Can. J. Math. Vol. **50** (4), 1998 pp. 863–896.

SMOOTH FORMAL EMBEDDINGS AND THE RESIDUE COMPLEX

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ABSTRACT. Let $\pi: X \longrightarrow S$ be a finite type morphism of noetherian schemes. A *smooth formal embedding* of X (over S) is a bijective closed immersion $X \subset \mathfrak{X}$, where $\mathfrak{X}$ is a noetherian formal scheme, formally smooth over S. An example of such an embedding is the formal completion $\mathfrak{X} = Y_{/X}$ where $X \subset Y$ is an algebraic embedding. Smooth formal embeddings can be used to calculate algebraic De Rham (co)homology.

Our main application is an explicit construction of the Grothendieck residue complex when S is a regular scheme. By definition the residue complex is the Cousin complex of $\pi^! \mathcal{O}_S$, as in [RD]. We start with I-C. Huang's theory of pseudofunctors on modules with 0-dimensional support, which provides a graded sheaf $\bigoplus_q \mathcal{K}^q_{X/S}$. We then use smooth formal embeddings to obtain the coboundary operator $\delta: \mathcal{K}^q_{X/S} \longrightarrow \mathcal{K}^{q+1}_{X/S}$. We exhibit a canonical isomorphism between the complex $(\mathcal{K}^{\cdot}_{X/S}, \delta)$ and the residue complex of [RD]. When π is equidimensional of dimension n and generically smooth we show that $\mathrm{H}^{-n} \mathcal{K}^{\cdot}_{X/S}$ is canonically isomorphic to to the sheaf of regular differentials of Kunz-Waldi [KW].

Another issue we discuss is Grothendieck Duality on a noetherian formal scheme $\mathfrak{X}$. Our results on duality are used in the construction of $\mathcal{K}^{\cdot}_{X/S}$.

0. **Introduction.** It is sometimes the case in algebraic geometry, that in order to define an object associated to a singular variety X, one first embeds X into a nonsingular variety Y. One such instance is algebraic De Rham cohomology $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X) = \mathrm{H}^{\cdot}(Y, \hat{\Omega}^{\cdot})$, where $\hat{\Omega}^{\cdot}$ is the completion along X of the De Rham complex $\Omega^{\cdot}_{Y/k}$ (relative to a base field k of characteristic 0; *cf.* [Ha]). Now $\hat{\Omega}^{\cdot}$ coincides with the complete De Rham complex $\hat{\Omega}^{\cdot}_{\mathfrak{X}/k}$, where $\mathfrak{X}$ is the formal scheme $Y_{/X}$. It is therefore reasonable to ask what sort of embedding $X \subset \mathfrak{X}$ into a formal scheme would give rise to the same cohomology.

The answer we provide in this paper is that any *smooth formal embedding* works. Let us define this notion. Suppose S is a noetherian base scheme and $\pi: X \longrightarrow S$ is a finite type morphism. A smooth formal embedding of X consists of morphisms $X \longrightarrow \mathfrak{X} \longrightarrow S$, where $X \longrightarrow \mathfrak{X}$ is a closed immersion of X into a noetherian formal scheme $\mathfrak{X}$, which is a homeomorphism of the underlying topological spaces; and $\mathfrak{X} \longrightarrow S$ is a *formally smooth* morphism. A smooth formal embedding $X \subset \mathfrak{X} = Y_{/X}$ like in the previous paragraph is said to be *algebraizable*. But in general $X \subset \mathfrak{X}$ will not be algebraizable.

Smooth formal embeddings enjoy a few advantages over algebraic embeddings. First consider an embedding $X \subset \mathfrak{X}$ and an étale morphism $U \longrightarrow X$. Then it is pretty clear (*cf.* Proposition 2.4) that there is an étale morphism of formal schemes $\mathfrak{U} \longrightarrow \mathfrak{X}$ and a smooth

Received by the editors October 2, 1997.
Incumbent of the Anna and Maurice Boukstein Career Development Chair.
AMS subject classification: Primary: 14B20; secondary: 14F10, 14B15, 14F20.

formal embedding $U \subset \mathfrak{U}$, s.t. $U \cong \mathfrak{U} \times_{\mathfrak{X}} X$. Next suppose $X \subset \mathfrak{X}, \mathfrak{Y}$ are two smooth formal embeddings, and we are given either a closed immersion $\mathfrak{X} \longrightarrow \mathfrak{Y}$ or a formally smooth morphism $\mathfrak{Y} \longrightarrow \mathfrak{X}$, which restrict to the identity on X. Then locally on X,

$$\mathfrak{Y} \cong \mathfrak{X} \times \operatorname{Spf} \mathbb{Z}[[t_1, \ldots, t_n]] \tag{0.1}$$

(Theorem 2.6).

As mentioned above, De Rham cohomology can be calculated by smooth formal embeddings. Indeed, when $\operatorname{char} S = 0$, $\mathrm{H}^q_{\mathrm{DR}}(X/S) = \mathrm{R}\pi_* \hat{\Omega}^{\cdot}_{\mathfrak{X}/S}$, where $X \subset \mathfrak{X}$ is any smooth formal embedding (Corollary 2.8). Moreover, in [Ye3] it is proved that De Rham homology $\mathrm{H}^{\mathrm{DR}}_{\cdot}(X)$ can also be calculated by smooth formal embeddings, when $S = \operatorname{Spec} k$, k a field. According to the preceding paragraph, given an étale morphism $g\colon U \longrightarrow X$ there is a homomorphism $g^*\colon \mathrm{H}^{\mathrm{DR}}_{\cdot}(X) \longrightarrow \mathrm{H}^{\mathrm{DR}}_{\cdot}(U)$, and we conclude that homology is contravariant w.r.t. étale morphisms. See Remark 2.11 for an application to $\mathcal{D}$-modules on singular varieties.

The main application of smooth formal embeddings in the present paper is for an *explicit construction of the Grothendieck residue complex* $\mathcal{K}^{\cdot}_{X/S}$, when S is any regular scheme. By definition $\mathcal{K}^{\cdot}_{X/S}$ is the Cousin complex $\mathrm{E}\pi^! \mathcal{O}_S$, in the notation of [RD] Sections IV.3 and VII.3.

Recall that Grothendieck Duality, as developed by Hartshorne in [RD], is an abstract theory, stated in the language of derived categories. Even though this abstraction is suitable for many important applications, often one wants more explicit information. In particular a significant amount of work was directed at finding an explicit presentation of duality in terms of differential forms and residues. Mostly the focus was on the dualizing sheaf ω_X, in various circumstances. The structure of ω_X as a coherent $\mathcal{O}_X$-module and its variance properties are thoroughly understood by now, thanks to an extended effort including [KW], [Li], [HK1], [HK2], [LS1] and [HS]. Regarding an explicit presentation of the full duality theory of dualizing complexes, there have been some advances in recent years, notably in the papers [Ye1], [SY], [Hu], [Hg1] [Sa] and [Ye3]. The later papers [Hg2], [Hg3] and [LS2] somewhat overlap our present paper in their results, but their methods are quite distinct; specifically, they do not use formal schemes.

We base our construction of $\mathcal{K}^{\cdot}_{X/S}$ on I-C. Huang's theory of pseudofunctors on modules with zero dimensional support (see [Hg1]). Suppose $\phi\colon A \longrightarrow B$ is a residually finitely generated homomorphism between complete noetherian local rings, and M is a discrete A-module (*i.e.* $\dim \operatorname{supp} M = 0$). Then according to [Hg1] there is a discrete B-module $\phi_{\#} M$, equipped with certain variance properties (*cf.* Theorem 6.2). In particular when ϕ is residually finite there is a map $\operatorname{Tr}_{\phi}\colon \phi_{\#} M \longrightarrow M$. Huang's theory is developed using only methods of commutative algebra.

Now given a point $x \in X$ with $s := \pi(x) \in S$, consider the local homomorphism $\phi\colon \hat{\mathcal{O}}_{S,s} \longrightarrow \hat{\mathcal{O}}_{X,x}$. Define $\mathcal{K}_{X/S}(x) := \phi_{\#} \mathrm{H}^d_{\mathfrak{m}_s} \hat{\mathcal{O}}_{S,s}$, where $d := \dim \hat{\mathcal{O}}_{S,s}$, $\mathfrak{m}_s$ is the maximal ideal and $\mathrm{H}^d_{\mathfrak{m}_s}$ is local cohomology. Then $\mathcal{K}_{X/S}(x)$ is an injective hull of $k(x)$ as $\mathcal{O}_{X,x}$-module. As a graded $\mathcal{O}_X$-module we take $\mathcal{K}^{\cdot}_{X/S} := \bigoplus_{x \in X} \mathcal{K}_{X/S}(x)$, with the obvious grading. Then for any scheme morphism $f: X \longrightarrow Y$, we deduce from Huang's theory a homomorphism of graded sheaves $\mathrm{Tr}_f: f_* \mathcal{K}^{\cdot}_{X/S} \longrightarrow \mathcal{K}^{\cdot}_{Y/S}$.

The problem is to exhibit a coboundary operator $\delta: \mathcal{K}^q_{X/S} \longrightarrow \mathcal{K}^{q+1}_{X/S}$, and to determine that the complex we obtain is indeed isomorphic to $\mathrm{E}\pi^! \mathcal{O}_S$. For this we use smooth formal embeddings, as explained below.

In Section 5 we discuss Grothendieck Duality on formal schemes, extending the theory of [RD]. We propose a definition of dualizing complex $\mathcal{R}^{\cdot}$ on a noetherian formal scheme (Definition 5.2), and prove its uniqueness (Theorem 5.6). It is important to note that the cohomology sheaves $\mathrm{H}^q \mathcal{R}^{\cdot}$ are discrete quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-modules, and in general *not coherent*. We define the Cousin functor E associated to $\mathcal{R}^{\cdot}$, and show that $\mathrm{E}\mathcal{R}^{\cdot} \cong \mathcal{R}^{\cdot}$ in the derived category, and $\mathrm{E}\mathcal{R}^{\cdot}$ is a residual complex. On a regular formal scheme $\mathfrak{X}$ the (surprising) fact is that $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{X}}$ is a dualizing complex, and not $\mathcal{O}_{\mathfrak{X}}$ (Theorem 5.14).

Now let $U \subset X$ be an affine open set and suppose $U \subset \mathfrak{U}$ is a smooth formal embedding. Say $n := \mathrm{rank}\, \hat{\Omega}^1_{\mathfrak{U}/S}$, so $\hat{\Omega}^n_{\mathfrak{U}/S}$ is a locally free $\mathcal{O}_{\mathfrak{U}}$-module of rank 1, and $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{U}/S}[n]$ is a dualizing complex. Since the Cousin complex is a sum of local cohomology modules, there is a natural identification of graded $\mathcal{O}_{\mathfrak{U}}$-modules $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{U}/S}[n] \cong \mathcal{K}^{\cdot}_{\mathfrak{U}/S}$. This makes $\mathcal{K}^{\cdot}_{\mathfrak{U}/S}$ into a complex. Since $\mathcal{K}^{\cdot}_{U/S} \cong \mathcal{H}om_{\mathfrak{U}}(\mathcal{O}_U, \mathcal{K}^{\cdot}_{\mathfrak{U}/S})$ we come up with an operator δ on $\mathcal{K}^{\cdot}_{U/S} = \mathcal{K}^{\cdot}_{X/S}|_U$.

Given another smooth formal embedding $U \subset \mathfrak{V}$ we have to compare the complexes $\mathcal{K}^{\cdot}_{\mathfrak{U}/S}$ and $\mathcal{K}^{\cdot}_{\mathfrak{V}/S}$. This is rather easy to do using the following trick. Choosing a sequence $\underline{a}$ of generators of some defining ideal of $\mathfrak{U}$, and letting $\mathbf{K}^{\cdot}_{\infty}(\underline{a})$ be the associated Koszul complex, we obtain an explicit presentation of the dualizing complex, namely

$$\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{U}/S}[n] \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \hat{\Omega}^n_{\mathfrak{U}/S}[n]$$

(*cf.* Lemma 4.5). By the structure of smooth formal embeddings we may assume there is a morphism $f: \mathfrak{U} \longrightarrow \mathfrak{V}$ which is either formally smooth or a closed immersion. Then choosing relative coordinates (*cf.* formula 0.1) and using Koszul complexes we produce a morphism $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{U}/S}[n] \longrightarrow \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^m_{\mathfrak{V}/S}[m]$. Applying the Cousin functor E we recover $\mathrm{Tr}_f: \mathcal{K}^{\cdot}_{\mathfrak{U}/S} \longrightarrow \mathcal{K}^{\cdot}_{\mathfrak{V}/S}$ as a map of complexes! We conclude that δ is independent of $\mathfrak{U}$ and hence it glues to a global operator (Theorem 6.14).

If $f: X \longrightarrow Y$ is a finite morphism, then the trace map $\mathrm{Tr}_f: f_* \mathcal{K}^{\cdot}_{X/S} \longrightarrow \mathcal{K}^{\cdot}_{Y/S}$, which is provided by Huang's theory, is actually a homomorphism of complexes (Theorem 7.1). We show this by employing the same trick as above of going from Koszul complexes to Cousin complexes, this time inserting a "Tate residue map" into the picture. We use Theorem 7.1 to prove directly that if $\pi: X \longrightarrow S$ is equidimensional of dimension n and generically smooth, then $\mathrm{H}^{-n} \mathcal{K}^{\cdot}_{X/S}$ coincides with the sheaf of regular differentials $\tilde{\omega}^n_{X/S}$ of Kunz-Waldi [KW] (Theorem 7.10).

Finally in Theorem 8.1 we exhibit a canonical isomorphism ζ_X between the complex $\mathcal{K}^{\cdot}_{X/S}$ constructed here and the complex $\pi^{\triangle} \mathcal{O}_S = \mathrm{E}\pi^! \mathcal{O}_S$ of [RD]. Given a morphism of schemes $f: X \longrightarrow Y$ the isomorphisms ζ_X and ζ_Y send Huang's trace map $\mathrm{Tr}_f: f_* \mathcal{K}^{\cdot}_{X/S} \longrightarrow$

$\mathcal{K}^{\cdot}_{Y/S}$ to the trace $\mathrm{Tr}^{\mathrm{RD}}_f : f_* \mathrm{E}\pi^!_X \mathcal{O}_S \longrightarrow \mathrm{E}\pi^!_Y \mathcal{O}_S$ of [RD] Section VI.4. In particular it follows that for f proper, Tr_f is a homomorphism of complexes (Corollary 8.3).

Sections 1 and 3 of the paper contain the necessary supplements to [EGA]. Perhaps the most noteworthy result there is Theorem 1.22, which states that formally finite type morphisms are stable under base change. This was also proved in [AJL2].

ACKNOWLEDGMENTS. The author wishes to thank L. Alonso, I-C. Huang, R. Hübl, A. Jeremías, J. Lipman and P. Sastry for helpful discussions, some of which took place during a meeting in Oberwolfach in May 1996.

1. **Formally finite type morphisms.** In this section we define formally finite type morphisms between noetherian formal schemes. This mild generalization of the finite type morphism of [EGA] I Section 10 has the advantage that it includes the completion morphism $\mathfrak{X} \longrightarrow \mathfrak{X}_{/Z}$ (*cf.* Proposition 1.21), and still is preserved under base change (Theorem 1.22).

We follow the conventions of [EGA] 0_{I} Section 7 on *adic* rings. Thus an adic ring is a commutative ring A which is complete and separated in the $\mathfrak{a}$-adic topology, for some ideal $\mathfrak{a} \subset A$. As for formal schemes, we follow the conventions of [EGA] I Section 10. Throughout the paper all formal schemes are by default noetherian (adic) formal schemes.

We write $A[\underline{t}] = A[t_1, \ldots, t_n]$ for the polynomial algebra with variables $t_1, \ldots, t_n$ over a ring A. The easy lemma below is taken from [AJL2].

LEMMA 1.1. *Let $A \longrightarrow B$ be a continuous homomorphism between noetherian adic rings, and let $\mathfrak{b} \subset B$ be a defining ideal. Then the following are equivalent:*

(i) $A \longrightarrow B/\mathfrak{b}$ is a finite type homomorphism.

(ii) For some homomorphism $f : A[\underline{t}] \longrightarrow B$ extending $A \longrightarrow B$ one has $\mathfrak{b} = B \cdot f^{-1}(\mathfrak{b})$ and $A[\underline{t}] \longrightarrow B/\mathfrak{b}$ is surjective.

PROOF. (i) $\Rightarrow$ (ii): Say $b_1, \ldots, b_m$ generate $\mathfrak{b}$ as a B-module, and the images of $b_{m+1}, \ldots, b_n$ generate $B/\mathfrak{b}$ as an A-algebra. Then the homomorphism $A[\underline{t}] \longrightarrow B$, $t_i \longrightarrow b_i$ has the required properties.

(ii) $\Rightarrow$ (i): Trivial. ∎

DEFINITION 1.2. Let $A \longrightarrow B$ be a continuous homomorphism between adic noetherian rings. We say that $A \longrightarrow B$ is of *formally finite type* (f.f.t.) if the equivalent conditions of Lemma 1.1 hold. We shall also say that B is a formally finite type A-algebra.

EXAMPLE 1.3. Let $I \subset A$ be any open ideal, and let $B := \lim_{\leftarrow i} A/I^i$. Then $A \longrightarrow B$ is f.f.t.

Recall that if A' and B are adic A-algebras, with defining ideals $\mathfrak{a}'$ and $\mathfrak{b}$, the complete tensor product $A' \hat{\otimes}_A B$ is the completion of $A' \otimes_A B$ w.r.t. the topology defined by the image of $(\mathfrak{a}' \otimes_A B) \oplus (A' \otimes_A \mathfrak{b})$.

PROPOSITION 1.4. *Let A, A' and B be noetherian adic rings, $A \longrightarrow B$ a* f.f.t. *homomorphism, and $A \longrightarrow A'$ any continuous homomorphism. Then $B' := A' \hat{\otimes}_A B$ is a noetherian adic ring, and $A' \longrightarrow B'$ is a* f.f.t. *homomorphism.*

PROOF. Choose a homomorphism $f: A[\underline{t}] \to B$ satisfying condition (ii) of Lemma 1.1. Let $\mathfrak{b} \subset B$ and $\mathfrak{a}' \subset A'$ be defining ideals. Write $C := A' \otimes_A B$ and $\mathfrak{c} := \mathfrak{a}' \cdot C + C \cdot \mathfrak{b}$, so $B' = \lim_{\leftarrow i} C/\mathfrak{c}^i$. Consider the homomorphism $f': A'[\underline{t}] \to C$, and let $\mathfrak{c}' := f'^{-1}(\mathfrak{c})$ and $\widehat{A'[\underline{t}]} := \lim_{\leftarrow i} A'[\underline{t}]/\mathfrak{c}'^i$. Since $\mathfrak{c} = C \cdot \mathfrak{c}'$, it follows from [CA] Section III.2.11 Proposition 14 that $\widehat{A'[\underline{t}]} \to B'$ is surjective. Hence B' is a noetherian adic ring with the $\mathfrak{b}'$-adic topology, where $\mathfrak{b}' = B' \cdot \mathfrak{c}$. Furthermore $A'[\underline{t}] \to B'/\mathfrak{b}'$ is surjective, and we conclude that $A' \to B'$ is f.f.t. ∎

In the next three examples A is an adic ring with defining ideal $\mathfrak{a}$.

EXAMPLE 1.5. Recall that for $a \in A$, the complete ring of fractions $A_{\{a\}}$ is the completion of the localized ring A_a w.r.t. the $\mathfrak{a}_a$-adic topology. Then $A_{\{a\}} \cong A \hat{\otimes}_{\mathbb{Z}[t]} \mathbb{Z}[t, t^{-1}]$, which proves that $A \to A_{\{a\}}$ is f.f.t.

EXAMPLE 1.6. Given indeterminates $t_1, \ldots, t_n$, the ring of restricted formal power series $A\{\underline{t}\} = A\{t_1, \ldots, t_n\}$ is the completion of the polynomial ring $A[\underline{t}]$ w.r.t. the $(A[\underline{t}] \cdot \mathfrak{a})$-adic topology. Hence $A\{\underline{t}\} \cong A \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[\underline{t}]$, which demonstrates that $A \to A\{\underline{t}\}$ is f.f.t.

EXAMPLE 1.7. Consider the adic ring $A \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[\underline{t}]]$, where $\mathbb{Z}[[\underline{t}]] = \mathbb{Z}[[t_1, \ldots, t_n]]$ is the ring of formal power series, with the $(\underline{t})$-adic topology. Since inverse limits commute, we see that $A \hat{\otimes}_{\mathbb{Z}} \mathbb{Z}[[\underline{t}]] \cong A[[\underline{t}]]$, the ring of formal power series over A, endowed with the $\big(A[[\underline{t}]] \cdot (\mathfrak{a} + \underline{t})\big)$-adic topology. By Proposition 1.4, $A \to A[[\underline{t}]]$ is f.f.t.

Let $A \to B$ be a f.f.t homomorphism between adic rings. Choose a defining ideal $\mathfrak{b} \subset B$, and set $B_i := B/\mathfrak{b}^{i+1}$. For $n \geq 0$ define

$$\hat{\Omega}^n_{B/A} := \lim_{\leftarrow i} \Omega^n_{B_i/A} \cong \lim_{\leftarrow i} B_i \otimes_B \Omega^n_{B/A}$$

(*cf.* [EGA] 0_{IV} 20.7.14). Let $\hat{\Omega}^{\cdot}_{B/A} := \oplus_{n \geq 0} \hat{\Omega}^n_{B/A}$, which is a topological DGA (differential graded algebra), with $\hat{\Omega}^0_{B/A} = B$. This definition is independent of the ideal $\mathfrak{b}$. Since $\Omega^n_{B_i/A}$ is finite over B_i it follows that $\hat{\Omega}^n_{B/A}$ is finite over B.

REMARK 1.8. If $A \to B$ is f.f.t. then $\hat{\Omega}^{\cdot}_{B/A} \cong \Omega^{\cdot,\mathrm{sep}}_{B/A}$, where $\Omega^{\cdot,\mathrm{sep}}_{B/A}$ is the separated algebra of differentials defined in [Ye1] Section 1.5 for semi-topological algebras. Also $\hat{\Omega}^{\cdot}_{B/A}$ is the universally finite differential algebra in the sense of [Ku].

PROPOSITION 1.9. *Let $L \to A \to B$ be* f.f.t. *homomorphisms between adic noetherian rings.*

1. *$A \to B$ is formally smooth relative to L iff the sequence*

$$0 \to B \otimes_A \hat{\Omega}^1_{A/L} \xrightarrow{v} \hat{\Omega}^1_{B/L} \xrightarrow{u} \hat{\Omega}^1_{B/A} \to 0$$

   is split exact.
2. *$A \to B$ is formally étale relative to L iff $B \otimes_A \hat{\Omega}^1_{A/L} \to \hat{\Omega}^1_{B/L}$ is bijective.*

PROOF. Use the results of [EGA] 0_{IV} Section 20.7, together the fact that these are finite B-modules. ∎

PROPOSITION 1.10. *Let $f\colon A \longrightarrow B$ be a formally smooth,* f.f.t. *homomorphism between noetherian adic rings. Then B is flat over A and $\hat{\Omega}^1_{B/A}$ is a projective finitely generated B-module.*

PROOF. For flatness it suffices to show that if $\mathfrak{n}$ is a maximal ideal of B and $\mathfrak{m} := f^{-1}(\mathfrak{n})$, then $\hat{A}_{\mathfrak{m}} \longrightarrow \hat{B}_{\mathfrak{n}}$ is flat ($\hat{B}_{\mathfrak{n}}$ is the completion of $B_{\mathfrak{n}}$ with the $\mathfrak{n}$-adic topology). Now $\mathfrak{n}$ is open, and hence so is $\mathfrak{m}$. Both $A \longrightarrow \hat{A}_{\mathfrak{m}}$ and $B \longrightarrow \hat{B}_{\mathfrak{n}}$ are formally étale, therefore $\hat{A}_{\mathfrak{m}} \longrightarrow \hat{B}_{\mathfrak{n}}$ is formally smooth. Because $A \longrightarrow B$ is f.f.t. it follows that $A/\mathfrak{m} \longrightarrow B/\mathfrak{n}$ is finite type, and hence finite (and $\mathfrak{m}$ is a maximal ideal). By [EGA] 0_{IV} Theorem 19.7.1, $\hat{B}_{\mathfrak{n}}$ is flat over $\hat{A}_{\mathfrak{m}}$.

The second assertion follows from [EGA] 0_{IV} Theorem 20.4.9. ∎

PROPOSITION 1.11. *Let $f\colon A \longrightarrow B$ be a* f.f.t.*, formally smooth homomorphism of noetherian adic rings, and let $\mathfrak{q} \in \operatorname{Spf} B$. Suppose* $\operatorname{rank} \hat{\Omega}^1_{\hat{B}_{\mathfrak{q}}/A} = n$*. Then:*

1. *For some $b \in B - \mathfrak{q}$ there is a formally étale homomorphism $\tilde{f}\colon A[\underline{t}] = A[t_1, \ldots, t_n] \longrightarrow B_{\{b\}}$ extending f.*
2. *For any $\mathfrak{q}' \in \operatorname{Spf} B_{\{b\}}$ let $\mathfrak{r} := \tilde{f}^{-1}(\mathfrak{q}')$. Then $\widehat{A[\underline{t}]}_{\mathfrak{r}} \longrightarrow \hat{B}_{\mathfrak{q}'}$ is finite étale.*
3. *Let $\mathfrak{p} := f^{-1}(\mathfrak{q})$. Assume $\hat{A}_{\mathfrak{p}}$ is regular of dimension m, and* $\operatorname{tr}\deg_{k(\mathfrak{p})} k(\mathfrak{q}) = l$*. Then $\hat{B}_{\mathfrak{q}}$ is regular of dimension $n + m - l$.*

PROOF. 1. By Proposition 1.10 we can find b s.t. $\hat{\Omega}^1_{B_{\{b\}}/A} \cong B_{\{b\}} \otimes_B \hat{\Omega}^1_{B/A}$ is free, say with basis $\mathrm{d}b_1, \ldots, \mathrm{d}b_n$. Then we get a homomorphism $A[\underline{t}] \longrightarrow B_{\{b\}}$, $t_i \longmapsto b_i$. In order to stay inside the category of adic rings we may replace $A[\underline{t}]$ with its completion $A\{\underline{t}\}$ (*cf.* Examples 1.5–1.7 for the notation). According to Proposition 1.9 we see that $A[\underline{t}] \longrightarrow B_{\{b\}}$ is formally étale relative to A. But since $A \longrightarrow B_{\{b\}}$ is formally smooth, this implies that $A[\underline{t}] \longrightarrow B_{\{b\}}$ is actually (absolutely) formally étale.

2. Consider the formally étale homomorphism $k(\mathfrak{r}) \longrightarrow \hat{B}_{\mathfrak{q}'}/\mathfrak{r}\hat{B}_{\mathfrak{q}'}$. Since $\mathfrak{q}'$ is an open prime ideal it follows that $A \longrightarrow B/\mathfrak{q}'$ is a finite type homomorphism, so the field extension $k(\mathfrak{r}) \longrightarrow k(\mathfrak{q}')$ is finitely generated. By [Hg1] Lemma 3.9 we see that in fact $\hat{B}_{\mathfrak{q}'}/\mathfrak{r}\hat{B}_{\mathfrak{q}'} = k(\mathfrak{q}')$, so $k(\mathfrak{r}) \longrightarrow k(\mathfrak{q}')$ is finite separable. Hence $\widehat{A[\underline{t}]}_{\mathfrak{r}} \longrightarrow \hat{B}_{\mathfrak{q}'}$ is finite étale.

3. Take $\mathfrak{q}' := \mathfrak{q}$. Under the assumption the ring $\widehat{A[\underline{t}]}_{\mathfrak{r}}$ is regular, and according to [Ma] Section 14.c Theorem 23, $\dim \widehat{A[\underline{t}]}_{\mathfrak{r}} = m + n - l$. By part 2, $\hat{B}_{\mathfrak{q}}$ is also regular, and $\dim \hat{B}_{\mathfrak{q}} = \dim \widehat{A[\underline{t}]}_{\mathfrak{r}}$. ∎

Let us now pass to formal schemes.

Given a noetherian formal scheme $\mathfrak{X}$, choose a defining ideal $I \subset \mathcal{O}_{\mathfrak{X}}$, and set

$$X_n := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / I^{n+1}). \tag{1.12}$$

X_n is a noetherian (usual) scheme, and $\mathfrak{X} \cong \lim_{n\to} X_n$ in the category of formal schemes. One possible choice for I is the largest defining ideal, in which case one has $X_0 = \mathfrak{X}_{\mathrm{red}}$, the reduced closed subscheme (*cf.* [EGA] I Section 10.5).

LEMMA 1.13. *Suppose* $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ *is a morphism between noetherian formal schemes. There are defining ideals* $I \subset O_{\mathfrak{X}}$ *and* $\mathcal{J} \subset O_{\mathfrak{Y}}$ s.t. $f^{-1}\mathcal{J} \cdot O_{\mathfrak{X}} \subset I$. *Letting* X_n *and* Y_n *be the corresponding schemes (cf. (1.12)), we get morphisms of schemes* $f_n : X_n \longrightarrow Y_n$, *with* $f = \lim_{n\to} f_n$.

PROOF. See [EGA] I Section 10.6. For instance, one could take I to be the largest defining ideal and $\mathcal{J}$ arbitrary. ∎

DEFINITION 1.14. Let $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ be a morphism of noetherian (adic) formal schemes. We say that f is of *formally finite type* (or that $\mathfrak{X}$ is a formally finite type formal scheme over $\mathfrak{Y}$) if the morphism $f_0 : X_0 \longrightarrow Y_0$ in Lemma 1.13 is finite type, for some choice of defining ideals of $\mathfrak{X}$ and $\mathfrak{Y}$.

Observe that if the morphism f_0 is finite type then so are all the f_n, and the definition doesn't depend on the defining ideals chosen.

REMARK 1.15. The definition of f.f.t. morphism we gave in an earlier version of the paper was more cumbersome, though equivalent. The present Definition 1.14 is taken from [AJL2], where the name is "pseudo-finite type morphism", and I wish to thank A. Jeremías for bringing it to my attention.

Here are a couple of examples of f.f.t. morphisms:

EXAMPLE 1.16. A finite type morphism $\mathfrak{X} \longrightarrow \mathfrak{Y}$ (in the sense of [EGA] I Section 10.13) is f.f.t.

EXAMPLE 1.17. Let X be a scheme of finite type over a noetherian scheme S, and let $X_0 \subset X$ be a locally closed subset. Then the completion $\mathfrak{X} = X_{/X_0}$ (see [EGA] I Section 10.8) is of f.f.t. over S. Such a formal scheme is called *algebraizable*.

DEFINITION 1.18. A f.f.t. morphism $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$ is called *formally finite* (resp. *formally proper*) if the morphism $f_0 : X_0 \longrightarrow Y_0$ in Lemma 1.4 is finite (resp. proper), for some choice of defining ideals.

EXAMPLE 1.19. If in Example 1.17 the subset $X_0 \subset X$ is closed, then $\mathfrak{X} \longrightarrow X$ is formally finite. If $X_0 \longrightarrow S$ is proper, then $\mathfrak{X} \longrightarrow S$ is formally proper.

PROPOSITION 1.20. *1. An immersion* $\mathfrak{X} \longrightarrow \mathfrak{Y}$ *is* f.f.t.

2. If $\mathfrak{X} \longrightarrow \mathfrak{Y}$ *and* $\mathfrak{Y} \longrightarrow \mathfrak{Z}$ *are* f.f.t., *then so is* $\mathfrak{X} \longrightarrow \mathfrak{Z}$.

3. Let $\mathfrak{U} = \operatorname{Spf} B$ *and* $\mathfrak{V} = \operatorname{Spf} A$. *Then a morphism* $\mathfrak{U} \longrightarrow \mathfrak{V}$ *is* f.f.t. *iff the ring homomorphism* $A \longrightarrow B$ *is* f.f.t.

PROOF. Consider morphisms of schemes $X_0 \longrightarrow Y_0$ *etc.* as in Lemma 1.13. For part 3 use condition (i) of Lemma 1.1. ∎

PROPOSITION 1.21. *Let* $\mathfrak{X}$ *be a noetherian formal scheme and* $Z \subset \mathfrak{X}$ *a locally closed subset. Then there is a noetherian formal scheme* $\mathfrak{X}_{/Z}$, *with underlying topological space* Z, *and the natural morphism* $\mathfrak{X}_{/Z} \longrightarrow \mathfrak{X}$ *is* f.f.t.

PROOF. Pick an open subset $\mathfrak{U} \subset \mathfrak{X}$ s.t. $Z \subset \mathfrak{U}$ is closed, and choose a defining ideal $\mathcal{I}$ of Z. Let $\mathcal{O}_{\mathfrak{Z}} := \lim_{\leftarrow i} \mathcal{O}_{\mathfrak{U}} / \mathcal{I}^i$. According to [EGA] I Section 10.6, $\mathfrak{X}_{/Z} := (Z, \mathcal{O}_{\mathfrak{Z}})$ is a noetherian formal scheme. Clearly $\mathfrak{X}_{/Z} \to \mathfrak{X}$ is f.f.t. ∎

In [EGA] I Section 10.3 it is shown that finite type morphisms between noetherian formal schemes are preserved by base change. This is true also for f.f.t. morphisms:

THEOREM 1.22. *Suppose $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Y}'$ are noetherian formal schemes, $\mathfrak{X} \to \mathfrak{Y}$ is a* f.f.t. *morphism, and $\mathfrak{Y}' \to \mathfrak{Y}$ is an arbitrary morphism. Then $\mathfrak{X}' := \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ is also noetherian, and the morphism $\mathfrak{X}' \to \mathfrak{Y}'$ is* f.f.t.

PROOF. First note that the formal scheme $\mathfrak{X}' = \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{Y}'$ exists ([EGA] I Section 10.7). For any affine open sets $\mathfrak{U} = \operatorname{Spf} B \subset \mathfrak{X}$, $\mathfrak{V}' = \operatorname{Spf} A' \subset \mathfrak{Y}'$ and $\mathfrak{V} = \operatorname{Spf} A \subset \mathfrak{Y}$ such that $\mathfrak{U} \to \mathfrak{V}$ and $\mathfrak{V}' \to \mathfrak{V}$, one has $\mathfrak{U}' = \mathfrak{U} \times_{\mathfrak{V}} \mathfrak{V}' = \operatorname{Spf} B \hat{\otimes}_A A'$, and $\mathfrak{U}' \subset \mathfrak{X}'$ is open. By Propositions 1.4 and 1.20, $\mathfrak{U}'$ is a noetherian formal scheme, and $\mathfrak{U}' \to \mathfrak{V}'$ is f.f.t. But finitely many such $\mathfrak{U}'$ cover $\mathfrak{X}'$. ∎

COROLLARY 1.23. *If $\mathfrak{X}_1$, $\mathfrak{X}_2$ and $\mathfrak{Y}$ are noetherian and $\mathfrak{X}_i \to \mathfrak{Y}$ are* f.f.t. *morphisms, then $\mathfrak{X}_3 := \mathfrak{X}_1 \times_{\mathfrak{Y}} \mathfrak{X}_2$ is also noetherian, and $\mathfrak{X}_3 \to \mathfrak{Y}$ is* f.f.t.

REMARK 1.24. I do not know an example of a f.f.t. formal scheme $\mathfrak{X}$ over a scheme S which is not locally algebraizable. (Locally algebraizable means there is an open covering $\mathfrak{X} = \bigcup \mathfrak{U}_i$, with $\mathfrak{U}_i \to S$ algebraizable, in the sense of Example 1.17.)

DEFINITION 1.25. A morphism of formal schemes $\mathfrak{X} \to \mathfrak{Y}$ is said to be *formally smooth* (resp. *formally étale*) if, given a (usual) affine scheme Z, a morphism $Z \to \mathfrak{Y}$ and a closed subscheme $Z_0 \subset Z$ defined by a nilpotent ideal, the map $\mathcal{H}om_{\mathfrak{Y}}(Z, \mathfrak{X}) \to \mathcal{H}om_{\mathfrak{Y}}(Z_0, \mathfrak{X})$ is surjective (resp. bijective).

This is the definition of formal smoothness used in [EGA] IV Section 17.1. We shall also require the next notion.

DEFINITION 1.26. A morphism $g\colon \mathfrak{X} \to \mathfrak{Y}$ between noetherian formal schemes is called *étale* if it is of finite type (see [EGA] I Section 10.13) and formally étale.

Note that if $\mathfrak{Y}$ is a usual scheme, then so is $\mathfrak{X}$, and g is an étale morphism of schemes. According to [EGA] I Proposition 10.13.5 and by the obvious properties of formally étale morphisms, if $\mathfrak{U} \to \mathfrak{X}$ and $\mathfrak{V} \to \mathfrak{X}$ are étale, then so is $\mathfrak{U} \times_{\mathfrak{X}} \mathfrak{V} \to \mathfrak{X}$. Hence for fixed $\mathfrak{X}$, the category of all étale morphisms $\mathfrak{U} \to \mathfrak{X}$ forms a site (*cf.* [Mi] Chapter II Section 1). We call this site the small étale site on $\mathfrak{X}$, and denote it by $\mathfrak{X}_{\text{et}}$.

2. Smooth formal embeddings and De Rham cohomology.

Fix a noetherian base scheme S and a finite type S-scheme X.

DEFINITION 2.1. A *smooth formal embedding* (s.f.e.) of X (over S) is the following data:

(i) A noetherian formal scheme $\mathfrak{X}$.

(ii) A formally finite type, formally smooth morphism $\mathfrak{X} \to S$.

(iii) An S-morphism $X \longrightarrow \mathfrak{X}$, which is a closed immersion and a homeomorphism between the underlying topological spaces.

We shall refer to this by writing "$X \subset \mathfrak{X}$ is a s.f.e."

EXAMPLE 2.2. Suppose Y is a smooth S-scheme, $X \subset Y$ a locally closed subset, and $\mathfrak{X} = Y_{/X}$ the completion. Then $X \subset \mathfrak{X}$ is a smooth formal embedding. Such an embedding is called an *algebraizable embedding* (*cf.* Remark 1.24).

The smooth formal embeddings of X form a category, in which a morphism of embeddings is an S-morphism of formal schemes $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ inducing the identity on X. Note that any morphism of embeddings $f: \mathfrak{X} \longrightarrow \mathfrak{Y}$ is affine (*cf.* [EGA] I Proposition 10.6.12), and the functor $f_*: \mathsf{Mod}(\mathfrak{X}) \longrightarrow \mathsf{Mod}(\mathfrak{Y})$ is exact. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be two smooth formal embeddings of X. Consider the formal scheme $\mathfrak{X} \times_S \mathfrak{Y}$. Then the diagonal $\Delta: X \longrightarrow \mathfrak{X} \times_S \mathfrak{Y}$ is an immersion (we do not assume our formal schemes are separated!).

PROPOSITION 2.3. *The completion $(\mathfrak{X} \times_S \mathfrak{Y})_{/X}$ of $\mathfrak{X} \times_S \mathfrak{Y}$ along $\Delta(X)$ is a smooth formal embedding of X, and moreover it is a product of $\mathfrak{X}$ and $\mathfrak{Y}$ in the category of smooth formal embeddings.*

PROOF. By Theorem 1.22 and Proposition 1.21 it follows that $(\mathfrak{X} \times_S \mathfrak{Y})_{/X}$ is formally finite type over S, so in particular it is noetherian. Clearly $(\mathfrak{X} \times_S \mathfrak{Y})_{/X} \longrightarrow S$ is formally smooth. ∎

The benefit of using formal rather than algebraic embeddings is in:

PROPOSITION 2.4. *Let $X \subset \mathfrak{X}$ be a smooth formal embedding (over S) and $g: U \longrightarrow X$ an étale morphism. Then there exists a noetherian formal scheme $\mathfrak{U}$ and an étale morphism $\hat{g}: \mathfrak{U} \longrightarrow \mathfrak{X}$* s.t. $U \cong \mathfrak{U} \times_{\mathfrak{X}} X$. $\hat{g}: \mathfrak{U} \longrightarrow \mathfrak{X}$ *is unique (up to a unique isomorphism), and moreover $U \longrightarrow \mathfrak{U}$ is a smooth formal embedding.*

PROOF. This is essentially the "topological invariance of étale morphisms", (*cf.* [EGA] IV Section 18.1 or [Mi] Chapter I Theorem 3.23). Let $I := \operatorname{Ker}(\mathcal{O}_{\mathfrak{X}} \longrightarrow \mathcal{O}_X)$ and $X_i := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / I^{i+1})$; so $X = X_0$. For every i there is a unique étale morphism $g_i: U_i \longrightarrow X_i$ s.t. $U \cong U_i \times_{X_i} X$. Identifying the underlying topological spaces of U_i and U we get an inverse system of sheaves $\{\mathcal{O}_{U_i}\}$ on U. Setting $\mathcal{O}_{\mathfrak{U}} := \lim_{\leftarrow i} \mathcal{O}_{U_i}$ we get a noetherian formal scheme $\mathfrak{U}$ having the proclaimed properties (*cf.* [EGA] I Section 10.6). ∎

Thus we can consider $\mathfrak{X}_{\mathrm{et}}$ as a "smooth formal embedding" of X_{et}. If $\mathcal{M}$ is a sheaf on X_{et} and $U \longrightarrow X$ is an étale morphism, we denote by $\mathcal{M}|_U$ the restriction of $\mathcal{M}$ to U_{Zar}.

COROLLARY 2.5. *Let $X \subset \mathfrak{X}$ be a smooth formal embedding over S. Then there is a sheaf of* DGA*s $\hat{\Omega}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}/S}$ on X_{et}, with the property that for each $g: U \longrightarrow X$ in X_{et} and corresponding $\hat{g}: \mathfrak{U} \longrightarrow \mathfrak{X}$ in $\mathfrak{X}_{\mathrm{et}}$, one has $\hat{\Omega}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}/S}|_U \cong \hat{\Omega}^{\cdot}_{\mathfrak{U}/S}$.*

PROOF. By Proposition 1.9, $\hat{\Omega}^p_{\mathfrak{U}/S} \cong \hat{g}^* \hat{\Omega}^p_{\mathfrak{X}/S}$. Now $\hat{\Omega}^p_{\mathfrak{X}/S}$ is coherent, so we can use [Mi] Chapter II Corollary 1.6 (which applies to our étale site $\mathfrak{X}_{\mathrm{et}}$). ■

For smooth formal embeddings, closed immersions and smooth morphisms are locally trivial, in the following sense. Recall that for an adic algebra A, the ring of formal power series $A[[\underline{t}]] = A[[t_1, \dots, t_n]]$ is adic (*cf.* Example 1.7).

THEOREM 2.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of smooth formal embeddings of X over S. Assume f is a closed immersion (resp. formally smooth). Then, given a point $x \in X$, there are affine open sets $U \subset X$ and $W \subset S$, with $x \in U$ and $U \to W$, satisfying condition ($*$) below.*

($*$) Let $W = \operatorname{Spec} L$, and let $\operatorname{Spf} A \subset \mathfrak{Y}$ and $\operatorname{Spf} B \subset \mathfrak{X}$ be the affine formal schemes supported on U. Then there is an isomorphism of topological L-algebras $A \cong B[[\underline{t}]]$ (resp. $B \cong A[[\underline{t}]]$) such that $f^* : A \to B$ is projection modulo $(\underline{t})$ (resp. the inclusion).

PROOF. 1. Assume f is a closed immersion. According to [EGA] 0_{IV} Theorem 19.5.3 and Corollary 20.7.9, by choosing $U = \operatorname{Spec} C$ small enough, and setting $I := \operatorname{Ker}(f^* : A \to B)$, we obtain an exact sequence

$$0 \to I/I^2 \to B \otimes_A \hat{\Omega}^1_{A/L} \to \hat{\Omega}^1_{B/L} \to 0$$

of free B-modules. Choose $a_1, \dots, a_n, b_1, \dots, b_m \in A$ s.t. $\{a_i\}$ is a B-basis of I/I^2, and $\{\mathrm{d}b_i\}$ is a B-basis of $\hat{\Omega}^1_{B/L}$.

By the proof of Proposition 1.11 the homomorphisms $L[\underline{s}] \to B$, $L[\underline{s}, \underline{t}] \to A$ and $L[\underline{s}, \underline{t}] \to B[[\underline{t}]]$, sending $s_i \mapsto b_i$ and $t_i \mapsto a_i$, are all formally étale. Take $\mathfrak{a} := \operatorname{Ker}(A \to C)$, which is a defining ideal of A, containing $A \cdot (\underline{t}) = I$. Let $\mathfrak{b} := \mathfrak{a} \cdot B$, which is a defining ideal of B. Hence the ideal $\mathfrak{c} = B[[\underline{t}]] \cdot (\mathfrak{b}, \underline{t})$ is a defining ideal of $B[[\underline{t}]]$. By formal étaleness of $L[\underline{s}, \underline{t}] \to A$ and $L[\underline{s}, \underline{t}] \to B[[\underline{t}]]$, the isomorphism $A/\mathfrak{a} \cong B[[\underline{t}]]/\mathfrak{c} \cong C$ lifts uniquely to an isomorphism $A \cong B[[\underline{t}]]$.

2. Now assume f is formally smooth. Let $\mathfrak{b} := \operatorname{Ker}(B \to C)$, which is a defining ideal of B. Since $A \to B/\mathfrak{b}$ is surjective it follows that $(B/\mathfrak{b}) \otimes_B \hat{\Omega}^1_{B/A}$ is generated by $\mathrm{d}(\mathfrak{b})$. By Nakayama's Lemma we see that $\hat{\Omega}^1_{B/A} = B \cdot \mathrm{d}(\mathfrak{b})$. After shrinking U sufficiently we get $\hat{\Omega}^1_{B/A} = \bigoplus_{i=1}^n B \cdot \mathrm{d}b_i$ with $b_i \in \mathfrak{b}$, and the homomorphism $A[[\underline{t}]] \to B$, $t_i \mapsto b_i$, is formally étale. Continuing like in part 1 of the proof we conclude that this is actually an isomorphism. ■

THEOREM 2.7. *Suppose S is a noetherian scheme of characteristic 0, and X is a finite type S-scheme. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of smooth formal embeddings of X. Then the* DGA *homomorphism $f^* : \hat{\Omega}^{\cdot}_{\mathfrak{Y}/S} \to \hat{\Omega}^{\cdot}_{\mathfrak{X}/S}$ is a quasi-isomorphism. Moreover, if $g : \mathfrak{X} \to \mathfrak{Y}$ is any other morphism, then $\mathrm{H}(f^*) = \mathrm{H}(g^*)$.*

PROOF. The assertions of the theorem are both local, and they will be proved in three steps.

STEP 1. Assume f is a closed immersion. By Theorem 2.6 it suffices to check the case $f\colon \operatorname{Spf} B = \mathfrak{U} \longrightarrow \operatorname{Spf} A = \mathfrak{V}$ with $A \cong B[[\underline{t}]]$ as topological L-algebras. We must show that $\hat{\Omega}^{\cdot}_{A/L} \longrightarrow \hat{\Omega}^{\cdot}_{B/L}$ is a quasi-isomorphism. But since $\mathbb{Q} \subset L$, this is the well known Poincaré Lemma for formal power series (*cf.* [Ha] Chapter II Proposition 1.1, or [Ye3] Lemma 7.5).

STEP 2. Suppose $f_1, f_2 \colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ are two morphisms. We wish to show that $\mathrm{H}(f_1^*) = \mathrm{H}(f_2^*)$. First consider

$$\mathfrak{Y} \xrightarrow{\text{diag}} (\mathfrak{Y} \times_k \mathfrak{Y})_{/X} \xrightarrow{p_i} \mathfrak{Y}.$$

Since the diagonal immersion is closed, we can apply the result of the previous paragraph to it. We conclude that $\mathrm{H}(p_1^*) = \mathrm{H}(p_2^*)$, and that these are isomorphisms. But looking at

$$\mathfrak{X} \xrightarrow{\text{diag}} (\mathfrak{X} \times_k \mathfrak{X})_{/X} \xrightarrow{f_1 \times f_2} (\mathfrak{Y} \times_k \mathfrak{Y})_{/X} \xrightarrow{p_i} \mathfrak{Y}$$

we see that our claim is proved.

STEP 3. Consider an arbitrary morphism $f\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$. Take any affine open set $U \subset X$, with corresponding affine formal schemes $\operatorname{Spf} B = \mathfrak{U} \subset \mathfrak{X}$ and $\operatorname{Spf} A = \mathfrak{V} \subset \mathfrak{Y}$. The definition of formal smoothness implies there is some morphism of embeddings $g\colon \mathfrak{V} \longrightarrow \mathfrak{U}$. This morphism will not necessarily be an inverse of $f|_{\mathfrak{U}}$, but nonetheless, according to Step 2, $\mathrm{H}(g^*)$ and $\mathrm{H}(f|_{\mathfrak{U}}^*)$ will be isomorphisms between $\mathrm{H}\hat{\Omega}^{\cdot}_{\mathfrak{U}/S}$ and $\mathrm{H}\hat{\Omega}^{\cdot}_{\mathfrak{V}/S}$, inverse to each other. ∎

In [Ha] the relative De Rham cohomology $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/S)$ was defined. In the situation of Example 2.2, where $X \subset Y$ is a smooth algebraic embedding of S-schemes, $\mathfrak{X} = Y_{/X}$ and $\pi\colon \mathfrak{X} \longrightarrow S$ is the structural morphism, the definition is $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/S) = \mathrm{H}^{\cdot}\mathrm{R}\pi_* \hat{\Omega}^{\cdot}_{\mathfrak{X}/S}$. Even if X is not globally embeddable, $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/S)$ can still be defined, by taking a system of local embeddings $\{U_i \subset V_i\}$, $X = \bigcup U_i$, and putting together a "Čech-De Rham" complex (*cf.* [Ha] pp. 28–29; it seems one should also assume X separated and the U_i are affine).

COROLLARY 2.8. *Suppose S has characteristic* 0. *Let $X \subset \mathfrak{X}$ be any smooth formal embedding (not necessarily algebraizable). Then* $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/S) = \mathrm{H}^{\cdot}\mathrm{R}\pi_* \hat{\Omega}^{\cdot}_{\mathfrak{X}/S}$ *as graded $\mathcal{O}_S$-algebras.*

PROOF. Assume for simplicity that a global smooth algebraic embedding exists. The general case, involving a system of embeddings, only requires more bookkeeping. Say $X \subset Y$ is the given algebraic embedding, and let $\mathfrak{Y} := Y_{/X}$. Now the two formal embeddings $\mathfrak{X}$ and $\mathfrak{Y}$ are comparable: their product $(\mathfrak{X} \times_S \mathfrak{Y})_{/X}$ maps to both. By the theorem we get quasi-isomorphic DGAs on X. ∎

REMARK 2.9. From Corollaries 2.5 and 2.8 we see that there is a sheaf of DGAs $\hat{\Omega}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}/S}$ on X_{et}, with the property that for any $U \longrightarrow X$ étale, $\mathrm{H}^{\cdot}_{\mathrm{DR}}(U/S) = \mathrm{H}^{\cdot}\Gamma(U, \hat{\Omega}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}/S})$. As will be shown in [Ye4], the DGA $\hat{\Omega}^{\cdot}_{\mathfrak{X}/S}$ has an adelic resolution $\mathcal{A}^{\cdot}_{\mathfrak{X}/S}$, where $\mathcal{A}^{p,q}_{\mathfrak{X}/S} = \underline{\mathbb{A}}^{q}_{\mathrm{red}}(\hat{\Omega}^{p}_{\mathfrak{X}/S})$, Beilinson's sheaf of adeles. The adeles calculate cohomology: $\mathrm{H}^{\cdot}_{\mathrm{DR}}(X/S) = \mathrm{H}^{\cdot}\Gamma(X, \mathcal{A}^{\cdot}_{\mathfrak{X}/S})$. Furthermore the adeles extend to an étale sheaf $\mathcal{A}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}/S}$.

REMARK 2.10. Suppose $S = \operatorname{Spec} k$, a field of characteristic 0. In [Ye3] a complex $\mathcal{F}^{\cdot}_{\mathfrak{X}}$, called the *De Rham-residue complex*, is defined. One has $\mathrm{H}^i(X, \mathcal{F}^{\cdot}_{\mathfrak{X}}) = \mathrm{H}^{\mathrm{DR}}_{-i}(X)$, the De Rham homology. Moreover there is a sheaf $\mathcal{F}^{\cdot}_{\mathfrak{X}_{\mathrm{et}}}$ on X_{et}, which directly implies that the De Rham homology is contravariant for étale morphisms. Furthermore $\mathcal{F}^{\cdot}_{\mathfrak{X}}$ is naturally a DG $\mathcal{A}^{\cdot}_{\mathfrak{X}}$-module,

REMARK 2.11. Smooth formal embeddings can be also used to define the category of $\mathcal{D}$-modules on a singular scheme X (in characteristic 0). Say $X \subset \mathfrak{X}$ is such an embedding. Then a formal version of Kashiwara's Theorem (*cf.* [Bo] Theorem VI.7.11) implies that $\mathsf{Mod}_{\mathrm{disc}}(\mathcal{D}_{\mathfrak{X}})$, the category of discrete modules over the ring of differential operators $\mathcal{D}_{\mathfrak{X}}$ is, as an abelian category, independent of $\mathfrak{X}$.

3. **Quasi-coherent sheaves on formal schemes.** Let $\mathfrak{X}$ be a noetherian (adic) formal scheme. By definition, a quasi-coherent sheaf on $\mathfrak{X}$ is an $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$, such that on sufficiently small open sets $\mathfrak{U} \subset \mathfrak{X}$ there are exact sequences $\mathcal{O}_{\mathfrak{U}}^{(J)} \longrightarrow \mathcal{O}_{\mathfrak{U}}^{(I)} \longrightarrow \mathcal{M}|_{\mathfrak{U}} \longrightarrow 0$, for some indexing sets I, J (*cf.* [EGA] 0_{I} Section 5.1). We shall denote by $\mathsf{Mod}(\mathfrak{X})$ (resp. $\mathsf{Coh}(\mathfrak{X})$, resp. $\mathsf{QCo}(\mathfrak{X})$) the category of $\mathcal{O}_{\mathfrak{X}}$-modules (resp. the full subcategory of coherent, resp. quasi-coherent, modules). It seems that the only important quasi-coherent sheaves are the coherent and the discrete ones (Definition 3.7). Nevertheless we shall consider all quasi-coherent sheaves, at the price of a little extra effort.

REMARK 3.1. There is some overlap between results in this section and [AJL2].

Let A be a noetherian adic ring, and let $\mathfrak{U} := \operatorname{Spf} A$ be the affine formal scheme. Then there is an exact functor $M \longmapsto M^{\triangle}$ from the category $\mathsf{Mod}_{\mathrm{f}}(A)$ of finitely generated A-modules to $\mathsf{Mod}(\mathfrak{U})$. It is an equivalence between $\mathsf{Mod}_{\mathrm{f}}(A)$ and $\mathsf{Coh}(\mathfrak{U})$ (see [EGA] I Section 10.10).

PROPOSITION 3.2. *The functor $M \longmapsto M^{\triangle}$ extends uniquely to a functor* $\mathsf{Mod}(A) \longrightarrow \mathsf{Mod}(\mathfrak{U})$*, which is exact and commutes with direct limits. The $\mathcal{O}_{\mathfrak{U}}$-module $M^{\triangle}$ is quasi-coherent. For any $\mathcal{O}_{\mathfrak{U}}$-module $\mathcal{M}$ the following are equivalent:*

(i) $\mathcal{M} \cong M^{\triangle}$ for some A-module M.

(ii) $\mathcal{M} \cong \lim_{\alpha \to} \mathcal{M}_{\alpha}$ for some directed system $\{\mathcal{M}_{\alpha}\}$ of coherent $\mathcal{O}_{\mathfrak{U}}$-modules.

(iii) For every affine open set $\mathfrak{V} = \operatorname{Spf} B \subset \mathfrak{U}$, one has $\Gamma(\mathfrak{V}, \mathcal{M}) \cong B \otimes_A \Gamma(\mathfrak{U}, \mathcal{M})$.

PROOF. Take any A-module M and write it as $M = \lim_{\alpha \to} M_{\alpha}$ with finitely generated modules M_{α}. Define a presheaf $M^{\triangle}$ on $\mathfrak{U}$ by $\Gamma(\mathfrak{V}, M^{\triangle}) := \lim_{\alpha \to} \Gamma(\mathfrak{V}, M_{\alpha}^{\triangle})$, for $\mathfrak{V} \subset \mathfrak{U}$ open. Since $\mathfrak{U}$ is a noetherian topological space it follows that $M^{\triangle}$ is actually a sheaf. By construction $M \longmapsto M^{\triangle}$ commutes with direct limits. Since the functor is exact on $\mathsf{Mod}_{\mathrm{f}}(A)$, it's also exact on $\mathsf{Mod}(A)$.

The implication (i) $\Rightarrow$ (ii) is because $M_{\alpha}^{\triangle}$ is coherent. (ii) $\Rightarrow$ (iii): for such B one has $\Gamma(\mathfrak{V}, \mathcal{M}_{\alpha}) \cong B \otimes_A \Gamma(\mathfrak{U}, \mathcal{M}_{\alpha})$; now apply $\lim_{\alpha \to}$. (iii) $\Rightarrow$ (i): set $M := \Gamma(\mathfrak{U}, \mathcal{M})$. Then for every affine $\mathfrak{V}$ we have $\Gamma(\mathfrak{V}, \mathcal{M}) = B \otimes_A M = \Gamma(\mathfrak{V}, M^{\triangle})$, so $\mathcal{M} = M^{\triangle}$.

Finally the module M has a presentation $A^{(I)} \longrightarrow A^{(J)} \longrightarrow M \longrightarrow 0$. By exactness we get a presentation for $M^{\triangle}$. ∎

It will be convenient to write $\mathcal{O}_{\mathfrak{U}} \otimes_A M$ instead of $M^{\triangle}$.

REMARK 3.3. I do not know whether Serre's Theorem holds, namely whether *every* quasi-coherent $\mathcal{O}_{\mathfrak{U}}$-module $\mathcal{M}$ is of the form $\mathcal{M} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A M$. Thus it may be that $\mathsf{QCo}(\mathfrak{U})$ is not closed under direct limits in $\mathsf{Mod}(\mathfrak{U})$ (*cf.* Lemma 4.1).

COROLLARY 3.4. *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module and $x \in \mathfrak{X}$ a point. Then there is an open neighborhood* $\mathfrak{U} = \operatorname{Spf} A$ *of* x s.t. $\mathcal{M}|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A \Gamma(\mathfrak{U}, \mathcal{M})$. *For such* $\mathfrak{U}$ *one has* $\mathrm{H}^1(\mathfrak{U}, \mathcal{M}) = 0$.

PROOF. Choose $\mathfrak{U}$ affine such that $\mathcal{M}|_{\mathfrak{U}}$ has a presentation $\mathcal{O}_{\mathfrak{U}}^{(J)} \xrightarrow{\phi} \mathcal{O}_{\mathfrak{U}}^{(I)} \xrightarrow{\psi} \mathcal{M}|_{\mathfrak{U}} \to 0$. Define $M := \operatorname{Coker}(\phi\colon A^{(I)} \to A^{(J)})$. Applying the exact functor $\mathcal{O}_{\mathfrak{U}} \otimes_A$ to $A^{(I)} \xrightarrow{\phi} A^{(J)} \to M \to 0$ we get $\mathcal{M}|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A M$. By the proposition $M \cong \Gamma(\mathfrak{U}, \mathcal{M})$. As for $\mathrm{H}^1(\mathfrak{U}, -)$, use the fact that it vanishes on coherent sheaves. ∎

PROPOSITION 3.5. *Let $\mathcal{M}$ be coherent and $\mathcal{N}$ quasi-coherent (resp. coherent). Then $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N})$ is quasi-coherent (resp. coherent).*

PROOF. For small enough $\mathfrak{U} = \operatorname{Spf} A$ we get $\mathcal{M}|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A M$ and $\mathcal{N}|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A N$. Now for any $\mathfrak{V} = \operatorname{Spf} B \subset \mathfrak{U}$, $A \to B$ is flat; so

$$\operatorname{Hom}_B(B \otimes_A M, B \otimes_A N) \cong B \otimes_A \operatorname{Hom}_A(M, N).$$

Hence

$$\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{M}, \mathcal{N})|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A \operatorname{Hom}_A(M, N). \qquad \blacksquare$$

Recall that a subcategory B of an abelian category A is called a *thick abelian subcategory* if for any exact sequence $M_1 \to M_2 \to N \to M_3 \to M_4$ in A with $M_i \in \mathsf{B}$, also $N \in \mathsf{B}$.

PROPOSITION 3.6. *The category* $\mathsf{QCo}(\mathfrak{X})$ *is a thick abelian subcategory of* $\mathsf{Mod}(\mathfrak{X})$.

PROOF. First observe that the kernel and cokernel of a homomorphism $\mathcal{M} \to \mathcal{N}$ between quasi-coherent sheaves is also quasi-coherent. This is immediate from Corollary 3.4 and Proposition 3.2. So it suffices to prove: $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ exact, $\mathcal{M}', \mathcal{M}''$ quasi-coherent $\Rightarrow$ $\mathcal{M}$ quasi-coherent. For a sufficiently small affine open formal subscheme $\mathfrak{U} = \operatorname{Spf} A$ we will get, by Corollary 3.4, that $\mathrm{H}^1(\mathfrak{U}, \mathcal{M}') = 0$. Hence the sequence

$$0 \to \Gamma(\mathfrak{U}, \mathcal{M}') \to M = \Gamma(\mathfrak{U}, \mathcal{M}) \to \Gamma(\mathfrak{U}, \mathcal{M}'') \to 0$$

is exact. This implies that $\mathcal{M}|_{\mathfrak{U}} \cong \mathcal{O}_{\mathfrak{U}} \otimes_A M$. ∎

DEFINITION 3.7. Let $\mathcal{M}$ be an $\mathcal{O}_{\mathfrak{X}}$-module. Define

$$\underline{\Gamma}_{\mathrm{disc}} \mathcal{M} := \lim_{n \to} \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{\mathfrak{X}} / \mathcal{I}^n, \mathcal{M}) \subset \mathcal{M}$$

where $\mathcal{I} \subset \mathcal{O}_{\mathfrak{X}}$ is any defining ideal. $\mathcal{M}$ is called *discrete* if $\underline{\Gamma}_{\mathrm{disc}} \mathcal{M} = \mathcal{M}$.

PROPOSITION 3.8. *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module. Then $\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}$ is quasi-coherent, and in fact is a direct limit of discrete coherent $\mathcal{O}_{\mathfrak{X}}$-modules.*

PROOF. Let X_n be as in formula (1.12) and $\mathcal{M}_n := \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{O}_{X_n}, \mathcal{M})$, so $\underline{\Gamma}_{\mathrm{disc}} \mathcal{M} = \lim_{n \to} \mathcal{M}_n$. If $\mathcal{M}$ is quasi-coherent, then $\mathcal{M}_n$ is a quasi-coherent $\mathcal{O}_{X_n}$-module (by Proposition 3.5), and hence is a direct limit of coherent modules. ∎

4. **Some derived functors of $\mathcal{O}_{\mathfrak{X}}$-modules.** Denote by $\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ (resp. $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$) the full subcategory of $\mathsf{Mod}(\mathfrak{X})$ consisting of discrete modules (resp. discrete quasi-coherent modules). These are thick abelian subcategories. In this section we study injective objects in the category $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$, and introduce the discrete Cousin functor $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}$.

LEMMA 4.1. $\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ *is a locally noetherian category, with enough injectives.*

PROOF. A family of noetherian generators consists of the sheaves $\mathcal{O}_U$, where $X \subset \mathfrak{X}$ is a closed subscheme, $U \subset X$ is an open set, and $\mathcal{O}_U$ is extended by 0 to all of X (*cf.* [RD] Theorem II.7.8). If $\mathcal{J} \in \mathsf{Mod}(\mathfrak{X})$ is injective then $\underline{\Gamma}_{\mathrm{disc}} \mathcal{J}$ is injective in $\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$. ∎

Given a point $x \in \mathfrak{X}$ let $J(x)$ be an injective hull of the residue field $k(x)$ over the local ring $\mathcal{O}_{\mathfrak{X},x}$, and let $\mathcal{J}(x)$ be the corresponding $\mathcal{O}_{\mathfrak{X}}$-module. Then $\mathcal{J}(x)$ is a discrete quasi-coherent sheaf, constant on $\overline{\{x\}}$, and it is injective in $\mathsf{Mod}(\mathfrak{X})$.

PROPOSITION 4.2. *1.* $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$ *is a locally noetherian category with enough injectives.*

2. Let $\mathcal{J} \in \mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$ *be an injective object. Then* $\mathcal{J}$ *is injective in* $\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ *and injective on* $\mathsf{Coh}(\mathfrak{X})$. *For any* $\mathcal{M} \in \mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ *or* $\mathcal{M} \in \mathsf{Coh}(\mathfrak{X})$ *the sheaf* $\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}, \mathcal{J})$ *is flasque.*

PROOF. 1. Let $\mathcal{N} \in \mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$. Choose a defining ideal $\mathcal{I}$ of $\mathfrak{X}$ and let X_0 be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{I})$. Define $\mathcal{N}_0 := \mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{X_0}, \mathcal{N})$, which is a quasi-coherent $\mathcal{O}_{X_0}$-module. Then the injective hull of $\mathcal{N}_0$ in $\mathsf{Mod}(X_0)$ is isomorphic to $\bigoplus_\alpha \mathcal{J}_0(x_\alpha)$ for some $x_\alpha \in X_0$. According to Proposition 3.8, $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$ is locally noetherian, and this implies that $\bigoplus_\alpha \mathcal{J}(x_\alpha)$ is an injective object in it. Now $\mathcal{N}_0 \subset \mathcal{N}$ and $\mathcal{N}_0 \subset \bigoplus_\alpha \mathcal{J}(x_\alpha)$ are essential submodules, so there is some homomorphism $\mathcal{N} \to \bigoplus_\alpha \mathcal{J}(x_\alpha)$, which is necessarily injective and essential.

2. If $\mathcal{N} = \mathcal{J}$ is injective in $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$, it follows that $\mathcal{J} \to \bigoplus_\alpha \mathcal{J}(x_\alpha)$ is an isomorphism. Since $\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ is locally noetherian it follows that $\mathcal{J}$ is injective in it. Given $\mathcal{M} \in \mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ and open sets $\mathfrak{V} \subset \mathfrak{U} \subset \mathfrak{X}$ consider the sheaves $\mathcal{M}|_{\mathfrak{V}} \subset \mathcal{M}|_{\mathfrak{U}} \subset \mathcal{M}$ (extension by 0). Then $\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}|_{\mathfrak{U}}, \mathcal{J}) \to \mathcal{H}om_{\mathfrak{X}}(\mathcal{M}|_{\mathfrak{V}}, \mathcal{J})$ is surjective.

The category $\mathsf{Coh}(\mathfrak{X})$ is noetherian, and therefore the functor $\mathcal{H}om_{\mathfrak{X}}(-, \mathcal{J})$ is exact on it. Given $\mathcal{M} \in \mathsf{Coh}(\mathfrak{X})$ we have $\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}, \mathcal{J}) \cong \bigoplus \mathcal{H}om_{\mathfrak{X}}\big(\mathcal{M}, \mathcal{J}(x_\alpha)\big)$ which is clearly flasque. ∎

COROLLARY 4.3. *Let* $\mathcal{J}^{\cdot} \in \mathsf{D}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big)$ *be a complex of injectives. Then for any* $\mathcal{M}^{\cdot} \in \mathsf{D}^{-}\big(\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})\big)$ *or* $\mathcal{M}^{\cdot} \in \mathsf{D}^{-}\big(\mathsf{Coh}(\mathfrak{X})\big)$ *one has*

$$\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{J}^{\cdot}) \cong \mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{J}^{\cdot})$$
$$\mathrm{R}\operatorname{Hom}_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{J}^{\cdot}) \cong \operatorname{Hom}_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{J}^{\cdot}) \cong \Gamma\big(\mathfrak{X}, \mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{J}^{\cdot})\big).$$

PROOF. The first equality follows from Proposition 4.2 (*cf.* [RD] Section I.6). Since each sheaf $\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^p, \mathcal{J}^q)$ is flasque we obtain the second equality. ∎

The functor $\underline{\Gamma}_{\mathrm{disc}} \colon \mathsf{Mod}(\mathfrak{X}) \longrightarrow \mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})$ has a derived functor

$$\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \colon \mathsf{D}^+\big(\mathsf{Mod}(\mathfrak{X})\big) \longrightarrow \mathsf{D}^+\big(\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})\big),$$

which is calculated by injective resolutions.

There is another way to compute cohomology with supports. Let t be an indeterminate. Define $\mathbf{K}^{\cdot}(t)$ to be the Koszul complex $\mathbb{Z}[t] \xrightarrow{\cdot t} \mathbb{Z}[t]$, in dimensions 0 and 1, and let $\mathbf{K}^{\cdot}_{\infty}(t) := \lim_{i\to} \mathbf{K}^{\cdot}(t^i)$. Given a sequence $\underline{t} = (t_1, \ldots, t_n)$ define $\mathbf{K}^{\cdot}_{\infty}(\underline{t}) := \mathbf{K}^{\cdot}_{\infty}(t_1) \otimes \cdots \otimes \mathbf{K}^{\cdot}_{\infty}(t_n)$, a complex of flat $\mathbb{Z}[\underline{t}]$-modules (in fact it's a commutative DGA). If A is a noetherian commutative ring and $\underline{a} = (a_1, \ldots, a_n) \in A^n$, then we write $\mathbf{K}^{\cdot}_{\infty}(\underline{a})$ instead of $\mathbf{K}^{\cdot}_{\infty}(\underline{t}) \otimes_{\mathbb{Z}[\underline{t}]} A$. Now suppose $\mathfrak{a} \subset A$ is an ideal, and $\underline{a}$ are generators of $\mathfrak{a}$. Then for any $M^{\cdot} \in \mathsf{D}^+\big(\mathsf{Mod}(A)\big)$ there is a natural isomorphism

$$\mathrm{R}\Gamma_{\mathfrak{a}} M^{\cdot} \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes M^{\cdot} \tag{4.4}$$

in $\mathsf{D}\big(\mathsf{Mod}(A)\big)$. We refer to [LS1], [Hg1] and [AJL1] for full details and proofs. For sheaves one has:

LEMMA 4.5. *Suppose $\underline{a} \in \Gamma(\mathfrak{U}, \mathcal{O}_{\mathfrak{U}})^n$ generates a defining ideal of the formal scheme $\mathfrak{U}$. Then for any $\mathcal{M}^{\cdot} \in \mathsf{D}^+\big(\mathsf{Mod}(\mathfrak{U})\big)$ there is a natural isomorphism*

$$\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot} \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \mathcal{M}^{\cdot}.$$

PROOF. Let $I := \mathcal{O}_{\mathfrak{U}} \cdot \underline{a}$. Then $\underline{\Gamma}_{\mathrm{disc}} = \underline{\Gamma}_I$, and we may use [AJL1] Lemma 3.1.1. ∎

PROPOSITION 4.6. *Let X be a noetherian scheme, $X_0 \subset X$ a closed subset, $\mathfrak{X} = X_{/X_0}$ and $g \colon \mathfrak{X} \longrightarrow X$ the completion morphism. Then for any $\mathcal{M}^{\cdot} \in \mathsf{D}^+_{\mathrm{qc}}\big(\mathsf{Mod}(X)\big)$ there is a natural isomorphism $g^* \mathrm{R}\underline{\Gamma}_{X_0} \mathcal{M}^{\cdot} \cong \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{M}^{\cdot}$. In particular for a single quasi-coherent sheaf $\mathcal{M}$ one has $g^* \underline{\Gamma}_{X_0} \mathcal{M} \cong \underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{M}$.*

PROOF. Let $\mathcal{M}^{\cdot} \longrightarrow \mathcal{J}^{\cdot}$ be a resolution by quasi-coherent injectives. Since g is flat we get

$$\phi \colon g^* \mathrm{R}\underline{\Gamma}_{X_0} \mathcal{M}^{\cdot} = g^* \underline{\Gamma}_{X_0} \mathcal{J}^{\cdot} \longrightarrow \underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{J}^{\cdot} \longrightarrow \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{J}^{\cdot} = \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{M}^{\cdot}.$$

Locally on any affine open $U \subset X$, with $U_0 = U \cap X_0$ and $\mathfrak{U} = U_{/U_0}$, we can find $\underline{a}$ in $\Gamma(U, \mathcal{O}_U)$ which define U_0. It's known that $\underline{\Gamma}_{U_0}(\mathcal{J}^{\cdot}|_U) \longrightarrow \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes (\mathcal{J}^{\cdot}|_U)$ is a quasi-isomorphism. Since g is flat we obtain quasi-isomorphisms

$$\phi|_{\mathfrak{U}} \colon g^* \underline{\Gamma}_{U_0}(\mathcal{J}^{\cdot}|_U) \longrightarrow g^*\big(\mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes (\mathcal{J}^{\cdot}|_U)\big) \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes g^*(\mathcal{J}^{\cdot}|_U) = \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} g^*(\mathcal{J}^{\cdot}|_U).$$

It follows that ϕ is an isomorphism. ∎

Denote by $\mathsf{D}^+_{\mathrm{d}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ the subcategory of complexes with discrete cohomologies.

LEMMA 4.7. *1. If $\mathcal{M}^{\cdot} \in \mathsf{D}^+_{\mathrm{d}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ then $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M} \longrightarrow \mathcal{M}$ is an isomorphism.*
2. If $\mathcal{M}^{\cdot} \in \mathsf{D}^+_{\mathrm{qc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ then $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M} \in \mathsf{D}^+_{\mathrm{qc}}\big(\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})\big)$.

PROOF. From Lemma 4.5 we see that the functor $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}$ has finite cohomological dimension. By way-out reasons (*cf.* [RD] Section I.7) we may assume $\mathcal{M}^{\cdot}$ is a single discrete (resp. quasi-coherent) sheaf. Then the claims are obvious (use Proposition 3.8 for 2). ∎

THEOREM 4.8. *The identity functor* $\mathsf{D}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big) \to \mathsf{D}^{+}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ *is an equivalence of categories. In particular any* $\mathcal{M}^{\cdot} \in \mathsf{D}^{+}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ *is isomorphic to a complex of injectives* $\mathcal{J}^{\cdot} \in \mathsf{D}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big)$.

PROOF. According to Lemma 4.7 we see that $\mathsf{D}^{+}_{\mathrm{qc}}\big(\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})\big) \to \mathsf{D}^{+}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ is an equivalence with quasi-inverse $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}$. Next, by Proposition 4.2 and by [RD] Proposition I.4.8, the functor $\mathsf{D}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big) \to \mathsf{D}^{+}_{\mathrm{qc}}\big(\mathsf{Mod}_{\mathrm{disc}}(\mathfrak{X})\big)$ is an equivalence. ∎

REMARK 4.9. In [AJL2] it is proved that $\mathsf{D}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big) \to \mathsf{D}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ is an equivalence, using the quasi-coherator functor.

Suppose there is a codimension function $d\colon \mathfrak{X} \to \mathbb{Z}$, *i.e.* a function satisfying $d(y) = d(x)+1$ whenever (x, y) is an immediate specialization pair. Then there is a filtration $\cdots \supset Z^{p} \supset Z^{p+1} \supset \cdots$ of $\mathfrak{X}$, with $Z^{p} := \{F \subset \mathfrak{X} \mid F \text{ closed}, d(F) \geq p\}$. Here $d(F) := \min\{d(x) \mid x \in F\}$. This filtration determines a Cousin functor

$$\mathrm{E}\colon \mathsf{D}^{+}\big(\mathsf{Ab}(\mathfrak{X})\big) \to \mathsf{C}^{+}\big(\mathsf{Ab}(\mathfrak{X})\big) \tag{4.10}$$

where C^{+} denotes the abelian category of bounded below complexes (*cf.* [RD] Section IV.1).

Given a point $x \in \mathfrak{X}$ and a sheaf $\mathcal{M} \in \mathsf{Ab}(\mathfrak{X})$ we let $\Gamma_{x}\mathcal{M} := (\underline{\Gamma}_{\overline{\{x\}}}\mathcal{M})_{x} \subset \mathcal{M}_{x}$. The derived functor $\mathrm{R}\Gamma_{x}\colon \mathsf{D}^{+}\big(\mathsf{Ab}(\mathfrak{X})\big) \to \mathsf{D}(\mathsf{Ab})$ is calculated by flasque sheaves. Let us write $\mathrm{H}^{q}_{x}\mathcal{M} := \mathrm{H}^{q}\mathrm{R}\Gamma_{x}\mathcal{M}$, the local cohomology, and let $i_{x}\colon \{x\} \to \mathfrak{X}$ be the inclusion.

According to [RD] Section IV.1 Motif F one has a natural isomorphism

$$\mathrm{E}^{p}\mathcal{M}^{\cdot} = \mathcal{H}^{p}_{Z^{p}/Z^{p+1}}\mathcal{M}^{\cdot} \cong \bigoplus_{d(x)=p} i_{x*}\mathrm{H}^{p}_{x}\mathcal{M}^{\cdot}. \tag{4.11}$$

Observe that if $\mathcal{M} \in \mathsf{D}^{+}\big(\mathsf{Mod}(\mathfrak{X})\big)$ then $\mathrm{E}\mathcal{M}^{\cdot} \in \mathsf{C}^{+}\big(\mathsf{Mod}(\mathfrak{X})\big)$ and $\mathrm{R}\Gamma_{x}\mathcal{M} \in \mathsf{D}^{+}\big(\mathsf{Mod}(\mathcal{O}_{\mathfrak{X},x})\big)$.

Unlike an ordinary scheme, on a formal scheme the topological support of a quasi-coherent sheaf does not coincide with its algebraic support. But for discrete sheaves these two notions of support do coincide. This suggests:

DEFINITION 4.12. Given $\mathcal{M} \in \mathsf{D}^{+}\big(\mathsf{Mod}(\mathfrak{X})\big)$ its *discrete Cousin complex* is $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\mathcal{M}^{\cdot}$.

THEOREM 4.13. *For any* $\mathcal{M}^{\cdot} \in \mathsf{D}^{+}_{\mathrm{qc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ *the complex* $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\mathcal{M}^{\cdot}$ *consists of discrete quasi-coherent sheaves. So we get a functor*

$$\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\colon \mathsf{D}^{+}_{\mathrm{qc}}\big(\mathsf{Mod}(\mathfrak{X})\big) \to \mathsf{C}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big).$$

PROOF. According to Theorem 4.8 we may assume $\mathcal{N}^{\cdot} = \mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\mathcal{M}^{\cdot}$ is in $\mathsf{D}^{+}\big(\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})\big)$. On any open formal subscheme $\mathfrak{U} = \operatorname{Spf} A$ we get $\mathcal{N}^{\cdot} = \mathcal{O}_{\mathfrak{U}} \otimes_{A} N^{\cdot}$, where $N^{q} = \Gamma(\mathfrak{U}, N^{q})$ (*cf.* Propositions 3.8 and 3.2). Then for $x \in \mathfrak{U}$,

$$\mathrm{R}\Gamma_{x}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\mathcal{M}^{\cdot} = \mathrm{R}\Gamma_{x}\mathcal{N}^{\cdot} = \mathrm{R}\Gamma_{\mathfrak{p}}N^{\cdot}_{\mathfrak{p}}$$

where $\mathfrak{p} \subset A$ is the prime ideal of x. Hence $\mathrm{H}^{q}_{x}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}}\mathcal{M}^{\cdot} = \mathrm{H}^{q}_{\mathfrak{p}}N^{\cdot}_{\mathfrak{p}}$ is $\mathfrak{p}$-torsion. So the sheaf corresponding to x in (4.11) is quasi-coherent and discrete. ∎

5. **Dualizing complexes on formal schemes.** In this section we propose a theory of duality on noetherian formal schemes. There is a fundamental difference between this theory and the duality theory on schemes, as developed in [RD]. A dualizing complex $\mathcal{R}^{\cdot}$ on a scheme X has coherent cohomology sheaves; this will not be true on a general formal scheme $\mathfrak{X}$, where $\mathrm{H}^{q}\mathcal{R}^{\cdot}$ are discrete quasi-coherent sheaves (Definition 5.2). We prove uniqueness of dualizing complexes (Theorem 5.6), and existence in some cases (Proposition 5.11 and Theorem 5.14).

Before we begin here is an instructive example due to J. Lipman.

EXAMPLE 5.1. Consider the ring $A = k[[t]]$ of formal power series over a field k. Let $\mathfrak{X} := \operatorname{Spf} A$, which has a single point. The modules A and $J = \mathrm{H}^{1}_{(t)}A$ both have finite injective dimension and satisfy $\operatorname{Hom}_{A}(A, A) = \operatorname{Hom}_{A}(J, J) = A$. Which one is a dualizing complex on $\mathfrak{X}$? We will see that J is the correct answer (Definition 5.2), and A is a "fake" dualizing complex (Theorem 5.14). The relevant relation between them is: $J = \mathrm{R}\Gamma_{\mathrm{disc}}A[1]$.

Suppose $\mathcal{N}^{\cdot} \in \mathsf{D}^{+}\big(\mathsf{Mod}(\mathfrak{X})\big)$. We say $\mathcal{N}^{\cdot}$ has finite injective dimension on $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$ if there is an integer q_0 s.t. for all $q > q_0$ and $\mathcal{M} \in \mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$, $\mathrm{H}^{q}\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}, \mathcal{N}^{\cdot}) = 0$.

DEFINITION 5.2. A *dualizing complex* on $\mathfrak{X}$ is a complex $\mathcal{R}^{\cdot} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ satisfying:

(i) $\mathcal{R}^{\cdot}$ has finite injective dimension on $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$.
(ii) The adjunction morphism $\mathcal{O}_{\mathfrak{X}} \to \mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{R}^{\cdot}, \mathcal{R}^{\cdot})$ is an isomorphism.
(iii) For some defining ideal $\mathcal{I}$ of $\mathfrak{X}$, $\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{\mathfrak{X}}/\mathcal{I}, \mathcal{R}^{\cdot})$ has coherent cohomology sheaves.

LEMMA 5.3. *Let* $\mathcal{N}^{\cdot} \in \mathsf{D}^{+}_{\mathrm{dqc}}\big(\mathsf{Mod}(\mathfrak{X})\big)$. *Then* $\mathcal{N}^{\cdot}$ *has finite injective dimension on* $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$ *iff it is isomorphic to a bounded complex of injectives in* $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$.

PROOF. Because of Theorem 4.8 and Corollary 4.3, the proof is just like [RD] Proposition I.7.6. ∎

In light of this, we can, when convenient, assume the dualizing complex $\mathcal{R}^{\cdot}$ is a bounded complex of discrete quasi-coherent injectives.

PROPOSITION 5.4. *Let* $\mathcal{R}^{\cdot}$ *be a dualizing complex on* $\mathfrak{X}$. *Then for any* $\mathcal{M}^{\cdot} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}\big(\mathsf{Mod}(\mathfrak{X})\big)$ *the morphism of adjunction*

$$\mathcal{M}^{\cdot} \to \mathrm{R}\mathcal{H}om_{\mathfrak{X}}\big(\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{R}^{\cdot}), \mathcal{R}^{\cdot}\big)$$

is an isomorphism.

PROOF. We can assume $\mathfrak{X}$ is affine, and so replace $\mathcal{M}^{\cdot}$ with a complex of coherent sheaves. By "way-out" arguments (*cf.* [RD] Section I.7) we reduce to the case $\mathcal{M}^{\cdot} = \mathcal{O}_{\mathfrak{X}}$, to which property (ii) applies. ∎

LEMMA 5.5. *Suppose $\mathcal{R}^{\cdot}$ is a dualizing complex on $\mathfrak{X}$. Let $\mathcal{I}$ be any defining ideal of $\mathfrak{X}$, and let X_0 be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{I})$. Then $\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{X_0}, \mathcal{R}^{\cdot})$ is a dualizing complex on X_0.*

PROOF. We can assume $\mathcal{R}^{\cdot}$ is a bounded complex of injectives in $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$, so $\mathcal{R}_0^{\cdot} := \mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{X_0}, \mathcal{R}^{\cdot})$ is a complex of injectives on X_0. Property (iii) implies that $\mathcal{R}_0^{\cdot}$ has coherent cohomology sheaves. Now

$$\mathcal{H}om_{X_0}(\mathcal{R}_0^{\cdot}, \mathcal{R}_0^{\cdot}) \cong \mathcal{H}om_{\mathfrak{X}}\big(\mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{X_0}, \mathcal{R}^{\cdot}), \mathcal{R}^{\cdot}\big) \cong \mathcal{O}_{X_0},$$

so $\mathcal{R}_0^{\cdot}$ is dualizing. ∎

THEOREM 5.6 (UNIQUENESS). *Suppose $\mathcal{R}^{\cdot}$ and $\tilde{\mathcal{R}}^{\cdot}$ are dualizing complexes and $\mathfrak{X}$ is connected. Then $\tilde{\mathcal{R}}^{\cdot} \cong \mathcal{R}^{\cdot} \otimes \mathcal{L}[n]$ in $\mathsf{D}\big(\mathsf{Mod}(\mathfrak{X})\big)$, for some invertible sheaf $\mathcal{L}$ and integer n.*

PROOF. We can assume both $\mathcal{R}^{\cdot}$ and $\tilde{\mathcal{R}}^{\cdot}$ are bounded complexes of injectives in $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$. Choose a defining ideal $\mathcal{I}$ and let X_m be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{I}^{m+1})$. Define a complex $\mathcal{R}_m^{\cdot} := \mathcal{H}om_{\mathfrak{X}}(\mathcal{O}_{X_m}, \mathcal{R}^{\cdot})$ and likewise $\tilde{\mathcal{R}}_m^{\cdot}$. These are dualizing complexes on X_m, so by [RD] Theorem IV.3.1 there is an isomorphism

$$\phi_m \colon \mathcal{R}_m^{\cdot} \otimes \mathcal{L}_m[n_m] \longrightarrow \tilde{\mathcal{R}}_m^{\cdot}$$

in $\mathsf{D}\big(\mathsf{Mod}(X_m)\big)$, for some invertible sheaf $\mathcal{L}_m$ and integer n_m. Writing $\mathcal{M}_m^{\cdot} := \mathcal{H}om_{X_m}(\mathcal{R}_m^{\cdot}, \tilde{\mathcal{R}}_m^{\cdot})$ we have $\mathcal{M}_m^{\cdot} \cong \mathcal{L}_m[n_m]$ in $\mathsf{D}\big(\mathsf{Mod}(X_m)\big)$. Now

$$\mathcal{M}_m^{\cdot} \cong \mathcal{H}om_{X_{m+1}}(\mathcal{H}om_{X_{m+1}}(\mathcal{O}_{X_m}, \mathcal{R}_{m+1}^{\cdot}), \mathcal{R}_{m+1}^{\cdot})) \otimes \mathcal{L}_{m+1}[n_{m+1}]$$

as complexes of $\mathcal{O}_{X_{m+1}}$-modules, so by the dualizing property of $\mathcal{R}_{m+1}^{\cdot}$ we deduce an isomorphism $\mathcal{M}_m^{\cdot} \cong \mathcal{O}_{X_m} \otimes \mathcal{L}_{m+1}[n_{m+1}]$ in $\mathsf{D}\big(\mathsf{Mod}(X_{m+1})\big)$. We conclude that $n_m = n_{m+1}$ and $\mathcal{L}_m \cong \mathcal{O}_{X_m} \otimes \mathcal{L}_{m+1}$. Set $n := n_m$ and $\mathcal{L} := \lim_{\leftarrow m} \mathcal{L}_m$.

Next, since $\mathcal{R}_m^q \subset \mathcal{R}_{m+1}^q$ and $\tilde{\mathcal{R}}_{m+1}^q$ is injective in $\mathsf{Mod}(X_{m+1})$, we see that $\mathcal{M}_{m+1}^q \longrightarrow \mathcal{M}_m^q$ is surjective for all q, m. Furthermore, $\mathrm{H}^q \mathcal{M}_{m+1}^{\cdot} \longrightarrow \mathrm{H}^q \mathcal{M}_m^{\cdot}$ is also surjective, since $\mathrm{H}^q \mathcal{M}_m^{\cdot} = \mathcal{L}_m$ or 0. Define

$$\mathcal{M}^{\cdot} := \mathcal{H}om_{\mathfrak{X}}(\mathcal{R}^{\cdot}, \tilde{\mathcal{R}}^{\cdot}) \cong \lim_{\leftarrow m} \mathcal{M}_m^{\cdot}.$$

According to [Ha] Corollary I.4.3 and Proposition I.4.4 it follows that $\mathrm{H}^q \mathcal{M}^{\cdot} = \lim_{\leftarrow m} \mathrm{H}^q \mathcal{M}_m^{\cdot}$. This implies that $\mathcal{H}om_{\mathfrak{X}}(\mathcal{R}^{\cdot} \otimes \mathcal{L}[n], \tilde{\mathcal{R}}^{\cdot})) \cong \mathcal{O}_{\mathfrak{X}}$ in $\mathsf{D}\big(\mathsf{Mod}(\mathfrak{X})\big)$, so by Corollary 4.3

$$\mathrm{H}^0\mathrm{Hom}_{\mathfrak{X}}(\mathcal{R}^{\cdot} \otimes \mathcal{L}[n], \tilde{\mathcal{R}}^{\cdot}) \cong \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}).$$

Choose a homomorphism of complexes $\phi \colon \mathcal{R}^{\cdot} \otimes \mathcal{L}[n] \longrightarrow \tilde{\mathcal{R}}^{\cdot}$ corresponding to $1 \in \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. Backtracking we see that for every m, ϕ induces a homomorphism $\mathcal{R}_m^{\cdot} \otimes \mathcal{L}[n] \longrightarrow \tilde{\mathcal{R}}_m^{\cdot}$ which represents ϕ_m in $\mathsf{D}\big(\mathsf{Mod}(X_m)\big)$. So $\phi = \lim_{m \rightarrow} \phi_m$ is a quasi-isomorphism. ∎

PROBLEM 5.7. Let $\mathcal{R}^{\cdot}$ be a dualizing complex. Is it true that the following conditions on $\mathcal{N}^{\cdot} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{dqc}}\big(\mathsf{Mod}\,(\mathfrak{X})\big)$ are equivalent?

(i) $\mathcal{N}^{\cdot} \cong \mathrm{R}\mathcal{H}om^{\cdot}_{\mathfrak{X}}(\mathcal{M}^{\cdot}, \mathcal{R}^{\cdot})$ for some $\mathcal{M}^{\cdot} \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}\big(\mathsf{Mod}\,(\mathfrak{X})\big)$.

(ii) For any $\mathcal{M}$ discrete coherent, $\mathrm{R}\mathcal{H}om^{\cdot}_{\mathfrak{X}}(\mathcal{M}, \mathcal{N}^{\cdot}) \in \mathsf{D}^{\mathrm{b}}_{\mathrm{c}}\big(\mathsf{Mod}\,(\mathfrak{X})\big)$.

Recall that for a point $x \in \mathfrak{X}$ we denote by $J(x)$ an injective hull of $k(x)$ over $\mathcal{O}_{\mathfrak{X},x}$, and $\mathcal{J}(x)$ is the corresponding quasi-coherent sheaf.

LEMMA 5.8. *Suppose $\mathcal{R}^{\cdot}$ is a dualizing complex on $\mathfrak{X}$. For any $x \in \mathfrak{X}$ there is a unique integer $d(x)$* s.t.

$$\mathrm{H}^q_x\mathcal{R}^{\cdot} \cong \begin{cases} J(x) & \text{if } q = d(x) \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore d is a codimension function.

PROOF. We can assume $\mathcal{R}^{\cdot}$ is a bounded complex of injectives in $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$. Then as seen before $\mathrm{H}^q_x\mathcal{R}^{\cdot} = \mathrm{H}^q\Gamma_x\mathcal{R}^{\cdot}$. Define schemes X_m and complexes $\mathcal{R}^{\cdot}_m$ like in the proof of Theorem 5.6. Since $\mathcal{R}^{\cdot}_m$ is dualizing it determines a codimension function d_m on X_m (*cf.* [RD] Chapter V Section 7). But the arguments used before show that $d_m = d_{m+1}$. Finally $\mathrm{H}^q\Gamma_x\mathcal{R}^{\cdot} = \lim_{m\to} \mathrm{H}^q\Gamma_x\mathcal{R}^{\cdot}_m$, and $\mathrm{H}^q\Gamma_x\mathcal{R}^{\cdot}_m \cong J_m(x)$, an injective hull of $k(x)$ over $\mathcal{O}_{X_m,x}$.

DEFINITION 5.9. A residual complex on the noetherian formal scheme $\mathfrak{X}$ is a dualizing complex $\mathcal{K}^{\cdot}$ which is isomorphic, as $\mathcal{O}_{\mathfrak{X}}$-module, to $\bigoplus_{x\in\mathfrak{X}} \mathcal{J}(x)$.

PROPOSITION 5.10. *Say $\mathcal{R}^{\cdot}$ is a dualizing complex on $\mathfrak{X}$. Let d be the codimension function above, and let* E *be the associated Cousin functor. Then $\mathcal{R}^{\cdot} \cong \mathrm{E}\mathcal{R}^{\cdot}$ in $\mathsf{D}\big(\mathsf{Mod}\,(\mathfrak{X})\big)$, and $\mathrm{E}\mathcal{R}^{\cdot}$ is a residual complex.*

PROOF. By Lemma 5.8 $\mathcal{R}^{\cdot}$ is a Cohen-Macaulay complex, in the sense of [RD] p. 247, Definition. So there exists some isomorphism $\mathcal{R}^{\cdot} \to \mathrm{E}\mathcal{R}^{\cdot}$ in $\mathsf{D}^{\mathrm{b}}\big(\mathsf{Mod}\,(\mathfrak{X})\big)$. ∎

To conclude this section we consider some situations where a dualizing complex exists. If $f: \mathfrak{X} \to \mathfrak{Y}$ is a morphism then $(\mathfrak{Y}, f_*\mathcal{O}_{\mathfrak{X}})$ is a ringed space, and $\bar{f}: \mathfrak{X} \to (\mathfrak{Y}, f_*\mathcal{O}_{\mathfrak{X}})$ is a morphism of ringed spaces.

PROPOSITION 5.11. *Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a formally finite morphism, and assume $\mathcal{K}^{\cdot}$ is a residual complex on $\mathfrak{Y}$. Then $\bar{f}^*\mathcal{H}om_{\mathfrak{Y}}(f_*\mathcal{O}_{\mathfrak{X}}, \mathcal{K}^{\cdot})$ is a residual complex on $\mathfrak{X}$.*

PROOF. Let $f_n: X_n \to Y_n$ be morphisms as in Lemma 1.13, and let $\mathcal{K}^{\cdot}_n := \mathcal{H}om_{\mathfrak{Y}}(\mathcal{O}_{Y_n}, \mathcal{K}^{\cdot})$. Since f_n is a finite morphism, $\bar{f}^*_n\mathcal{H}om_{Y_n}(f_{n*}\mathcal{O}_{X_n}, \mathcal{K}^{\cdot}_n)$ is a residual complex on X_n. As in the proof of Theorem 5.6,

$$\bar{f}^*\mathcal{H}om_{\mathfrak{Y}}(f_*\mathcal{O}_{\mathfrak{X}}, \mathcal{K}^{\cdot}) \cong \lim_{n\to} \bar{f}^*_n\mathcal{H}om_{Y_n}(f_{n*}\mathcal{O}_{X_n}, \mathcal{K}^{\cdot}_n)$$

is residual. ∎

EXAMPLE 5.12. Suppose $X_0 \subset X$ is closed, $\mathfrak{X} = X_{/X_0}$ and $g: \mathfrak{X} \to X$ is the completion morphism. Let $\mathcal{K}^{\cdot}$ be a residual complex on X. In this case $g = \bar{g}$, and by Proposition 4.6

$$g^* \mathcal{H}om_X(g_* \mathcal{O}_{\mathfrak{X}}, \mathcal{K}^{\cdot}) \cong \lim_{n \to} g^* \mathcal{K}^{\cdot}_n \cong g^* \underline{\Gamma}_{X_0} \mathcal{K}^{\cdot} \cong \underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{K}^{\cdot}$$

is a residual complex. We see that if $\mathcal{R}^{\cdot}$ is any dualizing complex on X then $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} g^* \mathcal{R}^{\cdot}$ is dualizing on $\mathfrak{X}$.

We call a formal scheme $\mathfrak{X}$ *regular* of all its local rings $\mathcal{O}_{\mathfrak{X},x}$ are regular.

LEMMA 5.13. *Suppose $\mathfrak{X}$ is a regular formal scheme. Then $d(x) := \dim \mathcal{O}_{\mathfrak{X},x}$ is a bounded codimension function on $\mathfrak{X}$.*

PROOF. Let $\mathfrak{U} = \operatorname{Spf} A \subset \mathfrak{X}$ be a connected affine open set. If $x \in \mathfrak{U}$ is the point corresponding to an open prime ideal $\mathfrak{p}$, then $\hat{A}_{\mathfrak{p}} \cong \hat{\mathcal{O}}_{\mathfrak{X},x}$. Therefore $A_{\mathfrak{p}}$ is a regular local ring. Now in the adic noetherian ring A any maximal ideal $\mathfrak{m}$ is open. Hence, by [Ma] Section 18 Lemma 5(III), A is a regular ring, of finite global dimension equal to its Krull dimension.

Now let $U := \operatorname{Spec} A$, so as a topological space, $\mathfrak{U} \subset U$ is the closed set defined by any defining ideal $I \subset A$. Since U is a regular scheme, $\mathcal{O}_U$ is a dualizing complex on it. The codimension function d' corresponding to $\mathcal{O}_U$ satisfies $d'(y) = \dim \mathcal{O}_{U,y}$. Thus $0 \leq d'(y) \leq \dim U$. But clearly $d|_{\mathfrak{U}} = d'|_{\mathfrak{U}}$. By covering $\mathfrak{X}$ with finitely many such $\mathfrak{U}$ this implies that d is a bounded codimension function. ∎

THEOREM 5.14. *Suppose $\mathfrak{X}$ is a regular formal scheme. Then $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{X}}$ is a dualizing complex on $\mathfrak{X}$.*

PROOF. By the proof of Theorem 4.13 and known properties of regular local rings, for any $x \in \mathfrak{X}$

$$\mathrm{H}^q_x \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{X}} \cong \mathrm{H}^q_{\mathfrak{m}_x} \hat{\mathcal{O}}_{\mathfrak{X},x} \cong \begin{cases} J(x) & \text{if } q = d(x) \\ 0 & \text{otherwise} \end{cases}$$

where $\mathfrak{m}_x \subset \hat{\mathcal{O}}_{\mathfrak{X},x}$ is the maximal ideal, and $J(x)$ is an injective hull of $k(x)$. Since d is bounded it follows that $\mathcal{K}^{\cdot} := \mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{X}}$ is a bounded complex of injectives in $\mathsf{QCo}_{\mathrm{disc}}(\mathfrak{X})$. Like in the proof of Proposition 5.10, $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{X}} \cong \mathcal{K}^{\cdot}$ in $\mathsf{D}(\mathsf{Mod}(\mathfrak{X}))$.

To complete the proof it suffices to show that for any affine open set $\mathfrak{U} = \operatorname{Spf} A \subset \mathfrak{X}$ the complex $\mathcal{K}^{\cdot}|_{\mathfrak{U}}$ is residual on $\mathfrak{U}$. Let $U := \operatorname{Spec} A$ and let $g: \mathfrak{U} \to U$ be the canonical morphism. Let $U_0 \subset U$ be the closed set $g(\mathfrak{U})$, so that $\mathfrak{U} \cong U_{/U_0}$. Define $\mathcal{K}^{\cdot}_U := \mathrm{E}\mathcal{O}_U$, which is a residual complex on U. Then according to Proposition 4.6

$$\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{U}} \cong g^* \mathrm{R}\underline{\Gamma}_{U_0} \mathcal{O}_U \cong g^* \underline{\Gamma}_{U_0} \mathcal{K}^{\cdot}_U.$$

As in Example 5.12 this is a dualizing complex, so $\mathcal{K}^{\cdot}|_{\mathfrak{U}} \cong \mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{O}_{\mathfrak{U}}$ is a residual complex. ∎

REMARK 5.15. According to [RD] Theorem VI.3.1, if $f: X \to Y$ is a finite type morphism between finite dimensional noetherian schemes, and if $\mathcal{K}^{\cdot}$ is a residual complex on Y, then there is a residual complex $f^{\triangle}\mathcal{K}^{\cdot}$ on X. Now suppose $f\colon \mathfrak{X} \longrightarrow \mathfrak{Y}$ is a f.f.t. morphism and $f_n\colon X_n \longrightarrow Y_n$ are like in Lemma 1.13. In the same fashion as in Proposition 5.11 we set $f^{\triangle}\mathcal{K}^{\cdot} := \lim_{n\to} f_n^{\triangle}\mathcal{K}_n^{\cdot}$. This is a residual complex on $\mathfrak{X}$. If f is formally proper then $\mathrm{Tr}_f = \lim_{n\to} \mathrm{Tr}_{f_n}$ induces a duality

$$\mathrm{R}f_*\mathcal{M}^{\cdot} \longrightarrow \mathrm{R}\mathcal{H}om_{\mathfrak{Y}}\big(\mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathfrak{X}}(\mathcal{M}^{\cdot}, f^{\triangle}\mathcal{K}^{\cdot}), \mathcal{K}^{\cdot}\big)$$

for every $\mathcal{M}^{\cdot} \in \mathsf{D}^{\mathrm{b}}\big(\mathsf{Coh}\,(\mathfrak{X})\big)$. The proofs are standard, given the results of this section.

6. **Construction of the complex $\mathcal{K}^{\cdot}_{X/S}$.** In this section we work over a regular noetherian base scheme S. We construct the relative residue complex $\mathcal{K}^{\cdot}_{X/S}$ on any finite type S-scheme X. The construction is explicit and does not rely on [RD].

Let A, B be complete local rings, with maximal ideals $\mathfrak{m}, \mathfrak{n}$. Recall that a local homomorphism $\phi\colon A \longrightarrow B$ is called *residually finitely generated* if the field extension $A/\mathfrak{m} \longrightarrow B/\mathfrak{n}$ is finitely generated. Denote by $\mathsf{Mod}_{\mathrm{disc}}(A)$ the category of $\mathfrak{m}$-torsion A-modules (equivalently, modules with 0-dimensional support).

Suppose $A[\underline{t}] = A[t_1, \ldots, t_n]$ is a polynomial algebra and $\mathfrak{p} \subset A[\underline{t}]$ is some maximal ideal. Then $A \longrightarrow B = \widehat{A[\underline{t}]}_{\mathfrak{p}}$ is formally smooth of relative dimension n and residually finite. Let $b_i \in B/\mathfrak{n}$ be the image of t_i and $\bar{q}_i \in (A/\mathfrak{m})[b_1, \ldots, b_{i-1}][t_i]$ the monic irreducible polynomial of b_i, of degree d_i. Choose a monic lifting $q_i \in A[t_1, \ldots, t_i]$. Then for a discrete A-module M one has

$$\mathrm{H}^n_{\mathfrak{p}}(\hat{\Omega}^n_{B/A} \otimes_A M) \cong \bigoplus_{1 \leq i_l} \bigoplus_{0 \leq j_l < d_l} \frac{t_1^{j_1} \cdots t_n^{j_n} \mathrm{d}t_1 \cdots \mathrm{d}t_n}{q_1^{i_1} \cdots q_n^{i_n}} \otimes M.$$

As in [Hg1] Section 7 define the Tate residue

$$\mathrm{res}_{t_1,\ldots,t_n;A,B}\colon \mathrm{H}^n_{\mathfrak{p}}(\hat{\Omega}^n_{B/A} \otimes_A M) \longrightarrow M \tag{6.1}$$

by the rule

$$\frac{t_1^{j_1} \cdots t_n^{j_n} \mathrm{d}t_1 \cdots \mathrm{d}t_n}{q_1^{i_1} \cdots q_n^{i_n}} \otimes m \longmapsto \begin{cases} m & \text{if } i_l = 1, j_l = d_l - 1 \\ 0 & \text{otherwise} \end{cases}$$

(*cf.* [Ta]). Observe that any residually finite homomorphism $A \longrightarrow C$ factors into some $A \longrightarrow B = \widehat{A[\underline{t}]}_{\mathfrak{p}} \longrightarrow C$.

THEOREM 6.2 (HUANG). *Consider the category* Loc *of complete noetherian local rings and residually finitely generated local homomorphisms. Then:*

1. *For any morphism $\phi\colon A \longrightarrow B$ in* Loc *there is a functor*

$$\phi_{\#}\colon \mathsf{Mod}_{\mathrm{disc}}(A) \longrightarrow \mathsf{Mod}_{\mathrm{disc}}(B).$$

For composable morphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ there is an isomorphism $(\psi\phi)_{\#} \cong \psi_{\#}\phi_{\#}$, and $(1_A)_{\#} \cong 1_{\mathsf{Mod}_{\mathrm{disc}}(A)}$. These data form a pseudofunctor on Loc *(cf. [Hg1] Definition 4.1).*

2. If $\phi: A \longrightarrow B$ is formally smooth of relative dimension q, and $n = \operatorname{rank} \hat{\Omega}^1_{B/A}$, then there is an isomorphism, functorial in $M \in \mathsf{Mod}_{\mathrm{disc}}(A)$,

$$\phi_\# M \cong \mathrm{H}^q_{\mathfrak{n}}(\hat{\Omega}^n_{B/A} \otimes_A M).$$

3. If $\phi: A \longrightarrow B$ is residually finite then there is an A-linear homomorphism, functorial in $M \in \mathsf{Mod}_{\mathrm{disc}}(A)$,

$$\mathrm{Tr}_\phi: \phi_\# M \longrightarrow M,$$

which induces an isomorphism $\phi_\# M \cong \mathrm{Hom}^{\mathrm{cont}}_A(B, M)$. For composable homomorphisms $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ one has $\mathrm{Tr}_{\psi\phi} = \mathrm{Tr}_\phi \mathrm{Tr}_\psi$ under the isomorphism of part 1.

4. If $B = \widehat{A[\underline{t}]}_{\mathfrak{p}}$ then $\mathrm{Tr}_\phi = \mathrm{res}_{t_1,\dots,t_n;A,B}$ under the isomorphism of part 2.

PROOF. Parts 1 and 2 are [Hg1] Theorem 6.12. Parts 3 and 4 follow from [Hg1] Section 7. ■

DEFINITION 6.3. Suppose L is a regular local ring of dimension q, with maximal ideal $\mathfrak{r}$. Given a homomorphism $\phi: L \longrightarrow A$ in Loc, define

$$\mathcal{K}(A/L) := \phi_\# \mathrm{H}^q_{\mathfrak{r}} L,$$

the *dual module of A relative to L*.

Since $\mathrm{H}^q_{\mathfrak{r}} L$ is an injective hull of the field $L/\mathfrak{r}$, it follows that $\mathcal{K}(A/L)$ is an injective hull of $A/\mathfrak{m}$ (*cf.* [Hg1] Corollary 3.10).

COROLLARY 6.4. *If $\psi: A \longrightarrow B$ is a residually finite homomorphism, then there is an A-linear homomorphism*

$$\mathrm{Tr}_\psi = \mathrm{Tr}_{B/A}: \mathcal{K}(B/L) \longrightarrow \mathcal{K}(A/L).$$

Given another such homomorphism $B \longrightarrow C$, one has $\mathrm{Tr}_{C/A} = \mathrm{Tr}_{B/A} \mathrm{Tr}_{C/B}$.

REMARK 6.5. One can show that when L is a perfect field, there is a functorial isomorphism between $\mathcal{K}(A/L) = \phi_\# L$ above and the dual module $\mathcal{K}(A)$ of [Ye2], which was defined via Beilinson completion algebras.

Suppose $\pi: \mathfrak{X} \longrightarrow S$ is a formally finite type (f.f.t.) formally smooth morphism. According to Proposition 1.11, $\mathfrak{X}$ is a regular formal scheme. When we write $n = \operatorname{rank} \hat{\Omega}^1_{\mathfrak{X}/S}$ we mean that n is a locally constant function $n: \mathfrak{X} \longrightarrow \mathbb{N}$.

LEMMA 6.6. *Given a* f.f.t. *morphism $\pi: \mathfrak{X} \longrightarrow S$ and a point $x \in \mathfrak{X}$, let $s := \pi(x)$, and define*

$$d_S(x) := \dim \hat{\mathcal{O}}_{S,s} - \operatorname{tr} \deg_{k(s)} k(x).$$

Then:

1. d_S is a codimension function.

2. If π is formally smooth then

$$d_S(x) = \dim \hat{\mathcal{O}}_{\mathfrak{X},x} - \operatorname{rank} \hat{\Omega}^1_{\mathfrak{X}/S}.$$

PROOF. We shall prove 2 first. Let $L := \hat{O}_{S,s}$ and $A := \hat{O}_{\mathfrak{X},x}$. By Proposition 1.11,

$$n := \operatorname{rank} \hat{\Omega}^1_{A/L} = \dim A - \dim L + \operatorname{tr\,deg}_{L/\mathfrak{r}} A/\mathfrak{m}.$$

We see that d_S is the codimension function associated with the dualizing complex $\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} O_{\mathfrak{X}}[n]$ (see Theorem 5.14).

As for 1, the property of being a codimension function is local. But locally there is always a closed immersion $\mathfrak{X} \subset \mathfrak{Y}$ with $\mathfrak{Y} \to S$ formally smooth. ■

We shall use the codimension function d_S by default.

DEFINITION 6.7. Let $\pi: \mathfrak{X} \to S$ be a formally finite type morphism. Given a point $x \in \mathfrak{X}$, consider $\phi: L = \hat{O}_{S,\pi(x)} \to A = \hat{O}_{\mathfrak{X},x}$, which is a morphism in Loc. Since L is a regular local ring, the dual module $\mathcal{K}(A/L)$ is defined. Let $\mathcal{K}_{\mathfrak{X}/S}(x)$ be the quasi-coherent sheaf which is constant on $\overline{\{x\}}$ with group of sections $\mathcal{K}(A/L)$, and define

$$\mathcal{K}^q_{\mathfrak{X}/S} := \bigoplus_{d_S(x)=q} \mathcal{K}_{\mathfrak{X}/S}(x).$$

In Theorem 6.14 we are going to prove that on the graded sheaf $\mathcal{K}^{\cdot}_{X/S}$ there is a canonical coboundary operator δ which makes it into residual complex.

DEFINITION 6.8. Let $f: \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes over S. Define a homomorphism of graded $O_{\mathfrak{Y}}$-modules $\mathrm{Tr}_f: f_* \mathcal{K}^{\cdot}_{\mathfrak{X}/S} \to \mathcal{K}^{\cdot}_{\mathfrak{Y}/S}$ as follows. If $x \in \mathfrak{X}$ is closed in its fiber and $y = f(x)$, then $A = \hat{O}_{\mathfrak{Y},y} \to B = \hat{O}_{\mathfrak{X},x}$ is a residually finite L-algebra homomorphism. The homomorphism $\mathrm{Tr}_{B/A}: \mathcal{K}(B/L) \to \mathcal{K}(A/L)$ of Corollary 6.4 gives a map of sheaves

$$\mathrm{Tr}_f: f_* \mathcal{K}_{\mathfrak{X}/S}(x) \to \mathcal{K}_{\mathfrak{Y}/S}(y).$$

If x is not closed in its fiber, we let Tr_f vanish on $f_* \mathcal{K}_{\mathfrak{X}/S}(x)$.

PROPOSITION 6.9. *1.* Tr_f *is functorial: if* $g: \mathfrak{Y} \to \mathfrak{Z}$ *is another morphism, then* $\mathrm{Tr}_{gf} = \mathrm{Tr}_g \mathrm{Tr}_f$.

2. If f is formally finite (see Definition 1.18), then Tr_f *induces an isomorphism of graded sheaves*

$$f_* \mathcal{K}^{\cdot}_{\mathfrak{X}/S} \cong \mathcal{H}om_{\mathfrak{Y}}(f_* O_{\mathfrak{X}}, \mathcal{K}^{\cdot}_{\mathfrak{Y}/S}).$$

3. If $g: \mathfrak{U} \to \mathfrak{X}$ is an open immersion, then there is a natural isomorphism $\mathcal{K}^{\cdot}_{\mathfrak{U}/S} \cong g^* \mathcal{K}^{\cdot}_{\mathfrak{X}/S}$.

PROOF. Part 3 is trivial. Part 1 is a consequence of Corollary 6.4. As for part 2, f is an affine morphism, and fibers of f are all finite, so all points of X are closed in their fibers. ■

Suppose $\underline{a} = (a_1, \ldots, a_n)$ is a sequence of elements in the noetherian ring A. Let us write $\tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a})$ for the subcomplex $\mathbf{K}^{\geq 1}_{\infty}(\underline{a})$, so we get an exact sequence

$$0 \to \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a}) \to \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \to A \to 0. \tag{6.10}$$

For any $M^{\cdot} \in \mathsf{D}^{+}\big(\mathsf{Mod}\,(A)\big)$ let $\mathcal{M}^{\cdot}$ be the complex of sheaves $\mathcal{O}_X \otimes M^{\cdot}$ on $X := \operatorname{Spec} A$, and let $U \subset X$ be the open set $\bigcup\{a_i \neq 0\}$. Then

$$\mathrm{R}\Gamma(U, \mathcal{M}^{\cdot}) \cong \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a})[1] \otimes M^{\cdot}$$

in $\mathsf{D}\big(\mathsf{Mod}\,(A)\big)$. In fact $\tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a}) \otimes \mathcal{O}_X$ is a shift by 1 of the Čech complex corresponding to the open cover of U.

LEMMA 6.11. *Let A be an adic noetherian ring and $M^{\cdot} \in \mathsf{D}^{+}\big(\mathsf{Mod}\,(A)\big)$. Define $\mathfrak{U} := \operatorname{Spf} A$ and $\mathcal{M}^{\cdot} := \mathcal{O}_{\mathfrak{U}} \otimes M^{\cdot}$.*

1. Let $x \in \mathfrak{U}$ with corresponding open prime ideal $\mathfrak{p} \subset A$. Suppose the sequence $\underline{a}$ generates $\mathfrak{p}$. Then

$$\mathrm{R}\Gamma_x \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot} \cong \mathrm{R}\Gamma_{\mathfrak{p}} M^{\cdot}_{\mathfrak{p}} \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes M^{\cdot}_{\mathfrak{p}}$$

in $\mathsf{D}^{+}\big(\mathsf{Mod}\,(A_{\mathfrak{p}})\big)$.

2. Suppose $y \in \mathfrak{U}$ is an immediate specialization of x, and its ideal $\mathfrak{q}$ has generators $\underline{a}, \underline{b}$. Then

$$\mathrm{R}\Gamma_x \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot} \cong \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{b})[1] \otimes M^{\cdot}_{\mathfrak{q}}$$

in $\mathsf{D}^{+}\big(\mathsf{Mod}\,(A_{\mathfrak{q}})\big)$.

3. Assume d is a codimension function on $\mathfrak{U}$. Then in the Cousin complex $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot}$ the map

$$\mathrm{H}^{d(x)}_x \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot} \to \mathrm{H}^{d(y)}_y \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \mathcal{M}^{\cdot}$$

is given by applying $\mathrm{H}^{d(y)}$ to

$$\big(\mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{b}) \to \mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{b})\big) \otimes M^{\cdot}_{\mathfrak{q}}.$$

PROOF. Part 1 follows immediately from formula (4.4). Parts 2 and 3 are true because $\operatorname{Spec}(A/\mathfrak{p})_{\mathfrak{q}} = \{\mathfrak{p}, \mathfrak{q}\}$. ∎

As a warm up for Theorem 6.14, here is:

PROPOSITION 6.12. *If $\pi: \mathfrak{X} \to S$ is formally smooth, with $n = \operatorname{rank} \hat{\Omega}^1_{\mathfrak{X}/S}$, then there is a canonical isomorphism of graded sheaves*

$$\mathcal{K}^{\cdot}_{\mathfrak{X}/S} \cong \mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{X}/S}[n].$$

This makes $\mathcal{K}^{\cdot}_{\mathfrak{X}/S}$ into a residual complex.

PROOF. Take any point x, and with the notation of Definition 6.7 let $p := \dim L$ and $q := \dim A$. Then by Lemma 6.11 part 1 and [Hg1] Proposition 2.6 we have a canonical isomorphism

$$\mathrm{H}^{d(x)}_x \mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{X}/S}[n] \cong \mathrm{H}^q_{\mathfrak{m}} \hat{\Omega}^n_{A/L} \cong \mathrm{H}^{q-p}_{\mathfrak{m}}(\hat{\Omega}^n_{A/L} \otimes_L \mathrm{H}^p_{\mathfrak{r}} L) \cong \mathcal{K}(A/L).$$

According to Theorem 5.14 and Proposition 5.10, $\mathrm{E}\mathrm{R}\underline{\Gamma}_{\mathrm{disc}} \hat{\Omega}^n_{\mathfrak{X}/S}[n]$ is a residual complex. ∎

In particular taking $\mathfrak{X} = S$ we get $\mathcal{K}^{\cdot}_{S/S} = \mathrm{E}\mathcal{O}_S$.

LEMMA 6.13. *Suppose $X \subset \mathfrak{X}$ and $X \subset \mathfrak{Y}$ are* s.f.e.*'s and $f: \mathfrak{X} \to \mathfrak{Y}$ is a morphism of embeddings. Then $\operatorname{Tr}_f: \mathcal{K}^{\cdot}_{\mathfrak{X}} \to \mathcal{K}^{\cdot}_{\mathfrak{Y}}$ is a homomorphism of complexes.*

PROOF. Factoring f through $(\mathfrak{X} \times_S \mathfrak{Y})_{/X}$ we can assume that f is either a closed immersion, or that it is formally smooth. At any rate f is an affine morphism, so we can take $\mathfrak{X} = \operatorname{Spf} B$, $\mathfrak{Y} = \operatorname{Spf} A$ and $S = \operatorname{Spec} L$. By Theorem 2.6 we can suppose one of the following holds: (i) $B \cong A[[\underline{t}]]$ for a sequence of indeterminates $\underline{t} = (t_1, \dots, t_l)$, and $A \to B$ is the inclusion; or (ii) $A \cong B[[\underline{t}]]$ and $A \to B$ is the projection modulo $\underline{t}$. We shall treat each case separately.

(i) Choose generators $\underline{a}$ for a defining ideal of A. Let $m := \operatorname{rank} \hat{\Omega}^1_{A/L}$ and $n := \operatorname{rank} \hat{\Omega}^1_{B/L}$, so $n = m + l$. Define an A-linear map $\rho \colon \mathbf{K}^{\cdot}_{\infty}(\underline{t}) \otimes \hat{\Omega}^l_{B/A}[l] \to A$ by $\rho(\underline{t}^{(-1,\dots,-1)}\mathrm{d}\underline{t}) = 1$ and $\rho(\underline{t}^{\underline{i}}\mathrm{d}\underline{t}) = 0$ if $\underline{i} \neq (-1, \dots, -1)$. Extend ρ linearly to

$$\rho \colon \mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{t}) \otimes \hat{\Omega}^n_{B/L}[n] \to \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \hat{\Omega}^m_{A/L}[m].$$

This ρ sheafifies to give a map of complexes in $\mathsf{Ab}(X)$

$$\tilde{\rho} \colon \mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{t}) \otimes \hat{\Omega}^n_{\mathfrak{X}/S}[n] \to \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \hat{\Omega}^m_{\mathfrak{Y}/S}[m].$$

By Lemma 6.11 and [Hg1] Section 5, for any point $x \in X$, $\mathrm{H}^{d(x)}_x(\tilde{\rho})$ recovers $\mathrm{Tr}_f \colon \mathcal{K}_{\mathfrak{X}/S}(x) \to \mathcal{K}_{\mathfrak{Y}/S}(x)$. Thus $\mathrm{Tr}_f = \mathrm{E}(\tilde{\rho})$ is a homomorphism of complexes.

(ii) Now $l = m - n$. Take $\underline{a}$ to be generators of a defining ideal of B. Define a B-linear map $\rho' \colon B \to \mathbf{K}^{\cdot}_{\infty}(\underline{t}) \otimes \hat{\Omega}^l_{A/L}[l]$ by $\rho'(1) = \underline{t}^{(-1,\dots,-1)}\mathrm{d}\underline{t}$. Extend ρ linearly to

$$\rho' \colon \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \hat{\Omega}^n_{B/L}[n] \to \mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{t}) \otimes \hat{\Omega}^m_{A/L}[m].$$

Again this extends to a map of complexes of sheaves $\tilde{\rho}'$ in $\mathsf{Ab}(X)$, and checking punctually we see that $\mathrm{Tr}_f = \mathrm{E}(\tilde{\rho}')$. ∎

THEOREM 6.14. *Suppose $X \to S$ is a finite type morphism. There is a unique operator $\delta \colon \mathcal{K}^q_{X/S} \to \mathcal{K}^{q+1}_{X/S}$, satisfying the following local condition:*

(LE) *Suppose $U \subset X$ is an open subset, and $U \subset \mathfrak{U}$ is a smooth formal embedding. By Proposition 6.9 there is an inclusion of graded $\mathcal{O}_U$-modules $\mathcal{K}^{\cdot}_{X/S}|_U \subset \mathcal{K}^{\cdot}_{\mathfrak{U}/S}$. Then $\delta|_U$ is compatible with the coboundary operator on $\mathcal{K}^{\cdot}_{\mathfrak{U}/S}$ coming from Proposition 6.12.*

Moreover $(\mathcal{K}^{\cdot}_{X/S}, \delta)$ is a residual complex on X.

PROOF. Define $\delta|_U$ using (LE). According to Lemma 6.13, $\delta|_U$ is independent of $\mathfrak{U}$, so it glues. We get a bounded complex of quasi-coherent injectives on X. By Proposition 6.12 it follows that it is residual. ∎

REMARK 6.15. This construction of $\mathcal{K}^{\cdot}_{X/S}$ actually allows a computation of the operator δ, given the data of a local embedding. The formula is in part 3 of Lemma 6.11, with $M^{\cdot} = \hat{\Omega}^n_{A/L}[n]$. The formula for changing the embedding can be extracted from the proof of Lemma 6.13. Of course when $\operatorname{rank} \hat{\Omega}^1_{\mathfrak{X}/S}$ is high these computations can be nasty.

REMARK 6.16. The recent papers [Hg2], [Hg3] and [LS2] also use the local theory of [Hg1] as a starting point for explicit constructions of Grothendieck Duality. Their constructions are more general than ours: Huang constructs $f^! \mathcal{M}^{\cdot}$ for a finite type morphism $f: X \longrightarrow Y$ and a residual complex complex $\mathcal{M}^{\cdot}$; and Lipman-Sastry even allow $\mathcal{M}^{\cdot}$ to be any Cousin complex.

7**. The trace for finite morphisms.** In this section we prove that Tr_f is a homomorphism of complexes when f is a finite morphism. The proof is by a self contained calculation involving Koszul complexes and a comparison of global and local Tate residue maps. In Theorem 7.10 we compare the complex $\mathcal{K}^{\cdot}_{X/S}$ to the sheaf of regular differentials of Kunz-Waldi. Throughout S is a regular noetherian scheme.

THEOREM 7.1. *Suppose $f: X \longrightarrow Y$ is finite. Then $\mathrm{Tr}_f: f_* \mathcal{K}^{\cdot}_{X/S} \longrightarrow \mathcal{K}^{\cdot}_{Y/S}$ is a homomorphism of complexes.*

The proof appears after some preparatory work, based on and inspired by [Hg1] Section 7.

REMARK 7.2. In Section 8 we prove a much stronger result, namely Corollary 8.3, but its proof is indirect and relies on the Residue Theorem of [RD] Chapter VII. We have decided to include Theorem 7.1 because of its direct algebraic proof.

Let A be an adic noetherian ring with defining ideal $\mathfrak{a}$. Suppose $p \in A[t]$ is a monic polynomial of degree $e > 0$. Define an A-algebra

$$B := \lim_{\leftarrow i} A[t] / A[t] \cdot p^i. \tag{7.3}$$

Let $\mathfrak{b} := B\mathfrak{a} + Bp$; then $B \cong \lim_{\leftarrow i} B / \mathfrak{b}^i$, so that B is an adic ring with the $\mathfrak{b}$-adic topology. The homomorphism $\phi: A \longrightarrow B$ is f.f.t. and formally smooth, and $\hat{\Omega}^1_{B/A} = B \cdot \mathrm{d}t$. Furthermore $p \in B$ is a non-zero-divisor, and by long division we obtain an isomorphism

$$\mathrm{H}^1_{(p)} B = \mathrm{H}^1\big(\mathbf{K}^{\cdot}_{\infty}(p) \otimes B\big) \cong \bigoplus_{1 \leq i} \bigoplus_{0 \leq j < e} A \cdot \frac{t^j}{p^i}. \tag{7.4}$$

Define an A-linear homomorphism $\mathrm{Res}_{B/A}: \mathrm{H}^1_{(p)} \hat{\Omega}^1_{B/A} \longrightarrow A$ by

$$\mathrm{Res}_{B/A}\Big(\frac{t^j \mathrm{d}t}{p^i}\Big) := \begin{cases} 1 & \text{if } i = 1, j = e - 1 \\ 0 & \text{otherwise.} \end{cases}$$

We call $\mathrm{Res}_{B/A}$ the *global Tate residue*. It gives rise to a map of complexes in $\mathsf{Mod}\,(A)$:

$$\mathrm{Res}_{B/A}: \mathbf{K}^{\cdot}_{\infty}(p)[1] \otimes \hat{\Omega}^1_{B/A} \longrightarrow A. \tag{7.5}$$

Note that both the algebra B and the map $\mathrm{Res}_{B/A}$ depend on t and p.

Suppose $\mathfrak{q} \subset B$ is an open prime ideal and $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \subset A$. Then the local homomorphism $\phi_{\mathfrak{q}}: \hat{A}_{\mathfrak{p}} \longrightarrow \hat{B}_{\mathfrak{q}}$ is formally smooth of relative dimension 1 and residually finite.

Let $\tilde{\mathfrak{q}} := \mathfrak{q} \cap \hat{A}_{\mathfrak{p}}[t]$, and denote by $\bar{\mathfrak{q}}$ the image of $\tilde{\mathfrak{q}}$ in $k(\mathfrak{p})[t]$, so $k(\mathfrak{p})[t]/\bar{\mathfrak{q}} = k(\mathfrak{q})$. For a polynomial $q \in \hat{A}_{\mathfrak{p}}[t]$ let $\bar{q}$ be its image in $k(\mathfrak{p})[t]$. Suppose q satisfies:

$$q \text{ is monic, and the ideal } (\bar{q}) \subset k(\mathfrak{p})[t] \text{ is } \bar{\mathfrak{q}}\text{-primary.} \tag{7.6}$$

Then $\hat{B}_{\mathfrak{q}} \cdot \mathfrak{q} = \sqrt{\hat{B}_{\mathfrak{q}} \cdot (\mathfrak{p}, q)} \subset \hat{B}_{\mathfrak{q}}$, and

$$\hat{B}_{\mathfrak{q}} \cong \lim_{\leftarrow i} \hat{A}_{\mathfrak{p}}[t]/\tilde{\mathfrak{q}}^i \cong \lim_{\leftarrow i} \hat{A}_{\mathfrak{p}}[t]/\hat{A}_{\mathfrak{p}}[t] \cdot q^i.$$

Hence q is a non-zero-divisor in $\hat{B}_{\mathfrak{q}}$ and $\hat{B}_{\mathfrak{q}}/\hat{B}_{\mathfrak{q}} \cdot q$ is a free $\hat{A}_{\mathfrak{p}}$-module with basis $1, t, \dots, t^{d-1}$, where $d = \deg q$. We see that a decomposition like (7.4) exists for $\mathrm{H}^1_{(q)}\hat{B}_{\mathfrak{q}}$.

Suppose we are given a discrete $\hat{A}_{\mathfrak{p}}$-module M. Then one gets

$$\mathrm{H}^1_{\mathfrak{q}}(\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}} \otimes_{\hat{A}_{\mathfrak{p}}} M) \cong (\mathrm{H}^1_{(q)}\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}) \otimes_{\hat{A}_{\mathfrak{p}}} M \cong \bigoplus_{1 \le i} \bigoplus_{0 \le j < d} \frac{t^j \mathrm{d}t}{q^i} \otimes M$$

(*cf.* [Hg1] pp. 41–42). Define the *local Tate residue map*

$$\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}} \colon \mathrm{H}^1_{\mathfrak{q}}(\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}} \otimes_{\hat{A}_{\mathfrak{p}}} M) \longrightarrow M$$

by

$$\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}\Big(\frac{t^j \mathrm{d}t \otimes m}{q^i}\Big) := \begin{cases} m & \text{if } i = 1, j = d-1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$ is functorial in M, and it depends on t.

LEMMA 7.7. $\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$ *is independent of* q*. It coincides with the residue map* $\mathrm{res}_{t;\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}}}$ *of (6.1), i.e. of [Hg1] Definition 8.1.*

PROOF. Suppose the polynomials $q_1, q_2 \in \hat{A}_{\mathfrak{p}}[t]$ satisfy (7.6). Then so does $q_3 := q_1 q_2$. Let $\deg q_h = d_h$, and let $\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_h}$ be the residue map determined by q_h. Pick any $1 \le i$ and $0 \le j < d_1$, and write $q_2^i = \sum_{l=0}^{id_2} a_l t^l$, so $a_{id_2} = 1$. By the rules for manipulating generalized fractions (*cf.* [Hg1] Section 1) we have

$$\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_3}\Big(\frac{t^j \mathrm{d}t \otimes m}{q_1^i}\Big) = \sum_{l=0}^{id_2} \mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_3}\Big(\frac{t^{l+j} \mathrm{d}t \otimes a_l m}{q_3^i}\Big). \tag{7.8}$$

If $i \ge 2$ or $j \le d_1 - 2$ one has $l + j \le id_3 - 2$, and therefore each summand of the right side of (7.8) is 0. When $i = 1$ and $j = d_1 - 1$ the only possible nonzero residue there is for $l = d_2$, and this residue is m. We conclude that $\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_3} = \mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_1}$. Clearly also $\mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_3} = \mathrm{Res}_{\hat{B}_{\mathfrak{q}}/\hat{A}_{\mathfrak{p}};q_2}$.

If we take q such that $(\bar{q}) = \bar{\mathfrak{q}}$, this is by definition the residue map of (6.1). ∎

LEMMA 7.9. *Let F be the set of prime ideals in $B/(p)$ lying over $\mathfrak{p}$. Then for any $M \in \mathsf{Mod}_{\mathrm{disc}}(\hat{A}_{\mathfrak{p}})$ one has*

$$(\mathrm{H}^1_{(p)}\hat{\Omega}^1_{B/A}) \otimes_A M \cong \bigoplus_{\mathfrak{q}' \in F} \mathrm{H}^1_{\mathfrak{q}'}(\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}'}/\hat{A}_{\mathfrak{p}}} \otimes_{\hat{A}_{\mathfrak{p}}} M),$$

and w.r.t. *this isomorphism,*

$$\mathrm{Res}_{B/A} \otimes 1 = \sum_{\mathfrak{q}' \in F} \mathrm{Res}_{\hat{B}_{\mathfrak{q}'}/\hat{A}_{\mathfrak{p}}}.$$

PROOF. The isomorphism of modules is not hard to see. Let $\bar{p} = \prod_{\mathfrak{q}' \in F} \bar{p}_{\mathfrak{q}'}$ be the primary decomposition in $k(\mathfrak{p})[t]$ (all the $\bar{p}_{\mathfrak{q}'}$ monic). By Hensel's Lemma this decomposition lifts to $p = \prod_{\mathfrak{q}' \in F} p_{\mathfrak{q}'}$ in $\hat{A}_{\mathfrak{p}}[t]$. Since each polynomial $p_{\mathfrak{q}'}$ satisfies condition (7.6) for the prime ideal $\mathfrak{q}'$, we can use it to calculate $\operatorname{Res}_{\hat{B}_{\mathfrak{q}'}/\hat{A}_{\mathfrak{p}}}$. ∎

PROOF OF THEOREM 7.1. This claim is local on Y, so we may assume X, Y and S are affine, say $X = \operatorname{Spec} \bar{B}$, $Y = \operatorname{Spec} \bar{A}$ and $S = \operatorname{Spec} L$. By the functoriality of Tr we can assume $\bar{B} = \bar{A}[b]$ for some element $b \in \bar{B}$. It will suffice to find suitable s.f.e.'s $X \subset \mathfrak{X}$ and $Y \subset \mathfrak{Y}$ with a morphism $\hat{f}: \mathfrak{X} \to \mathfrak{Y}$ extending f, and to check that $\operatorname{Tr}_{\hat{f}}: \hat{f}_* \mathcal{K}^{\cdot}_{\mathfrak{X}/S} \to \mathcal{K}^{\cdot}_{\mathfrak{Y}/S}$ commutes with δ.

Pick any s.f.e. $Y \subset \mathfrak{Y} = \operatorname{Spf} A$, so $\mathfrak{a} := \operatorname{Ker}(A \to \bar{A})$ is a defining ideal. Let $A[t] \to \bar{B}$ be the homomorphism $t \mapsto b$. Choose any monic polynomial $p(t) \in A[t]$ s.t. $p(b) = 0$, and define the adic ring B as in formula (7.3). So $\mathfrak{X} := \operatorname{Spf} B$ is the s.f.e. of X we want.

Let (y_0, y_1) be an immediate specialization pair in Y, and let $F_i := f^{-1}(y_i) \subset X$. Let $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset A$ be the prime ideals corresponding to (y_0, y_1). Pick a sequence of generators $\underline{a}$ for $\mathfrak{p}_0$, and generators $(\underline{a}, \underline{a}')$ for $\mathfrak{p}_1$. Let $m := \operatorname{rank} \hat{\Omega}^1_{A/L}$.

Consider the commutative diagram of complexes

$$\begin{array}{ccc}
\mathbf{K}^{\cdot}_{\infty}(\underline{a}, p)[1] \otimes \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a}') \otimes (\hat{\Omega}^{m+1}_{B/L})_{\mathfrak{p}_1} & \xrightarrow{\operatorname{Res}_{B/A} \otimes 1} & \mathbf{K}^{\cdot}_{\infty}(\underline{a}) \otimes \tilde{\mathbf{K}}^{\cdot}_{\infty}(\underline{a}') \otimes (\hat{\Omega}^{m}_{A/L})_{\mathfrak{p}_1} \\
\downarrow & & \downarrow \\
\mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{a}', p)[1] \otimes (\hat{\Omega}^{m+1}_{B/L})_{\mathfrak{p}_1} & \xrightarrow{\operatorname{Res}_{B/A} \otimes 1} & \mathbf{K}^{\cdot}_{\infty}(\underline{a}, \underline{a}') \otimes (\hat{\Omega}^{m}_{A/L})_{\mathfrak{p}_1}
\end{array}$$

gotten from tensoring the map $\operatorname{Res}_{B/A}$ of (7.5) with $A_{\mathfrak{p}_1} \otimes \hat{\Omega}^m_{A/L}$ and the various $\mathbf{K}^{\cdot}_{\infty}$. Applying H^i to this diagram, where $i := \dim \hat{A}_{\mathfrak{p}_1}$, and using Lemmas 6.11 and 7.9 we obtain a commutative diagram

$$\begin{array}{ccc}
\bigoplus_{\mathfrak{q}_0 \in F_0} \mathrm{H}^1_{\mathfrak{q}_0}(\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}_0}/\hat{A}_{\mathfrak{p}_0}} \otimes \mathrm{H}^{i-1}_{\mathfrak{p}_0} \hat{\Omega}^m_{\hat{A}_{\mathfrak{p}_0}/L}) & \xrightarrow{\sum \operatorname{Res}} & \mathrm{H}^{i-1}_{\mathfrak{p}_0} \hat{\Omega}^m_{\hat{A}_{\mathfrak{p}_0}/L} \\
\downarrow & & \downarrow \\
\bigoplus_{\mathfrak{q}_1 \in F_1} \mathrm{H}^1_{\mathfrak{q}_1}(\hat{\Omega}^1_{\hat{B}_{\mathfrak{q}_1}/\hat{A}_{\mathfrak{p}_1}} \otimes \mathrm{H}^{i}_{\mathfrak{p}_1} \hat{\Omega}^m_{\hat{A}_{\mathfrak{p}_1}/L}) & \xrightarrow{\sum \operatorname{Res}} & \mathrm{H}^{i}_{\mathfrak{p}_1} \hat{\Omega}^m_{\hat{A}_{\mathfrak{p}_1}/L}.
\end{array}$$

In this diagram $\operatorname{Res} = \operatorname{Res}_{\hat{B}_{\mathfrak{q}_0}/\hat{A}_{\mathfrak{p}_0}}$ *etc.* Using the definitions this is the same as

$$\begin{array}{ccc}
\bigoplus_{x_0 \in F_0} f_* \mathcal{K}_{\mathfrak{X}/S}(x_0) & \xrightarrow{\operatorname{Tr}_f} & \mathcal{K}_{\mathfrak{Y}/S}(y_0) \\
{\scriptstyle\delta}\downarrow & & {\scriptstyle\delta}\downarrow \\
\bigoplus_{x_1 \in F_1} f_* \mathcal{K}_{\mathfrak{X}/S}(x_1) & \xrightarrow{\operatorname{Tr}_f} & \mathcal{K}_{\mathfrak{Y}/S}(y_1).
\end{array}$$

∎

According to [KW], if $\pi: X \to S$ is equidimensional of dimension n and generically smooth, and X is integral, then the *sheaf of regular differentials* $\tilde{\omega}^n_{X/S}$ (relative to the DGA $\mathcal{O}_S$) exists. It is a coherent subsheaf of $\Omega^n_{k(X)/k(S)}$.

THEOREM 7.10. *Suppose $\pi\colon X \longrightarrow S$ is equidimensional of dimension n and generically smooth, and X is integral. Then $\mathcal{K}^{-n}_{X/S} = \Omega^n_{k(X)/k(S)}$, and*

$$\tilde{\omega}^n_{X/S} = \mathrm{H}^{-n}\mathcal{K}^{\cdot}_{X/S}.$$

First we need:

LEMMA 7.11. *Suppose $L_0 \longrightarrow A_0 \longrightarrow B_0$ are finitely generated field extensions, with $L_0 \longrightarrow A_0$ and $L_0 \longrightarrow B_0$ separable, $A_0 \longrightarrow B_0$ finite, and $\operatorname{tr deg}_{L_0} A_0 = n$. Then $\mathcal{K}(A_0/L_0) = \Omega^n_{A_0/L_0}$, $\mathcal{K}(B_0/L_0) = \Omega^n_{B_0/L_0}$, and $\operatorname{Tr}_{B_0/A_0}\colon \mathcal{K}(B_0/L_0) \longrightarrow \mathcal{K}(A_0/L_0)$ coincides with $\sigma^{L_0}_{B_0/A_0}\colon \Omega^n_{B_0/L_0} \longrightarrow \Omega^n_{A_0/L_0}$ of [Ku] Section 16.*

PROOF. Since $L_0 \longrightarrow A_0$ is formally smooth, we get $\mathcal{K}(A_0/L_0) = \Omega^n_{A_0/L_0}$. The same for B_0. Consider the trivial DGA L_0. Then the universal B_0-extension of $\Omega^{\cdot}_{A_0/L_0}$ is $\Omega^{\cdot}_{B_0/L_0}$, so $\sigma^{L_0}_{B_0/A_0}$ makes sense. To check that $\sigma^{L_0}_{B_0/A_0} = \operatorname{Tr}_{B_0/A_0}$ we may reduce to the cases $A_0 \longrightarrow B_0$ separable, or purely inseparable of prime degree, and then use the properties of the trace. ■

PROOF OF THE THEOREM. Given any point $x \in X$ there is an open neighborhood U of x which admits a factorization $\pi|_U = hgf$, with $f\colon U \longrightarrow Y$ an open immersion; $g\colon Y \longrightarrow Z$ finite; and $h\colon Z \longrightarrow S$ smooth of relative dimension n (in fact one can take Z open in $\mathbf{A}^n \times S$). This follows from quasi-normalization ([Ku] Theorem B20) and Zariski's Main Theorem ([EGA] IV 8.12.3; *cf.* [Ku] Theorem B16). We can also assume Y, Z, S are affine, say $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} A$ and $S = \operatorname{Spec} L$. Let us write $\tilde{\omega}^n_{B/L} := \Gamma(Y, \tilde{\omega}^n_{Y/S})$ and $\mathcal{K}^{\cdot}_{B/L} := \Gamma(Y, \mathcal{K}^{\cdot}_{Y/S})$. Also let us write $B_0 := k(Y)$, $A_0 := k(Z)$ and $L_0 := k(S)$.

By [KW] Section 4,

$$\tilde{\omega}^n_{B/L} = \{\beta \in \Omega^n_{B_0/L_0} \mid \sigma^{L_0}_{B_0/A_0}(b\beta) \in \Omega^n_{A/L} \text{ for all } b \in B\}.$$

One has

$$\mathcal{K}^{-n}_{B/L} = \mathcal{K}(B_0/L_0) = \Omega^n_{A_0/L_0}$$

and the same for A. According to Proposition 6.12 there is a quasi-isomorphism $\Omega^n_{A/L}[n] \to \mathcal{K}^{\cdot}_{A/L}$. From the commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^{-n}\mathcal{K}^{\cdot}_{B/L} & \longrightarrow & \mathcal{K}^{-n}_{B/L} & \xrightarrow{\delta} & \mathcal{K}^{-n+1}_{B/L} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \operatorname{Tr}_g} & & \downarrow{\scriptstyle \operatorname{Tr}_g} \\
0 & \longrightarrow & \Omega^n_{A/L} & \longrightarrow & \mathcal{K}^{-n}_{A/L} & \xrightarrow{\delta} & \mathcal{K}^{-n+1}_{A/L}
\end{array}$$

and the isomorphism

$$\mathcal{K}^{-n+1}_{B/L} \cong \operatorname{Hom}_A(B, \mathcal{K}^{-n+1}_{A/L})$$

induced by Tr_g we conclude that $\tilde{\omega}^n_{B/L} = \mathrm{H}^{-n}\mathcal{K}^{\cdot}_{B/L}$. Since $\tilde{\omega}^n_{Y/S}$ and $\mathrm{H}^{-n}\mathcal{K}^{\cdot}_{Y/S}$ are coherent sheaves and $f\colon U \longrightarrow Y$ is an open immersion, this shows that $\tilde{\omega}_{U/S} = \mathrm{H}^{-n}\mathcal{K}^{\cdot}_{U/S}$. ■

COROLLARY 7.12. *If X is a Cohen-Macaulay scheme then the sequence*

$$0 \to \tilde{\omega}^n_{X/S} \to \mathcal{K}^{-n}_{X/S} \to \cdots \to \mathcal{K}^m_{X/S} \to 0$$

($m = \dim S$) is exact.

PROOF. X is Cohen-Macaulay iff any dualizing complex has a single nonzero cohomology sheaf. ∎

EXAMPLE 7.13. Suppose X is an $(n+1)$-dimensional integral scheme and $\pi: X \longrightarrow \operatorname{Spec} \mathbb{Z}$ is a finite type dominant morphism (*i.e.* X has mixed characteristics). Then π is flat, equidimensional of dimension n and generically smooth. So

$$\tilde{\omega}^n_{X/\mathbb{Z}} = \mathrm{H}^{-n} \mathcal{K}^{\cdot}_{X/\mathbb{Z}} \subset \Omega^n_{k(X)/\mathbb{Q}}.$$

REMARK 7.14. In the situation of Theorem 7.10 there is a homomorphism

$$\mathrm{C}_X: \Omega^n_{X/S} \longrightarrow \mathcal{K}^{-n}_{X/S}$$

called the *fundamental class of* X/S. According to [KW], when π is flat one has $\mathrm{C}_X(\Omega^n_{X/S}) \subset \tilde{\omega}^n_{X/S}$; so $\mathrm{C}_X: \Omega^n_{X/S}[n] \longrightarrow \mathcal{K}^{\cdot}_{X/S}$ is a homomorphism of complexes.

REMARK 7.15. In [LS2] Theorem 11.2 we find a stronger statement than our Theorem 7.10: S is only required to be an excellent equidimensional scheme without embedded points, satisfying Serre's condition S_2; and π is finite type, equidimensional and generically smooth. Moreover, for π proper, the trace is compared to the integral of [HS] (*cf.* Remark 8.4). The price of this generality is that the proofs in [LS2] are not self-contained but rely on rather complicated results from other papers.

8. The isomorphism $\mathcal{K}^{\cdot}_{X/S} \cong \pi^! \mathcal{O}_S$.

In this section we describe the canonical isomorphism between the complex $\mathcal{K}^{\cdot}_{X/S}$ constructed in Section 6, and the twisted inverse image $\pi^! \mathcal{O}_S$ of [RD]. Recall that for residual complexes there is an inverse image $\pi^{\triangle}$, and $\pi^{\triangle} \mathcal{K}^{\cdot}_{S/S} = \mathrm{E}\pi^! \mathcal{O}_S$, where E is the Cousin functor corresponding to the dualizing complex $\pi^! \mathcal{O}_S$. For an S-morphism $f: X \longrightarrow Y$ denote by $\mathrm{Tr}^{\mathrm{RD}}_f$ the homomorphism of graded sheaves

$$\mathrm{Tr}^{\mathrm{RD}}_f: f_* \pi^{\triangle}_X \mathcal{K}^{\cdot}_{S/S} \cong f_* f^{\triangle} \pi^{\triangle}_Y \mathcal{K}^{\cdot}_{S/S} \longrightarrow \pi^{\triangle}_Y \mathcal{K}^{\cdot}_{S/S}$$

of [RD] Section VI.4.

THEOREM 8.1. *Let $\pi: X \longrightarrow S$ be a finite type morphism. Then there exists a unique isomorphism of complexes*

$$\zeta_X: \mathcal{K}^{\cdot}_{X/S} \longrightarrow \pi^{\triangle} \mathcal{K}^{\cdot}_{S/S}$$

such that for every morphism $f: X \longrightarrow Y$ the diagram

$$\begin{array}{ccc} f_* \mathcal{K}^{\cdot}_{X/S} & \xrightarrow{\mathrm{Tr}_f} & \mathcal{K}^{\cdot}_{Y/S} \\ {\scriptstyle f_*(\zeta_X)} \downarrow & & \downarrow {\scriptstyle \zeta_Y} \\ f_* \pi^{\triangle}_X \mathcal{K}^{\cdot}_{S/S} & \xrightarrow{\mathrm{Tr}^{\mathrm{RD}}_f} & \pi^{\triangle}_Y \mathcal{K}^{\cdot}_{S/S} \end{array} \tag{8.2}$$

is commutative.

The proof of Theorem 8.1 is given later in this section, after some preparation. Here is one corollary:

COROLLARY 8.3. *If $f: X \longrightarrow Y$ is proper then Tr_f is a homomorphism of complexes, and for any $\mathcal{M}^{\boldsymbol{\cdot}} \in \mathsf{D}^-_{\mathrm{qc}}\big(\mathsf{Mod}(X)\big)$ the induced morphism*

$$f_* \mathcal{H}om_X(\mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{K}^{\boldsymbol{\cdot}}_{X/S}) \longrightarrow \mathcal{H}om_Y(\mathrm{R}f_* \mathcal{M}^{\boldsymbol{\cdot}}, \mathcal{K}^{\boldsymbol{\cdot}}_{Y/S})$$

is an isomorphism.

PROOF. Use [RD] Theorem VII.2.1 and Corollary VII.3.4. ∎

REMARK 8.4. In [Hg3] and [LS2] the authors prove that in their respective constructions the trace $\mathrm{Tr}_f: f_* f^! \mathcal{N}^{\boldsymbol{\cdot}} \longrightarrow \mathcal{N}^{\boldsymbol{\cdot}}$ is a homomorphism of complexes for any proper morphism f and residual (resp. Cousin) complex $\mathcal{N}^{\boldsymbol{\cdot}}$ (*cf.* Remark 6.16).

Let $Y = \operatorname{Spec} A$ be an affine noetherian scheme, $X := \mathbf{A}^n \times Y = \operatorname{Spec} A[t_1, \ldots, t_n]$ and $f: X \longrightarrow Y$ the projection. Fix a point $x \in X$, and let $y := f(x)$, $Z_0 := \overline{\{x\}}_{\mathrm{red}}$. Assume $Z_0 \longrightarrow Y$ is finite.

LEMMA 8.5. *There exists an open set $U \subset Y$ containing y and a flat finite morphism $g: Y' \longrightarrow U$ s.t.:*

(i) $g^{-1}(y)$ is one point, say y'.

(ii) Define $X' := \mathbf{A}^n \times Y'$, and let $f': X' \longrightarrow Y'$, $h: X' \longrightarrow X$. Then for every point $x' \in h^{-1}(x)$ there is some section $\sigma_{x'}: Y' \longrightarrow X'$ of f' with $x' \in \sigma_{x'}(Y')$.

PROOF. Choose any finite normal field extension K of $k(y)$ containing $k(x)$. Define recursively open sets $U_i = \operatorname{Spec} A_i \subset Y$ and finite flat morphisms $g_i: Y_i = \operatorname{Spec} A'_i \longrightarrow U_i$ s.t. $g_i^{-1}(y) = \{y_i\}$ and $k(y_i) \subset K$, as follows. Start with $U_0 = Y_0 := Y$ and $A'_0 = A_0 := A$. If $k(y_i) \neq K$ take some $\bar{b} \in K - k(y_i)$ and let $\bar{p} \in k(y_i)[t]$ be the monic irreducible polynomial of $\bar{b}$. Choose a monic polynomial $p \in \mathcal{O}_{Y_i, y_i}[t]$ lifting $\bar{p}$. There is some open set $U_{i+1} = \operatorname{Spec} A_{i+1} \subset U_i$ s.t. $p \in (A'_i \otimes_{A_i} A_{i+1})[t]$. Define $A'_{i+1} := (A'_i \otimes_{A_i} A_{i+1})[t]/(p)$ and $Y_{i+1} = \operatorname{Spec} A'_{i+1}$. For $i = r$ this stops, and $k(y_r) = K$.

For every point $x' \in \operatorname{Spec}\big(K \otimes_{k(y)} k(x)\big)$ and $1 \leq i \leq n$ let $\bar{a}_{i,x'} \in k(x') \cong k(y_r)$ be the image of t_i, and let $a_{i,x'} \in \mathcal{O}_{Y_r, y_r}$ be a lifting. Take an open set $U = \operatorname{Spec} A_{r+1} \subset U_r$ s.t. each $a_{i,x'} \in A' = (A'_r \otimes_{A_r} A_{r+1})$, and define $Y' := \operatorname{Spec} A'$. So for each x' the homomorphism $B' = A'[t] \longrightarrow A'$, $t_i \longmapsto a_{i,x'}$ gives the desired section $\sigma_{x'}: Y' \longrightarrow X'$. ∎

Let Z_i be the i-th infinitesimal neighborhood of Z_0 in X, so $f_i: Z_i \longrightarrow Y$ is a finite morphism. Suppose we are given a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{M}$ which is supported on $\overline{\{y\}}$. One has

$$\mathcal{H}^n_{Z_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M}) \cong \lim_{i \to} \mathcal{E}xt^n_X(\mathcal{O}_{Z_i}, \Omega^n_{X/Y} \otimes f^* \mathcal{M})$$

and by [RD] Theorem VI.3.1

$$\mathcal{E}xt^n_X(\mathcal{O}_{Z_i}, \Omega^n_{X/Y} \otimes f^* \mathcal{M}) = \mathcal{H}^0 f_i^! \mathcal{M}.$$

Note that we can also factor f_i through $\mathbf{P}^n \times Y$, so f_i is projectively embeddable, and by [RD] Theorem III.10.5 we have a map

$$\mathrm{Tr}^{\mathrm{RD}}_f: f_* \mathcal{H}^n_{Z_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M}) \longrightarrow \mathcal{M}. \tag{8.6}$$

Now define $\hat{A} := \hat{O}_{Y,y}$ and $\hat{B} := \hat{O}_{X,x}$, with $\mathfrak{n} \subset \hat{B}$ the maximal ideal and $\phi = f^*: \hat{A} \longrightarrow \hat{B}$. Set $M := \mathcal{M}_y$, which is a discrete $\hat{A}$-module. We then have a natural isomorphism of $\hat{A}$-modules

$$\left(f_* \mathcal{H}^n_{Z_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M})\right)_y \cong \mathrm{H}^n_{\mathfrak{n}}(\hat{\Omega}^n_{\hat{B}/\hat{A}} \otimes_{\hat{A}} M) \cong \phi_\# M. \tag{8.7}$$

LEMMA 8.8. *Under the isomorphism (8.7),*

$$\mathrm{Tr}^{\mathrm{RD}}_f = \mathrm{Tr}_\phi : \phi_\# M \longrightarrow M.$$

PROOF. The proof is in two steps.

STEP 1. Assume there is a section $\sigma: Y \longrightarrow X$ to f with $x \in W_0 = \sigma(Y)$. The homomorphism $\sigma^*: B = A[\underline{t}] \longrightarrow A$ chooses $a_i = \sigma^*(t_i) \in A$, so after the linear change of variables $t_i \longmapsto t_i - a_i$ we may assume that σ is the 0-section (*i.e.* $O_{W_0} = O_X / O_X \cdot \underline{t}$). Let W_i be the i-th infinitesimal neighborhood of W_0. Since $f: W_i \longrightarrow Y$ is projectively embeddable, there is a trace map

$$\mathrm{Tr}^{\mathrm{RD}}_f : f_* \mathcal{H}^n_{W_0} \Omega^n_{X/Y} \longrightarrow O_Y.$$

For any $a \in A$ one has

$$\mathrm{Tr}^{\mathrm{RD}}_f \left(\frac{a \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1^{i_1} \cdots t_n^{i_n}} \right) = \begin{cases} a & \text{if } \underline{i} = (1, \ldots, 1) \\ 0 & \text{otherwise.} \end{cases} \tag{8.9}$$

This follows from properties R6 (normalization) and R7 (intersection) of the residue symbol ([RD] Section III.9). Alternatively this can be checked as follows. Note that $\mathrm{Tr}^{\mathrm{RD}}_f$ factors through $\mathrm{R}f_* \Omega^n_{\mathbf{P}^n_Y / Y}$. For the case $\underline{i} = (1, \ldots, 1)$ use [RD] Proposition III.10.1. For $\underline{i} \neq (1, \ldots, 1)$ consider a change of coordinates $t_i \longmapsto \lambda_i t_i$, $\lambda_i \in A$. By [RD] Corollary III.10.2, $\mathrm{Tr}^{\mathrm{RD}}_f$ is independent of homogeneous coordinates, so it must be 0.

Now since $W_0 \cap f^{-1}(y) = Z_0$ we have

$$\mathcal{H}^n_{Z_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M}) \cong \mathcal{H}^n_{W_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M})$$

and so the formula for $\mathrm{Tr}^{\mathrm{RD}}_f$ in (8.6) is given by (8.9). But the same formula is used in [Hg1] to define Tr_ϕ.

STEP 2. The general situation: take $g: Y' \longrightarrow Y$ as in Lemma 8.5, and set $Z'_0 := Z_0 \times_Y Y'$. The flatness of g implies there is a natural isomorphism of $O_{Y'}$-modules

$$g^* f_* \mathcal{H}^n_{Z_0}(\Omega^n_{X/Y} \otimes f^* \mathcal{M}) \cong f'_* \mathcal{H}^n_{Z'_0}(\Omega^n_{X'/Y'} \otimes f'^* \mathcal{M}')$$

(where $\mathcal{M}' := g^* \mathcal{M}$) and by [RD] Theorem III.10.5 property TRA4 we have

$$g^*(\mathrm{Tr}^{\mathrm{RD}}_f) = \mathrm{Tr}^{\mathrm{RD}}_{f'} . \tag{8.10}$$

Let $\hat{A}' := \hat{O}_{Y',y'} \cong A' \otimes_A \hat{A}$, so $\hat{A} \longrightarrow \hat{A}'$ is finite flat. Therefore

$$\hat{A}' \otimes_{\hat{A}} \mathrm{H}^n_{\mathfrak{n}}(\hat{\Omega}^n_{\hat{B}/\hat{A}} \otimes_{\hat{A}} M) \cong \bigoplus_{\mathfrak{n}' \in Z'_0} \mathrm{H}^n_{\mathfrak{n}'}(\hat{\Omega}^n_{\hat{B}_{\mathfrak{n}'}/\hat{A}'} \otimes_{\hat{A}'} M'). \tag{8.11}$$

Here $M' := \mathcal{M}'_{y'} \cong \hat{A}' \otimes_{\hat{A}} M$ and $\prod_{\mathfrak{n}' \in Z'_0} \hat{B}_{\mathfrak{n}'}$ is the decomposition of $A' \otimes_A \hat{B}$ to local rings. Write $\phi'_{\mathfrak{n}'}: \hat{A}' \longrightarrow \hat{B}_{\mathfrak{n}'}$. Direct verification shows that under the isomorphism (8.11),

$$1 \otimes \mathrm{Tr}_{\phi} = \sum_{\mathfrak{n}' \in Z'_0} \mathrm{Tr}_{\phi'_{\mathfrak{n}'}} . \tag{8.12}$$

Since $\hat{A} \longrightarrow \hat{A}'$ is faithfully flat it follows that $M \longrightarrow M'$ is injective. In view of the equalities (8.10) and (8.12), we conclude that it suffices to check for each $\mathfrak{n}' = x' \in Z_0$ that $\mathrm{Tr}_{\phi'_{\mathfrak{n}'}} = \mathrm{Tr}^{\mathrm{RD}}_{f'}$ on $\mathrm{H}^n_{\mathfrak{n}'}(\hat{\Omega}^n_{\hat{B}_{\mathfrak{n}'}/\hat{A}'} \otimes_{\hat{A}'} M')$. But there is a section $\sigma_{x'}: Y' \longrightarrow X'$, so we can apply Step 1. ∎

PROOF OF THEOREM 8.1.

STEP 1 (UNIQUENESS). Suppose $\zeta'_X: \mathcal{K}^{\cdot}_{X/S} \longrightarrow \pi^{\triangle} \mathcal{K}^{\cdot}_{S/S}$ is another isomorphism satisfying $\mathrm{Tr}_{\pi} = \mathrm{Tr}^{\mathrm{RD}}_{\pi} \pi_*(\zeta'_X)$. Then $\zeta'_X = a\zeta_X$ for some $a \in \Gamma(X, \mathcal{O}^*_X)$, and by assumption for any closed point $x \in X$ and $\alpha \in \mathcal{K}_{X/S}(x)$ there is equality $\mathrm{Tr}_{\pi}(\alpha) = \mathrm{Tr}_{\pi}(a\alpha)$. Now writing $s := \pi(x)$, it's known that

$$\mathrm{Hom}_{\mathcal{O}_{S,s}}\big(\mathcal{K}_{X/S}(x), \mathcal{K}_{S/S}(s)\big)$$

is a free $\hat{\mathcal{O}}_{X,x}$-module with basis Tr_{π}. Therefore $a = 1$ in $\hat{\mathcal{O}}_{X,x}$. Because this is true for all closed points we see that $a = 1$.

STEP 2. Assume $X = \mathbf{A}^n \times S$ and $f = \pi$. In this case there is a canonical isomorphism of complexes

$$\mathcal{K}^{\cdot}_{X/S} \cong \mathrm{E}\Omega^n_{X/S}[n] \cong \mathrm{E}\pi^! \mathcal{O}_S \cong \pi^{\triangle} \mathcal{K}^{\cdot}_{S/S}$$

(*cf.* [RD] Theorem VI.3.1 and our Proposition 6.12), which we use to define $\zeta_X: \mathcal{K}^{\cdot}_{X/S} \longrightarrow \pi^{\triangle} \mathcal{K}^{\cdot}_{S/S}$. Consider $x \in X$, $Z := \overline{\{x\}}_{\mathrm{red}}$, $s := \pi(x)$ and assume x is closed in $\pi^{-1}(s)$. By replacing S with a suitable open neighborhood of s we can assume $Z \longrightarrow S$ is finite. Then we are allowed to apply Lemma 8.8 with $Y = S$, $\mathcal{M} = \mathcal{K}_{S/S}(s)$. It follows that (8.2) commutes on $\pi_* \mathcal{K}_{X/S}(x) \subset \pi_* \mathcal{K}^{\cdot}_{X/S}$.

STEP 3. Let X be any finite type S-scheme. For every affine open subscheme $U \subset X$ we can find a closed immersion $h: U \longrightarrow \mathbf{A}^n_S$. Write $Y := \mathbf{A}^n_S$ and let π_U and π_Y be the structural morphisms. Now Tr_h induces an isomorphism

$$\mathcal{K}^{\cdot}_{U/S} \cong \mathcal{H}om_Y(\mathcal{O}_U, \mathcal{K}^{\cdot}_{Y/S}),$$

and $\mathrm{Tr}^{\mathrm{RD}}_h$ induces an isomorphism

$$\pi^{\triangle}_U \mathcal{K}^{\cdot}_{S/S} \cong \mathcal{H}om_Y(\mathcal{O}_U, \pi^{\triangle}_Y \mathcal{K}^{\cdot}_{S/S}).$$

So the isomorphism ζ_Y of Step 2 induces an isomorphism $\zeta_U: \mathcal{K}^{\cdot}_{U/S} \longrightarrow \pi^{\triangle}_U \mathcal{K}^{\cdot}_{S/S}$, which satisfies $\mathrm{Tr}_{\pi_U} = \mathrm{Tr}^{\mathrm{RD}}_{\pi_U} \pi_{U*}(\zeta_U)$. According to Step 1 the local isomorphisms ζ_U can be glued to a global isomorphism ζ_X.

STEP 4. Let $f: X \longrightarrow Y$ be any S-morphism. To check (8.2) we may assume X and Y are affine, and in view of Step 3 we may in fact assume $Y = \mathbf{A}^m \times S$ and $X = \mathbf{A}^n \times Y \cong \mathbf{A}^{n+m} \times S$. Now apply Lemma 8.8 with $x \in X$ closed in its fiber and $\mathcal{M} := \mathcal{K}_{Y/S}(y)$. ∎

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