}^{\cdot}_{A^\circ}(P^{\cdot}, A') \\
&\quad \cong \mathrm{Ext}^q_{A^\circ}(D(M^{\cdot}), A') \quad \blacksquare
\end{aligned}$$

An A-module M is called cofinite if $M \cong \mathrm{Hom}_k(N, k)$ for some finitely presented A°-module N.

COROLLARY 4.20. *Under the hypotheses of the theorem, for each q the module $\mathrm{H}^q_m(M^{\cdot})$ is cofinite.*

Proof. Using the natural isomorphism $\mathrm{Hom}_{A^\circ}(-, A') \cong \mathrm{Hom}_k(-, k)$ we get

$$\begin{aligned}
\mathrm{H}^q(\theta)\colon \mathrm{H}^q_m(M^{\cdot}) &\xrightarrow{\cong} \mathrm{H}^q\mathrm{Hom}_{A^\circ}(D(M^{\cdot}), A') \\
&\cong \mathrm{H}^q\mathrm{Hom}_k(D(M^{\cdot}), k) \\
&\cong \mathrm{Hom}_k(\mathrm{H}^{-q}D(M^{\cdot}), k).
\end{aligned}$$

But $\mathrm{H}^{-q}D(M^{\cdot})$ is coherent over A°. $\blacksquare$

COROLLARY 4.21. *There is an isomorphism $R^{\cdot} \cong \mathrm{Hom}^{\cdot}_A(\mathrm{R}\Gamma_m(A), A')$ in $\mathbf{D}(A^e)$. Hence $\mathrm{H}^q R^{\cdot} \cong \mathrm{Hom}_k(\mathrm{H}_m^{-q}(A), k)$.*

Proof. Taking $M^{\cdot} = A$ in the theorem we get an isomorphism

$$\theta\colon \mathrm{R}\Gamma_m(A) \xrightarrow{\cong} \mathrm{Hom}^{\cdot}_{A^\circ}(R^{\cdot}, A').$$

Now since $R^{\cdot}$ is in $\mathbf{D}_c^+(A)$ the module $\mathrm{H}^q(R^{\cdot})_n$ is finite dimensional over k for all q, n. This implies that Matlis duality holds for $R^{\cdot}$:

$$R^{\cdot} \cong \mathrm{Hom}^{\cdot}_A(\mathrm{Hom}^{\cdot}_{A^\circ}(R^{\cdot}, A')A') \cong \mathrm{Hom}^{\cdot}_k(\mathrm{Hom}^{\cdot}_k(R^{\cdot}, k), k).$$

Upon applying $\mathrm{Hom}_A(-, A')$ to θ we get the required isomorphism. $\blacksquare$

5. Finite Algebra Homomorphisms

In this section we consider the relations between dualizing complexes over two algebras, given a finite homomorphism between them. Throughout this section the algebras A and B are noetherian positively graded k-algebras satisfying assumptions (1.4). A homomorphism of graded k-algebras $f: A \to B$ is said to be finite if B is finitely generated as an A-module and as an A°-module.

The algebra homomorphism $f: A \to B$ induces exact restriction functors $\mathrm{Res}_{B \otimes A}$, Res_A, etc. These restriction functors do not in general send projectives to projectives, nor do they send injectives to injectives. However, the functor $\mathrm{Hom}_A(B, -)$ does send injective A-modules to injective B-modules. To see this, consider the natural isomorphism

$$\mathrm{Hom}_B(M, \mathrm{Hom}_A(B, N)) \cong \mathrm{Hom}_A(M, N)$$

for a B-moldule M and an A-module N. If N is A-injective then these two functors of M are exact, so $\mathrm{Hom}_A(B, N)$ is B-injective. From this we get a natural isomorphism of derived functors

$$\mathrm{RHom}^{\cdot}_B(M^{\cdot}, \mathrm{RHom}^{\cdot}_A(B, N^{\cdot})) \cong \mathrm{RHom}^{\cdot}_A(M^{\cdot}, N^{\cdot}) \tag{5.1}$$

for $M^{\cdot} \in \mathbf{D}(B)$ and $N^{\cdot} \in \mathbf{D}^{+}(A)$. This isomorphism of functors is valid for all combinations of categories for which it is defined (cf. Thm. 2.2).

Proposition 5.2. *Let $f: A \to B$ be a finite k-algebra homomorphism, and let $R^{\cdot}$ be a dualizing complex over A. Assume that there exists a complex $S^{\cdot} \in \mathbf{D}^{+}(B^{e})$ and isomorphisms*

$$\phi: \mathrm{Res}_{B \otimes A^{\circ}}(S^{\cdot}) \xrightarrow{\simeq} \mathrm{RHom}^{\cdot}_A(B, R^{\cdot})$$

in $\mathbf{D}(B \otimes A^{0})$ and

$$\phi^{\circ}: \mathrm{Res}_{A \otimes B^{\circ}}(S^{\cdot}) \xrightarrow{\simeq} \mathrm{RHom}^{\cdot}_{A^{\circ}}(B, R^{\cdot})$$

in $\mathbf{D}(A \otimes B^{\circ})$. Then $S^{\cdot}$ is a dualizing complex over B.

Proof. Since B is finitely generated over A°, it follows that $\mathrm{Res}_A(S^{\cdot}) \cong \mathrm{RHom}^{\cdot}_{A^{\circ}}(\mathrm{Res}_{A^{\circ}}(B), R^{\cdot})$ is in $\mathbf{D}^{+}_{c}(A)$, and hence $\mathrm{Res}_B(S^{\cdot}) \in \mathbf{D}^{+}_{c}(B)$. By symmetry we get $\mathrm{Res}_{B^{\circ}}(S^{\cdot}) \in \mathbf{D}^{+}_{c}(B^{\circ})$.

For any B module M there is an isomorphism

$$\mathrm{RHom}^{\cdot}_B(M, S^{\cdot}) \xrightarrow{\cong} \mathrm{RHom}^{\cdot}_A(M, R^{\cdot})$$

induced by ϕ. This implies that $S^{\cdot}$ has finite injective dimension over B. Symmetrically, it has finite injective dimension over B°.

In order to prove that the canonical morphism $\Phi: B \to \mathrm{RHom}^{\cdot}_B(S^{\cdot}, S^{\cdot})$ is an isomorphism, we first produce an isomorphism $B \cong \mathrm{RHom}^{\cdot}_B(S^{\cdot}, S^{\cdot})$ in $\mathbf{D}(B \otimes A^{\circ})$ as follows:

$$\begin{aligned} \mathrm{RHom}^{\cdot}_B(S^{\cdot}, S^{\cdot}) &\cong \mathrm{RHom}^{\cdot}_B(S^{\cdot}, \mathrm{RHom}^{\cdot}_A(B, R^{\cdot})) && [\text{via } \phi] \\ &\cong \mathrm{RHom}^{\cdot}_A(S^{\cdot}, R^{\cdot})) && [\text{by } (5.1)] \\ &\cong \mathrm{RHom}^{\cdot}_A(\mathrm{RHom}^{\cdot}_{A^{\circ}}(B, R^{\cdot}), R^{\cdot}) && [\text{via } \phi^{\circ}] \\ &\cong B. \end{aligned} \tag{5.3}$$

Now the canonical morphism Φ induces a k-algebra homomorphism $\mathrm{H}^0\Phi: B \to \mathrm{Ext}^0_B(S^{\cdot}, S^{\cdot})$ (cf. [RD, Chap. I, Thm. 6.4]). From (5.3) we know that B and $\mathrm{Ext}^0_B(S^{\cdot}, S^{\cdot})$ are isomorphic as B-modules, so $\mathrm{H}^0\Phi$ is actually an isomorphism. By symmetry the other canonical morphism $\Phi^{\circ}: B \to \mathrm{RHom}^{\cdot}_{B^{\circ}}(S^{\cdot}, S^{\cdot})$ is also an isomorphism. ∎

THEOREM 5.4. *Let $f: A \to B$ be a finite k-algebra homomorphism and let $R^{\cdot}_A$ be a balanced dualizing complex over A. Assume that the conditions of Proposition 5.2 hold for $R^{\cdot}_A$ and for some complex $S^{\cdot} \in \mathbf{D}^+(B^e)$. Then B has a balanced dualizing complex $R^{\cdot}_B$, and there are isomorphisms*

$$\phi: R^{\cdot}_B \xrightarrow{\simeq} \mathrm{RHom}^{\cdot}_A(B, R^{\cdot}_A)$$

in $\mathbf{D}(B \otimes A^{\circ})$ and

$$\phi^{\circ}: R^{\cdot}_B \xrightarrow{\simeq} \mathrm{RHom}^{\cdot}_{A^{\circ}}(B, R^{\cdot}_A)$$

in $\mathbf{D}(A \otimes B^{\circ})$.

Proof. According to Proposition 5.2, $S^{\cdot}$ is a dualizing complex over B. Furthermore, from the isomorphism

$$\mathrm{RHom}^{\cdot}_B(k, S^{\cdot}) \cong \mathrm{RHom}^{\cdot}_A(k, R^{\cdot}) \cong k$$

it follows that $S^{\cdot}$ is pre-balanced. By Corollary 4.10, B has a balanced dualizing complex $R^{\cdot}_B$, and we know that $R^{\cdot}_B \cong S^{\cdot} \otimes_B L \cong L \otimes_B S^{\cdot}$ in $\mathbf{D}(B^e)$ for some invertible B-bimodule L (see Cor. 4.11). To obtain ϕ (resp. ϕ°) it suffices to prove that $L \cong B$ over $B \otimes A^{\circ}$ (resp. $A \otimes B^{\circ}$).

Now $L \cong B$ over $B \otimes A^{\circ}$ iff $B/m^n_B \otimes_B L \cong B/m^n_B$ for all $n \geq 1$. Let N be the B^e-module $\mathrm{Hom}_k(B/m^n_B, k)$. Consider the isomorphisms in $\mathbf{D}(B \otimes A^{\circ})$:

$$\begin{aligned} B/m_B^n &\cong \mathrm{Hom}_k(N, k) \\ &\cong \mathrm{RHom}^{\cdot}_B(N, R^{\cdot}_B) \\ &\cong \mathrm{RHom}^{\cdot}_B(N, L\otimes_B S^{\cdot}) \\ &\cong \mathrm{RHom}^{\cdot}_B(L^{\vee}\otimes_B N, S^{\cdot}) \\ &\cong \mathrm{RHom}^{\cdot}_A(L^{\vee}\otimes_B N, R^{\cdot}_A) \\ &\cong \mathrm{Hom}_k(L^{\vee}\otimes_B N, k) \\ &\cong B/m_B^n\otimes_B L. \end{aligned}$$

By symmetry we get $L\otimes_B B/m_B^n \cong B/m_B^n$ over $A\otimes B^{\circ}$ for all $n\geqslant 1$. ∎

COROLLARY 5.5. *If $f\colon A\to B$ is surjective then the complex $S^{\cdot}$ of the theorem is itself a balanced dualizing complex.*

Proof. In the proof of the theorem we get $L\cong B$ over $B\otimes A^{\circ}$. Since f is surjective, this implies that $L\cong B$ over B^{e}, so $S^{\cdot}\cong R^{\cdot}_B$ in $\mathsf{D}(B^{e})$. ∎

At this stage it is possible to prove the existence of a balanced dualizing complex for an algebra finite over its center. First note that for a weighted commutative polynomial ring $A=k[X_1, ..., X_n]$, with $\deg X_i = d_i$, the complex $R^{\cdot}_A := A(-\sum_{i=1}^{n} d_i)[n]$ is a balanced dualizing complex. If B is an algebra finite over its center, then the center of B is a commutative finitely generated k-algebra. The center is therefore the image of some weighted polyomial ring A. Since in this case there is an isomorphism $\mathrm{RHom}^{\cdot}_A(B, R^{\cdot}_A)\cong \mathrm{RHom}^{\cdot}_{A^{\circ}}(B, R^{\cdot}_A)$ in $\mathsf{D}(B^{e})$, we have

COROLLARY 5.6. *A k-algebra finite over its center has a balanced dualizing complex.*

Remark 5.7. Given a balanced dualizing complex $R^{\cdot}_A$ over the algebra A, define $D_A := \mathrm{RHom}^{\cdot}_A(-, R^{\cdot}_A)$ and $D^{\circ}_A := \mathrm{RHom}^{\cdot}_{A^{\circ}}(-, R^{\cdot}_A)$. An isomorphism $\tau_A\colon k \xrightarrow{\simeq} D_A(k)$ in $\mathsf{D}(A^{e})$ gives rise to an isomorphism $D^{\circ}_A(\tau_A)\colon k\xrightarrow{\simeq} D^{\circ}_A(k)$, so the situation is symmetric.

A pair $(R^{\cdot}_A, \tau_A)$ is called a rigidified balanced dualizing complex. It is unique up to a unique isomorphism. If $f\colon A\to B$ is a finite homomorphism for which Theorem 5.4 holds, and if $(R^{\cdot}_B, \tau_B)$ is a rigidified balanced dualizing complex over B, then there exist unique isomorphisms $\phi\colon R^{\cdot}_B \xrightarrow{\simeq} D_A(B)$ in $\mathsf{D}(B\otimes A^{\circ})$ and $\varphi^{\circ}\colon R^{\cdot}_B\xrightarrow{\simeq} D^{\circ}_A(B)$ in $\mathsf{D}(A\otimes B^{\circ})$ which are compatible with τ_A and τ_B. In this way the homomorphism $f\colon A\to B$ induces two canonical trace morphisms

$$\mathrm{Tr}_{B/A}\colon R^{\cdot}_B \xrightarrow{\phi} D_A(B) \xrightarrow{D_A(f)} D_A(A) = R^{\cdot}_A$$

and

$$\mathrm{Tr}^{\circ}_{B/A}: R^{\cdot}_B \xrightarrow{\phi^{\circ}} D^{\circ}_A(B) \xrightarrow{D^{\circ}_A(f)} D^{\circ}_A(A) = R^{\cdot}_A$$

in $\mathbf{D}(A^{e})$.

It is not clear whether the two traces are equal in general. However, When A is commutative and f is a centralizing homomorphism (i.e., $B = f(A) \cdot C_B(f(A))$), we can identify $D_A(B) = D^{\circ}_A(B)$ in $\mathbf{D}(B^{e})$. The two isomorphisms ϕ and ϕ° will then coincide.

Let A be a k-algebra which has a balanced dualizing complex $R^{\cdot}_A$. Borrowing from the commutative terminology, we will call A a Cohen–Macaulay (resp. Gorenstein) algebra if $R^{\cdot}_A$ is isomorphic to a single bimodule (resp. an invertible bimodule) concentrated in some dimension.

PROPOSITION 5.8. *Let B be a k-algebra, and suppose there is a polynomial subalgebra A of the center of B such that B is finite over A. Then:*

(a) *B is a Cohen–Macaulay algebra iff* $\mathrm{Hom}_A(B, A)$ *is a dualizing complex over B, iff B is projective over A.*

(b) *B is a Gorenstein algebra iff B is projective over A and* $\mathrm{Hom}_A(B, A)$ *is an invertible bimodule over B.*

Proof. (a) By definition B is Cohen–Macaulay iff $\mathrm{RHom}^{\cdot}_A(B, A)$, which is a dualizing complex over B, is isomorphic in $\mathbf{D}(B^{e})$ to a single bimodule. This bimodule must be $\mathrm{H}^0\mathrm{RHom}^{\cdot}_A(B, A) = \mathrm{Hom}_A(B, A)$. Since $\mathrm{H}^q\mathrm{RHom}^{\cdot}_A(B, A) = 0$ for all $q \neq 0$, and since A has finite global dimension, B is projective over A. The converse is clear.

(b) Follows immediately from (a). ∎

6. SKEW HOMOGENEOUS COORDINATE RINGS

A skew (or twisted) homogeneous coordinate ring is a noncommutative analogue of the homogeneous coordinate ring of a projective variety. Skew homogeneous coordinate rings were introduced by M. Artin, J. Tate, and M. Van den Bergh in their study of three dimensional regular algebras (see [ATV1, Sect. 6]).

A triple $(X, \sigma, \mathscr{L})$ over the field k consists, by definition, of a scheme X, proper over k; a k-automorphism σ of X; and an invertible sheaf $\mathscr{L}$. We assume the following conventions, unless stated otherwise: k is algebraically closed; X is reduced, connected, and projective over k; and $\dim X > 0$. Fix such a triple $(X, \sigma, \mathscr{L})$. Exponents will be used to denote the action of σ on the category $\mathbf{Mod}(X)$ of $\mathcal{O}_X$-modules, by setting $\mathscr{F}^{\sigma} := \sigma^* \mathscr{F}$. In this section, an unadorned tensor symbol wil mean tensor over $\mathcal{O}_X$.

DEFINITION 6.1. For any integer n define the nth twist of $\mathcal{O}_X$ with respect to σ and $\mathcal{L}$ to be

$$\mathcal{O}(n, \sigma, \mathcal{L}) := \begin{cases} \mathcal{L} \otimes \mathcal{L}^{\sigma} \otimes \cdots \otimes \mathcal{L}^{\sigma^{n-1}} & \text{if } n > 0 \\ \mathcal{O}_X & \text{if } n = 0 \\ \mathcal{L}^{-\sigma^n} \otimes \cdots \otimes \mathcal{L}^{-\sigma^{-2}} \otimes \mathcal{L}^{-\sigma^{-1}} & \text{if } n < 0. \end{cases}$$

For any $\mathcal{O}_X$-module $\mathcal{F}$ define

$$\mathcal{F}(n, \sigma, \mathcal{L}) := \mathcal{O}(n, \sigma, \mathcal{L}) \otimes \mathcal{F}^{\sigma^n}.$$

Usually this will be the only twisting occurring, so we will use the shorthand $\mathcal{F}(n)$ for $\mathcal{F}(n, \sigma, \mathcal{L})$. Writing elements of the integral group ring $\mathbf{Z}\langle\sigma\rangle$ in the exponent one has $\mathcal{O}(n) = \mathcal{L}^{(1-\sigma^n)/(1-\sigma)}$ for all $n \in \mathbf{Z}$. Hence the identiy $\mathcal{F}(m+n) = \mathcal{F}(m)(n)$ holds for all $m, n \in \mathbf{Z}$ and all $\mathcal{O}_X$-modules $\mathcal{F}$.

Given an $\mathcal{O}_X$-module $\mathcal{F}$, set

$$\Gamma_*(\mathcal{F}) := \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathcal{F}(n)) \tag{6.2}$$

which is a k-module (recall the convention that a k-module is a graded k-vector space). There is a k-linear isomorphism

$$\sigma^*: \Gamma(X, \mathcal{F}) \xrightarrow{\simeq} \Gamma(X, \mathcal{F}^{\sigma}), \tag{6.3}$$

sending $a \in \Gamma(X, \mathcal{F})$ to $1 \otimes a \in \Gamma(X, \mathcal{O}_X \otimes_{\sigma^{-1}\mathcal{O}_X} \sigma^{-1}\mathcal{F}) = \Gamma(X, \mathcal{F}^{\sigma})$. This allows us to define a product

$$\begin{aligned} \Gamma(X, \mathcal{O}(m)) \times \Gamma(X, \mathcal{F}(n)) &\xrightarrow{1 \times (\sigma^*)^m} \Gamma(X, \mathcal{O}(m)) \times \Gamma(X, \mathcal{F}(n)^{\sigma^m}) \\ &\xrightarrow{\otimes} \Gamma(X, \mathcal{O}(m) \otimes \mathcal{F}(n)^{\sigma^m}) \\ &= \Gamma(X, \mathcal{F}(m+n)) \end{aligned} \tag{6.4}$$

for every $m, n \in \mathbf{Z}$. Summing over all degrees we get a graded product

$$\Gamma_*(\mathcal{O}_X) \times \Gamma_*(\mathcal{F}) \to \Gamma_*(\mathcal{F}).$$

LEMMA 6.5. *Taking $\mathcal{F} = \mathcal{O}_X$, the product above makes $\Gamma_*(\mathcal{O}_X)$ into a graded k-algebra. For any $\mathcal{O}_X$-module $\mathcal{F}$, $\Gamma_*(\mathcal{F})$ becomes a (graded left) $\Gamma_*(\mathcal{O}_X)$-module.*

Proof. The only thing to check is the associativity of the product. Given $a \in B_l$, $b \in B_m$, and $c \in \Gamma(X, \mathcal{F}(n))$, both $(a \cdot b) \cdot c$ and $a \cdot (b \cdot c)$ are equal to

$$a \otimes (\sigma^*)^l (b) \otimes (\sigma^*)^{l+m} (c) \in \Gamma(X, \mathcal{F}(l+m+n)). \quad \blacksquare$$

DEFINITION 6.6. The skew homogeneous coordinate ring associated to the triple $(X, \sigma, \mathscr{L})$ is the graded k-algebra $B = B(X, \sigma, \mathscr{L}) := \Gamma_*(\mathcal{O}_X)$.

The next definition generalizes the notion of an ample invertible sheaf.

DEFINITION 6.7. The sheaf $\mathscr{L}$ is said to be σ-ample if for every coherent sheaf $\mathscr{F}$ and for every $q > 0$, there exists an integer n_0 such that $\mathrm{H}^q(X, \mathscr{F}(n, \sigma, \mathscr{L})) = 0$ if $n \geqslant n_0$.

LEMMA 6.8. *Assume $\mathscr{L}$ is σ-ample.*

(a) *Given an exact sequence of coherent sheaves*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

there is an integer n such that the sequence of B-modules

$$0 \to \Gamma_{\geqslant n}(\mathscr{F}') \to \Gamma_{\geqslant n}(\mathscr{F}) \to \Gamma_{\geqslant n}(\mathscr{F}'') \to 0$$

is exact. Here $\Gamma_{\geqslant n}(-) := \bigoplus_{i \geqslant n} \Gamma(X, (-)(i))$.

(b) *If $\mathscr{F}$ is coherent then $\Gamma_{\geqslant n}(\mathscr{F})$ is a finitely generated B-module for any n.*

Proof. See [AV, Prop. 3.2].

LEMMA 6.9. *If $\mathscr{L}$ is σ-ample then $B_0 \cong k$ and $B_n = 0$ for $n < 0$.*

Proof. Because X is connected, reduced, and proper over k, $B_0 = \Gamma(X, \mathcal{O}_X)$ is a finite field extension of k. But k is algebraically closed, so $B_0 = k$.

Let $x_1, \ldots, x_r$ be distinct closed points in X (recall that $\dim X > 0$) and let $\mathscr{F} := \bigoplus_{j=1}^{r} k(x_j)$ be the sum of the residue fields. From the surjection $\mathcal{O} \to \mathscr{F}$ we get, for $n \gg 0$, that $B_n \to \Gamma(X, \mathscr{F}(n)) \cong k^r$ is surjective. Therefore $\lim_{n \to \infty} \dim_k B_n = \infty$. Let $\{Z_i\}$ be the irreducible components of X, with the reduced induced subscheme structures. Since the x_j can be distributed on the various components, we actually have

$$\lim_{n \to \infty} \dim_k \Gamma(Z_i, \mathcal{O}_{Z_i} \otimes \mathcal{O}(n)) = \infty \tag{6.10}$$

for every component Z_i.

Suppose now that there exists some nonzero $b \in B_{-m}$ with $m > 0$. Because X is reduced, $Z_i \subset \operatorname{supp}(b)$ for some i. For any $n \geqslant 0$, multiplication on the left by b is an injection

$$\Gamma(\sigma^{-m}(Z_i), \mathcal{O}_{\sigma^{-m}(Z_i)} \otimes \mathcal{O}(n)) \xrightarrow{b \cdot} \Gamma(Z_i, \mathcal{O}_{Z_i} \otimes \mathcal{O}(n-m)),$$

because it amounts to multiplication of global sections of locally free sheaves on the integral scheme Z_i. But this is a contradiction to (6.10). ▮

Next we bring two theorems of Artin and Van den Bergh, the first of which extends Serre's classical theorem about commutative graded algebras. Let m–**tors** denote the full subcategory of **GrMod**(B) consisting of m-torsion modules. This is a localizing subcategory. Also, let **QCoh**(X) denote the category of quasi-coherent $\mathcal{O}_X$-modules.

THEOREM 6.11 [AV, Thm. 1.3]. *Assume the sheaf $\mathscr{L}$ is σ-ample. Then the functor Γ_* induces an equivalence of categories between* **QCoh**(X) *and the quotient category* **GrMod**(B)/m–**tors**.

THEOREM 6.12 [AV, Thm. 1.4]. *Assume the sheaf $\mathscr{L}$ is σ-ample. Then B is a noetherian finitely generated k-algebra.*

The left adjoint $M \mapsto \tilde{M}$ of Γ_* is defined as follows. For a projective B-module of the form $P = \bigoplus B(n_i)$ set $\tilde{P} := \bigoplus \mathcal{O}(n_i)$. A homomorphism $\phi: P \to Q$ determines a homomorphism $\tilde{\phi}: \tilde{P} \to \tilde{Q}$ in the obvious way. Extend the functor to all B-modules as a left derived functor. Theorem 6.11 assert that the m-torsion modules are precisely those B-modules M such that $\tilde{M} = 0$. It also asserts that $M \mapsto \tilde{M}$ is an exact functor. Observe that the equivalence of categories is compatible with the two twisting operations. Let us now give a sufficient condition for $\mathscr{L}$ to be σ-ample.

PROPOSITION 6.13. *Assume that for some integers $m, n > 0$ the two conditions hold*:

(i) $\mathcal{O}(n) = \mathscr{L} \otimes \mathscr{L}^{\sigma} \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}}$ *is ample (in the usual sense)*;

(ii) *the class of the invertible sheaf $(\mathcal{O}(n) \otimes \mathcal{O}(-n))^m$ is in* $\mathrm{Pic}^0 X$, *the identity component of the Picard scheme of X.*

Then $\mathscr{L}$ is both σ-ample and σ^{-1}-ample.

Proof. This is an immediate consequence of [AV, Lemma 4.1, Prop. 1.5], since

$$(\mathcal{O}(n) \otimes \mathcal{O}(-n))^m = (\mathcal{O}(n)^{\sigma^n} \otimes \mathcal{O}(n)^{-1})^{m\sigma^{-n}}$$

and σ acts on $\mathrm{Pic}^0 X$. ▮

Condition (ii) is satisfied when σ^n is algebraically equivalent to the identity automorphism. This happens, for instance, when $\dim X = 1$.

LEMMA 6.14. *Assume $\mathscr{L}$ is σ-ample, and let $\mathscr{I}$ be a quasi-coherent injective $\mathcal{O}_X$-module. Then $\Gamma_*(\mathscr{I})$ is an injective B-module.*

Proof. This is true because $\Gamma_*: \mathbf{QCoh}(X) \to \mathbf{GrMod}(B)$ has an exact left adjoint (cf. Lemma 2.1). ∎

The functor $\Gamma_*: \mathbf{Mod}(X) \to \mathbf{GrMod}(B)$ has a derived functor $\mathrm{R}\Gamma_*: \mathbf{D}^+(X) \to \mathbf{D}(B)$, which is computed by flasque $\mathcal{O}_X$-modules. Given an $\mathcal{O}_X$-module $\mathscr{F}$, we have $\mathrm{H}^q\mathrm{R}\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbf{Z}} \mathrm{H}^q(X, \mathscr{F}(n))$. Thus we see that $\mathscr{L}$ is σ-ample iff $\mathrm{H}^q\mathrm{R}\Gamma_*(\mathscr{F})$ is a right limited B-module for every coherent sheaf $\mathscr{F}$ and every $q > 0$.

So far we have related $\mathcal{O}_X$-modules to B-modules. Next we consider σ-equivariant $\mathcal{O}_X$-modules and show how they correspond to graded B–B-bimodules, i.e., to B^{e}-modules. A σ-equivariant $\mathcal{O}_X$-module is a pair $(\mathscr{F}, e)$, where $\mathscr{F}$ is an $\mathcal{O}_X$-module and $e: \sigma^*\mathscr{F} \xrightarrow{\simeq} \mathscr{F}$ is an isomorphism of $\mathcal{O}_X$-modules. e is called an equivariance for $\mathscr{F}$. We will use the term "equivariant sheaf" as a synonym for "σ-equivariant $\mathcal{O}_X$-module." A morphism of equivariant sheaves $\phi: (\mathscr{F}, e) \to (\mathscr{F}', e')$ is an $\mathcal{O}_X$-module homomorphism $\phi: \mathscr{F} \to \mathscr{F}'$ s.t. $\phi e = e'\phi^\sigma$. The category of equivariant sheaves is abelian, and the operations $\otimes_{\mathcal{O}_X}$ and $\mathrm{Hom}_{\mathcal{O}_X}(-,-)$ are defined on it.

The sheaf $\mathcal{O}_X$ has a canonical equivariance, namely the identity $1: \sigma^*\mathcal{O}_X \xrightarrow{\simeq} \mathcal{O}_X$. Any other equivariance of $\mathcal{O}_X$ is obtained from 1 by multiplication with $\lambda \in k^*$ and is denoted by $(\mathcal{O}_X, \lambda)$. If X is smooth of dimension n over k, let $\omega_X = \Omega^n_{X/k}$ be the dualizing sheaf on X. The canonical equivariance of ω_X is denoted by ε. More generally, ε will denote the canonical equivariance of the residue complex $\mathscr{K}^{\cdot}_X$ (see formula (7.1)).

Let $(\mathscr{F}, e)$ be an equivariant sheaf. There is a k-linear isomorphism

$$e\sigma^*: \Gamma(X, \mathscr{F}) \xrightarrow{\sigma^*} \Gamma(X, \mathscr{F}^\sigma) \xrightarrow{e} \Gamma(X, \mathscr{F}),$$

which is the action of σ on the global sections of $(\mathscr{F}, e)$. Observe that a morphism $\phi: (\mathcal{O}_X, \lambda) \to (\mathscr{F}, e)$ corresponds to an eigenvector $\phi(1)$ of $e\sigma^*$ with eigenvalue λ.

An equivariant sheaf $(\mathscr{F}, e)$ induces, by pullback, an equivariance $e^{\sigma^n}: \mathscr{F}^{\sigma^{n+1}} \xrightarrow{\simeq} \mathscr{F}^{\sigma^n}$ for every integer n. Set $e^n_{n+1} := e^{\sigma^n}$. For any integer m let $e^n_{n+m}: \mathscr{F}^{\sigma^{n+m}} \xrightarrow{\simeq} \mathscr{F}^{\sigma^n}$ be the equivariance such that the recursive condition $e^n_{n+m+1} = e^n_{n+m} \circ e^{n+m}_{n+m+1}$ is satisfied. With this notation, we have $e = e^0_1$.

Fix an equivariant sheaf $(\mathscr{F}, e)$. For any $m, n \in \mathbf{Z}$ consider the product

$$\begin{aligned} \Gamma(X, \mathscr{F}(m)) \times \Gamma(X, \mathcal{O}(n)) &\xrightarrow{1 \times (\sigma^*)^m} \Gamma(X, \mathscr{F}(m)) \times \Gamma(X, \mathcal{O}(n)^{\sigma^m}) \\ &\xrightarrow{\otimes} \Gamma(X, \mathcal{O}(m+n) \otimes \mathscr{F}^{\sigma^m}) \\ &\xrightarrow{1 \otimes e^{m+n}_m} \Gamma(X, \mathcal{O}(m+n) \otimes \mathscr{F}^{\sigma^{m+n}}) \\ &= \Gamma(X, \mathscr{F}(m+n)). \end{aligned} \tag{6.15}$$

Summing up on all degrees, one gets a graded product

$$\Gamma_*(\mathscr{F}) \times B \to \Gamma_*(\mathscr{F}).$$

LEMMA 6.16. *The product above, together with the product* (6.4), *makes* $\Gamma_*(\mathscr{F})$ *into a* B^e*-module* (*i.e., a graded B–B-bimodule*).

Proof. We must check two things:

(i) The two maps $\Gamma_*(\mathscr{F}) \times B \times B \to \Gamma_*(\mathscr{F})$ are equal (associativity).

(ii) The two maps $B \times \Gamma_*(\mathscr{F}) \times B \to \Gamma_*(\mathscr{F})$ are equal (the left and right actions commute).

To prove (i), we note that given $a \in \Gamma(X, \mathscr{F}(l))$, $b \in B_m$, and $c \in B_n$, both maps send (a, b, c) to

$$(1 \otimes e_l^{l+m+n})(a) \otimes (\sigma^*)^l (b) \otimes (\sigma^*)^{l+m} (c).$$

The proof of (ii) is similar. ∎

We denote this B^e-module by $\Gamma_*(\mathscr{F}, e)$. Thus Γ_* becomes a functor from equivariant sheaves to B^e-modules.

Given an $\mathscr{O}_X$-module F, its nth opposite twist is defined to be the module $\mathscr{F}^{\circ}(n) = \mathscr{F}^{\circ}(n, \sigma, \mathscr{L}) := \mathscr{F}(n, \sigma^{-1}, \mathscr{L})$. Set $\Gamma^{\circ}_*(\mathscr{F}) := \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{F}^{\circ}(n))$ and $B^{\circ} := \Gamma^{\circ}_*(\mathscr{O}_X)$. Formula (6.4), with σ^{-1} instead of σ, makes B° into a graded k-algebra, and it makes $\Gamma^{\circ}_*(\mathscr{F})$ into a (left) B°-module.

Now let $(\mathscr{F}, e)$ be a σ-equivariant sheaf. Then $(\mathscr{F}, e^0_{-1})$ is a σ^{-1}-equivariant sheaf. For any $n \in \mathbf{Z}$ consider the k-linear isomorphism

$$\begin{aligned}\tau_n \colon \Gamma(X, \mathscr{F}(n)) &= \Gamma(X, \mathscr{L}^{(1-\sigma^n)/(1-\sigma)} \otimes \mathscr{F}^{\sigma^n}) \\ &\xrightarrow{(\sigma^*)^{-n+1}} \Gamma(X, \mathscr{L}^{(1-\sigma^{-n})/(1-\sigma^{-1})} \otimes \mathscr{F}^{\sigma}) \\ &\xrightarrow{1 \otimes e_1^{-n}} \Gamma(X, \mathscr{L}^{(1-\sigma^{-n})/(1-\sigma^{-1})} \otimes \mathscr{F}^{\sigma^{-n}}) \\ &= \Gamma(X, \mathscr{F}^{\circ}(n)).\end{aligned}$$

The graded isomorphism $\tau \colon \Gamma_*(\mathscr{F}) \xrightarrow{\simeq} \Gamma^{\circ}_*(\mathscr{F})$ obtained by summing over all degrees is called the transposition associated to e.

PROPOSITION 6.17. *The transposition* $\tau \colon B \to B^{\circ}$ *associated to the equivariance* $1 \colon \sigma^*\mathscr{O}_X \xrightarrow{\simeq} \mathscr{O}_X$ *is an anti-isomorphism of k-algebras. Thus* B° *can be identified with the opposite algebra of B. For any σ-equivariant sheaf* $(\mathscr{F}, e)$, *the transposition* $\tau \colon \Gamma_*(\mathscr{F}) \xrightarrow{\simeq} \Gamma^{\circ}_*(\mathscr{F})$ *is an anti-isomorphism of B–B-bimodules.*

Proof. The claim is that for any $a \in B_l$, $b \in \Gamma(X, \mathscr{F}(m))$, and $c \in B_n$ the equality

$$\tau_{l+m+n}(a \cdot b \cdot c) = \tau_n(c) \cdot \tau_m(b) \cdot \tau_l(a)$$

holds. Taking $(\mathscr{F}, e) = (\mathscr{O}_X, 1)$ and $b = 1 \in B_0$ shows that τ is a ring anti-isomorphism. The equality is verified by direct computation: both sides are equal to

$$(\sigma^*)^{-l-m-n+1}(a) \otimes (\sigma^*)^{-n+1}(c) \otimes (\sigma^*)^{-m-n+1}(1 \otimes e_m^{-l-1})(b)$$

in $\Gamma(X, \mathscr{F}^{\circ}(l+m+n))$. ∎

COROLLARY 6.18. *Assume the invertible sheaf $\mathscr{L}$ is both σ-ample and σ^{-1}-ample. Let $(\mathscr{I}, e)$ be a σ-equivariant sheaf, with $\mathscr{I}$ a quasi-coherent injective $\mathscr{O}_X$-module. Then the B^e-module $\Gamma_*(\mathscr{I}, e)$ is injective both over B and over B°.*

Proof. This is a consequence of Proposition 6.17 and Lemma 6.14. ∎

It may happen that the sheaf $\mathscr{L}$ itself admits a σ-equivariance—which is the same as saying that σ fixes the class of $\mathscr{L}$ in Pic X. When this occurs, B-modules can be "untwisted." Choose an equivariance $e: \sigma^*\mathscr{L} \xrightarrow{\simeq} \mathscr{L}$. For any $n \in \mathbf{Z}$ define

$$\begin{aligned} U_n: \Gamma(X, \mathscr{F}(n, \sigma, \mathscr{L})) &= \Gamma(X, \mathscr{L} \otimes \cdots \otimes \mathscr{L}^{\sigma^{n-1}} \otimes \mathscr{F}^{\sigma^n}) \\ &\xrightarrow{(\sigma^*)^{-n}} \Gamma(X, \mathscr{L}^{\sigma^{-n}} \otimes \cdots \otimes \mathscr{L}^{\sigma^{-1}} \otimes \mathscr{F}) \\ &\xrightarrow{e^0_{-n} \otimes \cdots \otimes e^0_{-1} \otimes 1} \Gamma(X, \mathscr{L}^n \otimes \mathscr{F}). \end{aligned}$$

Summing over all n gives an isomorphism of k-modules $U: \Gamma_*(\mathscr{F}) \xrightarrow{\simeq} \bigoplus_{n \in \mathbf{Z}} \Gamma(X, \mathscr{L}^n \otimes \mathscr{F})$. For any $a \in B_l$ and $b \in \Gamma(X, \mathscr{F}(m))$ the identity

$$U_{l+m}(a \cdot b) = U_l([(e_1^0 \otimes \cdots \otimes e_l^{l-1})\sigma^*]^{-m}(a)) \otimes U_m(b) \tag{6.19}$$

holds.

If $(\mathscr{F}, f)$ is an equivariant sheaf, then for any $a \in B_l$, $b \in \Gamma(X, \mathscr{F}(m))$, and $c \in B_n$, one has

$$\begin{aligned} U_{l+m+n}&(a \cdot b \cdot c) \\ &= U_l([(e_1^0 \otimes \cdots \otimes e_l^{l-1})\sigma^*]^{-(m+n)}(a)) \\ &\quad \otimes U_m([(e_1^0 \otimes \cdots \otimes e_m^{m-1} \otimes f^m_{m+1})\sigma^*]^{-n}(b)) \otimes U_n(c). \end{aligned} \tag{6.20}$$

The untwisting U shows that $\mathscr{L}$ is σ-ample iff it is ample in the usual sense (cf. Prop. 6.13); the same is true for σ^{-1}. Since U sends left ideals of

$B = B(X, \sigma, \mathscr{L})$ to ideals of the commutative k-algebra $\bigoplus_n \Gamma(X, \mathscr{L}^n)$, we get an elementary proof that B is noetherian in the ample case. Untwisting also enables a description of B in terms of generators and relations, as demonstrated in the following:

EXAMPLE 6.21 ("Quantum plane"). Let X be the projective line $\mathbf{P}^1_k$, let σ be any automorphism of X, and let $\mathscr{L}$ be the sheaf of linear forms $\mathscr{O}_X(1)$. Choose an equivariance $e: \sigma^*\mathscr{L} \xrightarrow{\simeq} \mathscr{L}$ and a k-basis x, y for $\Gamma(X, \mathscr{L})$ such that

$$e\sigma^* \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In the first case we get $U(x) = q^{-1}x$ and $U(y) = y$ and by (6.19)

$$U(xy) = U((e\sigma^*)^{-1}(x)) \otimes U(y) = q^{-2}x \otimes y$$
$$U(yx) = U((e\sigma^*)^{-1}(y)) \otimes U(x) = q^{-1}y \otimes x$$

so that $U(yx - qxy) = 0$. A dimension comparison shows that

$$B = k\langle x, y\rangle/(yx - qxy).$$

In the second case $U(x) = x$, $U(y) = y + x$, and

$$U(x^2) = U((e\sigma^*)^{-1}(x)) \otimes U(x) = x \otimes x$$
$$U(xy) = U((e\sigma^*)^{-1}(x)) \otimes U(y) = x \otimes y + x \otimes x$$
$$U(yx) = U((e\sigma^*)^{-1}(y)) \otimes U(x) = y \otimes x + 2x \otimes x.$$

Therefore $U(yx - xy - x^2) = 0$ and

$$B = k\langle x, y\rangle/(yx - xy - x^2).$$

7. Dualizing Complexes over Skew Homogeneous Coordinate Rings

In this section we retain the conventions of the previous sections regarding the triple $(X, \sigma, \mathscr{L})$. In addition, we assume that the invertible sheaf $\mathscr{L}$ is both σ-ample and σ^{-1}-ample. Thus $B = B(X, \sigma, \mathscr{L})$ is noetherian positively graded and $B_0 \cong k$. Let $\pi: X \to \operatorname{Spec} k$ be the structural morphism, and let $\mathscr{K}^{\cdot}_X$ be the residue (or residual) complex associated to $\pi^! k$ (see [RD, Chap. VI, Sect. 1). This is a complex of quasi-coherent injective $\mathscr{O}_X$-modules with coherent cohomology sheaves. Now $\pi\sigma = \pi$, so according to [RD, Chap. VI, Thm. 3.1], there is a canonical isomorphism $\sigma^! \pi^! k \cong \pi^! k$ in $\mathbf{D}(X) = \mathbf{D}(\mathbf{Mod}(X))$. Since $\sigma^! = \sigma^*$ this isomorphism induces a canonical equivariance of complexes of $\mathcal{O}_X$-modules

$$\varepsilon \colon \sigma^* \mathcal{K}^{\cdot}_X \xrightarrow{\simeq} \mathcal{K}^{\cdot}_X. \tag{7.1}$$

Upon applying Γ_* we get a complex of B^e-modules $\Gamma_*(\mathcal{K}^{\cdot}_X, \varepsilon)$. Note that $\pi_* = \Gamma(X, -)$ as functors, so for any $\mathcal{O}_X$-module $\mathcal{F}$, $\Gamma_*(\mathcal{F}) = \bigoplus_n \pi_* \mathcal{F}(n)$.

DEFINITION 7.2. Let $\mathcal{K}^{\cdot}_B$ be the complex of B^e-modules

$$\mathcal{K}^q_B := \begin{cases} \Gamma_*(\mathcal{K}^{q+1}_X, \varepsilon) & \text{if } q < 0 \\ \mathrm{H}^0 \Gamma_*(\mathcal{K}^{\cdot}_X, \varepsilon) & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

with differential induced from the complex $\Gamma_*(\mathcal{K}^{\cdot}_X, \varepsilon)[1]$.

The main result of this section is:

THEOREM 7.3. *The complex $\mathcal{K}^{\cdot}_B$ is a balanced dualizing complex over B.*

Recall the Grothendieck duality theorem [RD, Chap. III, Thm. 11.1] for projectively embedable morphisms; Chap. VII, Thm. 3.3, for the general case. Let $\mathbf{D}^-_{\mathrm{qc}}(X)$ be the full subcategory of $\mathbf{D}^-(X)$ whose objects are the complexes with quasi-coherent cohomology sheaves. Given a complex $\mathcal{F}^{\cdot} \in \mathbf{D}^-_{\mathrm{qc}}(X)$, there is a natural isomorphism

$$\theta \colon \mathrm{R}\pi_* \mathrm{RHom}^{\cdot}_{\mathcal{O}_X}(\mathcal{F}^{\cdot}, \mathcal{K}^{\cdot}_X) \xrightarrow{\simeq} \mathrm{Hom}^{\cdot}_k(\mathrm{R}\pi_* \mathcal{F}^{\cdot}, k) \tag{7.4}$$

in $\mathbf{D}(k)$. This isomorphism is induced by the trace morphism $\mathrm{Tr}_\pi \colon \pi_* \mathcal{K}^{\cdot}_X \to k$.

Note that in $\mathbf{D}(k)$, a complex can be identified with its cohomology, so θ induces isomorphisms

$$\theta^q \colon \mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{F}^{\cdot}, \mathcal{K}^{\cdot}_X) \xrightarrow{\simeq} \mathrm{Hom}_k(\mathrm{H}^{-q}(X, \mathcal{F}^{\cdot}), k). \tag{7.5}$$

When $\mathcal{F}^{\cdot} \in \mathbf{D}^-_{\mathrm{c}}(X)$, i.e., when the cohomology sheaves of $\mathcal{F}^{\cdot}$ are coherent, these k-vector spaces are finite dimensional, so the Yoneda pairing

$$\mathrm{H}^{-q}(X, \mathcal{F}^{\cdot}) \times \mathrm{Ext}^q_{\mathcal{O}_X}(\mathcal{F}^{\cdot}, \mathcal{K}^{\cdot}_X) \to \mathrm{H}^0(X, \mathcal{K}^{\cdot}_X) \xrightarrow{\mathrm{Tr}_\pi} k \tag{7.6}$$

is perfect. Taking $\mathcal{F}^{\cdot} = \mathcal{O}_X$ in (7.5) gives the isomorphism

$$\theta^0 = \mathrm{Tr}_\pi \colon \mathrm{H}^0(X, \mathcal{K}^{\cdot}_X) \xrightarrow{\simeq} \mathrm{Hom}_k(\mathrm{H}^0(X, \mathcal{O}_X), k) \cong k.$$

Let $\mathrm{Tr}_\sigma : \sigma_* \sigma^! \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_X$ be the trace associated to σ considered as a finite morphism (see [RD, Chap. III, Sect. 6]. Then

$$\mathrm{Tr}_\sigma : \pi_* \sigma^! \mathcal{K}^{\cdot}_X = \pi_* \sigma_* \sigma^! \mathcal{K}^{\cdot}_X \to \pi_* \mathcal{K}^{\cdot}_X$$

is the inverse of the map

$$\sigma^* : \Gamma(X, \mathcal{K}^{\cdot}_X) \to \Gamma(X, \sigma^* \mathcal{K}^{\cdot}_X)$$

of (6.3). According to condition (TRA1) of RD, Chap. III, Thm. 10.5] we get a commutative diagram

$$\begin{array}{ccc} \pi_* \mathcal{K}^{\cdot}_X & \xrightarrow{\mathrm{Tr}_\pi} & k \\ \uparrow{\scriptstyle \varepsilon} & & \uparrow{\scriptstyle \mathrm{Tr}_\pi} \\ \pi_* \sigma_* \sigma^! \mathcal{K}^{\cdot}_X & \xrightarrow{\mathrm{Tr}_\sigma} & \pi_* \mathcal{K}^{\cdot}_X \end{array}$$

and thus deduce the important formula

$$\mathrm{Tr}_\pi = \mathrm{Tr}_\pi \circ (\varepsilon \sigma^*) : \Gamma(X, \mathcal{K}^{\cdot}_X) \to k. \tag{7.7}$$

Remark 7.8. An explicit construction of the residue complex $\mathcal{K}^{\cdot}_X$, using local fields, can be found in the author's thesis [Ye]. This construction exhibits the trace maps Tr_π and Tr_σ on the level of differential forms. The equivariant structure of $\mathcal{K}^{\cdot}_X$ becomes combinatorial, with σ permuting the components of the complex.

According to Corollary 6.18, the B^e-modules $\mathcal{K}^q_B$, $q \neq 0$, are injective both over B and over B°. The next lemma implies that the same is true for $\mathcal{K}^0_B$.

LEMMA 7.9. *There is a canonical isomorphism of B^e-modules $\Theta : \mathcal{K}^0_B \xrightarrow{\simeq} B' = \mathrm{Hom}_k(B, k)$.*

Proof. It suffices to prove that $\mathcal{K}^0_B \cong B'$ as B-modules and that the B^e-module $\mathrm{Hom}_k(\mathcal{K}^0_B, k)$ is generated over B° by a central element. With the degree 0 part $(\mathcal{K}^0_B)_0$ of $\mathcal{K}^0_B$ identified with $\mathrm{H}^0(X, \mathcal{K}^{\cdot}_X)$, the trace map Tr_π gives rise to a canonical k-linear isomorphism $t : (\mathcal{K}^0_B)_0 \xrightarrow{\simeq} k$.

For any integer n consider the pairing

$$B_n \times (\mathcal{K}^0_B)_{-n} \to k, \qquad \langle b, \phi \rangle = t(b \cdot \phi).$$

It factors as

$$\begin{aligned} &\mathrm{H}^0(X, \mathcal{O}(n)) \times \mathrm{H}^0(X, \mathcal{O}(-n) \otimes (K_X)^{\sigma^{-n}}) \\ &\xrightarrow{1 \times (\sigma^*)^n} \mathrm{H}^0(X, \mathcal{O}(n)) \times \mathrm{H}^0(X, \mathcal{O}(-n)^{-n} \otimes \mathscr{K}^{\cdot}_X) \\ &\cong \mathrm{H}^0(X, \mathcal{O}(n)) \times \mathrm{Ext}^0_{\mathcal{O}_X}(\mathcal{O}(n), \mathscr{K}^{\cdot}_X) \\ &\to \mathrm{H}^0(X, \mathscr{K}^{\cdot}_X) \xrightarrow{\mathrm{Tr}_\pi} k \end{aligned}$$

so it is a perfect pairing of k-modules. Thus $\phi \mapsto \langle -, \phi \rangle$ is an isomorphism of B-modules from $\mathscr{K}^0_B$ to B'. By dualizing this gives a B°-module isomorphism $B \xrightarrow{\simeq} \mathrm{Hom}_k(\mathscr{K}^0_B, k)$ which sends 1 to t.

It remains to show that t is central, i.e., that $b \cdot t = t \cdot b$ for all $b \in B$. Equivalently, one has to show that $t(b \cdot \phi) = t(\phi \cdot b)$ for all $b \in B_n$, $\phi \in (\mathscr{K}^0_B)_{-n}$. Since $\mathscr{K}^0_X$ is a flasque sheaf, we may write $\phi = af(\sigma^*)^{-n}(\psi)$ with a a local section of $\mathcal{O}(-n)$ and ψ a global section of $\mathscr{K}^0_X$. The two products to compare are

$$b \cdot \phi = [b \otimes (\sigma^*)^n (a)] \otimes \psi$$

and

$$\phi \cdot b = [a \otimes (\sigma^*)^{-n} (b)] \otimes (\varepsilon\sigma^*)^{-n} (\psi),$$

where $b \otimes (\sigma^*)^n (a)$ and $a \otimes (\sigma^*)^{-n} (b)$ are local sections of $\mathcal{O}_X$ (at different loci). Now Tr_π commutes with $\varepsilon\sigma^*$ (formula (7.7)) and ε is $\mathcal{O}_X$-linear. Therefore

$$\begin{aligned} \mathrm{Tr}_\pi(b \cdot \phi) &= \mathrm{Tr}_\pi((\varepsilon\sigma^*)^{-n} (b \cdot \phi)) \\ &= \mathrm{Tr}_\pi(a \otimes (\sigma^*)^{-n} (b) \otimes (\varepsilon\sigma^*)^{-n} (\psi)) \\ &= \mathrm{Tr}_\pi(\phi \cdot b). \quad \blacksquare \end{aligned}$$

LEMMA 7.10. $\mathrm{Res}_B(\mathscr{K}^{\cdot}_B) \in \mathbf{D}^+_c(B)$ *and* $\mathrm{Res}_{B^\circ}(\mathscr{K}^{\cdot}_B) \in \mathbf{D}^+_c(B^\circ)$.

Proof. By symmetry it suffices to prove the first assertion. We need to introduce an auxiliary category, **FH**. This is the full subcategory of **GrMod**(B) consisting of the modules M such that the submodule $\bigoplus_{n \geq 0} M_n$ is finitely generated over B. **FH** is a thick abelian subcategory (see Section 1).

If $\mathscr{F}$ is a coherent sheaf, then by Lemma 6.8(b), $\mathrm{H}^0\mathrm{R}\Gamma_*(\mathscr{F}) \in \mathbf{FH}$. From the definition of σ-ampleness and from the finiteness of $\mathrm{H}^q(X, \mathscr{F}(n))$ over k, it follows that $\mathrm{H}^q\mathrm{R}\Gamma_*(\mathscr{F}) \in \mathbf{FH}$ for $q > 0$. Thus $\mathrm{R}\Gamma_*(\mathscr{F})$ is in the full subcategory $\mathbf{D}^+_{\mathbf{FH}}(B)$ of $\mathbf{D}^+(B)$ consisting of the complexes $M^{\cdot}$ such that $\mathrm{H}^q M^{\cdot} \in \mathbf{FH}$ for all q. Since the functor $\mathrm{R}\Gamma_*$ is way out right, [RD, Chap. I, Prop. 7.3(ii)] says that $R\Gamma_*$ sends $\mathbf{D}_c^+(X)$ into $\mathbf{D}_{\mathsf{FH}}^+(B)$. In particular, $\Gamma_*(\mathscr{K}^\cdot_X) \in \mathbf{D}_{\mathsf{FH}}^+(B)$.

Now for any q and n, we have an isomorphism $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}(n), \mathscr{K}^q_X) \cong \Gamma(X, \mathscr{K}^q_X(-n))$. The Yoneda pairing (7.6) gives rise to a perfect pairing

$$\mathrm{H}^q(X, \mathscr{O}(n)) \times \mathrm{H}^{-q}(X, \mathscr{K}^\cdot_X(-n)) \to k.$$

Hence for $q > 0$, $\mathrm{H}^{-q}\Gamma_*(\mathscr{K}^\cdot_X)$ is a left limited B-module. Being in FH, it is then finitely generated over B. This proves that $\mathrm{H}^p\mathscr{K}^\cdot_B$ is finitely generated over B for $p < -1$. By definition, $\mathrm{H}^p\mathscr{K}^\cdot_B = 0$ for $p \geqslant -1$. ∎

LEMMA 7.11. *The canonical homomorphisms of complexes $\Phi: B \to \mathrm{Hom}^\cdot_B(\mathscr{K}^\cdot_B, \mathscr{K}^\cdot_B)$ and $\Phi^\circ: B \to \mathrm{Hom}^\cdot_{B^\circ}(\mathscr{K}^\cdot_B, \mathscr{K}^\cdot_B)$ are quasi-isomorphisms.*

Proof. By symmetry it suffices to consider the homomorphism Φ only. Fix a degree n and let $V^\cdot := \mathrm{Hom}^\cdot_B(\mathscr{K}^\cdot_B, \mathscr{K}^\cdot_B)_n$. We will show that $\Phi: B_n \to V^\cdot$ is a quasi-isomorphism. This will be done in three steps.

(1) Let $U^\cdot$ be the complex $\mathrm{Hom}^\cdot_k((\mathscr{K}^\cdot_B)_{-n}, k)$. Using the canonical isomorphism $\Theta: \mathscr{K}^0_B \xrightarrow{\simeq} B'$ of Lemma 7.9, we obtain isomorphisms

$$U^\cdot \cong \mathrm{Hom}^\cdot_B(\mathscr{K}^\cdot_B, B')_n \cong \mathrm{Hom}^\cdot_B(\mathscr{K}^\cdot_B, \mathscr{K}^0_B)_n,$$

which in turn induce a monomorphism of complexes of k-modules $P: U^\cdot \hookrightarrow V^\cdot$.

According to Theorem 6.11 and the definition of $\mathscr{K}^\cdot_B$, for any $p, q < 0$ there is an isomorphism

$$Q: \mathrm{Hom}_B(\mathscr{K}^p_B, \mathscr{K}^q_B)_n \xrightarrow{\simeq} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{K}^{p+1}_X(-n), \mathscr{K}^{q+1}_X).$$

Set $W^\cdot := \mathrm{Hom}^\cdot_{\mathscr{O}_X}(\mathscr{K}^\cdot_X(-n), \mathscr{K}^\cdot_X)$. Since $W^q = 0$ for $q < 0$, we get an exact sequence of complexes of k-modules

$$0 \longrightarrow U^\cdot \xrightarrow{\mathrm{P}} V^\cdot \xrightarrow{Q} W^\cdot \longrightarrow 0. \qquad (7.12)$$

(2) We claim that $\mathrm{H}^0U^\cdot = \mathrm{H}^1U^\cdot = 0$. In fact, by definition $\mathrm{H}^0\mathscr{K}^\cdot_B = H^{-1}\mathscr{K}^\cdot_B = 0$, and for any q, $\mathrm{H}^qU^\cdot = \mathrm{Hom}_k(\mathrm{H}^{-q}(\mathscr{K}^\cdot_B)_{-n}, k)$.

Next, we claim that for $q \geqslant 1$ the connecting homomorphism $\partial^q: \mathrm{H}^qW^\cdot \to \mathrm{H}^{q+1}U^\cdot$ of the long exact sequence derived from (7.12) is bijective. Because $\mathrm{H}^{-q}(X, \mathscr{K}^\cdot_X(-n)) = \mathrm{H}^{-q-1}(\mathscr{K}^\cdot_B)_{-n}$, taking $\mathscr{F}^\cdot = \mathscr{K}^\cdot_X(-n)$ in Eq. (7.5) gives an isomorphism

$$\theta^q: \mathrm{H}^qW^\cdot = \mathrm{Ext}^q_{\mathscr{O}_X}(\mathscr{K}^\cdot_X(-n), \mathscr{K}^\cdot_X) \xrightarrow{\simeq} \mathrm{H}^{q+1}U^\cdot$$

$$= \mathrm{Hom}_k(\mathrm{H}^{-q-1}(\mathscr{K}^\cdot_B)_{-n}, k).$$

Chasing diagrams we see that the two maps ∂^q and θ^q coincide.

(3) So far we know that $H^q V^{\cdot} = 0$ for $q \neq 0$ and that $H^0 Q: H^0 V^{\cdot} \xrightarrow{\simeq} H^0 W^{\cdot}$. According to (7.5) with $\mathscr{F}^{\cdot} = \mathscr{K}^{\cdot}_X(-n)$ we have an isomorphism

$$\theta^0: H^0 W^{\cdot} \xrightarrow{\simeq} \mathrm{Hom}^{\cdot}_k(H^0(X, \mathscr{K}_X(-n)), k) = U^0.$$

Denote by $\Theta': B_n \xrightarrow{\simeq} U^0$ the k-linear dual of the isomorphism $\Theta: (\mathscr{K}^0_B)_{-n} \xrightarrow{\simeq} (B')_{-n}$. Then the diagram

$$\begin{array}{ccc} B_n & \xrightarrow{H^0\Phi} & H^0 V^{\cdot} \\ \downarrow{\scriptstyle \Theta'} & & \downarrow{\scriptstyle H^0 Q} \\ U^0 & \xleftarrow{\theta^0} & H^0 W^{\cdot} \end{array}$$

is commutative, which implies that $\Phi: B_n \to V^{\cdot}$ is a quasi-isomorphism. ∎

Proof (*of Theorem* 7.3). According to Corollary 6.18 and Lemma 7.9, 7.10, and 7.11, $\mathscr{K}^{\cdot}_B$ is a dualizing complex over B. Since

$$\Gamma_m(\mathscr{K}^{\cdot}_B) = \Gamma_{m^0}(\mathscr{K}^{\cdot}_B) = \mathscr{K}^0_B \cong B'$$

it is in fact balanced. ∎

There is an interesting corollary to this theorem:

COROLLARY 7.13. *If B is Cohen–Macaulay (resp. Gorenstein) then so is X.*

Proof. Assume B is Cohen–Macaulay. Then there exists a B-module M s.t. $\mathscr{K}^{\cdot}_B \cong M[n+1]$ in $\mathbf{D}(B)$ for some n. Applying he exact functor $\tilde{}$ we get that $\mathscr{K}^{\cdot}_X \cong \tilde{M}[n]$, so X is a Cohen–Macaulay scheme with dualizing sheaf $\omega_X \cong \tilde{M}$. If B is Gorenstein then $M \cong B(m)$ for some m, so $\omega_X \cong \mathcal{O}(m)$ is an invertible sheaf and X is a Gorenstein scheme. ∎

We now look at some examples of skew homogeneous coordinate rings and examine their balanced dualizing complexes.

EXAMPLE 7.14. Let X be the projective space $\mathbf{P}^n$ $(n > 0)$ over k, let $\mathscr{L}$ be the sheaf of linear forms $\mathcal{O}_{\mathbf{P}^n}(1)$, and let σ be any k-automorphism of $\mathbf{P}^n$. There is a canonical isomorphism $\omega_X[n] \cong \mathscr{K}^{\cdot}_X$ in $\mathbf{D}(X)$. Since $H^q(X, \omega_X(i)) \cong H^q(X, \mathcal{O}(i-n-1)) = 0$ for $q \neq 0$, n and for all i, it follows that

$$\Gamma_*(\omega_X, \varepsilon)[n+1] \to \mathscr{K}^{\cdot}_B$$

is an isomorphism in $\mathbf{D}(B^e)$. The bimodule $\Gamma_*(\omega_X, \varepsilon)$ is invertible, so B is a Gorenstein algebra. In fact, B is a twist of the algebra $B(X, \mathrm{id}, \mathscr{L})$ which is a commutative polynomial ring in $n+1$ indeterminates, so B is a regular algebra (see [ATV2, Sect. 8]).

Choose an equivariance $e: \sigma^*\mathscr{L} \xrightarrow{\simeq} \mathscr{L}$ and a basis $t_0, ..., t_n$ for $\Gamma(X, \mathscr{L})$ s.t. $e\sigma^*(t_0) = t_0$. The section $t_0^{n+1} d(t_1/t_0) \wedge \cdots \wedge d(t_n/t_0)$ is a basis for $\Gamma(X, \omega_X)$. If $M = (m_{i,j})$ is the matrix for $e\sigma^*$ acting on $\Gamma(X, \mathscr{L})$ relative to the given basis, then M has the form

$$M = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ * & & & \\ \vdots & & \bar{M} & \\ * & & & \end{pmatrix}.$$

Let W be the affine open set $\{t_0 \neq 0\}$, which is invariant under σ. Then $\varepsilon\sigma^*$ acts on $\Gamma(W, \Omega^1_{X,k})$, and its action on the k-subspace spanned by $d(t_1/t_0), ..., d(t_n/t_0)$ is given by the matrix $\bar{M}$. Define an equivariance

$$f := e^{\otimes(n+1)} \otimes \left(\bigwedge^n \varepsilon\right) : \sigma^*(\mathscr{L}^{\otimes(n+1)} \otimes \omega_X) \xrightarrow{\simeq} \mathscr{L}^{\otimes(n+1)} \otimes \omega_X.$$

Then we get $f\sigma^*(t_0^{n+1} d(t_1/t_0) \wedge \cdots \wedge d(t_n/t_0)) = \lambda t_0^{n+1} d(t_1/t_0) \wedge \cdots \wedge d(t_n/t_0)$, where $\lambda = \det(\bar{M}) = \det(M)$.

Let $U: \Gamma_*(\omega_X) \xrightarrow{\simeq} \bigoplus_i \Gamma(X, \mathscr{L}^{\otimes i} \otimes \omega_X)$ be the untwisting map of Section 6. Set $\alpha := U^{-1}(t_0^{n+1} d(t_1/t_0) \wedge \cdots \wedge d(t_n/t_0)) \in \Gamma(X, \omega_X(n+1))$, and denote by f' the equivariance on $\omega_X(n+1)$ determined by e and ε. Then $(f\sigma^*) \circ U = U \circ (f'\sigma^*)$ and therefore $f'\sigma^*(\alpha) = \lambda\alpha$. For every $0 \leqslant i \leqslant n$ we have, by (6.20),

$$U(t_i \cdot \alpha) = U((e\sigma^*)^{-(n+1)}(t_i)) \otimes U(\alpha)$$
$$U(\alpha \cdot t_i) = U((f'\sigma^*)^{-1}(\alpha)) \otimes U(t_i) = \lambda^{-1} U(t_i) \otimes U(\alpha)$$

in $\Gamma(X, \mathscr{L}^{\otimes(n+2)} \otimes \omega_X)$. In other words, $\alpha \cdot t_i = \phi(t_i) \cdot \alpha$, where ϕ is the linear automorphism of B_1

$$\phi = \det(e\sigma^*)^{-1} (e\sigma^*)^{n+1}$$

(this formula was suggested to the author by J. Tate). ϕ determines an algebra automorphism of B, and in the notation of Section 1, one has

$$\mathscr{K}^{\cdot}_B \cong \Gamma_*(\omega_X, \varepsilon)[n+1] \cong B(\phi, -n-1)[n+1].$$

EXAMPLE 7.15. Again we take $X = \mathbf{P}^n$, but this time $\mathscr{L}$ is any ample invertible sheaf. Unless the degree of $\mathscr{L}$ divides the degree of ω_X, which is $-(n+1)$, the bimodule $\Gamma_*(\omega_X, \varepsilon)$ is not invertible. This is seen from the dimensions of the homogeneous components of the bimodule. In any case B is a Cohen–Macaulay algebra.

EXAMPLE 7.16. Let E be a smooth elliptic curve over k, let $\mathscr{L}$ be an invertible sheaf of positive degree, and let σ be any automorphism of E. According to Proposition 6.13, $\mathscr{L}$ is both σ-ample and σ^{-1}-ample. As in the previous examples we have $\mathscr{K}^{\cdot}_B \cong \Gamma_*(\omega_E, \varepsilon)[2]$ (this is true for any smooth curve) so B is a Gorenstein algebra. It is not regular, because the only 2 dimensional regular algebras generated in degree 1 are those discussed in Example 6.21 (B is generated in degree 1—see [ATV1, Thm. 6.6(i)]. Choose a basis α for $\Gamma(E, \omega_E)$, and let λ be the eigenvalue of $(\varepsilon\sigma^*)^{-1}$ on this space. For $b \in B_n$ we get

$$b \cdot \alpha = b \otimes (\sigma^*)^n (\alpha)$$
$$\alpha \cdot b = b \otimes \varepsilon_1^n(\alpha) = b \otimes (\sigma^*)^n (\varepsilon\sigma^*)^{-n} (\alpha) = \lambda^n b \otimes (\sigma^*)^n (\alpha)$$

in $\Gamma(E, \omega_E(n))$. If we denote by ϕ_λ the automorphism of B s.t. $\phi_\lambda(b) = \lambda^n b$ for $b \in B_n$, then

$$\mathscr{K}^{\cdot}_B \cong \Gamma_*(\omega_E, \varepsilon) \cong B(\phi_\lambda, 0)[2].$$

EXAMPLE 7.17. Let (X, P_0) be an abelian variety over k of dimension n, let σ be the translation by some point $P \in X$, and let $\mathscr{L}$ be an ample invertible sheaf on X. Since σ is algebraically equivalent to the identity automorphism, $\mathscr{L}$ is σ-ample (see [AV, Cor. 1.6]). According to [Mu, Sect. 13 Cor. 2], $\dim_k \mathrm{H}^q(X, \omega_X) = \binom{n}{q}$. Therefore if $n \geq 2$ the balanced dualizing complex $\mathscr{K}^{\cdot}_B$ is not isomorphic to a single bimodule, and B is not Cohen–Macaulay.

To conclude the paper, we describe the balanced dualizing complex $R^{\cdot}_A$ over a 3 dimensional regular graded algebra A generated in degree 1. The existence of this complex was established in Corollary 4.14. Let $(E, \sigma, \mathscr{L})$ be the triple associated to A (see [ATV1, Sect. 1]). If $\dim E = 2$ then $A \cong B(E, \sigma, \mathscr{L})$, E is either $\mathbf{P}^2$ or $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathscr{L}$ is either $\mathcal{O}(1)$ or $\mathcal{O}(1, 0)$, respectively. The complex $R^{\cdot}_A$ is then isomorphic to $\mathscr{K}^{\cdot}_B$ and hence to $\Gamma_*(\omega_E, \varepsilon)[3]$.

If $\dim E = 1$ then E is either a divisor of degree 3 in $\mathbf{P}^2$ or a divisor of bidegree (2, 2) in $\mathbf{P}^1 \times \mathbf{P}^1$. In this case A is called an elliptic algebra. There exists a normalizing nonzero-divisor $g \in A_{s+1}$ ($s = 2$ or 3), such that $B = B(E, \sigma, \mathscr{L}) \cong A/(g)$. If E is smooth, it is an elliptic curve; even if not, it is Gorenstein scheme with $\omega_E \cong \mathcal{O}_E$. Hence the calculations of Example 7.16 are valid. Let λ and ϕ_λ be as in that example. Let ϕ_g be the automorphism of A which is "conjugation by g," i.e., $\phi_g(a)\, g = ga$, $a \in A$. In the invertible bimodule notation we have $(g) \cong A(\phi_g, -s-1)$. Observe that the two automorphisms ϕ_g and ϕ_λ commute.

THEOREM 7.18. *Let A be an elliptic algebra. Then the complex $A(\phi_g \phi_\lambda, -s-1)[3]$ is a balanced dualizing complex over A.*

Proof. Since A has finite global dimension, any invertible bimodule is a dualizing complex. To show that $R_A^{\cdot} := A(\phi_g \phi_\lambda, -s-1)[3]$ is balanced, it is enough to show that

$$\mathrm{RHom}_A^{\cdot}(A/m_A^n) \cong \mathrm{Hom}_k(A/m_A^n, k)$$

in $\mathsf{D}(A^e)$ for $n = 1, 2$ (see Thm. 4.8). Since $g \in m_A^2$, it follows that $A/m_A^n \cong B/m_B^n$ as A^e-modules for these values of n. Consider the exact sequence of A^e-modules

$$0 \to (g) \to A \to B \to 0. \tag{7.19}$$

The functor $\mathrm{RHom}_A^{\cdot}(-,-)$ can be computed by A-projectives. Since

$$\mathrm{Hom}_A(A(\phi_g, -s-1), A(\phi_g \phi_\lambda, -s-1)) \cong A(\phi_\lambda, 0)$$

we get

$$\begin{aligned}
\mathrm{RHom}_A^{\cdot}(B, R_A^{\cdot}) &= \mathrm{RHom}_A^{\cdot}(B, A(\phi_g \phi_\lambda, -s-1)) \\
&\cong \mathrm{coker}(A(\phi_g \phi_\lambda, -s-1) \to A(\phi_\lambda, 0))[2] \\
&\qquad [\text{sequence } (7.19)] \\
&\cong B \otimes_A A(\phi_\lambda, 0)[2] \\
&\cong B(\phi_\lambda, 0)[2]
\end{aligned}$$

in $\mathsf{D}(B \otimes_k A^{\circ})$. In Example 7.16 it was shown that $R_B^{\cdot} := B(\phi_\lambda, 0)[2]$ is a balanced dualizing complex over B. Using the isomorphism (5.1) we carry out the following computation in $\mathsf{D}(A^e)$, for $n = 1, 2$,

$$\begin{aligned}
\mathrm{RHom}_A^{\cdot}(A/m_A^n, R_A^{\cdot}) &\cong \mathrm{RHom}_B^{\cdot}(A/m_A^n, R_B^{\cdot}) \\
&\cong \mathrm{Hom}_k(A/m_A^n, k),
\end{aligned}$$

which is what we had to prove. ∎

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