ELSEVIER

Journal of Pure and Applied Algebra 150 (2000) 85–93

JOURNAL OF PURE AND APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

The rigid dualizing complex of a universal enveloping algebra

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Received 12 October 1998; received in revised form 12 December 1998
Communicated by C.A. Weibel

Abstract

Let k be a field and A a noetherian (noncommutative) k-algebra. The rigid dualizing complex of A was introduced by Van den Bergh. When $A = \mathrm{U}(\mathfrak{g})$, the enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$, Van den Bergh conjectured that the rigid dualizing complex is $(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g})[n]$, where $n = \dim \mathfrak{g}$. We prove this conjecture, and give a few applications in representation theory and Hochschild cohomology.

MSC: Primary 16D90; secondary 16E40; 16E30; 17B55

Dualizing complexes were introduced as part of Grothendieck Duality Theory on schemes, in [3], and the noncommutative version was first studied in [8]. The basic change is that a dualizing complex over a noncommutative ring is a complex of bimodules. For technical reasons we work with noetherian algebras over a base field k, and abbreviate $\otimes := \otimes_k$. Given an algebra A, we write A° for the opposite algebra, and $A^{\mathrm{e}} := A \otimes A^\circ$. We consider left modules by default. A dualizing complex R is an object in the bounded derived category of bimodules $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}\, A^{\mathrm{e}})$, of finite injective dimension on both sides, such that the functors $\mathrm{R}\,\mathrm{Hom}_A(-, R)$ and $\mathrm{R}\,\mathrm{Hom}_{A^\circ}(-, R)$ induce a duality (i.e. a contravariant equivalence) between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A^\circ)$. The subscript f denotes complexes with finitely generated cohomologies. See [7,8] for details on noncommutative Grothendieck duality.

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0022-4049/00/$ - see front matter
PII: S0022-4049(99)00032-8

In the fundamental paper [5], Van den Bergh defined the *rigid dualizing complex* of a k-algebra A. A dualizing complex R is rigid if there exists an isomorphism

$$\rho : R \xrightarrow{\simeq} \mathrm{R}\,\mathrm{Hom}_{A^e}(A, R \otimes R) \tag{1}$$

in $\mathsf{D}(\mathsf{Mod}\, A^e)$, which we shall call a *rigidifying isomorphism*. According to [5], a rigid dualizing complex R, if it exists, is unique up to isomorphism. Moreover it turns out that rigid dualizing complexes are functorial with respect to finite homomorphisms of k-algebras (under some technical restrictions; cf. Theorem 1.2).

For instance, if A is a commutative finite type k-algebra, $\pi : X = \operatorname{Spec} A \to \operatorname{Spec} k$ is the structural morphism and $\pi^! : \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, k) \to \mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{Mod}\, A)$ is the twisted inverse image of [3], then $R := \pi^! k$ is a rigid dualizing complex, and ρ is the fundamental class of the diagonal $X \hookrightarrow X \times X$.

Regarding existence of rigid dualizing complexes, Van den Bergh proved the following result: if A is filtered such that $B := \operatorname{gr} A$ is a connected graded noetherian k-algebra, and B has a *balanced dualizing complex* in the sense of [7], then A has a rigid dualizing complex. In particular this holds for $A = \mathrm{U}(\mathfrak{g})$, the universal enveloping algebra of a finite dimensional Lie algebra $\mathfrak{g}$.

Our main result verifies a conjecture of Van den Bergh (Private communication, 1996):

Theorem A. *Let* $\mathfrak{g}$ *be a finite dimensional Lie algebra over k. Then the rigid dualizing complex of the universal enveloping algebra* $\mathrm{U}(\mathfrak{g})$ *is*

$$R = \left(\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g} \right) [n],$$

where $n = \dim \mathfrak{g}$, *and we consider* $\bigwedge^n \mathfrak{g}$ *as a* $\mathrm{U}(\mathfrak{g})$*-bimodule with trivial action from the left and adjoint action from the right.*

Observe that in the two extreme cases – $\mathfrak{g}$ abelian or semisimple – the adjoint representation on $\bigwedge^n \mathfrak{g}$ is trivial. But for a solvable Lie algebra we can get something nontrivial, as shown in Example 2.5. The semisimple case was already known to Van den Bergh (cf. [6, Corollary 6]).

An indication that Theorem A should be true can be seen by deforming $\mathfrak{g}$ to an abelian Lie algebra. In the abelian case $A = \mathrm{U}(\mathfrak{g})$ is a commutative polynomial algebra, and there is a canonical isomorphism $\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g} \cong \Omega^n_{A/k}$. As mentioned before, the complex $\Omega^n_{A/k}[n] = \pi^! k$ is the rigid dualizing complex of A (cf. Remark 2.8).

The proof of Theorem A is at the end of Section 1. In Section 2 we give a few corollaries of Theorem A, and also an analogous result for a ring $\mathscr{D}(C)$ of differential operators over a smooth commutative k-algebra C.

1. Proof of main result

Let us start with some general facts about rigid dualizing complexes of filtered k-algebras.

If γ is an automorphism of a ring A then the twist of a right module M by γ is M_γ, where the new action is via γ. In particular the twisted bimodule A_γ has basis 1_γ, and $1_\gamma \cdot a = \gamma(a) \cdot 1_\gamma$ for $a \in A$. The shift by $i \in \mathbb{Z}$ of a graded module M is denoted by $M(i)$, whereas the shift of a complex $M^{\bullet}$ is $M^{\bullet}[i]$.

Proposition 1.1. *Let A be a filtered k-algebra, and assume $\operatorname{gr} A$ is a connected graded, noetherian, Artin–Schelter Gorenstein algebra.*

1. *A has a rigid dualizing complex $R_A = \omega_A[n]$ for some integer n and invertible bimodule ω_A. Furthermore $\omega_A \cong A_\gamma$ where γ is a filtered k-algebra automorphism of A.*
2. *The balanced dualizing complex of $\operatorname{gr} A$ is $R_{\operatorname{gr} A} = \omega_{\operatorname{gr} A}[n]$, and $\omega_{\operatorname{gr} A} \cong (\operatorname{gr} A)_{\operatorname{gr}(\gamma)}(m)$ for some integer m.*

Proof. (Cf. [8, Proposition 6.18].) Let $\tilde{A} := \operatorname{Rees} A \subset A[t, t^{-1}]$ denote the Rees algebra. Recall that t is a central variable and $(\operatorname{Rees} A)_i = F_i A \cdot t^i$. Since $\tilde{A}$ is also AS-Gorenstein its balanced dualizing complex is $R_{\tilde{A}} = \tilde{A}_{\tilde{\gamma}}(m-1)[n+1]$, where $\tilde{\gamma}$ is a graded k-algebra automorphism and $m, n \in \mathbb{Z}$. Because $\tilde{A}_{\tilde{\gamma}}$ is $k[t]$-central, $\tilde{\gamma}$ is in fact a $k[t]$-algebra automorphism. Now by [8, Theorem 6.2], $R_A \cong (\tilde{A}_{\tilde{\gamma}} \otimes_{\tilde{A}} A)[n]$. On the other hand, using the exact sequence $0 \to \tilde{A}(-1) \xrightarrow{t} \tilde{A} \to \operatorname{gr} A \to 0$ we get

$$R_{\operatorname{gr} A} \cong \operatorname{R Hom}_{\tilde{A}}(\operatorname{gr} A, \tilde{A}_{\tilde{\gamma}}(m-1)[n+1]) \cong (\tilde{A}_{\tilde{\gamma}} \otimes_{\tilde{A}} \operatorname{gr} A)(m)[n]. \qquad \square$$

We call ω_A the *dualizing bimodule* of A and γ is the *dualizing automorphism.*

Next let us quote a result from [8]. A filtration $\{F_i A\}$ is said to be noetherian connected if $\operatorname{gr}^F A$ is a noetherian connected graded k-algebra. A ring homomorphism $A \to B$ is finite centralizing if $B = \sum_{i=1}^{l} A \cdot b_i$ for some elements $b_1, \ldots, b_l \in B$ that commute with A.

Theorem 1.2 (Yekutieli and Zhang [8, Theorem 6.17]). *Let $A \to B$ be a finite centralizing homomorphism of k-algebras. Suppose A has a noetherian connected filtration $\{F_i A\}$ and $\operatorname{gr}^F A$ has a balanced dualizing complex. Then the algebras A and B have rigid dualizing complexes R_A and R_B respectively, and the trace morphism $\operatorname{Tr}_{B/A} : R_B \to R_A$ in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$ exists. The trace induces isomorphisms*

$$R_B \cong \operatorname{R Hom}_A(B, R_A) \cong \operatorname{R Hom}_{A^{\circ}}(B, R_A)$$

in $\mathsf{D}(\mathsf{Mod}\, A^{\mathrm{e}})$.

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over the field k, let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra, and denote by $\mathbf{K}_{\bullet}(\mathfrak{h})$ the Chevalley–Eilenberg complex of $\mathrm{U}(\mathfrak{h})$, namely the free resolution of the trivial $\mathfrak{h}$-module k (cf. [2, Section XIII.7] or [4, Section 10.1.3]). Recall that for any i one has $\mathbf{K}_i(\mathfrak{h}) := \mathrm{U}(\mathfrak{h}) \otimes \bigwedge^i \mathfrak{h}$, a free left $\mathrm{U}(\mathfrak{h})$-module (the action on the exterior power $\bigwedge^i \mathfrak{h}$ is trivial). The boundary operator $\delta : \mathbf{K}_i(\mathfrak{h}) \to \mathbf{K}_{i-1}(\mathfrak{h})$ is

$$\begin{aligned}\delta(1 \otimes x_1 \wedge \cdots \wedge x_i) = {} & \sum_{p=1}^{i} (-1)^{p+1} x_p \otimes x_1 \wedge \cdots \widehat{x}_p \cdots \wedge x_i \\ & + \sum_{1 \le p < q \le i} (-1)^{p+q} \otimes [x_p, x_q] \wedge x_1 \wedge \cdots \widehat{x}_p \cdots \widehat{x}_q \cdots \wedge x_i\end{aligned}$$

for $x_1,\dots,x_i \in \mathfrak{h}$. Define

$$\mathbf{K}_i(\mathfrak{g};\mathfrak{h}):=\mathrm{U}(\mathfrak{g}) \otimes_{\mathrm{U}(\mathfrak{h})} \mathbf{K}_i(\mathfrak{h}) \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^i \mathfrak{h},$$

so that $(\mathbf{K}_{\bullet}(\mathfrak{g};\mathfrak{h}),\delta)$ is a complex of free left $\mathrm{U}(\mathfrak{g})$-modules. As usual for any two $\mathrm{U}(\mathfrak{g})$-modules M, N the tensor product $M \otimes N$ is also a $\mathrm{U}(\mathfrak{g})$-module by the coproduct.

Lemma 1.3. *Suppose* $\mathfrak{h} \subset \mathfrak{g}$ *is an ideal, and consider* $\bigwedge^i \mathfrak{h}$ *as a right* $\mathrm{U}(\mathfrak{g})$*-module by the adjoint action, so that* $\mathbf{K}_i(\mathfrak{g};\mathfrak{h})$ *becomes a* $\mathrm{U}(\mathfrak{g})$-bimodule.
1. *The boundary operator* $\delta : \mathbf{K}_i(\mathfrak{g};\mathfrak{h}) \to \mathbf{K}_{i-1}(\mathfrak{g};\mathfrak{h})$ *commutes with the right* $\mathrm{U}(\mathfrak{g})$*-action.*
2. *There is a quasi-isomorphism of complexes of* $\mathrm{U}(\mathfrak{g})$*-bimodules* $\mathbf{K}^{\bullet}(\mathfrak{g};\mathfrak{h}) \to \mathrm{U}(\mathfrak{g}/\mathfrak{h})$.

Proof. 1. Since $\bigwedge^i \mathfrak{h} \subset \bigwedge^i \mathfrak{g}$ is a $\mathrm{U}(\mathfrak{g})$-submodule for the adjoint action, it follows that $\mathbf{K}_i(\mathfrak{g};\mathfrak{h}) \subset \mathbf{K}_i(\mathfrak{g})$ is a sub $\mathrm{U}(\mathfrak{g})$-bimodule. Hence we may assume that $\mathfrak{h}=\mathfrak{g}$ and $\mathbf{K}_{\bullet}(\mathfrak{g};\mathfrak{h}) = \mathbf{K}_{\bullet}(\mathfrak{g})$. But then the assertion is [4, Proposition 10.1.7]. (I wish to thank P. Smith for referring me to [4].)

2. As usual we let $\mathbf{K}^i(\mathfrak{g};\mathfrak{h}):=\mathbf{K}_{-i}(\mathfrak{g};\mathfrak{h})$, and the coboundary operator is $(-1)^{i+1}\delta : \mathbf{K}^i(\mathfrak{g};\mathfrak{h}) \to \mathbf{K}^{i+1}(\mathfrak{g};\mathfrak{h})$. Since $\mathrm{U}(\mathfrak{h}) \to \mathrm{U}(\mathfrak{g})$ is flat we get $\mathrm{H}^i\mathbf{K}^{\bullet}(\mathfrak{g};\mathfrak{h}) = 0$ if $i<0$. For $i=0$ we note that $\mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h} = \mathfrak{h} \cdot \mathrm{U}(\mathfrak{g})$ is a two-sided ideal, and

$$\mathrm{U}(\mathfrak{g}/\mathfrak{h}) \cong \mathrm{U}(\mathfrak{g})/\mathrm{U}(\mathfrak{g}) \cdot \mathfrak{h} \cong \mathrm{H}^0\mathbf{K}^{\bullet}(\mathfrak{g};\mathfrak{h})$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. □

For any k-module M let $M^*:=\mathrm{Hom}_k(M,k)$. We consider $\bigwedge^n \mathfrak{g}^*$ as a right $\mathrm{U}(\mathfrak{g})$-module with the coadjoint action, and a left $\mathrm{U}(\mathfrak{g})$-module with the trivial action.

Lemma 1.4. *Let* $\mathfrak{h} \subset \mathfrak{g}$ *be an ideal, with* $\dim_k \mathfrak{h} = m$. *Assume that* $\gamma(\mathrm{U}(\mathfrak{g})\cdot\mathfrak{h}) = \mathrm{U}(\mathfrak{g})\cdot\mathfrak{h}$, *where* γ *is the dualizing automorphism of* $\mathrm{U}(\mathfrak{g})$. *Then*

$$\mathrm{Ext}^q_{\mathrm{U}(\mathfrak{g})}(\mathrm{U}(\mathfrak{g}/\mathfrak{h}),\mathrm{U}(\mathfrak{g})) \cong \begin{cases} \mathrm{U}(\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^m \mathfrak{h}^* & \text{if } q=m, \\ 0 & \text{if } q \neq m \end{cases}$$

as $\mathrm{U}(\mathfrak{g})$*-bimodules.*

Proof. Since $\mathrm{gr}\,\mathrm{U}(\mathfrak{g})$ is a commutative polynomial algebra in n variables we know that its balanced dualizing complex is $R_{\mathrm{gr}\,\mathrm{U}(\mathfrak{g})} \cong (\mathrm{gr}\,\mathrm{U}(\mathfrak{g})(-n)[n]$. Therefore by Proposition 1.1 the rigid dualizing complexes of $\mathrm{U}(\mathfrak{g})$ and $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$ are $R_{\mathrm{U}(\mathfrak{g})} \cong \mathrm{U}(\mathfrak{g})_{\gamma}[n]$ and $R_{\mathrm{U}(\mathfrak{g}/\mathfrak{h})} \cong \mathrm{U}(\mathfrak{g}/\mathfrak{h})_{\tau}[n-m]$, respectively, where τ is the dualizing automorphism of $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$. According to Theorem 1.2 we get the vanishing of all Ext^q, $q \neq m$, and

$$M:=\mathrm{Ext}^m_{\mathrm{U}(\mathfrak{g})}(\mathrm{U}(\mathfrak{g}/\mathfrak{h}),\mathrm{U}(\mathfrak{g})) \cong \mathrm{U}(\mathfrak{g}/\mathfrak{h})_{\tau\gamma^{-1}}$$

as $\mathrm{U}(\mathfrak{g})$-bimodules.

According to Lemma 1.3 we get

$$M = \mathrm{H}^m \operatorname{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathbf{K}^{\bullet}(\mathfrak{g};\mathfrak{h}), \mathrm{U}(\mathfrak{g})),$$

so the bimodule M is a quotient of $\mathrm{U}(\mathfrak{g}) \otimes \bigwedge^m \mathfrak{h}^*$. Let α be any k-basis of $\bigwedge^m \mathfrak{h}^*$, and let β be the image of $1 \otimes \alpha \in \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^m \mathfrak{h}^*$ in the $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$-bimodule M. Hence for any $x \in \mathfrak{g}$ we have

$$\beta \cdot x = (x - \operatorname{tr}(\operatorname{ad}_{\wedge^m \mathfrak{h}^*} x)) \cdot \beta.$$

Since M is free of rank 1 on either side as $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$-module, and since $\mathrm{U}(\mathfrak{g}/\mathfrak{h})$ is an integral domain, it follows that the generator β is a basis of M. Sending $\beta \mapsto 1 \otimes \alpha \in \mathrm{U}(\mathfrak{g}/\mathfrak{h}) \otimes \bigwedge^m \mathfrak{h}^*$ is the desired isomorphism of $\mathrm{U}(\mathfrak{g})$-bimodules. □

Here is another result of Van den Bergh (cf. [6, Proof of Corollary 6]).

Lemma 1.5. *Let A be a positively filtered k-algebra such that $\operatorname{gr} A$ is commutative and $\operatorname{gr}_0 A = k$. Let $\mathfrak{g} := \operatorname{gr}_1 A$, so $\mathfrak{g}$ is a Lie algebra over k. Let γ be a filtered k-algebra automorphism of A such that $\operatorname{gr}(\gamma)$ is the identity. Then there is a Lie homomorphism $\lambda : \mathfrak{g} \to k$ such that $\gamma(a) = a + \lambda(\bar{a})$ for all $a \in F_1 A$, where $\bar{a} \in \mathfrak{g}$ is the symbol of a.*

Proof. Define $\lambda(a) := \gamma(a) - a$ for $a \in F_1 A$. It factors through $F_1 A \twoheadrightarrow \mathfrak{g} \to F_0 A \hookrightarrow F_1 A$, is easily seen to be k-linear, and $\lambda([a, b]) = 0$. □

At last here is the proof of our main result.

Proof of Theorem A. According to Proposition 1.1, the rigid dualizing complex of $\mathrm{U}(\mathfrak{g})$ is $R_{\mathrm{U}(\mathfrak{g})} \cong \mathrm{U}(\mathfrak{g})_{\gamma}[n]$; and $\operatorname{gr}(\gamma)$ is the identity. In view of Lemma 1.5, it remains to prove that $\lambda = -\operatorname{tr} \operatorname{ad}_{\wedge^n \mathfrak{g}}$. Since λ is a Lie homomorphism it has to vanish on the commutator ideal $\mathfrak{h} := [\mathfrak{g}, \mathfrak{g}]$, and so it factors through $\mathfrak{a} := \mathfrak{g}/\mathfrak{h}$. Therefore it suffices to prove that the induced automorphism $\bar{\gamma}$ of $\mathrm{U}(\mathfrak{a})$ satisfies $\bar{\gamma}(y) = y - \operatorname{tr}(\operatorname{ad}_{\wedge^n \mathfrak{g}} y)$ for $y \in \mathfrak{a}$.

The algebra $\mathrm{U}(\mathfrak{a})$ is a commutative polynomial algebra in $l = n - m$ variables, where $m = \dim_k \mathfrak{h}$, so its rigid dualizing complex is $\mathrm{U}(\mathfrak{a})[l]$. According to Lemma 1.4 and Theorem 1.2 we get

$$\mathrm{U}(\mathfrak{a}) \cong \operatorname{Ext}^m_{\mathrm{U}(\mathfrak{g})}(\mathrm{U}(\mathfrak{a}), \mathrm{U}(\mathfrak{g})_{\gamma}) \cong \mathrm{U}(\mathfrak{a})_{\gamma} \otimes \bigwedge^m \mathfrak{h}^*$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. Therefore $\mathrm{U}(\mathfrak{a})_{\bar{\gamma}} \cong \mathrm{U}(\mathfrak{a}) \otimes \bigwedge^m \mathfrak{h}$, so $\bar{\gamma}(y) = y - \operatorname{tr}(\operatorname{ad}_{\wedge^m \mathfrak{h}} y)$ for all $y \in \mathfrak{a}$. Finally, since $\bigwedge^{n-m} \mathfrak{a}$ is a trivial representation of $\mathfrak{g}$, one has $\bigwedge^m \mathfrak{h} \cong \bigwedge^n \mathfrak{g}$. □

Question 1.6. Suppose $\mathfrak{g}$ is semisimple and $\operatorname{char} k = 0$. Does the quantum enveloping algebra $\mathrm{U}_q(\mathfrak{g})$ admit a rigid dualizing complex? If so, what is it?

2. Some corollaries and complements

Corollary 2.1. *Let M be any finitely generated* $\mathrm{U}(\mathfrak{g})$*-module, pure of* GKdim $= m$, *and let $I := \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})} M$. Then*

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{g})^\circ} \mathrm{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g})) = \gamma(I) \subset \mathrm{U}(\mathfrak{g})^\circ,$$

where γ is the dualizing automorphism.

Proof. Let us view γ as an anti-isomorphism $\gamma : \mathrm{U}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g})^\circ$. Define $M' := \mathrm{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g}))$ and $I' := \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})^\circ} M'$. By [8, Proposition 6.18(4)] one has $\gamma(I) \subset I'$. Since M is pure, $M \subset M'' := \mathrm{Ext}^{n-m}_{\mathrm{U}(\mathfrak{g})^\circ}(M', \mathrm{U}(\mathfrak{g}))$. Hence $\gamma^{-1}(I') \subset \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})} M'' \subset I$. □

It is a standard fact that if M is a finite dimensional representation of $\mathfrak{g}$, then $\mathrm{Ext}^q_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g})) = 0$ for $q < n$. The group $\mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g}))$ is a right $\mathrm{U}(\mathfrak{g})$-module, but the structure is not obvious [1]. Since we can make M into a $\mathrm{U}(\mathfrak{g})$-bimodule with trivial right action, the next corollary gives the answer.

Corollary 2.2. *Suppose M is a finite dimensional k-central* $\mathrm{U}(\mathfrak{g})$*-bimodule. Then there is an isomorphism of* $\mathrm{U}(\mathfrak{g})$*-bimodules*

$$\mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}(M, \mathrm{U}(\mathfrak{g})) \cong M^* \otimes \bigwedge^n \mathfrak{g}^*,$$

which is functorial in M.

Proof. Let $I := \mathrm{Ann}_{\mathrm{U}(\mathfrak{g})} M$ and $B := \mathrm{U}(\mathfrak{g})/I$. Since $k \to B$ is a finite homomorphism the rigid dualizing complex of B is $B^* = \mathrm{Hom}_k(B, k)$. By [8, Proposition 3.9],

$$\mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})}\left(M, \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}\right) \cong \mathrm{Hom}_B(M, B^*) \cong M^*$$

as $\mathrm{U}(\mathfrak{g})$-bimodules. Now twist by $\bigwedge^n \mathfrak{g}^*$. □

Theorem A has an interpretation in terms of Hochschild cohomology. For a $\mathrm{U}(\mathfrak{g})$-bimodule M denote by $\mathrm{H}^q(\mathrm{U}(\mathfrak{g}), M)$ and $\mathrm{H}_q(\mathrm{U}(\mathfrak{g}), M)$ the Hochschild cohomology and homology, respectively.

Corollary 2.3. *There are* $\mathrm{U}(\mathfrak{g})$*-bimodule isomorphisms*

$$\mathrm{H}^q(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}) \cong \begin{cases} \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^* & \text{if } q = n, \\ 0 & \text{if } q \neq n. \end{cases}$$

Proof. Let us write $\omega := \omega_{\mathrm{U}(\mathfrak{g})}$ and $\omega^\vee := \mathrm{Hom}_{\mathrm{U}(\mathfrak{g})}(\omega, \mathrm{U}(\mathfrak{g}))$. By formula (1), $\omega \cong \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}(\mathrm{U}(\mathfrak{g}), \omega \otimes \omega)$ as bimodules, so applying the twist $- \otimes_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} (\omega^\vee \otimes \omega^\vee)$ we get $\omega^\vee \cong \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}})$. But by Theorem A, $\omega^\vee \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^*$. □

[1] The right module structure was calculated by S. Chemla [Bull. Soc. Math. France 122 (1994)].

In [6], Van den Bergh proves a Poincaré duality between the Hochschild cohomology and homology of certain Gorenstein algebras A. We obtain the following variation of his result.

Corollary 2.4. *Let M be any k-central* $\mathrm{U}(\mathfrak{g})$*-bimodule. Then*

$$\mathrm{H}^q(\mathrm{U}(\mathfrak{g}), M) \cong \mathrm{H}_{n-q}\left(\mathrm{U}(\mathfrak{g}), M \otimes \bigwedge^n \mathfrak{g}^*\right).$$

Proof. Corollary 2.3 says that

$$\mathrm{R}\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}})[n] \cong \omega^\vee \cong \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}^*$$

in $\mathsf{D}(\mathsf{Mod}\,\mathrm{U}(\mathfrak{g})^{\mathrm{e}})$. Copying the proof of [6, Theorem 1] we obtain

$$\begin{aligned}
\mathrm{H}^q(\mathrm{U}(\mathfrak{g}), M) &\cong \mathrm{H}^q \mathrm{R}\mathrm{Hom}_{\mathrm{U}}(\mathfrak{g})^{\mathrm{e}}(\mathrm{U}(\mathfrak{g}), M) \\
&\cong \mathrm{H}^q(\mathrm{R}\,\mathrm{Hom}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}(\mathrm{U}(\mathfrak{g}), \mathrm{U}(\mathfrak{g})^{\mathrm{e}}) \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} M) \\
&\cong \mathrm{H}^{q-n}(\omega^\vee \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} M) \\
&\cong \mathrm{H}^{q-n}(\mathrm{U}(\mathfrak{g}) \otimes^{\mathrm{L}}_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}} (M \otimes_{\mathrm{U}(\mathfrak{g})} \omega^\vee)) \\
&\cong \mathrm{H}_{n-q}\left(\mathrm{U}(\mathfrak{g}), M \otimes \bigwedge^n \mathfrak{g}^*\right). \qquad \square
\end{aligned}$$

Here is an easy example where the dualizing bimodule ω is not trivial.

Example 2.5. Let $\mathfrak{g}$ be the nonabelian 2-dimensional Lie algebra, with basis x, y such that $[x, y] = y$. Then $\mathrm{tr}(\mathrm{ad}_{\wedge^2 \mathfrak{g}} x) = 1$.

If $\mathrm{char}\, k = 0$ and C is a smooth, integral, commutative k-algebra then the ring of differential operators $\mathscr{D}(C)$ is noetherian and has finite global dimension. Since $\mathscr{D}(C)$ can be deformed to a smooth commutative k-algebra (namely the algebra of functions on the cotangent bundle of $\mathrm{Spec}\, C$), one could expect $\mathscr{D}(C)$ to have a rigid dualizing complex. This is indeed true, and follows from results in $\mathscr{D}$-module theory.

Theorem 2.6. *Let C be a smooth, integral, commutative k-algebra of dimension n, and assume* $\mathrm{char}\, k = 0$. *Let $\mathscr{D}(C)$ be the ring of differential operators. Then the rigid dualizing complex of $\mathscr{D}(C)$ is $\mathscr{D}(C)[2n]$.*

Proof. Let $X := \mathrm{Spec}\, C$ and $X^{\mathrm{e}} := X \times X \cong \mathrm{Spec}\, C^{\mathrm{e}}$. Then $\Gamma(X, \mathscr{D}_X) \cong \mathscr{D}(C)$, $\Gamma(X^{\mathrm{e}}, \mathscr{D}_{X^{\mathrm{e}}}) \cong \mathscr{D}(C) \otimes \mathscr{D}(C)$ and $\mathscr{D}(C)^\circ \cong \omega_C \otimes_C \mathscr{D}(C) \otimes_C \omega_C^\vee$.

The sheaf $\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee$ is filtered, and has two commuting left $\mathscr{D}_X$-module structures. The two structures coincide on $\mathrm{gr}\,(\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee) \cong (\mathrm{gr}\,\mathscr{D}_X) \otimes_{\mathscr{O}_X} \omega_X^\vee$. Hence there is an involution of $\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee$, which is the identity on the subsheaf $\omega_X^\vee = F_0(\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee)$, and exchanges the two $\mathscr{D}_X$-module structures.

Denote by $\mathbf{D}_X$ the duality functor on left $\mathscr{D}_X$-modules, namely $\mathbf{D}_X \mathscr{M} := \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee)[n]$; cf. [1, VI.3.6]. Let $f : X \hookrightarrow X^{\mathrm{e}}$ be the diagonal embedding. According to [1, Proposition VII.9.6] there is a functorial isomorphism $\mathbf{D}_{X^{\mathrm{e}}}\, f_+ \cong f_+\, \mathbf{D}_X$. We shall apply this isomorphism with the $\mathscr{D}_X$-module $\mathscr{O}_X$.

First note that $\mathbf{D}_X\, \mathscr{O}_X \cong \mathscr{O}_X$, as can be checked using the quasi-isomorphism $\Omega_X^{\boldsymbol{\cdot}}(\mathscr{D}_X)[n] \otimes_{\mathscr{O}_X} \omega_X^\vee \to \mathscr{O}_X$ in $\mathsf{Mod}\, \mathscr{D}_X$; cf. [1] VI.3.5. Next, by [1, Theorem VI.7.4(ii) and Theorem VI.7.11] (Kashiwara's Theorem) we see that $f_+\, \mathscr{O}_X \cong \mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee$ in $\mathsf{Mod}\, \mathscr{D}_{X^{\mathrm{e}}}$. Thus we have an isomorphism of $\mathscr{D}_{X^{\mathrm{e}}}$-modules

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee \cong \mathscr{E}xt^{2n}_{\mathscr{D}_{X^{\mathrm{e}}}}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \omega_X^\vee, \mathscr{D}_{X^{\mathrm{e}}} \otimes_{\mathscr{O}_{X^{\mathrm{e}}}} \omega_{X^{\mathrm{e}}}^\vee).$$

Passing to global sections, replacing $\mathscr{D}(C)$ by $\mathscr{D}(C)^\circ$ and using the involution of $\mathscr{D}(C) \otimes_C \omega_C^\vee$, we get

$$\begin{aligned}
&\mathscr{D}(C) \otimes_C \omega_C^\vee \\
&\quad\cong \mathrm{Ext}^{2n}_{\mathscr{D}(C)\otimes\mathscr{D}(C)}(\mathscr{D}(C) \otimes_C \omega_C^\vee, (\mathscr{D}(C) \otimes_C \omega_C^\vee) \otimes (\mathscr{D}(C) \otimes_C \omega_C^\vee)) \\
&\quad\cong \mathrm{Ext}^{2n}_{\mathscr{D}(C)\otimes\mathscr{D}(C)^\circ}(\mathscr{D}(C), (\mathscr{D}(C) \otimes_C \omega_C^\vee) \otimes \mathscr{D}(C)) \\
&\quad\cong \mathrm{Ext}^{2n}_{\mathscr{D}(C)^{\mathrm{e}}}(\mathscr{D}(C), \mathscr{D}(C) \otimes \mathscr{D}(C)) \otimes_C \omega_C^\vee.
\end{aligned}$$

Twisting by ω_C and shifting degrees we obtain an isomorphism

$$\mathscr{D}(C)[2n] \cong \mathrm{R}\,\mathrm{Hom}_{\mathscr{D}(C)^{\mathrm{e}}}(\mathscr{D}(C), \mathscr{D}(C)[2n] \otimes \mathscr{D}(C)[2n])$$

in $\mathsf{D}(\mathsf{Mod}\, \mathscr{D}(C)^{\mathrm{e}})$. $\square$

By the same arguments given for Corollaries 2.3 and 2.4, one has:

Corollary 2.7. *Let $\mathscr{D}(C)$ be as above. Then there are $\mathscr{D}(C)$-bimodule isomorphisms*

$$\mathrm{H}^q(\mathscr{D}(C), \mathscr{D}(C)^{\mathrm{e}}) \cong \begin{cases} \mathscr{D}(C) & \text{if } q = 2n, \\ 0 & \text{if } q \neq 2n. \end{cases}$$

For any k-central $\mathscr{D}(C)$-bimodule M one has

$$\mathrm{H}^q(\mathscr{D}(C), M) \cong \mathrm{H}_{2n-q}(\mathscr{D}(C), M).$$

Remark 2.8. One can show that there is a canonical choice for the rigidifying isomorphism ρ of the complex $R = \omega[n]$, $\omega = \mathrm{U}(\mathfrak{g}) \otimes \bigwedge^n \mathfrak{g}$. This amounts to choosing an isomorphism of bimodules $\rho : \omega \xrightarrow{\simeq} E^n(\mathrm{U}(\mathfrak{g}))$, where $E^n(\mathrm{U}(\mathfrak{g})) := \mathrm{Ext}^n_{\mathrm{U}(\mathfrak{g})^{\mathrm{e}}}(\mathrm{U}(\mathfrak{g}), \omega \otimes \omega)$. Here is a sketch of the proof. Let $A := \mathrm{gr}\, \mathrm{U}(\mathfrak{g}) = \mathrm{S}(\mathfrak{g})$. The bimodule ω is filtered, and there is a canonical isomorphism $\mathrm{gr}\, \omega \cong \Omega^n_{A/k}$. The standard spectral sequence of the filtration identifies $\mathrm{gr}\, E^n(\mathrm{U}(\mathfrak{g}))$ with $E^n(A) := \mathrm{Ext}^n_{A^{\mathrm{e}}}(A, \Omega^{2n}_{A^{\mathrm{e}}/k})$. But as mentioned in the Introduction, $\Omega^n_{A/k}[n]$ is the rigid dualizing complex of A, and it comes equipped with a canonical isomorphism $\Omega^n_{A/k} \xrightarrow{\simeq} E^n(A)$. This isomorphism determines ρ. A similar statement holds for Theorem 2.6. As a consequence the isomorphisms of Corollaries 2.3, 2.4 and 2.7 are canonical. (I thank Van den Bergh for mentioning this idea to me.)

Acknowledgements

I am grateful to Michel Van den Bergh for telling me about his conjecture and for many helpful suggestions. This paper was written during visits to MIT and the University of Washington in 1998, and I wish to thank the Departments of Mathematics at these universities for their hospitality, and especially Michael Artin.

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