# ON RESIDUE COMPLEXES, DUALIZING SHEAVES AND LOCAL COHOMOLOGY MODULES

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Abstract. According to Grothendieck Duality Theory [RD], on each variety $V$ over a field $k$, there is a canonical complex of $\mathcal{O}_V$-modules, the *residue complex* $\mathcal{K}_V^{\mathrm{RD}\cdot} \cong \pi^! k$. These complexes satisfy (and are characterized by) functorial properties in the category $\mathcal{V}$ of $k$-varieties. In [Ye] a complex $\mathcal{K}_V^{\cdot}$ is constructed explicitly (when the field $k$ is perfect). The main result of this paper is that the two families of complexes, $\{\mathcal{K}_V^{\mathrm{RD}\cdot}\}_{V \in \mathcal{V}}$ and $\{\mathcal{K}_V^{\cdot}\}_{V \in \mathcal{V}}$, which carry certain additional data (such as trace maps...), are uniquely isomorphic. As a corollary we recover Lipman's canonical dualizing sheaf of [Li], and we obtain formulas for residues of local cohomology classes of differential forms.

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## 0. Introduction and Statement of Results

**0.1. Introduction.** This paper is yet another step in the program, begun by J. Lipman, E. Kunz and others, to give a concrete realization of the Grothendieck Duality Theory. In [Li] we find a realization of the *dualizing sheaf* in the absolute setting: varieties over a perfect field. With the recent papers [HK1], [HK2], [LS] and [HS], one may regard the program, in its restricted version of relative dualizing sheaves for dominant equidimensional morphisms (over a wide class of base schemes), as complete.

This is hardly the case for the full theory of dualizing complexes. Here we only have a concrete realization of absolute duality, namely the *Grothendieck Residue Complex* $\mathcal{K}^{\cdot}_X$ of [Ye]. Let $k$ be a perfect field. In [Ye] Appendix it was shown that for a proper $k$-variety $\pi : X \to \operatorname{Spec} k$, the pair $(\mathcal{K}^{\cdot}_X, \operatorname{Tr}_\pi)$ is a residue pair (cf. §0.2), so it is a realization of the pair $(\pi^! k, \operatorname{Tr}_\pi)$ of [RD]. Missing from this realization is the connection to differential forms. Borrowing from the language of [Li] §0, what we find in [Ye] Appendix is an account of the *dualizing structure* on $\mathcal{K}^{\cdot}_X$, whereas [Ye] §4 gives an account of the *canonical structure* on it. The main result of the present paper connects the two structures.

At the same time we recover Lipman's canonical dualizing sheaf $\tilde{\omega}$ of [Li]. Suppose $\dim X = n$. From [Ye] Thm. 4.4.16 we know that the *sheaf of regular differentials* $\tilde{\omega}_X$ satisfies $\tilde{\omega}_X = \mathrm{H}^{-n} \mathcal{K}^{\cdot}_X \subset \underline{K}_X \otimes \Omega^n_{X/k}$, where $\underline{K}_X$ is the constant sheaf of meromorphic functions. Let $\eta_X : \tilde{\omega}_X[n] \to \mathcal{K}^{\cdot}_X$ be the corresponding homomorphism of complexes. Let $\tilde{\theta}_X : \mathrm{H}^n(X, \tilde{\omega}_X) \to k$ be the trace map of [Li] Thm. 0.6 (d). We prove that $\tilde{\theta}_X = (-1)^n \operatorname{Tr}_\pi \circ \mathrm{H}^0 \mathrm{R}\, \pi_*(\eta_X)$.

An intriguing problem, posed to us by S. Kleiman (private comm.), is to explain the relation between local cohomology residues (residue symbols) and Parshin Residues. We solve this problem here, using the methods of semi-topological rings and Beilinson completions. Lipman, in a private communication, conjectured a formula for the canonical map $\mathrm{H}^c_x(\eta_X) : \mathrm{H}^c_x(\tilde{\omega}_X) \to \mathcal{K}(x)$, where $x \in X$ has codimension $c$, and $\mathcal{K}(x) = \mathrm{H}^{c-n}_x(\mathcal{K}^{\cdot}_X)$ is the dual module of [Ye] Def. 4.3.10, a formula which we prove. The proof is based on coboundary calculations in the Koszul-residue double complex.

The results mentioned in the two preceding paragraphs have been obtained independently by R. Hübl [Hu2], for the most part only for Cohen-Macaulay varieties. Interestingly, the route taken by Hübl is almost the opposite of ours.

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**0.2. Statement of Results.** Let $X$ be an $n$-dimensional variety over the perfect field $k$, with structural morphism $\pi$. Consider the following three objects associated to $X$:

(1) The complex $\mathcal{K}^{\mathrm{RD}\cdot}_X := \pi^\Delta k$ (see [RD] Ch. VI). It is a residual complex on $X$ (Def. 1.1.1), concentrated in the dimension range $[-n, 0]$.

(2) The sheaf of *regular differential forms* $\tilde{\omega}_X$ of Kunz (see [Ku] p. 68). This is a coherent subsheaf of the sheaf of meromorphic differentials $\underline{K}_X \otimes \Omega^n_{X/k}$.
(3) The *residue complex* $\mathcal{K}^{\cdot}_X$ of [Ye] Thm. 4.3.20. This too is a residual complex on $X$, concentrated in the dimension range $[-n, 0]$. In dimension $-n$ one has $\mathcal{K}^{-n}_X = \underline{K}_X \otimes \Omega^n_{X/k}$.

The second and third objects are easily related: $\tilde{\omega}_X = \mathrm{H}^{-n}\, \mathcal{K}^{\cdot}_X$ as subsheaves of $\underline{K}_X \otimes \Omega^n_{X/k}$ (cf. [Ye] Thm. 4.4.16; or cf. Example 1.3.3 and [Li] Cor. 2.3). Hence there is an induced homomorphism of complexes

$$\eta_X : \tilde{\omega}_X[n] \to \mathcal{K}^{\cdot}_X \ ,$$

which is a quasi-isomorphism iff $X$ is a Cohen-Macaulay scheme (cf. [Ye] Cor. 4.5.7). The main result of this paper, namely Thm. 0.2.3 below, provides a natural isomorphism of complexes $\mathcal{K}^{\cdot}_X \cong \mathcal{K}^{\mathrm{RD}\cdot}_X$ on any variety $X$.

Let $\mathcal{V}$ be the category of $k$-varieties, and let $\mathcal{V}_{\mathrm{Zar}}$ be the subcategory with the same objects, but with open immersions as its morphisms. In §1 the notion of a *residue complex* on $\mathcal{V}$ is defined. It consists of a residual complex $\mathcal{R}^{\cdot} = \{\mathcal{R}^{\cdot}_X\}_{X \in \mathcal{V}}$ on $\mathcal{V}_{\mathrm{Zar}}$, together with a dualizing structure $\{\theta_X\}$ and a canonical structure $(\{\gamma_X\}, \{\theta_f\})$, which are compatible with each other. These definitions are adapted from Lipman's corresponding definitions for sheaves; cf. [Li] §0. We prove:

**Theorem 0.2.1.** (Rigidity) *Let* $(\mathcal{R}^{\cdot}, \{\theta_X\}, \{\gamma_X\}, \{\theta_f\})$ *and* $(\mathcal{R}'^{\cdot}, \{\theta'_X\}, \{\gamma'_X\}, \{\theta'_f\})$ *be two residue complexes on* $\mathcal{V}$. *Then there exists a unique isomorphism of complexes on* $\mathcal{V}_{\mathrm{Zar}}$, $\lambda : \mathcal{R}^{\cdot} \xrightarrow{\simeq} \mathcal{R}'^{\cdot}$, *which respects the dualizing structures and the canonical structures.*

Observe the similarity to Thm. 0.3B of [Li].

One residue complex on $\mathcal{V}$ is gotten from the pseudo-functor $\triangle$ of [RD] Ch. VI. Given $\pi : X \to \operatorname{Spec} k$ in $\mathcal{V}$, set $\mathcal{K}^{\mathrm{RD}\cdot}_X := \pi^{\triangle} k$, a residual complex on $X$. The variance data of ibid. Thm. 3.1 make $\mathcal{K}^{\mathrm{RD}\cdot} = \{\mathcal{K}^{\mathrm{RD}\cdot}_X\}_{X \in \mathcal{V}}$ a residual complex on $\mathcal{V}_{\mathrm{Zar}}$. From ibid. Thm. 3.1 and Thm. 4.2 we obtain families of homomorphisms $\{\theta^{\mathrm{RD}}_X\}$, $\{\gamma^{\mathrm{RD}}_X\}$ and $\{\theta^{\mathrm{RD}}_f\}$.

**Proposition 0.2.2.** $(\mathcal{K}^{\mathrm{RD}\cdot}, \{\theta^{\mathrm{RD}}_X\}, \{\gamma^{\mathrm{RD}}_X\}, \{\theta^{\mathrm{RD}}_f\})$ *is a residue complex on* $\mathcal{V}$.

Next, consider the residual complex $\mathcal{K}^{\cdot} = \{\mathcal{K}^{\cdot}_X\}_{X \in \mathcal{V}}$ on $\mathcal{V}_{\mathrm{Zar}}$, where for a variety $X$, $\mathcal{K}^{\cdot}_X$ is the complex of [Ye] §4.3. Let $\{\mathrm{Tr}_\pi\}$, $\{\mathrm{C}_X\}$ and $\{\mathrm{Tr}_f\}$ be the maps defined in [Ye] §4.4 and 4.5. The main result of the paper is:

**Theorem 0.2.3.** $(\mathcal{K}^{\cdot}, \{(-1)^{\dim X} \mathrm{Tr}_\pi\}, \{\mathrm{C}_X\}, \{\mathrm{Tr}_f\})$ *is a residue complex on* $\mathcal{V}$.

*Remark* 0.2.4. In [Ye], Appendix it was essentially proved that the dualizing complexes $(\{\mathcal{K}^{\mathrm{RD}\cdot}_X\}, \{\theta^{\mathrm{RD}}_X\})$ and $(\{\mathcal{K}^{\cdot}_X\}, \{\mathrm{Tr}_\pi\})$ are isomorphic.

*Remark* 0.2.5. It is somewhat disconcerting that the two residue complexes mentioned above, viz. $\mathcal{K}^{\mathrm{RD}\cdot}$ and $\mathcal{K}^{\cdot}$, have complicated constructions running into many pages in [RD] and [Ye], respectively. In Example 1.2.5 and Remark 1.4.2 we indicate a more accessible approach to the existence of dualizing and residue complexes on $\mathcal{V}$.

The equality of sheaves $\tilde{\omega}_X = \mathrm{H}^{-\dim X}\, \mathcal{K}^{\cdot}_X$ on any variety $X$ makes $\tilde{\omega} = \{\tilde{\omega}_X\}_{X \in \mathcal{V}}$ into a sheaf on $\mathcal{V}_{\mathrm{Zar}}$. A direct consequence of Theorem 0.2.3 is:

**Corollary 0.2.6.** *The data*

$$(\tilde{\omega}, \{(-1)^{\dim X} \operatorname{Tr}_\pi \circ \mathrm{H}^0 \mathrm{R}\, \pi_*(\eta_X)\}, \{\mathrm{H}^{-\dim X}(\eta_X^{-1} \circ \mathrm{C}_X)\}, \{\mathrm{H}^{-\dim X}(\operatorname{Tr}_f)\})$$

*is a canonical dualizing $\mathcal{O}$-module; i.e. it satisfies the conditions of* [Li] *Thm.* 0.3B.

The uniqueness part of [Li] Thm. 0.3B implies:

**Corollary 0.2.7.** *Suppose $X$ is proper of dimension $n$ over $k$. Let*

$$\tilde{\theta}_X : \mathrm{H}^n(X, \tilde{\omega}_X) = \mathrm{H}^0(X, \tilde{\omega}_X[n]) \to k$$

*be the the trace map of* [Li] *Thm.* 0.6 (d)*. Then*

$$\tilde{\theta}_X = (-1)^n \operatorname{Tr}_\pi \circ \mathrm{H}^0 \mathrm{R}\, \pi_*(\eta_X)\ .$$

The proofs of the statements above are in §1.4. The proof of Thm. 0.2.3 breaks down into a global part, involving categories and functors, and a local part, whose essence is Theorems 0.2.9 and 0.2.10 below. These theorems are of interest on their own.

Let $x \in X$ be a point of codimension $c$, and let $\sigma : K \to \hat{\mathcal{O}}_{X,x}$ be a pseudo-coefficient field (i.e. $[k(x) : K] < \infty$). In [Hu2] Def. 1.1 the residue map

$$\operatorname{Res}_{\hat{\mathcal{O}}_{X,x}/K, \mathfrak{m}_x} : \mathrm{H}^c_x(\tilde{\omega}_X) \to \Omega^{n-c}_{K/k}$$

is defined (cf. ibid. Remark (ii) ). Let us denote this map by $\operatorname{Res}^{\mathrm{LC}}_{x,K} = \operatorname{Res}^{\mathrm{LC}}_{x,\sigma}$. When $x$ is a closed point (or equivalently, when $[K : k] < \infty$), this is the well known residue map; cf. [Li] §7. The generalized fraction notation (cf. [Li] §7 and [HK1] §3) shall be used to represent local cohomology classes.

*Remark* 0.2.8. The generalized fraction notation used in [LS] differs from the one used here (i.e. that of [Li] §7) by a factor of $(-1)^c$; cf. [Sa].

Suppose $\underline{t} = (t_1, \ldots, t_c)$ is a system of parameters in $\mathcal{O}_{X,x}$. Given a chain $\xi = (x_0, \ldots, x_c)$ of points in $X$, we write $\xi | (x; \underline{t})$ if $x_c = x$, and if $t_i(x_i) = 0$ for $i = 1, \ldots, c$. Here $t_i(x_i)$ is the class of $t_i$ in the residue field $k(x_i)$. Such a chain is necessarily saturated, and there are only finitely many of them. Also note that $x_0$ is the generic point of $X$, and that $t_i(x_{i-1}) \neq 0$ for all $i$. Let $\operatorname{Res}_{\xi,K} = \operatorname{Res}_{\xi,\sigma} : \Omega^*_{k(X)/k} \to \Omega^*_{k(x)/k}$ be the Parshin residue map of [Ye] §4.1.

**Theorem 0.2.9.** (Cf. [Hu2] Cor. 2.5.) *Let $\underline{t} = (t_1, \ldots, t_c)$ be a system of parameters in $\mathcal{O}_{X,x}$. For any regular differential form $\alpha \in \tilde{\omega}_{X,x}$,*

$$\operatorname{Res}^{\mathrm{LC}}_{x,K} \begin{bmatrix} \alpha \\ t_1, \ldots, t_c \end{bmatrix} = (-1)^{\binom{c}{2}} \sum_{\xi | (x;\underline{t})} \operatorname{Res}_{\xi,K} \left( \frac{\alpha}{t_1 \cdots t_c} \right)$$

*in $\Omega^{n-c}_{K/k}$.*

The proof of the theorem is in §3.2. This solves the problem posed by S. Kleiman, regarding the relation between the two types of residues.

Recall that for every $q$, $\mathcal{K}^{-q}_X = \bigoplus_{y \in X_q} \mathcal{K}(y)$, where $X_q := \{y \in X \mid \dim \{y\}^- = q\}$, and $\mathcal{K}(y)$ is a skyscraper sheaf supported on $\{y\}^-$. Hence there is a canonical isomorphism of $\mathcal{O}_{X,y}$-modules $\mathcal{K}(y) \cong \mathrm{H}^{-q}_y(\mathcal{K}^{\cdot}_X)$. If $\xi = (x_0, \ldots, x_c)$ is a chain of points as above, then $\mathcal{K}(x_0) = k(X) \otimes \Omega^n_{X/k}$. Let $\delta_\xi : \mathcal{K}(x_0) \to \mathcal{K}(x_c)$ be the coboundary map of [Ye] Def. 4.3.10.

**Theorem 0.2.10.** *Let $\underline{t} = (t_1, \ldots, t_c)$ be a system of parameters in $\mathcal{O}_{X,x}$. For any regular differential form $\alpha \in \tilde{\omega}_{X,x}$ one has*

$$\mathrm{H}^c_x(\eta_X) \left( \begin{bmatrix} \alpha \\ t_1, \ldots, t_c \end{bmatrix} \right) = (-1)^{\binom{c}{2} + cn} \sum_{\xi | (x;\underline{t})} \delta_\xi \left( \frac{\alpha}{t_1 \cdots t_c} \right)$$

*in $\mathrm{H}^c_x(\mathcal{K}^{\cdot}_X[-n]) = \mathrm{H}^{c-n}_x(\mathcal{K}^{\cdot}_X) \cong \mathcal{K}(x)$.*

The proof of this theorem is in §2.3.

Suppose $\sigma : K \to \hat{\mathcal{O}}_{X,x}$ is a pseudo-coefficient field. There is a canonical isomorphism of $\mathcal{O}_{X,x}$-modules (see [Ye] Def. 4.3.10 and Remark 4.3.17):

$$\Phi_\sigma : \mathcal{K}(\sigma) = \operatorname{Hom}^{\mathrm{cont}}_K (\hat{\mathcal{O}}_{X,x}, \Omega^{n-c}_{K/k}) \xrightarrow{\simeq} \mathcal{K}(x) \ .$$

Putting Theorems 0.2.9 and 0.2.10 together, we get Lipman's conjectured description of the map $\mathrm{H}^c_x(\eta_X)$, up to a sign:

**Corollary 0.2.11.** (Cf. [Hu2] Thm. 2.2) *Let $\sigma : K \to \hat{\mathcal{O}}_{X,x}$ be a pseudo-coefficient field. Then the homomorphism*

$$\mathrm{H}^c_x(\tilde{\omega}_X) \xrightarrow{\mathrm{H}^c_x(\eta_X)} \mathcal{K}(x) \xrightarrow{\Phi^{-1}_\sigma} \mathcal{K}(\sigma)$$

*is given by*

$$\mathrm{H}^c_x(\eta_X) \left( \begin{bmatrix} \alpha \\ \underline{t} \end{bmatrix} \right) : a \mapsto (-1)^{cn} \operatorname{Res}^{\mathrm{LC}}_{x,K} \begin{bmatrix} a\alpha \\ \underline{t} \end{bmatrix} \ ; \ a \in \mathcal{O}_{X,x} \ .$$

Observe that this corollary directly implies Cor. 0.2.7.

*Remark* 0.2.12. It is not hard to extend most results of this paper to reduced equidimensional schemes of finite type over $k$. For the sake of clarity of the presentation we restricted ourselves to varieties.

## 1. Global Calculations: Sheaves of $\mathcal{O}$-Modules on $\mathcal{V}_{\mathrm{Zar}}$

**1.1. Residual Complexes on $\mathcal{V}_{\mathrm{Zar}}$.** Let $k$ be a perfect field, and let $\mathcal{V}$ be the category of varieties over $k$, i.e. reduced, irreducible, separated $k$-schemes of finite type. Let $\mathcal{V}_{\mathrm{Zar}}$ be the subcategory of $\mathcal{V}$ having the same objects, and having open immersions as its morphisms. We recall the notion of an $\mathcal{O}$-module on $\mathcal{V}_{\mathrm{Zar}}$ from [Li] pp. 28-30. Briefly, $\mathcal{V}_{\mathrm{Zar}}$ is a Grothendieck topology (with the obvious notion of a cover), and we may speak of a sheaf on this site (a contravariant functor to $\mathsf{Sets}$, satisfying the usual conditions for a presheaf to be a sheaf). Let $\mathcal{O}$ be the sheaf on $\mathcal{V}_{\mathrm{Zar}}$ given by $V \mapsto \Gamma(V, \mathcal{O}_V)$. Then $\mathcal{O}$ is a sheaf of rings, and one defines a sheaf of $\mathcal{O}$-modules on $\mathcal{V}_{\mathrm{Zar}}$ in the obvious way. The $\mathcal{O}$-modules on $\mathcal{V}_{\mathrm{Zar}}$ form an abelian category. This category is equivalent to the category whose objects are families of $\mathcal{O}_V$-modules $\{\mathcal{F}_V\}_{V \in \mathcal{V}}$, together with an isomorphism $\beta_g : g^* \mathcal{F}_V \xrightarrow{\simeq} \mathcal{F}_U$ for every open immersion $g : U \to V$, satisfying $\beta_{hg} = \beta_g \circ g^*(\beta_h)$ for every $U \xrightarrow{g} V \xrightarrow{h} W$.

An $\mathcal{O}$-module $\mathcal{F}$ is said to be coherent (resp. quasi-coherent) if $\mathcal{F}_V := \mathcal{F}|_V$ is coherent (resp. quasi-coherent) for every $V$; here $\mathcal{F}_V$ is an actual sheaf on $V$. If $\mathcal{F}$ and $\mathcal{G}$ are two $\mathcal{O}$-modules, their tensor product $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is the $\mathcal{O}$-module s.t. for every $V$, $(\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G})|_V = \mathcal{F}_V \otimes_{\mathcal{O}_V} \mathcal{G}_V$. Given a complex $\mathcal{F}^{\cdot}$ of $\mathcal{O}$-modules, each cohomology $\mathrm{H}^q(\mathcal{F}^{\cdot})$ is also an $\mathcal{O}$-module.

Here are three examples of $\mathcal{O}$-modules:

(1) $\Omega^{\dim}$ is the sheaf $V \mapsto \Gamma(V, \Omega^{\dim V}_{V/k})$.
(2) $\tilde{\omega}$ is the sheaf $V \mapsto \Gamma(V, \tilde{\omega}_V)$.
(3) $\underline{K}$ is the sheaf of meromorphic functions, $V \mapsto k(V)$.

The sheaf $\underline{K}$ is a constant sheaf, and by definition $\underline{K} \otimes_{\mathcal{O}} \Omega^{\dim} = \underline{K} \otimes_{\mathcal{O}} \tilde{\omega}$ as $\mathcal{O}$-modules. Formally, one may consider dim as a section of the constant sheaf $\mathbb{Z}$ on $\mathcal{V}_{\mathrm{Zar}}$, taking the value $n$ on the subcategory $\mathcal{V}^n_{\mathrm{Zar}}$ of $n$-dimensional varieties.

In [RD] ch. VI §1 we find the following definitions. Let $X$ be a locally noetherian scheme. For a point $x \in X$, let $I$ be an injective hull of $k(x)$ as an $\mathcal{O}_{X,x}$-module, and let $J(x)$ be the skyscraper sheaf which is $I$ on the closed set $\{x\}^-$ and 0 elsewhere. Then $J(x)$ is a quasi-coherent, injective $\mathcal{O}_X$-module.

**Definition 1.1.1.** (Cf. [RD] Ch. VI §1 Def.) A residual complex on $X$ is a complex $\mathcal{R}^{\cdot}$ of quasi-coherent, injective $\mathcal{O}_X$-modules, bounded below, with coherent cohomology sheaves, and such that there is an isomorphism of $\mathcal{O}_X$-modules

$$\bigoplus_{p \in \mathbb{Z}} \mathcal{R}^p \cong \bigoplus_{x \in X} J(x) .$$

**Definition 1.1.2.** A complex $\mathcal{R}^{\cdot}$ of $\mathcal{O}$-modules on $\mathcal{V}_{\mathrm{Zar}}$ is said to be *residual* if $\mathcal{R}^{\cdot}_V := \mathcal{R}^{\cdot}|_V$ is residual for every $V \in \mathcal{V}$.

*Example* 1.1.3. According to [Ye] Prop. 4.4.1, the family of complexes $\{\mathcal{K}^{\cdot}_X\}_{X \in \mathcal{V}}$, together with the isomorphisms $(\gamma^*_g)^{-1} : g^* \mathcal{K}^{\cdot}_Y \xrightarrow{\simeq} \mathcal{K}^{\cdot}_X$ for every open immersion $g : X \to Y$, forms a complex of $\mathcal{O}$-modules on $\mathcal{V}_{\mathrm{Zar}}$. By ibid. Cor. 4.5.6, this is in fact a residual complex, which we denote by $\mathcal{K}^{\cdot}$.

**1.2. Dualizing Structures.** Suppose $V \in \mathcal{V}$ is proper, with structural morphism $\pi$. As in [Ye] Appendix, we say that a pair $(\mathcal{R}^{\cdot}_V, \theta_V)$ is a *residue pair* if $\mathcal{R}^{\cdot}_V$ is a residual complex on $V$, and if $\theta_V : \Gamma(V, \mathcal{R}^{\cdot}_V) \to k$ is a map of complexes, such that for any $\mathcal{F}^{\cdot} \in \mathsf{D}_{\mathrm{qc}}(V)$, the $k$-linear homomorphism

$$\mathrm{Hom}_{\mathsf{D}(V)}(\mathcal{F}^{\cdot}, \mathcal{R}^{\cdot}_V) \to \mathrm{Hom}_{\mathsf{D}(k)}(\mathrm{R}\,\pi_* \mathcal{F}^{\cdot}, k)$$

induced by $\theta_V$ is an isomorphism.

Given a variety $V$, let $V_q := \{x \in V \mid \dim \{x\}^- = q\}$. Consider the decreasing filtration $V^{\cdot} = \{\cdots \supset V^{-2} \supset V^{-1} \supset V^0\}$, called the dimension filtration, with $V^{-p} := \bigcup_{q \leq p} V_q$. If $(\mathcal{R}^{\cdot}_V, \theta_V)$ is a residue pair on $V$, then by taking $\mathcal{F}^{\cdot} = k(x)$ for any closed point $x \in V$, we see that $\mathcal{R}^{\cdot}_V$ is a *Cousin complex* with respect to the filtration $V^{\cdot}$. This means that $\mathcal{R}^{-q}_V \cong \bigoplus_{x \in V_q} \mathcal{R}(x)$, where for $x \in V_q$, $\mathcal{R}(x)$ is the skyscraper sheaf with support $\{x\}^-$ and group of sections $\mathrm{H}^{-q}_x(\mathcal{R}^{\cdot}_V)$ (cf. [RD] Ch. IV §3).

**Lemma 1.2.1.** *Suppose*

$$\begin{tikzcd} U \arrow[r, "g"] \arrow[d, "="] & V \arrow[d, "f"] \\ U \arrow[r, "h"] & W \end{tikzcd} \tag{1.2.1}$$

*is a commutative diagram of morphisms in $\mathcal{V}$, with $g, h$ open immersions and with $V, W$ proper over $k$. Let $(\mathcal{R}^{\cdot}_V, \theta_V)$ and $(\mathcal{R}^{\cdot}_W, \theta_W)$ be residue pairs on $V$ and $W$, resp. Then there exists a homomorphism of complexes of $\mathcal{O}_W$-modules $\theta_f : f_* \mathcal{R}^{\cdot}_V \to \mathcal{R}^{\cdot}_W$, such that the diagram*

$$\begin{tikzcd} \rho_* f_* \mathcal{R}^{\cdot}_V \arrow[r, "\rho_*(\theta_f)"] \arrow[d, "="] & \rho_* \mathcal{R}^{\cdot}_W \arrow[d, "\theta_W"] \\ \pi_* \mathcal{R}^{\cdot}_V \arrow[r, "\theta_V"] & k \end{tikzcd} \tag{1.2.2}$$

*commutes. Here $\pi : V \to \operatorname{Spec} k$ and $\rho : W \to \operatorname{Spec} k$ are the structural morphisms. Moreover, the pullback homomorphism $h^*(\theta_f) : h^* f_* \mathcal{R}^{\cdot}_V \to h^* \mathcal{R}^{\cdot}_W$ is unique.*

*Proof.* The existence and uniqueness of a morphism $[\theta_f]$ in the derived category $\mathsf{D}(W)$ which makes the diagram (1.2.2) commute is a direct consequence of the fact that $(\mathcal{R}^{\cdot}_W, \theta_W)$ is a residue pair. According to [RD] Ch. I Prop. 4.7, because $\mathcal{R}^{\cdot}_W$ is a complex of injective $\mathcal{O}_W$-modules, $[\theta_f]$ is represented by an actual homomorphism of complexes, say $\theta_f$. Now $f_* \mathcal{R}^{\cdot}_V$ needn't be a Cousin complex on $W$, w.r.t. the dimension filtration $W^{\cdot}$. However, since the diagram (1.2.1) is cartesian, we have $h^* f_* \mathcal{R}^{\cdot}_V = g^* \mathcal{R}^{\cdot}_V$. The complexes $h^* f_* \mathcal{R}^{\cdot}_V$ and $h^* \mathcal{R}^{\cdot}_W$ are Cousin complexes on $U$, so we can use [RD] Ch. IV Lemma 3.2 a) to conclude that $h^*(\theta_f)$ is the unique representative of $h^*([\theta_f])$. □

*Example* 1.2.2. Take $W := \mathbf{A}^3_k$, let $P \in W$ be the origin, and let $U := W - \{P\}$. Let $f : V \to W$ be the blow up of $W$ along $\{P\}$. Then $\underline{\Gamma}_{\{P\}}(f_* \mathcal{R}^{\cdot}_V)$ is a complex concentrated in dimensions $-2, -1, 0$, so $f_* \mathcal{R}^{\cdot}_V$ is not a Cousin complex.

The following definition is modelled on the definition of a dualizing $\mathcal{O}$-module of [Li] Def. 4.1.

**Definition 1.2.3.** Let $\mathcal{R}^{\cdot} = \{\mathcal{R}^{\cdot}_V\}$ be a residual complex on $\mathcal{V}_{\mathrm{Zar}}$, and for each open immersion $g : U \to V$, let $\beta_g : g^* \mathcal{R}^{\cdot}_V \xrightarrow{\simeq} \mathcal{R}^{\cdot}_U$ be the restriction isomorphism. A *dualizing structure* on $\mathcal{R}^{\cdot}$ is a family of maps of complexes

$$\theta_V : \Gamma(V, \mathcal{R}^{\cdot}_V) \to k \ ,$$

one for each *proper* $k$-variety $V$, such that:

(i) The pair $(\mathcal{R}^{\cdot}_V, \theta_V)$ is a residue pair.
(ii) Given a commutative diagram of morphisms (1.2.1), the homomorphism of complexes $h^*(\theta_f) : h^* f_* \mathcal{R}^{\cdot}_V \to h^* \mathcal{R}^{\cdot}_W$ of Lemma 1.2.1 makes the following diagram commute:

$$\begin{tikzcd} h^* f_* \mathcal{R}^{\cdot}_V \arrow[r, "h^*(\theta_f)"] \arrow[d, "="] & h^* \mathcal{R}^{\cdot}_W \arrow[d, "\beta_h"] \\ g^* \mathcal{R}^{\cdot}_V \arrow[r, "\beta_g"] & \mathcal{R}^{\cdot}_U \end{tikzcd}$$

We call $(\mathcal{R}^{\cdot}, \{\theta_V\})$ a dualizing complex on $\mathcal{V}$.

*Example* 1.2.4. Let $\mathcal{K}^{\cdot}$ be the residual complex of example 1.1.3. Consider the family of maps $\mathrm{Tr}_\pi : \pi_* \mathcal{K}^{\cdot}_X \to k$ of [Ye] Thm. 4.4.14, where for each $X$ proper over $k$, $\pi : X \to \operatorname{Spec} k$ is the structural morphism. According to [Ye] Appendix Theorems 1 and 3, $(\mathcal{K}^{\cdot}, \{\mathrm{Tr}_\pi\})$ is a dualizing complex on $\mathcal{V}$.

*Example* 1.2.5. Here is an outline of much quicker construction of a dualizing complex on $\mathcal{V}$. Suppose for each proper variety $X$, with structural morphism $\pi$, we have a dualizing pair $(\mathcal{I}^{\cdot}_X, \psi_X)$, i.e. a complex $\mathcal{I}^{\cdot}_X \in \mathsf{D}^+_{\mathrm{c}}(X)$ and a morphism $\psi_X : \mathrm{R}\,\pi_* \mathcal{I}^{\cdot}_X \to k$ in $\mathsf{D}(k)$, which represent the functor $\mathcal{F}^{\cdot} \mapsto \mathrm{Hom}_{\mathsf{D}(k)}(\mathrm{R}\,\pi_* \mathcal{F}^{\cdot}, k)$ on $\mathsf{D}_{\mathrm{qc}}(X)$. It follows that $\mathcal{I}^{\cdot}_X$ is a dualizing complex on $X$ (in the sense of [RD]), so the associated Cousin complex $\mathcal{R}^{\cdot}_X := \mathrm{E}(\mathcal{I}^{\cdot}_X)$ is a residual complex (cf. [RD] Ch. VI Prop. 1.1), and we get a residue pair $(\mathcal{R}^{\cdot}_X, \theta_X)$.

It is possible to find a residual complex $\mathcal{R}^{\cdot}$ on $\mathcal{V}_{\mathrm{Zar}}$ such that $\mathcal{R}^{\cdot}|_X = \mathcal{R}^{\cdot}_X$ for $X$ proper, and such that $(\mathcal{R}^{\cdot}, \{\theta_X\})$ is a dualizing complex on $\mathcal{V}$. This may be achieved by following [Li] pp. 40-47, *mutatis mutandis*; e.g. replacing Prop. 4.3 of loc. cit. by [Ve] Thm. 2, esp. the proof of case 1 of this theorem.

As for getting a dualizing pair $(\mathcal{I}^{\cdot}_X, \psi_X)$ on a proper variety $X$, this can be done in the following way, which is implicit in no. 4 of Deligne's appendix to [RD]. For an affine open subscheme $g : U = \operatorname{Spec} B \hookrightarrow X$, let $\mathcal{I}$ be the quasi coherent sheaf on $U$ corresponding to the $B$-module $\mathrm{Hom}_k(B, k)$, and let $\mathcal{I}_U := g_* \mathcal{I}$. Then $\mathcal{I}_U$ is a quasi coherent injective $\mathcal{O}_X$-module. Choose a finite open affine cover $\{U_i\}$ of $X$. For $p \in \mathbb{N}$ define

$$\mathcal{I}_X^{-p} := \bigoplus_{i_0 < \cdots < i_p} \mathcal{I}_{U_{i_0} \cap \cdots \cap U_{i_p}} .$$

Then $\mathcal{I}_X^{\cdot}$ is a complex in a natural way, and there is a homomorphism $\psi_X : \pi_* \mathcal{I}_X^0 \to k$, arising from the evaluation at 1 map: $\mathrm{Hom}_k(B, k) \to k$. It is easy to see that the pair $(\mathcal{I}_X^{\cdot}, \psi_X)$ represents the aforementioned functor. A more subtle fact is that $\mathcal{I}_X^{\cdot} \in \mathsf{D}^+_{\mathrm{c}}(X)$, i.e. that it has coherent cohomologies; this follows from the "local nature of $\pi^!$", cf. [RD] Appendix no. 5, or [Ve] Lemma 1.

**Proposition 1.2.6.** *Let $(\mathcal{R}^{\cdot}, \{\theta_V\})$ and $(\mathcal{R}'^{\cdot}, \{\theta'_V\})$ be two dualizing complexes on $\mathcal{V}$. Then there is a unique isomorphism of complexes $\lambda : \mathcal{R}^{\cdot} \xrightarrow{\simeq} \mathcal{R}'^{\cdot}$ which is compatible with the dualizing structures. By this we mean that for each proper variety $V$ in $\mathcal{V}$, the diagram below commutes:*

$$\begin{tikzcd}
\Gamma(V, \mathcal{R}^{\cdot}_V) \arrow[r, "\lambda_V"] \arrow[d, "\theta_V"] & \Gamma(V, \mathcal{R}'^{\cdot}_V) \arrow[d, "\theta'_V"] \\
k \arrow[r, "="] & k
\end{tikzcd}$$

*Proof.* For $V$ proper define $\lambda_V$ so as to make the diagram above commute. If $g : U \to V$ is an open immersion, with $V$ proper, define $\lambda_U := \beta'_g \circ g^*(\lambda_V) \circ \beta_g^{-1}$. One checks, using the proof of [Ye] Appendix Thm. 3, that $\lambda_U$ is independent of the compactification $g$. In general, any variety $W$ is covered by compactifiable open sets, $W = \bigcup_i U_i$; as shown in loc. cit. Exercise, the isomorphisms $\lambda_{U_i}$ glue to give $\lambda_W$. We point out that in this manner it is possible to bypass Nagata's difficult theorem on compactifications in [Na]. $\square$

**1.3. Canonical Structures.** We introduce the notion of a canonical structure on a residual complex of $\mathcal{O}$-modules, along the lines of [Li] Def. 2.1. Denote by $\mathcal{V}^{\mathrm{sm}}_{\mathrm{Zar}}$ the full subcategory of $\mathcal{V}_{\mathrm{Zar}}$ consisting of smooth $k$-varieties.

**Definition 1.3.1.** Let $\mathcal{R}^{\cdot}$ be a residual complex on $\mathcal{V}_{\mathrm{Zar}}$. A *canonical structure* on $\mathcal{R}^{\cdot}$ consists of the following data:

(a) A quasi-isomorphism of complexes of $\mathcal{O}$-modules:

$$\gamma : \Omega^{\dim}[\dim]|_{\mathcal{V}^{\mathrm{sm}}_{\mathrm{Zar}}} \to \mathcal{R}^{\cdot}|_{\mathcal{V}^{\mathrm{sm}}_{\mathrm{Zar}}} ,$$

i.e. for each smooth variety $V$ of dimension $n$, a quasi-isomorphism $\gamma_V : \Omega^n_{V/k}[n] \to \mathcal{R}^{\cdot}_V$, compatible with open immersions.

(b) For each finite, generically étale morphism $f : V \to W$ in $\mathcal{V}$, a homomorphism of complexes of $\mathcal{O}_W$-modules

$$\theta_f : f_* \mathcal{R}^{\cdot}_V \to \mathcal{R}^{\cdot}_W .$$

These data are required to satisfy the conditions below, for any morphism $f : V \to W$ as in (b).

(i) $\theta_f$ induces an isomorphism of complexes $f_* \mathcal{R}^{\cdot}_V \to \underline{\mathrm{Hom}}_W(f_* \mathcal{O}_V, \mathcal{R}^{\cdot}_W)$.

(ii) Let $g : V^{\mathrm{sm}} \to V$ be the inclusion of the smooth locus, and let $v \in V^{\mathrm{sm}} \subset V$ be the generic point. Say $V$ has dimension $n$. Then $\gamma_{V^{\mathrm{sm}}}$ induces an isomorphism $\gamma_{V,v} : \Omega^n_{k(V)/k} \xrightarrow{\simeq} \mathcal{R}^{-n}_{V,v}$. The same happens on $W$, which has generic point $w$. The condition is that the diagram below commutes:

$$\begin{tikzcd}
\Omega^n_{k(V)/k} \arrow[r, "\gamma_{V,v}"] \arrow[d, "\mathrm{Tr}"] & \mathcal{R}^{-n}_{V,v} = (f_* \mathcal{R}^{-n}_V)_w \arrow[d, "\theta_f"] \\
\Omega^n_{k(W)/k} \arrow[r, "\gamma_{W,w}"] & \mathcal{R}^{-n}_{W,w}
\end{tikzcd}$$

We call $(\mathcal{R}^{\cdot}, \{\gamma_V\}, \{\theta_f\})$ a canonical complex on $\mathcal{V}$.

Observe that if $\mathcal{R}^{\cdot}$ admits a canonical structure, then on any variety $V$, $\mathcal{R}^{\cdot}_V$ is a Cousin complex w.r.t. the dimension filtration $V^{\cdot}$.

*Remark* 1.3.2. If $(\mathcal{R}^{\cdot}, \{\gamma_V\}, \{\theta_f\})$ is a canonical complex on $\mathcal{V}$, then $(\mathrm{H}^{-\dim} \mathcal{R}^{\cdot}, \{\mathrm{H}^{-\dim}(\gamma_V)\}, \{\mathrm{H}^{-\dim}(\theta_f)\})$ is a canonical $\mathcal{O}$-module, in the sense of [Li] Def. 2.1. It follows that under the isomorphism of $\mathcal{O}$-modules $\underline{K} \otimes_{\mathcal{O}} \Omega^{\dim} \xrightarrow{\simeq} \mathcal{R}^{-\dim}$ induced by $\{\gamma_V\}$, we have $\tilde{\omega} \xrightarrow{\simeq} \mathrm{H}^{-\dim} \mathcal{R}^{\cdot}$.

*Example* 1.3.3. Consider the residual complex $\mathcal{K}^{\cdot}$ of example 1.1.3. The quasi-isomorphisms $\mathrm{C}_X : \Omega^n_{X/k}[n] \to \mathcal{K}^{\cdot}_X$, for $X$ smooth over $k$, and the trace maps $\mathrm{Tr}_f : f_* \mathcal{K}^{\cdot}_X \to \mathcal{K}^{\cdot}_Y$, for $f : X \to Y$ finite, of [Ye] §4.4 and 4.5, are a canonical structure on $\mathcal{K}^{\cdot}$. This is evident from the definition of maps $\mathrm{Tr}_f$.

**Proposition 1.3.4.** (Cf. [Li] Cor. 2.3) *Let $(\mathcal{R}^{\cdot}, \{\gamma_V\}, \{\theta_f\})$ and $(\mathcal{R}'^{\cdot}, \{\gamma'_V\}, \{\theta'_f\})$ be two canonical complexes on $\mathcal{V}$. Then there exists a unique isomorphism of complexes of $\mathcal{O}$-modules $\lambda : \mathcal{R}^{\cdot} \xrightarrow{\simeq} \mathcal{R}'^{\cdot}$, compatible with $\{\gamma_V\}$ and $\{\gamma'_V\}$. The isomorphism $\lambda$ is also compatible with $\{\theta_f\}$ and $\{\theta'_f\}$.*

*Proof.* For $W$ smooth set $\lambda_W := \gamma'_W \circ (\gamma_W)^{-1}$; this is possible since $(\gamma_W)^{-1}$, a priori only defined in the derived category $\mathsf{D}(W)$, is actually a homomorphism of complexes (cf. [RD] Ch. IV Lemma 3.2). If $f : V \to W$ is a morphism as in (b), denote by $\gamma^{\flat}_f$ the isomorphism $\mathcal{R}^{\cdot}_V \xrightarrow{\simeq} f^{\flat} \mathcal{R}^{\cdot}_W$ induced by $\theta_f$; here $f^{\flat}$ is the functor of [Ye] Def. 4.4.2. Set $\lambda_V := (\gamma'^{\flat}_f)^{-1} \circ \gamma^{\flat}_f$. Since $\mathrm{End}_{\mathsf{D}(V)}(\mathcal{R}^{\cdot}_V) = \Gamma(V, \mathcal{O}_V) \subset k(V)$, we see that $\lambda_V$ is completely determined by its action on the generic stalk $\mathcal{R}^{-n}_V \cong \Omega^n_{k(V)/k}$ ($n = \dim V$).

Now suppose $W' \subset W$ is an open subset such that, setting $V' := f^{-1}(W')$, the morphism $f' := f|_{V'} : V' \to W'$ is étale (and finite). Then $V'$ is smooth over $k$. Condition (ii) implies that the two definitions of $\lambda_{V'}$ - the intrinsic one, and the restriction $\lambda_V|_{V'}$, coincide. This, and the fact that $\lambda_V$ is determined by its action on the generic stalk $\Omega^n_{k(V)/k}$, show that $\lambda_V$ is independent of $f$.

For the general case, cover $V$ by open subsets $\{V_\alpha\}$, such that each $V_\alpha$ admits a finite, generically étale morphism to some smooth $W_\alpha$. One may take $\{V_\alpha\}$ to be any affine cover, and $W_\alpha = \mathbf{A}^n_k$; cf. [Ma] Remark on p. 90, or [Li] Appendix A. By the above arguments, the resulting isomorphisms $\lambda_{V_\alpha}$ patch. $\square$

### 1.4. Residue Complexes on $\mathcal{V}$.

**Definition 1.4.1.** A *residue complex* on $\mathcal{V}$ is a residual complex $\mathcal{R}^{\cdot}$ on $\mathcal{V}_{\mathrm{Zar}}$, together with a dualizing structure $\{\theta_V\}$ and a canonical structure $(\{\gamma_V\}, \{\theta_f\})$, satisfying the following conditions:

(a) (Compatibility) For any finite, generically étale morphism $f : V \to W$ between proper $k$-varieties, the diagram below commutes:

$$\begin{array}{ccc} \mathrm{H}^0(W, f_* \mathcal{R}^{\cdot}_V) & \xrightarrow{\theta_f} & \mathrm{H}^0(W, \mathcal{R}^{\cdot}_W) \\ \Big\downarrow{=} & & \Big\downarrow{\theta_W} \\ \mathrm{H}^0(V, \mathcal{R}^{\cdot}_V) & \xrightarrow{\theta_V} & k \end{array}$$

(b) (Normalization) For any $n \geq 0$, let $\mathbf{P} := \mathbf{P}^n_k$, and let $\int_{\mathbf{P}} : \mathrm{H}^n(\mathbf{P}, \Omega^n_{\mathbf{P}/k}) \xrightarrow{\simeq} k$ be the canonical projective trace map of [Li] Prop. 8.4. Then

$$\theta_{\mathbf{P}} \circ \mathrm{H}^n(\mathbf{P}, \gamma_{\mathbf{P}}) = \int_{\mathbf{P}} .$$

Given a variety $X$ over $k$, with structural morphism $\pi$, set $\mathcal{K}^{\mathrm{RD}\cdot}_X := \pi^{\triangle} k$, where $(-)^{\triangle}$ is the pseudo-functor of [RD] Ch. VI. The variance data of ibid. Thm. 3.1 make $\mathcal{K}^{\mathrm{RD}\cdot} = \{\mathcal{K}^{\mathrm{RD}\cdot}_X\}_{X \in \mathcal{V}}$ into a residual complex on $\mathcal{V}_{\mathrm{Zar}}$. For $X$ smooth of dimension $n$ over $k$ define $\gamma^{\mathrm{RD}}_X : \Omega^n_{X/k}[n] \xrightarrow{\simeq} \mathcal{K}^{\mathrm{RD}\cdot}_X$ using the morphism $\phi_\pi$ of ibid. Thm. 3.1 d). For $f : X \to Y$ finite and generically étale, define $\theta^{\mathrm{RD}}_f : f_* \mathcal{K}^{\mathrm{RD}\cdot}_X \to \mathcal{K}^{\mathrm{RD}\cdot}_Y$ using $\mathrm{Tr}_f$ of ibid. Thm. 4.2. For $X$ proper define $\theta^{\mathrm{RD}}_X : \pi_* \mathcal{K}^{\mathrm{RD}\cdot}_X \to k$ using $\mathrm{Tr}_\pi$; this is a homomorphism of complexes by ibid. Ch. VII Thm. 2.1.

Here are the proofs of Prop. 0.2.2, Thm. 0.2.3 and Thm. 0.2.1:

*Proof.* (of Prop. 0.2.2) $(\mathcal{K}^{\mathrm{RD}\cdot}, \{\theta^{\mathrm{RD}}_X\})$ is a dualizing complex on $\mathcal{V}$, according to [RD] Ch. VII Thm. 3.3 and Ch. VI Thm. 5.6.

Let us show that $(\mathcal{K}^{\mathrm{RD}\cdot}, \{\gamma^{\mathrm{RD}}_X\}, \{\theta^{\mathrm{RD}}_f\})$ is a canonical complex. Condition (i) of Def. 1.3.1 follows from [RD] Ch. VI Thm. 4.2 condition **TRA 2**. As for condition (ii), let $f : X \to Y$ be finite and generically étale. By restricting $\theta^{\mathrm{RD}}_f$ to a smooth open subset $U \subset Y$ s.t. $f|_{f^{-1}(U)}$ is étale, we may assume that $Y$ is smooth over $k$ (say, of dimension $n$) and that $f$ is étale. By [RD] Ch. VI Thm. 3.1, Ch. III Cor. 8.3 and the subsequent remark, it follows that $\mathrm{Tr}_f : f_* \Omega^n_{X/k} \cong f_* \mathrm{H}^{-n} \mathcal{K}^{\mathrm{RD}\cdot}_X \to \Omega^n_{Y/k} \cong f_* \mathrm{H}^{-n} \mathcal{K}^{\mathrm{RD}\cdot}_Y$ is induced from the "classical" trace map $\mathrm{Tr}_f : f_* \mathcal{O}_X \to \mathcal{O}_Y$. Passing to generic stalks we deduce condition (ii).

The compatibility condition (a) of Def. 1.4.1 is a consequence of the transitivity of the trace, [RD] Ch. VI Thm. 4.2 condition **TRA 1**. The normalization condition (b) follows from ibid. Ch. III Thm. 10.5 condition **TRA 3**; see also Ch. VII Cor. 3.4 (b). □

*Proof.* (of Thm. 0.2.3, relying on Thm. 0.2.10, which itself is proved in §2.3) In examples 1.2.4 and 1.3.3 it was shown that $(\mathcal{K}^{\cdot}, \{(-1)^{\dim} \mathrm{Tr}_\pi\})$ and $(\mathcal{K}^{\cdot}, \{\mathrm{C}_X\}, \{\mathrm{Tr}_f\})$ are a dualizing complex and a canonical complex, respectively. Condition 1.4.1

(a) holds in virtue of [Ye] Cor. 4.4.12 (b). It remains to check the normalization condition (b).

Fix a natural number $n$. Let $T_0, \ldots, T_n$ be the homogeneous coordinates of the projective space $\mathbf{P} := \mathbf{P}^n_k$, in other words $\mathbf{P} = \operatorname{Proj} k[T_0, \ldots, T_n]$. Consider the chain of points $\xi := (x_0, \ldots, x_n)$, with $x_i$ being the homogeneous prime ideal $(T_1, \ldots, T_i) \subset k[T_0, \ldots, T_n]$. Introduce inhomogeneous coordinates $t_i := T_i/T_0$, $i = 1, \ldots, n$; these form a regular system of parameters at the point $x := x_n = (1, 0, \ldots, 0)$. Consider the local cohomology class

$$\alpha := \begin{bmatrix} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n \\ t_1, \ldots, t_n \end{bmatrix} \in \mathrm{H}^n_x(\Omega^n_{\mathbf{P}/k}) \ .$$

By [Li] Prop. 8.5, the composite map

$$\mathrm{H}^n_x(\Omega^n_{\mathbf{P}/k}) \xrightarrow{\text{canonical}} \mathrm{H}^n(\mathbf{P}, \Omega^n_{\mathbf{P}/k}) \xrightarrow{\int_{\mathbf{P}}} k$$

sends $\alpha \mapsto 1$. On the other hand, by Thm. 0.2.10, and by the definition of $\mathrm{Tr}_\pi$ in [Ye], we find that the map

$$\mathrm{H}^n_x(\Omega^n_{\mathbf{P}/k}) = \mathrm{H}^0_x(\Omega^n_{\mathbf{P}/k}[n]) \xrightarrow{\mathrm{C}_{\mathbf{P}}} \mathrm{H}^0_x(\mathcal{K}^{\cdot}_{\mathbf{P}}) = \mathcal{K}(x) \to \mathrm{H}^0(\mathbf{P}, \mathcal{K}^{\cdot}_{\mathbf{P}}) \xrightarrow{(-1)^n \mathrm{Tr}_\pi} k$$

sends

$$\begin{aligned} \alpha \mapsto (-1)^{\binom{n}{2}+n^2+n} \operatorname{Res}_{x,k} \circ \delta_\xi & \left( \frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1 \cdots t_n} \right) \\ & = (-1)^{\binom{n}{2}+n^2+n} \operatorname{Res}_{\xi,k} \left( \frac{\mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n}{t_1 \cdots t_n} \right) = 1 \ . \end{aligned}$$

□

*Remark* 1.4.2. Let us indicate another, more accessible, way of constructing a residue complex on $\mathcal{V}$. Let $(\mathcal{R}^{\cdot}, \{\theta_X\})$ be a dualizing complex on $\mathcal{V}$ - say the one provided by Example 1.2.5. Then the proof of [Li] Thm. 0.3B, *mutatis mutandis*, gives the existence of a canonical structure $(\{\gamma_V\}, \{\theta_f\})$ on $\mathcal{R}^{\cdot}$, compatible with $\{\theta_X\}$.

*Proof.* (of Thm. 0.2.1) Let $\lambda^{\mathrm{du}} : \mathcal{R}^{\cdot} \xrightarrow{\simeq} \mathcal{R}'^{\cdot}$ (resp. $\lambda^{\mathrm{can}} : \mathcal{R}^{\cdot} \xrightarrow{\simeq} \mathcal{R}'^{\cdot}$) be the isomorphism of complexes arising from the dualizing structures, cf. Prop. 1.2.6 (resp. the canonical structures, cf. Prop. 1.3.4). Let $\lambda$ be the automorphism $(\lambda^{\mathrm{can}})^{-1} \circ \lambda^{\mathrm{du}}$ of $\mathcal{R}^{\cdot}$. We must prove that $\lambda = 1$.

Fix a natural number $n$. For any $X \in \mathcal{V}^n_{\mathrm{Zar}}$, one has $\lambda_X \in \operatorname{End}_{\mathsf{D}(X)}(\mathcal{R}^{\cdot}_X) = \Gamma(X, \mathcal{O}_X) \subset k(X)$, since $\mathcal{R}^{\cdot}_X$ is residual. Given a finite, generically étale morphism $f : X \to Y$, $\lambda$ commutes with $\theta_f$. Because the trace pairing $k(X) \times \Omega^n_{k(X)/k} \to \Omega^n_{k(Y)/k}$ is nondegenerate, we conclude that $\lambda_X = f^*(\lambda_Y) \in k(X)$. If $g : U \to X$ is an open immersion then $\lambda_U = g^*(\lambda_X) \in k(U)$. But on $\mathbf{P}^n_k$ we have by definition $\lambda_{\mathbf{P}^n_k} = 1 \in k$. Now given any $X \in \mathcal{V}^n$, we can find an open immersion $g : U \to X$ and a finite, generically étale morphism $f : U \to \mathbf{A}^n_k$ (cf. proof of Prop. 1.3.4). Hence $\lambda_X = \lambda_{\mathbf{P}^n_k} = 1$. □

Since residue complexes on $\mathcal{V}$ exist, we can conclude:

**Corollary 1.4.3.** *Let $\mathcal{R}^{\cdot}$ be a residual complex on $\mathcal{V}_{\mathrm{Zar}}$. Then any dualizing structure on $\mathcal{R}^{\cdot}$ induces a unique canonical structure on it, to make $\mathcal{R}^{\cdot}$ into a residue complex; and vice versa.*

## 2. Local Cohomology and Residual Complexes

**2.1. A Coboundary Calculation.** Let $(A,\mathfrak{m})$ be an equidimensional, catenary, noetherian local ring, of dimension $d$. Let $(\mathcal{R}^{\cdot},\partial)$ be a residual complex over $A$. Assume that the codimension function associated to $\mathcal{R}^{\cdot}$ (in the sense of [RD] p. 282) equals the height function $\mathrm{ht} : \operatorname{Spec} A \to \mathbb{Z}$. This implies that the complex $\mathcal{R}^{\cdot}$ is concentrated in the dimension range $[0,d]$. Given a prime $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathrm{ht}(\mathfrak{p}) = c$, set $\mathcal{R}(\mathfrak{p}) := \mathrm{H}^c_{\mathfrak{p}}(\mathcal{R}^{\cdot})$. Then $\mathcal{R}(\mathfrak{p})$ is an injective hull of the residue field $k(\mathfrak{p})$ over the local ring $A_{\mathfrak{p}}$. For any integer $c$ we get a canonical decomposition $\mathcal{R}^c = \bigoplus_{\mathrm{ht}(\mathfrak{p})=c} \mathcal{R}(\mathfrak{p})$.

Given a saturated chain $(\mathfrak{p},\mathfrak{q})$ in $\operatorname{Spec} A$ (i.e. $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$), let $\partial_{(\mathfrak{p},\mathfrak{q})}$ be the operator

$$\partial_{(\mathfrak{p},\mathfrak{q})} : \mathcal{R}(\mathfrak{p}) \hookrightarrow \mathcal{R}^c \xrightarrow{\partial} \mathcal{R}^{c+1} \twoheadrightarrow \mathcal{R}(\mathfrak{q})$$

where $c = \mathrm{ht}(\mathfrak{p})$ (cf. [EZ] Ch. I §1.2). For an element $t \in A$ let $\partial_{[t]} : \mathcal{R}^{\cdot} \to \mathcal{R}^{\cdot}$ be the operator of degree 1:

$$\partial_{[t]} := \sum_{t \in \mathfrak{q}-\mathfrak{p}} \partial_{(\mathfrak{p},\mathfrak{q})} \tag{2.1.1}$$

where $(\mathfrak{p},\mathfrak{q})$ runs over the saturated chains.

Through the end of Section 2, $\underline{t} = (t_1,\ldots,t_d)$ is a fixed system of parameters in $A$. Since the ring $A$ is catenary, according to [Ma] (12.I) Thm. 18 and (12.K) Prop., the following condition is satisfied:

(2.1.2) Let $\mathfrak{a} \subset A$ be an ideal generated by $c$ elements of $\{t_1,\ldots,t_d\}$. Then any minimal prime $\mathfrak{p}$ over $\mathfrak{a}$ has $\mathrm{ht}(\mathfrak{p}) = c$.

Given an element $s \in A$, the localization of an $A$-module $M$ with respect to powers of $s$ shall be denoted by $M_s$. Consider the exact sequence of complexes

$$0 \to \Gamma_{(s)}\mathcal{R}^{\cdot} \to \mathcal{R}^{\cdot} \to \mathcal{R}^{\cdot}_s \to 0\ , \tag{2.1.3}$$

where $\Gamma_{(s)}$ is the submodule of elements annihilated by some power of the ideal $(s)$. The sequence (2.1.3) is canonically split (as graded $A$-modules, not as complexes !); in fact

$$\begin{aligned} \mathcal{R}^{\cdot}_s &\cong \textstyle\bigoplus_{s\notin\mathfrak{p}} \mathcal{R}(\mathfrak{p}) \\ \Gamma_{(s)}\mathcal{R}^{\cdot} &\cong \textstyle\bigoplus_{s\in\mathfrak{p}} \mathcal{R}(\mathfrak{p})\,. \end{aligned}$$

Let $\partial_s$ be the coboundary operator of the localized complex $\mathcal{R}^{\cdot}_s$. The operator $\partial_{[s]}$ of (2.1.1) is an $A$-linear homomorphism $\partial_{[s]} : \mathcal{R}^{\cdot}_s \to \Gamma_{(s)}\mathcal{R}^{\cdot}$; it "goes across the boundary" : from the open set $\{s \neq 0\}$ to the closed set $\{s = 0\}$ in $\operatorname{Spec} A$.

From the condition (2.1.2) on our $d$-tuple $\underline{t} = (t_1,\ldots,t_d)$, it follows that if $\mathfrak{p}$ is a prime of $A$ s.t. $\mathrm{ht}(\mathfrak{p}) = c$ and $t_1,\ldots,t_c \in \mathfrak{p}$, then $t_{c+1},\ldots,t_d \notin \mathfrak{p}$. Therefore

$$\Gamma_{(t_1,\ldots,t_c)}\mathcal{R}^c \subset \mathcal{R}^c_{t_{c+1}\cdots t_d}\ . \tag{2.1.4}$$

Suppose we are given an $A$-module $M$, and a homomorphism $\eta : M \to \mathcal{R}^0$, such that $\partial \circ \eta = 0$. We may view $\eta$ as a homomorphism of complexes $\eta : M \to \mathcal{R}^{\cdot}$.

Since no $t_i$ lies in any prime $\mathfrak{p}$ of $A$ with $\mathrm{ht}(\mathfrak{p}) = 0$, we can localize with respect to $t_1 \cdots t_d$ to get $\eta : M_{t_1 \cdots t_d} \to \mathcal{R}^0_{t_1 \cdots t_d} = \mathcal{R}^0$.

Choose some $u \in M_{t_1 \cdots t_d}$, and define

$$\begin{aligned} u_0 &:= \eta(u) \in \mathcal{R}^0 \\ u_i &:= \partial_{[t_i]}(u_{i-1}) \in \mathcal{R}^i \ ; \ i = 1, \ldots, d \ . \end{aligned} \tag{2.1.5}$$

Therefore $u_i = \partial_{[t_i]} \circ \cdots \circ \partial_{[t_1]} \circ \eta(u)$, and in particular $u_i \in \Gamma_{(t_1, \ldots, t_i)} \mathcal{R}^i \subset \mathcal{R}^i_{t_{i+1} \cdots t_d}$.

**Lemma 2.1.1.** *For $i = 1, \ldots, d$ one has $u_i = \partial_{t_{i+1} \cdots t_d}(u_{i-1}) \in \mathcal{R}^i_{t_{i+1} \cdots t_d}$.*

*Proof.* The proof proceeds by induction on $i$. If $i \geq 2$ we may assume that $u_{i-1} = \partial_{t_i \cdots t_d}(u_{i-2})$. Let $\mathfrak{p} \in \operatorname{Spec} A$ have $\mathrm{ht}(\mathfrak{p}) = i$, and let $\partial^{i-1}_{\mathfrak{p}} : \mathcal{R}^{i-1}_{\mathfrak{p}} \to \mathcal{R}(\mathfrak{p})$ be the localization of $\partial^{i-1} : \mathcal{R}^{i-1} \to \mathcal{R}^i$ at $\mathfrak{p}$. We must show that if $t_1, \ldots, t_{i-1} \in \mathfrak{p}$ but $t_i, \ldots, t_d \notin \mathfrak{p}$, then $\partial^{i-1}_{\mathfrak{p}}(u_{i-1}) = 0$. But $\partial^{i-1}_{\mathfrak{p}}$ factors through $\partial^{i-1}_{t_i \cdots t_d} : \mathcal{R}^{i-1}_{t_i \cdots t_d} \to \mathcal{R}^i_{t_i \cdots t_d}$. If $i = 1$ then

$$\partial^0_{t_1 \cdots t_d}(u_0) = \partial^0_{t_1 \cdots t_d} \circ \eta(u) = 0 \ ;$$

and if $i \geq 2$ then

$$\partial^{i-1}_{t_i \cdots t_d}(u_{i-1}) = \partial^{i-1}_{t_i \cdots t_d} \circ \partial^{i-2}_{t_i \cdots t_d}(u_{i-2}) = 0$$

since $(\mathcal{R}^{\cdot}_{t_i \cdots t_d}, \partial_{t_i \cdots t_d})$ is a complex. $\square$

**2.2. Koszul-Residual Complexes.** Let $(A, \mathfrak{m})$, $(\mathcal{R}^{\cdot}, \partial)$, $\underline{t} = (t_1, \ldots, t_d)$ and $\eta : M \to \mathcal{R}^0$ be as in Subsection 2.1. Let $\mathbf{K}^{\cdot}(t_i)$ be the complex $\cdots \to 0 \to A \xrightarrow{1} A_{t_i} \to 0 \to \cdots$ with $A$ in dimension 0. The complex $\mathbf{K}^{\cdot}(\underline{t})$ is the tensor product $\mathbf{K}^{\cdot}(t_1) \otimes_A \cdots \otimes_A \mathbf{K}^{\cdot}(t_d)$ (cf. [LS] §3). It is the direct limit of the Koszul complexes $K^{\cdot}(\underline{t}^m; A)$, as $m \to \infty$. Denote the coboundary operator of $\mathbf{K}^{\cdot}(\underline{t})$ by d. According to [LS] Lemma (3.2.1), for any complex $N^{\cdot} \in \mathsf{D}^+(A)$, the inclusion $\Gamma_{\mathfrak{m}} N^{\cdot} \subset N^{\cdot} \cong \mathbf{K}^0(\underline{t}) \otimes_A N^{\cdot}$ induces an isomorphism $\mathrm{R}\,\Gamma_{\mathfrak{m}} N^{\cdot} \xrightarrow{\simeq} \mathbf{K}^{\cdot}(\underline{t}) \otimes_A N^{\cdot}$ in $\mathsf{D}^+(A)$, natural in $N^{\cdot}$. Taking $N^{\cdot} = M$, one gets an isomorphism

$$\mathrm{H}^d_{\mathfrak{m}}(M) = \mathrm{H}^d\, \mathrm{R}\,\Gamma_{\mathfrak{m}} M \xrightarrow{\simeq} \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A M) \ .$$

It turns out that this isomorphism differs from the one in [LC] Thm. 2.3, by a factor of $(-1)^d$; see [Sa]. Hence given an element $m \in M$, the generalized fraction $\left[\begin{smallmatrix} m \\ \underline{t} \end{smallmatrix}\right] \in \mathrm{H}^d_{\mathfrak{m}}(M)$ is sent to the class $(-1)^d [t_1^{-1} \otimes \cdots \otimes t_d^{-1} \otimes m] \in \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A M)$.

We are interested in the complex $\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot}$. By definition,

$$(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})^n = \bigoplus_{i+j=n} \mathbf{K}^i(\underline{t}) \otimes_A \mathcal{R}^j = \bigoplus_{i+j=n} \ \bigoplus_{l_1 < \cdots < l_i} \mathcal{R}^j_{t_{l_1} \cdots t_{l_i}}$$

and the coboundary $D$ is given by:

$$D^n = \sum_{i+j=n} \mathrm{d}^i \otimes 1 + (-1)^i 1 \otimes \partial^j \ .$$

**Lemma 2.2.1.** *Let $u_0$, $u_d$ be the elements of $\mathcal{R}^{\cdot}$ defined in formula (2.1.5). Consider them as elements of $(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})^d$ as follows: $u_0 \in \mathcal{R}^0_{t_1 \cdots t_d} = \mathbf{K}^d(\underline{t}) \otimes_A \mathcal{R}^0$; and $u_d \in \mathcal{R}^d = \mathbf{K}^0(\underline{t}) \otimes_A \mathcal{R}^d$. Then*

$$u_0 - (-1)^{\binom{d+1}{2}} u_d \in (\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})^d$$

*is a coboundary.*

*Proof.* Consider all $u_i$ as elements of $(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})^d$:

$$u_i \in \mathcal{R}^i_{t_{i+1} \cdots t_d} \subset \mathbf{K}^{d-i}(\underline{t}) \otimes_A \mathcal{R}^i \ ; \ i = 0, \ldots, d \ .$$

Define

$$v_i := u_i \in \mathcal{R}^i_{t_{i+2} \cdots t_d} \subset \mathbf{K}^{d-i-1}(\underline{t}) \otimes_A \mathcal{R}^i \ ; \ i = 0, \ldots, d-1 \ .$$

By Lemma 2.1.1, $(1 \otimes \partial)(v_{i-1}) = u_i$. On the other hand, since $u_i \in \Gamma_{(t_1, \ldots, t_i)} \mathcal{R}^i$, it follows that

$$(\mathrm{d} \otimes 1)(v_i) \in \mathcal{R}^i_{t_{i+1} \cdots t_d} \subset \mathbf{K}^{i+1}(\underline{t}) \otimes_A \mathcal{R}^{d-i-1} \ ,$$

so $(\mathrm{d} \otimes 1)(v_i) = u_i$. Thus

$$D(v_i) = (\mathrm{d} \otimes 1)(v_i) + (-1)^i (1 \otimes \partial)(v_i) = u_{i+1} + (-1)^i u_i$$

and $u_{i+1} \equiv (-1)^{i+1} u_i$ (mod coboundaries). $\square$

**Theorem 2.2.2.** *Consider the homomorphism $\mathrm{H}^d_{\mathfrak{m}}(\eta) : \mathrm{H}^d_{\mathfrak{m}}(M) \to \mathrm{H}^d_{\mathfrak{m}}(\mathcal{R}^{\cdot}) = \mathcal{R}(\mathfrak{m})$ induced by $\eta : M \to \mathcal{R}^{\cdot}$. For any $m \in M$ one has*

$$\mathrm{H}^d_{\mathfrak{m}}(\eta) \left( \begin{bmatrix} m \\ \underline{t} \end{bmatrix} \right) = (-1)^{\binom{d}{2}} \partial_{[t_d]} \circ \cdots \circ \partial_{[t_1]} \circ \eta \left( \frac{m}{t_1 \cdots t_d} \right) \ .$$

*Proof.* Set $u := t_1^{-1} \cdots t_d^{-1} m \in M_{t_1 \cdots t_d}$ and define $u_0, \ldots, u_d$ as in formula (2.1.5). By definition,

$$\begin{bmatrix} m \\ \underline{t} \end{bmatrix} = (-1)^d [u] \in \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A M) \cong \mathrm{H}^d_{\mathfrak{m}}(M) \ ,$$

where $[u]$ is the cohomology class of the cocycle $u \in \mathbf{K}^d(\underline{t}) \otimes_A M$. Thus we see that $\mathrm{H}^d_{\mathfrak{m}}(\eta)(\left[\begin{smallmatrix} m \\ \underline{t} \end{smallmatrix}\right]) = (-1)^d [u_0] \in \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A M)$. According to Lemma 2.2.1, $[u_0] = (-1)^{\binom{d+1}{2}} [u_d] \in \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})$. But $u_d = \partial_{[t_d]} \circ \cdots \circ \partial_{[t_1]} \circ \eta(u) \in \mathcal{R}(\mathfrak{m})$, and under the isomorphism $\mathcal{R}(\mathfrak{m}) \xrightarrow{\simeq} \mathrm{H}^d(\mathbf{K}^{\cdot}(\underline{t}) \otimes_A \mathcal{R}^{\cdot})$, we have $u_d \mapsto [u_d]$. $\square$

**2.3. Proof of Theorem 0.2.10.** Let $A := \mathcal{O}_{X,x}$, which is a local domain of dimension $c$. Set $(\mathcal{R}^{\cdot}, \partial) := (\mathcal{K}^{\cdot}_{X,x}, \delta_{X,x})[-n]$; so $\mathcal{R}^i = \mathcal{K}^{i-n}_{X,x}$ and $\partial = (-1)^{-n}\delta_{X,x}$. This is a residual complex over $A$ satisfying the assumptions of §2.1. Take $M := \tilde{\omega}_{X,x}$ and let $\eta : M \to \mathcal{R}^0$ be $\eta_{X,x} : \tilde{\omega}_{X,x} \to \mathcal{K}^{-n}_{X,x}$. Now for any saturated chain $\xi = (x_0, \ldots, x_c)$ in $X$ we have $\partial_{(x_{c-1}, x_c)} \circ \cdots \circ \partial_{(x_0, x_1)} = (-1)^{cn}\delta_{\xi}$. So from Thm. 2.2.2 we get

$$\begin{aligned} \mathrm{H}^c_x(\eta_X)\left(\begin{bmatrix} \alpha \\ \underline{t} \end{bmatrix}\right) &= (-1)^{\binom{c}{2}} \partial_{[t_c]} \circ \cdots \partial_{[t_1]} \circ \eta \left(\frac{\alpha}{t_1 \cdots t_c}\right) \\ &= (-1)^{\binom{c}{2}+cn} \sum_{\xi | (x;\underline{t})} \delta_{\xi}\left(\frac{\alpha}{t_1 \cdots t_c}\right) . \end{aligned}$$

## 3. Comparing Local Cohomology Residues to Parshin Residues

**3.1. Fraction Fields of Complete Local Rings.** In this subsection we consider a field $L$, which is the fraction field of a complete, local, integral $k$-algebra $A$, and has the "$\mathfrak{m}$-adic topology". We prove that given a finite extension $K \to L$ of such fields, there is a trace map on separated differential forms: $\mathrm{Tr}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$. This trace map is compatible with other known trace maps. Some familiarity on the part of the reader with [Ye] Sections 1-2 is assumed.

Let $k$ be a perfect field, and let $(A, \mathfrak{m})$ be a complete, noetherian, local $k$-algebra. Assume that $A$ is residually of finite type, i.e. $A/\mathfrak{m}$ is a finitely generated field extension of $k$. Put on $A$ the $\mathfrak{m}$-adic topology. Then $A$ is a Zariski semi-topological (ST) ring, as defined in [Ye] Def. 3.2.10. In fact, on any finitely generated $A$-module, the fine $A$-module topology is the same as the $\mathfrak{m}$-adic topology. If $A \to B$ is a finite, local homomorphism between two such algebras, then as an $A$-module, $B$ has the fine $A$-module topology.

Let $\Omega^{*,\mathrm{sep}}_{A/k}$ be the separated algebra of differential forms of $A$ relative to $k$. Recall that $A$ is said to be differentially of finite type over $k$, if $\Omega^{1,\mathrm{sep}}_{A/k}$ is a finitely generated $A$-module, with the fine topology. Since the multiplication map $\Omega^{1,\mathrm{sep}}_{A/k} \otimes_A \cdots \otimes_A \Omega^{1,\mathrm{sep}}_{A/k} \to \Omega^{n,\mathrm{sep}}_{A/k}$ is a strict epimorphism for every $n$ (cf. [Ye] Def. 1.5.3), it follows, by ibid. Lemma 1.2.12, that $\Omega^{*,\mathrm{sep}}_{A/k}$ is finitely generated over $A$ and has the fine topology too.

**Lemma 3.1.1.** *The algebra $A$ is differentially of finite type over $k$. Hence $\Omega^{*,\mathrm{sep}}_{A/k}$ is the universally finite differential algebra of $A/k$, in the sense of* [KD].

*Proof.* Choose a surjective local homomorphism of $k$-algebras $f : K[[\underline{t}]] \to A$, where $K[[\underline{t}]] = K[[t_1, \dots, t_n]]$ is a ring of formal power series. By [Ye] Cor. 1.5.19, $K[[\underline{t}]]$ is differentially of finite type over $k$. Since $K[[\underline{t}]] \otimes_k K[[\underline{t}]] \twoheadrightarrow A \otimes_k A$ is a strict epimorphism of ST $k$-modules, so is $\Omega^{1,\mathrm{sep}}_{K[[\underline{t}]]/k} \to \Omega^{1,\mathrm{sep}}_{A/k}$. Therefore $\Omega^{1,\mathrm{sep}}_{A/k}$ is finitely generated and has the fine $A$-module topology. According to [Hu1] Remark 2.5, $\Omega^{*,\mathrm{sep}}_{A/k}$ is the universally finite differential algebra of $A/k$. $\square$

Now assume in addition that $A$ is an integral domain, and set $L := \mathrm{Frac}(A)$. Put on $L$ the fine $A$-module topology; this makes $L$ into a ST ring.

**Lemma 3.1.2.**

(1) *$L$ is a separated ST $k$-algebra.*
(2) *$\Omega^{*,\mathrm{sep}}_{L/k} = L \otimes_A \Omega^{*,\mathrm{sep}}_{A/k}$, and this is a free ST $L$-module.*
(3) *If* $\mathrm{char}\, k = p > 0$, *then* $\Omega^{*,\mathrm{sep}}_{L/k} = \Omega^{*}_{L/k} = \Omega^{*}_{L/\mathbb{Z}}$.
(4) $\mathrm{rank}_L\, \Omega^{1,\mathrm{sep}}_{L/k} = \dim A + \mathrm{rank}_{A/\mathfrak{m}}\, \Omega^{1}_{(A/\mathfrak{m})/k}$.

*Proof.* (1) Choose a noether normalization: a finite, injective, local, $k$-algebra homomorphism $K[[\underline{t}]] = K[[t_1, \dots, t_n]] \to A$, and let $M := \mathrm{Frac}(K[[\underline{t}]])$. Since $A$ has the fine $K[[\underline{t}]]$-module topology, it follows that $L$ has the fine $M$-module topology, so it is a free ST $M$-module. Therefore it suffices to prove that $M$ is separated. This in turn is a consequence of the existence of an injective, continuous homomorphism $M \to K((t_1, \dots, t_n))$, since $K((t_1, \dots, t_n))$ is known to be separated.

(2) The homomorphism $A \to L$ is topologically étale, by [Ye] Prop. 1.5.8. Now use Thm. 1.5.11 of [Ye], Lemma 3.1.1 and Part (1) of this lemma.

(3) Let $M$ be as above. Then $M^{(p/k)} = \mathrm{Frac}(K^{(p/k)}[[t_1^p, \dots, t_n^p]]) \subset L^{(p/k)}$ and $L$ has the fine $M^{(p/k)}$-module topology. From here on it is a standard positive characteristic argument (cf. [Ye] Cor. 2.1.15).

(4) The formula holds for $M$, and hence for any finite extension thereof (if $\operatorname{char} k = p$ use Part (3)). $\square$

**Proposition 3.1.3.** *Let $A$ and $B$ be integral, complete, noetherian, residually of finite type, local $k$-algebras, and let $A \to B$ be a finite, injective, local, $k$-algebra homomorphism. Set $K := \mathrm{Frac}(A)$ and $L := \mathrm{Frac}(B)$. Then there is a trace map $\mathrm{Tr}_{L/K} : \Omega^{*,\mathrm{sep}}_{L/k} \to \Omega^{*,\mathrm{sep}}_{K/k}$, satisfying axioms* **T1**, **T2** *and* **T3** *of* [Ye] *Prop.* 2.3.2. *Hence $\mathrm{Tr}_{L/K}$ coincides with Kunz's trace map of* [KD] §16 (*cf. Lemma* 3.1.1).

*Proof.* First note that any finite extension $K'$ of $K$ is of the form $\mathrm{Frac}(A')$ for some finite, local homomorphism $A \to A'$. Now we may use the previous lemmas and the proof of [Ye] Prop. 2.3.2 to define (and uniquely determine) $\mathrm{Tr}_{L/K}$. Since the trace map is compatible with the projection $\Omega^{*}_{-/k} \to \Omega^{*,\mathrm{sep}}_{-/k}$, it coincides with the trace of [KD]. $\square$

**3.2. Spectra of Complete Local Rings.** In this subsection we pass to topological local fields (TLFs). This is done by the Beilinson completion method, cf. [Ye] §3. Let $A$ and $B$ be complete, noetherian, residually of finite type, local $k$-algebras, and let $A \to B$ be a finite, local, $k$-algebra homomorphism. Set $\hat{X} := \operatorname{Spec} B$ and $\hat{Y} := \operatorname{Spec} A$. Given an $A$-module $M$ and a chain $\hat{\eta}$ in $\hat{Y}$, denote by $M_{\hat{\eta}}$ the Beilinson completion of the sheaf $\mathcal{O}_{\hat{Y}} \otimes M$ along $\hat{\eta}$. This is a ST $A$-module (cf. [Ye] §3.2). Say $\hat{y}$ is the closed point of $\hat{Y}$. Then $\mathcal{O}_{\hat{Y},(\hat{y})} \cong A$ as ST rings. Therefore, if $M$ is a ST $A$-module with the fine topology, and if $\hat{\eta} = (\dots, \hat{y})$, then the natural map $M \to M_{\hat{\eta}}$ is continuous. (In fact, $M \to M_{(\hat{y})}$ is a homeomorphism, by [Ye] Prop. 1.2.20 and Cor. 1.2.6.) Suppose $\hat{\eta} = (\hat{y}_0, \dots, \hat{y}_n)$ is a maximal chain in $\hat{Y}$, and suppose $\hat{\xi} = (\hat{x}_0, \dots, \hat{x}_n)$ is a chain in $\hat{X}$ lying over $\hat{\eta}$ (so $\hat{\xi}$ is necessarily maximal). The fields $k(\hat{y}_0)$ and $k(\hat{x}_0)$ are topologized, and by Prop. 3.1.3 there is a trace map $\mathrm{Tr}_{k(\hat{x}_0)/k(\hat{y}_0)} : \Omega^{*,\mathrm{sep}}_{k(\hat{x}_0)/k} \to \Omega^{*,\mathrm{sep}}_{k(\hat{y}_0)/k}$. On the other hand, by [Ye] formula (2.3.10), there exists a trace map on the clusters of TLFs, $\mathrm{Tr}_{k(\hat{\xi})/k(\hat{\eta})} : \Omega^{*,\mathrm{sep}}_{k(\hat{\xi})/k} \to \Omega^{*,\mathrm{sep}}_{k(\hat{\eta})/k}$.

**Proposition 3.2.1.** *The diagram below commutes:*

$$\begin{tikzcd}
\bigoplus_{\hat{x}_0 | \hat{y}_0} \Omega^{*,\mathrm{sep}}_{k(\hat{x}_0)/k} \arrow[r] \arrow[d, "\mathrm{Tr}"] & \bigoplus_{\hat{\xi} | \hat{\eta}} \Omega^{*,\mathrm{sep}}_{k(\hat{\xi})/k} \arrow[d, "\mathrm{Tr}"] \\
\Omega^{*,\mathrm{sep}}_{k(\hat{y}_0)/k} \arrow[r] & \Omega^{*,\mathrm{sep}}_{k(\hat{\eta})/k}
\end{tikzcd}$$

*Proof.* One has $\prod_{\hat{\xi}|\hat{\eta}} k(\hat{\xi}) \cong (\prod_{\hat{x}_0|\hat{y}_0}) k(\hat{x}_0) \otimes_{k(\hat{y}_0)} k(\hat{\eta})$ and $\Omega^{*,\mathrm{sep}}_{k(\hat{\xi})/k} = k(\hat{\xi}) \otimes_{k(\hat{x}_0)} \Omega^{*,\mathrm{sep}}_{k(\hat{x}_0)/k}$ for $\hat{\xi} = (\hat{x}_0, \ldots)$. Now use the axioms. $\square$

Finally, here is:

*Proof.* (of Thm. 0.2.9) Take $B := \mathcal{O}_{X,(x)} = \hat{\mathcal{O}}_{X,x}$ and $A := K[[\underline{t}]] = K[[t_1, \ldots, t_c]]$. There is a finite morphism $\hat{X} \to \hat{Y}$, and a flat morphism $\hat{X} \to X$. Let $\hat{y}_i \in \hat{Y}$ be the prime ideal $\hat{y}_i := (t_1, \ldots, t_i)$, so $\hat{\eta} := (\hat{y}_0, \ldots, \hat{y}_c)$ is a maximal chain in $\hat{Y}$, and $k(\hat{\eta}) = K((\underline{t})) = K((t_c)) \cdots ((t_1))$. A chain $\hat{\xi} = (\hat{x}_0, \ldots, \hat{x}_c)$ in $\hat{X}$ lies over the chain $\hat{\eta}$ in $\hat{Y}$ iff $\hat{\xi}$ lies over some chain $\xi$ in $X$ such that $\xi|(x; \underline{t})$. Cor. 3.3.13 of [Ye] says that there is a canonical isomorphism of clusters of TLFs:

$$\prod_{\xi|(x;\underline{t})} k(\xi) \cong \prod_{\xi|(x;\underline{t})} \prod_{\hat{\xi}|\xi} k(\hat{\xi}) = \prod_{\hat{\xi}|\hat{\eta}} k(\hat{\xi}) \ . \tag{3.2.1}$$

Since $B$ is a reduced ring, its total ring of fractions $\operatorname{Frac} B = k(\hat{X})$ equals $\prod_{\hat{x}_0} k(\hat{x}_0)$, the product running over the generic points of $\hat{X}$. According to [Hu2] §1, $\mathrm{Tr}_{k(\hat{X})/k(\hat{Y})}(\alpha) \in \Omega^{n,\mathrm{sep}}_{K[[\underline{t}]]/k}$. Therefore

$$\mathrm{Tr}_{k(\hat{X})/k(\hat{Y})}(\alpha) = \sum_{\hat{x}_0|\hat{y}_0} \mathrm{Tr}_{k(\hat{x}_0)/k(\hat{y}_0)}(\alpha) = \sum_{\underline{i} \in \mathbb{N}^c} \beta_{\underline{i}} \wedge \underline{t}^{\underline{i}} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_c$$

for some $\beta_{\underline{i}} \in \Omega^{n-c}_{K/k}$, where $\underline{t}^{\underline{i}} := t_1^{i_1} \cdots t_c^{i_c}$. By definition, $\mathrm{Res}^{\mathrm{LC}}_{x,K} \left[\begin{smallmatrix} \alpha \\ t_1, \ldots, t_c \end{smallmatrix}\right] = \beta_{(0,\ldots,0)}$. On the other hand,

$$\begin{aligned}
\textstyle\sum_{\xi|(x;\underline{t})} \mathrm{Res}_{\xi,K}\left(\frac{\alpha}{t_1 \cdots t_c}\right) &= \textstyle\sum_{\xi|(x;\underline{t})} \mathrm{Res}_{k(\xi)/K}\left(\frac{\alpha}{t_1 \cdots t_c}\right) \\
&= \textstyle\sum_{\hat{\xi}|\hat{\eta}} \mathrm{Res}_{k(\hat{\xi})/K}\left(\frac{\alpha}{t_1 \cdots t_c}\right) && \text{by (3.2.1)} \\
&= \mathrm{Res}_{k(\hat{\eta})/K} \circ \textstyle\sum_{\hat{x}_0|\hat{y}_0} \mathrm{Tr}_{k(\hat{x}_0)/k(\hat{y}_0)}\left(\frac{\alpha}{t_1 \cdots t_c}\right) \\
&= \mathrm{Res}_{K((\underline{t}))/K}(\beta_{(0,\ldots,0)} \wedge \mathrm{dlog}\, t_1 \wedge \cdots \wedge \mathrm{dlog}\, t_c) \\
&= (-1)^{\binom{c}{2}} \beta_{(0,\ldots,0)} \ .
\end{aligned}$$

$\square$

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