MIXED RESOLUTIONS AND SIMPLICIAL SECTIONS

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Abstract. We introduce the notions of mixed resolutions and simplicial sections, and prove a theorem relating them. This result is used (in another paper) to study deformation quantization in algebraic geometry.

0. Introduction

Let $\mathbb{K}$ be a field of characteristic 0. In this paper we present several technical results about the geometry of $\mathbb{K}$-schemes. These results were discovered in the course of work on deformation quantization in algebraic geometry, and they play a crucial role in [Ye3]. This role will be explained at the end of the introduction. The idea behind the constructions in this paper can be traced back to old work of Bott [Bo, HY].

Let $\pi : Z \to X$ be a morphism of $\mathbb{K}$-schemes, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an open covering of X. A *simplicial section* $\boldsymbol{\sigma}$ of π, based on the covering $\boldsymbol{U}$, consists of a family of morphisms $\sigma_{\boldsymbol{i}} : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}} \to Z$, where $\boldsymbol{i} = (i_0, \ldots, i_q)$ is a multi-index; $\boldsymbol{\Delta}^q_{\mathbb{K}}$ is the q-dimensional geometric simplex; and $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$. The morphisms $\sigma_{\boldsymbol{i}}$ are required to be compatible with π and to satisfy simplicial relations. See Definition 5.1 for details. An important example of a simplicial section is mentioned at the end of the introduction.

Another notion we introduce is that of *mixed resolution*. Here we assume the $\mathbb{K}$-scheme X is smooth and separated, and each of the open sets $U_{(i)}$ in the covering $\boldsymbol{U}$ is affine. Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we define its mixed resolution $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$. This is a complex of sheaves on X, concentrated in non-negative degrees. As the name suggests, this resolution mixes two distinct types of resolutions: a de Rham type resolution which is related to the sheaf $\mathcal{P}_X$ of principal parts of X and its Grothendieck connection, and a simplicial-Čech type resolution which is related to the covering $\boldsymbol{U}$. The precise definition is too complicated to state here – see Section 4.

Let $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$ denote the abelian category of bounded below complexes of quasi-coherent $\mathcal{O}_X$-modules. For any $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$ the mixed resolution $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is defined by totalizing the double complex $\bigoplus_{p,q} \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p)$. The derived category of $\mathbb{K}$-modules is denoted by $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

Theorem 0.1. *Let X be a smooth separated $\mathbb{K}$-scheme, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an affine open covering of X.*

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(1) *There is a functorial quasi-isomorphism* $\mathcal{M} \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ *for* $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$.
(2) *Given* $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$*, the canonical morphism* $\Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \to \mathrm{R}\Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})\big)$ *in* $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ *is an isomorphism.*
(3) *The quasi-isomorphism in part* (1) *induces a functorial isomorphism* $\Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \cong \mathrm{R}\Gamma(X, \mathcal{M})$ *in* $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

This is repeated as Theorem 4.15 in the body of the paper. Note that part (3) is a formal consequence of parts (1) and (2).

A useful corollary of the theorem is the following (see Corollary 4.16). Suppose $\mathcal{M}$ and $\mathcal{N}$ are two complexes in $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_X)$, and $\phi : \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N})$ is a $\mathbb{K}$-linear quasi-isomorphism. Then

$$\Gamma(X, \phi) : \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}} \mathcal{M})\big) \to \Gamma\big(X, \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N})\big)$$

is a quasi-isomorphism.

Here is the connection between simplicial sections and mixed resolutions.

Theorem 0.2. *Let* X *be a smooth separated* $\mathbb{K}$*-scheme, let* $\pi : Z \to X$ *be a morphism of schemes, and let* $\boldsymbol{U}$ *be an affine open covering of* X*. Suppose* $\boldsymbol{\sigma}$ *is a simplicial section of* π *based on* $\boldsymbol{U}$*. Let* $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ *be quasi-coherent* $\mathcal{O}_X$*-modules, and let*

$$\phi : \prod_{i=1}^{r} \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous $\mathcal{O}_Z$*-multilinear sheaf morphism on* Z*. Then there is an induced* $\mathbb{K}$*-multilinear sheaf morphism*

$$\boldsymbol{\sigma}^*(\phi) : \prod_{i=1}^{r} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}_i) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{N})$$

on X*.*

In the theorem, the continuity and the complete pullback $\pi^{\widehat{*}}$ refer to the dir-inv structures on these sheaves, which are explained in Section 1. A more detailed statement is Theorem 5.2 in the body of the paper.

Let us explain, in vague terms, how Theorem 0.2, or rather Theorem 5.2, is used in the paper [Ye3]. Let X be a smooth separated n-dimensional $\mathbb{K}$-scheme. As we know from the work of Kontsevich [Ko], there are two important sheaves of DG Lie algebras on X, namely the sheaf $\mathcal{T}_{\mathrm{poly},X}$ of poly derivations, and the sheaf $\mathcal{D}_{\mathrm{poly},X}$ of poly differential operators. Suppose $\boldsymbol{U}$ is some affine open covering of X. The inclusions $\mathcal{T}_{\mathrm{poly},X} \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X})$ and $\mathcal{D}_{\mathrm{poly},X} \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$ are then quasi-isomorphisms of sheaves of DG Lie algebras (cf. Theorem 0.1). The goal is to find an L_∞ quasi-isomorphism

$$\Psi : \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

between these sheaves of DG Lie algebras. Having such an L_∞ quasi-isomorphism pretty much implies the solution of the deformation quantization problem for X.

Let $\operatorname{Coor} X$ denote the coordinate bundle of X. This is an infinite dimensional bundle over X, endowed with an action of the group $\mathrm{GL}_n(\mathbb{K})$. Let $\operatorname{LCC} X$ be the quotient bundle $\operatorname{Coor} X \,/\, \mathrm{GL}_n(\mathbb{K})$. In [Ye4] we proved that if the covering $\boldsymbol{U}$ is fine enough (the condition is that each open set $U_{(i)}$ admits an étale morphism to $\mathbf{A}^n_{\mathbb{K}}$), then the projection $\pi : \mathrm{LCC}\, X \to X$ admits a simplicial section $\boldsymbol{\sigma}$.

Now the universal deformation formula of Kontsevich [Ko] gives rise to a continuous L_∞ quasi-isomorphism

$$\mathcal{U} : \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X})$$

on $\mathrm{LCC}\, X$. This means that there is a sequence of continuous $\mathcal{O}_{\mathrm{LCC}\, X}$-multilinear sheaf morphisms

$$\mathcal{U}_r : \textstyle\prod^r \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}_{\mathrm{poly},X}) \to \pi^{\widehat{*}}(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{\mathrm{poly},X}),$$

$r \geq 1$, satisfying very complicated identities. Using Theorem 5.2 we obtain a sequence of multilinear sheaf morphisms

$$\boldsymbol{\sigma}^*(\mathcal{U}_r) : \textstyle\prod^r \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

on X. After twisting these morphisms suitably (this is needed due to the presence of the Grothendieck connection; cf. [Ye2]) we obtain the desired L_∞ quasi-isomorphism Ψ.

We believe that mixed resolutions, and the results of this paper, shall have additional applications in algebraic geometry (e.g. algebro-geometric versions of results on index theorems in differential geometry, cf. [NT]; or a proof of Kontsevich's famous yet unproved claim on Hochschild cohomology of a scheme [Ko, Claim 8.4]).

1. Review of Dir-Inv Modules

We begin the paper with a review of the concept of dir-inv structure, which was introduced in [Ye2]. A dir-inv structure is a generalization of adic topology.

Let C be a commutative ring. We denote by $\mathsf{Mod}\, C$ the category of C-modules.

Definition 1.1. (1) Let $M \in \mathsf{Mod}\, C$. An *inv module structure* on M is an inverse system $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ of C-submodules of M. The pair $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is called an *inv C-module*.

(2) Let $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ and $(N, \{\mathrm{F}^i N\}_{i \in \mathbb{N}})$ be two inv C-modules. A function $\phi : M \to N$ (C-linear or not) is said to be *continuous* if for every $i \in \mathbb{N}$ there exists $i' \in \mathbb{N}$ such that $\phi(\mathrm{F}^{i'} M) \subset \mathrm{F}^i N$.

(3) Define $\mathsf{Inv}\,\mathsf{Mod}\, C$ to be the category whose objects are the inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

There is a full and faithful embedding of categories $\mathsf{Mod}\, C \hookrightarrow \mathsf{Inv}\,\mathsf{Mod}\, C$, $M \mapsto (M, \{\ldots, 0, 0\})$.

Recall that a directed set is a partially ordered set J with the property that for any $j_1, j_2 \in J$ there exists $j_3 \in J$ such that $j_1, j_2 \leq j_3$.

Definition 1.2. (1) Let $M \in \mathsf{Mod}\, C$. A *dir-inv module structure* on M is a direct system $\{\mathrm{F}_j M\}_{j \in J}$ of C-submodules of M, indexed by a nonempty directed set J, together with an inv module structure on each $\mathrm{F}_j M$, such that for every $j_1 \leq j_2$ the inclusion $\mathrm{F}_{j_1} M \hookrightarrow \mathrm{F}_{j_2} M$ is continuous. The pair $(M, \{\mathrm{F}_j M\}_{j \in J})$ is called a *dir-inv C-module*.

(2) Let $(M, \{\mathrm{F}_j M\})_{j \in J}$ and $(N, \{\mathrm{F}_k N\}_{k \in K})$ be two dir-inv C-modules. A function $\phi : M \to N$ (C-linear or not) is said to be *continuous* if for every $j \in J$ there exists $k \in K$ such that $\phi(\mathrm{F}_j M) \subset \mathrm{F}_k N$, and $\phi : \mathrm{F}_j M \to \mathrm{F}_k N$ is a continuous homomorphism between these two inv C-modules.

(3) Define $\mathsf{Dir\,Inv\,Mod}\,C$ to be the category whose objects are the dir-inv C-modules, and whose morphisms are the continuous C-linear homomorphisms.

An inv C-module M can be endowed with a dir-inv module structure $\{\mathrm{F}_j M\}_{j \in J}$, where $J := \{0\}$ and $\mathrm{F}_0 M := M$. Thus we get a full and faithful embedding $\mathsf{Inv\,Mod}\,C \hookrightarrow \mathsf{Dir\,Inv\,Mod}\,C$.

Inv modules and dir-inv modules come in a few "flavors": trivial, discrete and complete. A *discrete inv module* is one which is isomorphic, in $\mathsf{Inv\,Mod}\,C$, to an object of $\mathsf{Mod}\,C$ (via the canonical embedding above). A *complete inv module* is an inv module $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ such that the canonical map $M \to \lim_{\leftarrow i} \mathrm{F}^i M$ is bijective. A *discrete* (resp. *complete*) *dir-inv module* is one which is isomorphic, in $\mathsf{Dir\,Inv\,Mod}\,C$, to a dir-inv module $(M, \{\mathrm{F}_j M\}_{j \in J})$, where all the inv modules $\mathrm{F}_j M$ are discrete (resp. complete), and the canonical map $\lim_{j \to} \mathrm{F}_j M \to M$ in $\mathsf{Mod}\,C$ is bijective. A *trivial dir-inv module* is one which is isomorphic to an object of $\mathsf{Mod}\,C$. Discrete dir-inv modules are complete, but there are also other complete modules, as the next example shows.

Example 1.3. Assume C is noetherian and $\mathfrak{c}$-adically complete for some ideal $\mathfrak{c}$. Let M be a finitely generated C-module, and define $\mathrm{F}^i M := \mathfrak{c}^{i+1} M$. Then $\{\mathrm{F}^i M\}_{i \in \mathbb{N}}$ is called the $\mathfrak{c}$-*adic inv structure*, and of course $(M, \{\mathrm{F}^i M\}_{i \in \mathbb{N}})$ is a complete inv module. Next consider an arbitrary C-module M. We take $\{\mathrm{F}_j M\}_{j \in J}$ to be the collection of finitely generated C-submodules of M. This dir-inv module structure on M is called the $\mathfrak{c}$-*adic dir-inv structure*. Again $(M, \{\mathrm{F}_j M\}_{j \in J})$ is a complete dir-inv C-module. Note that a finitely generated C-module M is discrete as inv module iff $\mathfrak{c}^i M = 0$ for $i \gg 0$; and a C-module is discrete as dir-inv module iff it is a direct limit of discrete finitely generated modules.

The category $\mathsf{Dir\,Inv\,Mod}\,C$ is additive. Given a collection $\{M_k\}_{k \in K}$ of dir-inv modules, the direct sum $\bigoplus_{k \in K} M_k$ has a structure of dir-inv module, making it into the coproduct of $\{M_k\}_{k \in K}$ in the category $\mathsf{Dir\,Inv\,Mod}\,C$. Note that if the index set K is infinite and each M_k is a nonzero discrete inv module, then $\bigoplus_{k \in K} M_k$ is a discrete dir-inv module which is not trivial. The tensor product $M \otimes_C N$ of two dir-inv modules is again a dir-inv module. There is a completion functor $M \mapsto \widehat{M}$. (Warning: if M is complete then $\widehat{M} = M$, but it is not known if $\widehat{M}$ is complete for arbitrary M.) The completed tensor product is $M \widehat{\otimes}_C N := \widehat{M \otimes_C N}$. Completion commutes with direct sums: if $M \cong \bigoplus_{k \in K} M_k$ then $\widehat{M} \cong \bigoplus_{k \in K} \widehat{M}_k$. See [Ye2] for full details.

A *graded dir-inv module* (or graded object in $\mathsf{Dir\,Inv\,Mod}\,C$) is a direct sum $M = \bigoplus_{k \in \mathbb{Z}} M_k$, where each M_k is a dir-inv module. A *DG algebra* in $\mathsf{Dir\,Inv\,Mod}\,C$ is a graded dir-inv module $A = \bigoplus_{k \in \mathbb{Z}} A^k$, together with continuous C-(bi)linear functions $\mu : A \times A \to A$ and $\mathrm{d} : A \to A$, which make A into a DG C-algebra. If A is a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\,C$, and $\mathfrak{g}$ is a DG Lie Algebra in $\mathsf{Dir\,Inv\,Mod}\,C$, then $A \widehat{\otimes}_C \mathfrak{g}$ is a DG Lie Algebra in $\mathsf{Dir\,Inv\,Mod}\,C$.

Let A be a super-commutative associative unital DG algebra in $\mathsf{Dir\,Inv\,Mod}\,C$. A *DG A-module* in $\mathsf{Dir\,Inv\,Mod}\,C$ is a graded object M in $\mathsf{Dir\,Inv\,Mod}\,C$, together with continuous C-(bi)linear functions $\mu : A \times M \to M$ and $\mathrm{d} : M \to M$, which make M into a DG A-module in the usual sense. A *DG A-module Lie algebra* in $\mathsf{Dir\,Inv\,Mod}\,C$ is a DG Lie algebra $\mathfrak{g}$ in $\mathsf{Dir\,Inv\,Mod}\,C$, together with a continuous C-bilinear function $\mu : A \times \mathfrak{g} \to \mathfrak{g}$, such that such that $\mathfrak{g}$ becomes a DG A-module, and

$$[a_1\gamma_1, a_2\gamma_2] = (-1)^{i_2 j_1} a_1 a_2 \, [\gamma_1, \gamma_2]$$

for all $a_k \in A^{i_k}$ and $\gamma_k \in \mathfrak{g}^{j_k}$.

All the constructions above can be geometrized. Let $(Y, \mathcal{O})$ be a commutative ringed space over $\mathbb{K}$, i.e. Y is a topological space, and $\mathcal{O}$ is a sheaf of commutative $\mathbb{K}$-algebras on Y. We denote by $\mathsf{Mod}\,\mathcal{O}$ the category of $\mathcal{O}$-modules on Y.

Example 1.4. Geometrizing Example 1.3, let $\mathfrak{X}$ be a noetherian formal scheme, with defining ideal $\mathcal{I}$. Then any coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is an inv $\mathcal{O}_{\mathfrak{X}}$-module, with system of submodules $\{\mathcal{I}^{i+1}\mathcal{M}\}_{i\in\mathbb{N}}$, and $\mathcal{M} \cong \widehat{\mathcal{M}}$; cf. [EGA I]. We call an $\mathcal{O}_{\mathfrak{X}}$-module *dir-coherent* if it is the direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules. Any dir-coherent module is quasi-coherent, but it is not known if the converse is true. At any rate, a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is a dir-inv $\mathcal{O}_{\mathfrak{X}}$-module, where we take $\{\mathrm{F}_j\mathcal{M}\}_{j\in J}$ to be the collection of coherent submodules of $\mathcal{M}$. Any dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is then a complete dir-inv module. This dir-inv module structure on $\mathcal{M}$ is called the *$\mathcal{I}$-adic dir-inv structure.* Note that a coherent $\mathcal{O}_{\mathfrak{X}}$-module $\mathcal{M}$ is discrete as inv module iff $\mathcal{I}^i\mathcal{M} = 0$ for $i \gg 0$; and a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is discrete as dir-inv module iff it is a direct limit of discrete coherent modules.

If $f : (Y', \mathcal{O}') \to (Y, \mathcal{O})$ is a morphism of ringed spaces and $\mathcal{M} \in \mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then there is an obvious structure of dir-inv $\mathcal{O}'$ -module on $f^*\mathcal{M}$, and we define $f^{\widehat{*}}\mathcal{M} := \widehat{f^*\mathcal{M}}$. If $\mathcal{M}$ is a graded object in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then the inverse images $f^*\mathcal{M}$ and $f^{\widehat{*}}\mathcal{M}$ are graded objects in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. If $\mathcal{G}$ is an algebra (resp. a DG algebra) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}$, then $f^*\mathcal{G}$ and $f^{\widehat{*}}\,\mathcal{G}$ are algebras (resp. DG algebras) in $\mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$. Given $\mathcal{N} \in \mathsf{Dir\,Inv\,Mod}\,\mathcal{O}'$ there is an obvious dir-inv $\mathcal{O}$-module structure on $f_*\mathcal{N}$.

Example 1.5. Let $(Y, \mathcal{O})$ be a ringed space and $V \subset Y$ an open set. For a dir-inv $\mathcal{O}$-module $\mathcal{M}$ there is an obvious way to make $\Gamma(V, \mathcal{M})$ into a dir-inv $\mathcal{O}$-module. If $\mathcal{M}$ is a complete inv $\mathcal{O}$-module then $\Gamma(V, \mathcal{M})$ is a complete inv $\mathcal{O}$-module. If V is quasi-compact and $\mathcal{M}$ is a complete dir-inv $\mathcal{O}$-module, then $\Gamma(V, \mathcal{M})$ is a complete dir-inv $\Gamma(V, \mathcal{O})$-module.

2. Complete Thom-Sullivan Cochains

From here on $\mathbb{K}$ is a field of characteristic 0. Let us begin with some abstract notions about cosimplicial modules and their normalizations, following [HS] and [HY]. We use the notation $\mathsf{Mod}\,\mathbb{K}$ and $\mathsf{DGMod}\,\mathbb{K}$ for the categories of $\mathbb{K}$-modules and DG (differential graded) $\mathbb{K}$-modules respectively.

Let $\mathbf{\Delta}$ denote the category with objects the ordered sets $[q] := \{0, 1, \ldots, q\}$, $q \in \mathbb{N}$. The morphisms $[p] \to [q]$ are the order preserving functions, and we write $\mathbf{\Delta}_p^q := \mathrm{Hom}_{\mathbf{\Delta}}([p], [q])$. The i-th co-face map $\partial^i : [p] \to [p{+}1]$ is the injective function that does not take the value i; and the i-th co-degeneracy map $\mathrm{s}^i : [p] \to [p-1]$ is the surjective function that takes the value i twice. All morphisms in $\mathbf{\Delta}$ are compositions of various ∂^i and s^i.

An element of $\mathbf{\Delta}_p^q$ may be thought of as a sequence $\boldsymbol{i} = (i_0, \ldots, i_p)$ of integers with $0 \le i_0 \le \cdots \le i_p \le q$. Given $\boldsymbol{i} \in \mathbf{\Delta}_q^m$, $\boldsymbol{j} \in \mathbf{\Delta}_m^p$ and $\alpha \in \mathbf{\Delta}_p^q$, we sometimes write $\alpha_*(\boldsymbol{i}) := \boldsymbol{i} \circ \alpha \in \mathbf{\Delta}_p^m$ and $\alpha^*(\boldsymbol{j}) := \alpha \circ \boldsymbol{j} \in \mathbf{\Delta}_m^q$.

Let C be some category. A *cosimplicial object* in C is a functor $C : \mathbf{\Delta} \to \mathsf{C}$. We shall usually refer to the cosimplicial object as $C = \{C^p\}_{p\in\mathbb{N}}$, and for any $\alpha \in \mathbf{\Delta}_p^q$

the corresponding morphism in C will be denoted by $\alpha^* : C^p \to C^q$. A *simplicial object* in C is a functor $C : \mathbf{\Delta}^{\mathrm{op}} \to \mathsf{C}$. The notation for a simplicial object will be $C = \{C_p\}_{p\in\mathbb{N}}$ and $\alpha_* : C_q \to C_p$.

Suppose $M = \{M^q\}_{q\in\mathbb{N}}$ is a cosimplicial $\mathbb{K}$-module. The *standard normalization* of M is the DG module $\mathrm{N}M$ defined as follows: $\mathrm{N}^q M := \bigcap_{i=0}^{q-1} \operatorname{Ker}(\mathrm{s}^i : M^q \to M^{q-1})$. The differential is $\partial := \sum_{i=0}^{q+1} (-1)^i \partial^i : \mathrm{N}^q M \to \mathrm{N}^{q+1} M$. We get a functor $\mathrm{N} : \mathbf{\Delta}\, \mathsf{Mod}\, \mathbb{K} \to \mathsf{DGMod}\, \mathbb{K}$.

For any q let $\mathbf{\Delta}^q_{\mathbb{K}}$ be the *geometric q-dimensional simplex*

$$\mathbf{\Delta}^q_{\mathbb{K}} := \operatorname{Spec} \mathbb{K}[t_0, \ldots, t_q]/(t_0 + \cdots + t_q - 1).$$

The i-th vertex of $\mathbf{\Delta}^q_{\mathbb{K}}$ is the $\mathbb{K}$-rational point x such that $t_i(x) = 1$ and $t_j(x) = 0$ for all $j \neq i$. We identify the vertices of $\mathbf{\Delta}^q_{\mathbb{K}}$ with the ordered set $[q] = \{0, 1, \ldots, q\}$. For any $\alpha : [p] \to [q]$ in $\mathbf{\Delta}$ there is a unique linear morphism $\alpha : \mathbf{\Delta}^p_{\mathbb{K}} \to \mathbf{\Delta}^q_{\mathbb{K}}$ extending it, and in this way $\{\mathbf{\Delta}^q_{\mathbb{K}}\}_{q\in\mathbb{N}}$ is a cosimplicial scheme.

For a $\mathbb{K}$-scheme X we write $\Omega^p(X) := \Gamma(X, \Omega^p_{X/\mathbb{K}})$. Taking $X := \mathbf{\Delta}^q_{\mathbb{K}}$ we have a super-commutative associative unital DG $\mathbb{K}$-algebra $\Omega(\mathbf{\Delta}^q_{\mathbb{K}}) = \bigoplus_{p\in\mathbb{N}} \Omega^p(\mathbf{\Delta}^q_{\mathbb{K}})$, that is generated as $\mathbb{K}$-algebra by the elements $t_0, \ldots, t_q, \mathrm{d}t_0, \ldots, \mathrm{d}t_q$. The collection $\{\Omega(\mathbf{\Delta}^q_{\mathbb{K}})\}_{q\in\mathbb{N}}$ is a simplicial DG algebra, namely a functor from $\mathbf{\Delta}^{\mathrm{op}}$ to the category of DG $\mathbb{K}$-algebras.

In [HY] we made use of the Thom-Sullivan normalization $\tilde{\mathrm{N}}M$ of a cosimplicial $\mathbb{K}$-module M. For some applications (specifically [Ye3]) a complete version of this construction is needed. Recall that for $M, N \in \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K}$ we can define the complete tensor product $N \mathbin{\widehat{\otimes}} M$. The $\mathbb{K}$-modules $\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}})$ are always considered as discrete inv modules, so $\Omega(\mathbf{\Delta}^l_{\mathbb{K}})$ is a discrete dir-inv DG $\mathbb{K}$-algebra.

Definition 2.1. Suppose $M = \{M^q\}_{q\in\mathbb{N}}$ is a cosimplicial dir-inv $\mathbb{K}$-module, namely each $M^q \in \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K}$, and the morphisms $\alpha^* : M^p \to M^q$, for $\alpha \in \mathbf{\Delta}^q_p$, are continuous $\mathbb{K}$-linear homomorphisms. Let

$$\widehat{\tilde{\mathrm{N}}}{}^q M \subset \prod_{l=0}^{\infty} \Big(\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \mathbin{\widehat{\otimes}} M^l \Big) \tag{2.2}$$

be the submodule consisting of all sequences $(u_0, u_1, \ldots)$, with $u_l \in \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \mathbin{\widehat{\otimes}} M^l$, such that

$$(\mathbf{1} \otimes \alpha^*)(u_k) = (\alpha_* \otimes \mathbf{1})(u_l) \in \Omega^q(\mathbf{\Delta}^k_{\mathbb{K}}) \mathbin{\widehat{\otimes}} M^l \tag{2.3}$$

for all $k, l \in \mathbb{N}$ and all $\alpha \in \mathbf{\Delta}^l_k$. Define a coboundary operator $\partial : \widehat{\tilde{\mathrm{N}}}{}^q M \to \widehat{\tilde{\mathrm{N}}}{}^{q+1} M$ using the exterior derivative $\mathrm{d} : \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \to \Omega^{q+1}(\mathbf{\Delta}^l_{\mathbb{K}})$. The resulting DG $\mathbb{K}$-module $\big(\widehat{\tilde{\mathrm{N}}} M, \partial\big)$ is called the *complete Thom-Sullivan normalization of M*.

The $\mathbb{K}$-module $\widehat{\tilde{\mathrm{N}}} M = \bigoplus_{q\in\mathbb{N}} \widehat{\tilde{\mathrm{N}}}{}^q M$ is viewed as an abstract module. We obtain a functor

$$\widehat{\tilde{\mathrm{N}}} : \mathbf{\Delta}\, \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K} \to \mathsf{DGMod}\,\mathbb{K}.$$

Remark 2.4. In case each M^l is a discrete dir-inv module one has $\Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \mathbin{\widehat{\otimes}} M^l = \Omega^q(\mathbf{\Delta}^l_{\mathbb{K}}) \otimes M^l$, and therefore $\widehat{\tilde{\mathrm{N}}} M = \tilde{\mathrm{N}} M$.

The standard normalization $\mathrm{N}M$ also makes sense here, via the forgetful functor $\mathbf{\Delta}\, \mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K} \to \mathbf{\Delta}\, \mathsf{Mod}\,\mathbb{K}$. The two normalizations $\widehat{\tilde{\mathrm{N}}}$ and N are related as follows. Let $\int_{\mathbf{\Delta}^l} : \Omega(\mathbf{\Delta}^l_{\mathbb{K}}) \to \mathbb{K}$ be the $\mathbb{K}$-linear map of degree $-l$ defined by integration on the compact real l-dimensional simplex, namely $\int_{\mathbf{\Delta}^l} \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_l = \frac{1}{l!}$ etc. Suppose each dir-inv module M^l is complete, so that using [Ye2, Proposition 1.5] we get a functorial $\mathbb{K}$-linear homomorphism

$$\int_{\mathbf{\Delta}^l} : \Omega(\mathbf{\Delta}^l_{\mathbb{K}}) \mathbin{\widehat{\otimes}} M^l \to \mathbb{K} \mathbin{\widehat{\otimes}} M^l \cong M^l.$$

Proposition 2.5. *Suppose $M = \{M^q\}_{q \in \mathbb{N}}$ is a cosimplicial dir-inv $\mathbb{K}$-module, with all dir-inv modules M^q complete. Then the homomorphisms $\int_{\mathbf{\Delta}^l}$ induce a quasi-isomorphism*

$$\int_{\mathbf{\Delta}} : \widehat{\widetilde{\mathrm{N}}} M \to \mathrm{N} M$$

in $\mathsf{DGMod}\,\mathbb{K}$.

Proof. This is a complete version of [HY, Theorem 1.12]. Let $\mathbf{\Delta}^l$ be the simplicial set $\mathbf{\Delta}^l := \mathrm{Hom}_{\mathbf{\Delta}}(-, [l])$; so its set of p-simplices is $\mathbf{\Delta}^l_p$. Define C_l to be the algebra of normalized cochains on $\mathbf{\Delta}^l$, namely

$$C_l := \mathrm{N}\,\mathrm{Hom}_{\mathsf{Sets}}(\mathbf{\Delta}^l, \mathbb{K}) \cong \mathrm{Hom}_{\mathsf{Sets}}(\mathbf{\Delta}^{l,\mathrm{nd}}, \mathbb{K}).$$

Here $\mathbf{\Delta}^{l,\mathrm{nd}}$ is the (finite) set of nondegenerate simplices, i.e. those sequences $\boldsymbol{i} = (i_0, \ldots, i_p)$ satisfying $0 \leq i_0 < \cdots < i_p \leq l$. As explained in [HY, Appendix A] we have simplicial DG algebras $C = \{C_l\}_{l \in \mathbb{N}}$ and $\Omega(\mathbf{\Delta}_{\mathbb{K}}) = \{\Omega(\mathbf{\Delta}^l_{\mathbb{K}})\}_{l \in \mathbb{N}}$, and a homomorphism of simplicial DG modules $\rho : \Omega(\mathbf{\Delta}_{\mathbb{K}}) \to C$.

It turns out (this is work of Bousfield-Gugenheim) that ρ is a homotopy equivalence in $\mathbf{\Delta}^{\mathrm{op}}\,\mathsf{DGMod}\,\mathbb{K}$, i.e. there are simplicial homomorphisms $\phi : C \to \Omega(\mathbf{\Delta}_{\mathbb{K}})$, $h : C \to C$ and $h' : \Omega(\mathbf{\Delta}_{\mathbb{K}}) \to \Omega(\mathbf{\Delta}_{\mathbb{K}})$ such that $\mathbf{1} - \rho \circ \phi = h \circ \mathrm{d} + \mathrm{d} \circ h$ and $\mathbf{1} - \phi \circ \rho = h' \circ \mathrm{d} + \mathrm{d} \circ h'$.

Now for $M = \{M^q\} \in \mathbf{\Delta}\,\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ and $N = \{N_q\} \in \mathbf{\Delta}^{\mathrm{op}}\,\mathsf{Mod}\,\mathbb{K}$ let $N \mathbin{\widehat{\otimes}_{\leftarrow}} M$ be the complete version of [HY, formula (A.1)], so that in particular $\Omega(\mathbf{\Delta}_{\mathbb{K}}) \mathbin{\widehat{\otimes}_{\leftarrow}} M \cong \widehat{\widetilde{\mathrm{N}}} M$ and $C \mathbin{\widehat{\otimes}_{\leftarrow}} M \cong \mathrm{N} M$. Moreover

$$\rho \mathbin{\widehat{\otimes}_{\leftarrow}} \mathbf{1}_M = \int_{\mathbf{\Delta}} : \widehat{\widetilde{\mathrm{N}}} M \to \mathrm{N} M.$$

It follows that $\int_{\mathbf{\Delta}}$ is a homotopy equivalence in $\mathsf{DGMod}\,\mathbb{K}$. □

Suppose $A = \{A^q\}_{q \in \mathbb{N}}$ is a cosimplicial DG algebra in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$ (not necessarily associative nor commutative). This is a pretty complicated object: for every q we have a DG algebra $A^q = \bigoplus_{i \in \mathbb{Z}} A^{q,i}$ in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$. For every $\alpha \in \mathbf{\Delta}^q_p$ there is a continuous DG algebra homomorphism $\alpha^* : A^p \to A^q$, and the α^* have to satisfy the simplicial relations.

Anyhow, both $\widehat{\widetilde{\mathrm{N}}} A$ and $\mathrm{N} A$ are DG algebras. For $\widehat{\widetilde{\mathrm{N}}} A$ the DG algebra structure comes from that of the DG algebras $\Omega(\mathbf{\Delta}^l_{\mathbb{K}}) \mathbin{\widehat{\otimes}} A^l$, via the embeddings (2.2). In case each A^l is an associative super-commutative unital DG $\mathbb{K}$-algebra, then so is $\widehat{\widetilde{\mathrm{N}}} A$. Likewise for DG Lie algebras. (The algebra $\mathrm{N} A$, with its Alexander-Whitney product, is very noncommutative.)

Assume that each $A^{q,i}$ is complete, so that the integral $\int_{\mathbf{\Delta}} : \widehat{\widetilde{\mathrm{N}}} A \to \mathrm{N} A$ is defined. This is not a DG algebra homomorphism. However:

Proposition 2.6. *Suppose* $A = \{A^q\}_{q\in\mathbb{N}}$ *is a cosimplicial DG algebra in* $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}$*, with all* A^q *complete. Then the homomorphisms* $\int_{\mathbf{\Delta}^l}$ *induce an isomorphism of graded algebras*

$$\mathrm{H}(\int_{\mathbf{\Delta}}) : \mathrm{H}\widehat{\tilde{\mathrm{N}}}A \xrightarrow{\simeq} \mathrm{HN}A.$$

Proof. This is a complete variant of [HY, Theorem 1.13]. The proof is identical, after replacing "$\otimes$" with "$\widehat{\otimes}$" where needed; cf. proof of previous proposition. □

Remark 2.7. If A is associative then presumably $\int_{\mathbf{\Delta}}$ extends to an A_∞ quasi-isomorphism $\widehat{\tilde{\mathrm{N}}}A \to \mathrm{N}A$.

3. Commutative Čech Resolutions

In this section $\mathbb{K}$ is a field of characteristic 0 and X is a noetherian topological space. We denote by $\mathbb{K}_X$ the constant sheaf $\mathbb{K}$ on X. We will be interested in the category $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, whose objects are sheaves of $\mathbb{K}$-modules on X with dir-inv structures. Note that any open set $V \subset X$ is quasi-compact.

Let $X = \bigcup_{i=0}^m U_{(i)}$ be an open covering, which we denote by $\boldsymbol{U}$. For any $\boldsymbol{i} = (i_0, \ldots, i_q) \in \mathbf{\Delta}_q^m$ define $U_{\boldsymbol{i}} := U_{(i_0)} \cap \cdots \cap U_{(i_q)}$, and let $g_{\boldsymbol{i}} : U_{\boldsymbol{i}} \to X$ be the inclusion. Given a dir-inv $\mathbb{K}_X$-module $\mathcal{M}$ and natural number q we define a sheaf

$$\mathrm{C}^q(\boldsymbol{U}, \mathcal{M}) := \prod_{\boldsymbol{i}\in\mathbf{\Delta}_q^m} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\, \mathcal{M}.$$

This is a finite product. For an open set $V \subset X$ we then have

$$\Gamma\big(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M})\big) = \prod_{\boldsymbol{i}\in\mathbf{\Delta}_q^m} \Gamma(V \cap U_{\boldsymbol{i}}, \mathcal{M}).$$

For any $\boldsymbol{i}$ the $\mathbb{K}$-module $\Gamma(V\cap U_{\boldsymbol{i}}, \mathcal{M})$ has a dir-inv structure. Hence $\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))$ is a dir-inv $\mathbb{K}$-module. If $\mathcal{M}$ happens to be a complete dir-inv $\mathbb{K}_X$-module then $\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))$ is a complete dir-inv $\mathbb{K}$-module, since each $V\cap U_{\boldsymbol{i}}$ is quasi-compact.

Keeping V fixed we get a cosimplicial dir-inv $\mathbb{K}$-module $\big\{\Gamma(V, \mathrm{C}^q(\boldsymbol{U}, \mathcal{M}))\big\}_{q\in\mathbb{N}}$. Applying the functors N^q and $\widehat{\tilde{\mathrm{N}}}^q$ we obtain $\mathbb{K}$-modules $\mathrm{N}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$ and $\widehat{\tilde{\mathrm{N}}}^q\Gamma(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M}))$. As we vary V these become presheaves of $\mathbb{K}$-modules, which we denote by $\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ and $\widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M})$.

Recall that a simplex $\boldsymbol{i} = (i_0, \ldots, i_q)$ is nondegenerate if $i_0 < \cdots < i_q$. Let $\mathbf{\Delta}_q^{m,\mathrm{nd}}$ be the set of non-degenerate simplices inside $\mathbf{\Delta}_q^m$.

Lemma 3.1. *For every* q *the presheaves*

$$\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) : V \mapsto \mathrm{N}^q\Gamma\big(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$$

and

$$\widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) : V \mapsto \widehat{\tilde{\mathrm{N}}}^q\Gamma\big(V, \mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$$

are sheaves. There is a functorial isomorphism of sheaves

$$\mathrm{N}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \cong \prod_{\boldsymbol{i}\in\mathbf{\Delta}_q^{m,\mathrm{nd}}} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\, \mathcal{M}, \tag{3.2}$$

and functorial embeddings of sheaves

$$\tilde{\hat{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M}) \hookrightarrow \prod_{l \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_l^m} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M} \right) \tag{3.3}$$

and

$$\mathcal{M} \hookrightarrow \tilde{\hat{\mathrm{N}}}^0 \mathrm{C}(\boldsymbol{U}, \mathcal{M}). \tag{3.4}$$

Proof. Since $\{\mathrm{C}^q(\boldsymbol{U}, \mathcal{M})\}_{q \in \mathbb{N}}$ is a cosimplicial sheaf we get the isomorphism (3.2).

As for $\tilde{\hat{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M})$, consider the sheaf $\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M}$ on X. Take any open set $V \subset X$ and $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$. Since $V \cap U_{\boldsymbol{i}}$ is quasi-compact we have

$$\begin{aligned} \Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \Gamma(V \cap U_{\boldsymbol{i}}, \mathcal{M}) &\cong \Gamma\big(V \cap U_{\boldsymbol{i}}, \Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M}\big) \\ &= \Gamma\big(V, g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M}\right)\big). \end{aligned}$$

By Definition 2.1 there is an exact sequence of presheaves on X:

$$\begin{aligned} 0 \to \tilde{\hat{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M}) &\to \prod_{l \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_l^m} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M} \right) \\ &\xrightarrow{\mathbf{1} \otimes \alpha^* - \alpha_* \otimes \mathbf{1}} \prod_{k, l \in \mathbb{N}} \prod_{\alpha \in \boldsymbol{\Delta}^l_k} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}_l^m} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^k_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M} \right). \end{aligned}$$

Since the presheaves in the middle and on the right are actually sheaves, it follows that $\tilde{\hat{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is also a sheaf.

Finally the embedding (3.4) comes from the embeddings $\mathcal{M} \hookrightarrow \Omega^0(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} \mathcal{M}$, $w \mapsto 1 \otimes w$. □

Thus we have complexes of sheaves $\mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ and $\tilde{\hat{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. There are functorial homomorphisms $\mathcal{M} \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ and $\mathcal{M} \to \tilde{\hat{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$. Note that the complex $\Gamma(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M}))$ is nothing but the usual global Čech complex of $\mathcal{M}$ for the covering $\boldsymbol{U}$.

Definition 3.5. The complex $\tilde{\hat{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is called the *commutative Čech resolution* of $\mathcal{M}$.

The reason for the name is that $\tilde{\hat{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$ is a sheaf of super-commutative DG algebras, as can be seen from the next lemma.

Lemma 3.6. *Suppose $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ are dir-inv $\mathbb{K}_X$-modules, and $q_1, \ldots, q_r \in \mathbb{N}$. Let $q := q_1 + \cdots + q_r$. Suppose that for every $l \in \mathbb{N}$ and $\boldsymbol{i} \in \boldsymbol{\Delta}^m_l$ we are given $\mathbb{K}$-multilinear sheaf maps*

$$\begin{aligned} \phi_{q_1, \ldots, q_r, \boldsymbol{i}} : \big(\Omega^{q_1}(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} (\mathcal{M}_1|_{U_{\boldsymbol{i}}})\big) \times \cdots \times \big(\Omega^{q_r}(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} (\mathcal{M}_r|_{U_{\boldsymbol{i}}})\big) \\ \longrightarrow \Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \mathbin{\hat{\otimes}} (\mathcal{N}|_{U_{\boldsymbol{i}}}) \end{aligned}$$

that are continuous (*for the dir-inv module structures*)*, and are compatible with the simplicial structure as in Definition* 2.1*. Then there are unique $\mathbb{K}$-multilinear sheaf maps*

$$\phi_{q_1, \ldots, q_r} : \tilde{\hat{\mathrm{N}}}^{q_1} \mathrm{C}(\boldsymbol{U}, \mathcal{M}_1) \times \cdots \times \tilde{\hat{\mathrm{N}}}^{q_r} \mathrm{C}(\boldsymbol{U}, \mathcal{M}_r) \to \tilde{\hat{\mathrm{N}}}^{q} \mathrm{C}(\boldsymbol{U}, \mathcal{N})$$

that commute with the embeddings (3.3)*.*

Proof. Direct verification. □

Lemma 3.7. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be dir-inv $\mathbb{K}_X$-modules, and $\phi : \prod \mathcal{M}_i \to \mathcal{N}$ a continuous $\mathbb{K}$-multilinear sheaf homomorphism. Then there is an induced homomorphism of complexes of sheaves*

$$\phi : \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}_1) \otimes \cdots \otimes \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}_r) \to \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{N}).$$

Proof. Use Lemma 3.6. □

In particular, if $\mathcal{M}$ is a dir-inv $\mathcal{O}_X$-module then $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is a DG $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$-module.

If $\mathcal{M} = \bigoplus_p \mathcal{M}^p$ is a graded dir-inv $\mathbb{K}_X$-module then we define

$$\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})^i := \bigoplus\nolimits_{p+q=i} \hat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{M}^p)$$

and

$$\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}) := \bigoplus\nolimits_i \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})^i.$$

Due to Lemma 3.7, if $\mathcal{M}$ is a complex in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, then $\hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ is also a complex (in $\mathsf{Mod}\,\mathbb{K}_X$), and there is a functorial homomorphism of complexes $\mathcal{M} \to \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$.

Theorem 3.8. *Let X be a noetherian topological space, with open covering $\boldsymbol{U} = \{U_{(i)}\}_{i=0}^m$. Let $\mathcal{M}$ be a bounded below complex in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, and assume each $\mathcal{M}^p$ is a complete dir-inv $\mathbb{K}_X$-module. Then:*

(1) *For any open set $V \subset X$ the homomorphism*

$$\Gamma(V, \int_{\boldsymbol{\Delta}}) : \Gamma\big(V, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) \to \Gamma\big(V, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})\big)$$

is a quasi-isomorphism of complexes of $\mathbb{K}$-modules.

(2) *There are functorial quasi-isomorphism of complexes of $\mathbb{K}_X$-modules*

$$\mathcal{M} \to \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \xrightarrow{\int_{\boldsymbol{\Delta}}} \mathrm{NC}(\boldsymbol{U}, \mathcal{M}).$$

Proof. (1) Lemma 3.1 and Proposition 2.5 imply that for any p the homomorphism of complexes

$$\Gamma(V, \int_{\boldsymbol{\Delta}}) : \Gamma\big(V, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}^p)\big) \to \Gamma\big(V, \mathrm{NC}(\boldsymbol{U}, \mathcal{M}^p)\big)$$

is a quasi-isomorphism. Now use the standard filtration argument (the complexes in question are all bounded below).

(2) From (1) we deduce that

$$\Gamma(V, \int_{\boldsymbol{\Delta}}) : \Gamma\big(V, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) \to \Gamma\big(V, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})\big) \tag{3.9}$$

is a quasi-isomorphism. Hence

$$\int_{\boldsymbol{\Delta}} : \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M}) \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$$

is a quasi-isomorphism of complexes of sheaves.

It is a known fact that $\mathcal{M}^p \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M}^p)$ is a quasi-isomorphism of sheaves (see [Ha] Lemma 4.2). Again this implies that $\mathcal{M} \to \mathrm{NC}(\boldsymbol{U}, \mathcal{M})$ is a quasi-isomorphism. And therefore the homomorphism $\mathcal{M} \to \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})$ coming from (3.4) is also a quasi-isomorphism. □

Now let us look at a separated noetherian formal scheme $\mathfrak{X}$. Let $\mathcal{I}$ be some defining ideal of $\mathfrak{X}$, and let X be the scheme with structure sheaf $\mathcal{O}_X := \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$. So $\mathfrak{X}$ and X have the same underlying topological space. Recall that a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module is a quasi-coherent $\mathcal{O}_{\mathfrak{X}}$-module which is the union of its coherent submodules.

Corollary 3.10. *Let $\mathfrak{X}$ be a noetherian separated formal scheme over $\mathbb{K}$, with defining ideal $\mathcal{I}$ and underlying topological space X. Let $\boldsymbol{U} = \{U_{(i)}\}_{i=0}^{m}$ be an affine open covering of X. Let $\mathcal{M}$ be a bounded below complex of sheaves of $\mathbb{K}$-modules on X. Assume each $\mathcal{M}^p$ is a dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module, and the coboundary operators $\mathcal{M}^p \to \mathcal{M}^{p+1}$ are continuous for the $\mathcal{I}$-adic dir-inv structures (but not necessarily $\mathcal{O}_{\mathfrak{X}}$-linear). Then:*

(1) *The canonical morphism*

$$\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) \to \mathrm{R}\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$ is an isomorphism.

(2) *There is a functorial isomorphism*

$$\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) \cong \mathrm{R}\Gamma(X, \mathcal{M})$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$.

Proof. (1) Consider the commutative diagram

$$\begin{array}{ccc}
\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) & \xrightarrow{\Gamma(X, \int_{\boldsymbol{\Delta}})} & \Gamma\big(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})\big) \\
\downarrow & & \downarrow \\
\mathrm{R}\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big) & \xrightarrow{\mathrm{R}\Gamma(X, \int_{\boldsymbol{\Delta}})} & \mathrm{R}\Gamma\big(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})\big)
\end{array}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$, in which the vertical arrows are the canonical morphisms. By part (1) of the theorem (with $V = X$) the top arrow is a quasi-isomorphism. And by part (2) the bottom arrow is an isomorphism. Hence it is enough to prove that the right vertical arrow is an isomorphism.

Using a filtration argument we may assume that $\mathcal{M}$ is a single dir-coherent $\mathcal{O}_{\mathfrak{X}}$-module. Now $\Gamma\big(X, \mathrm{NC}(\boldsymbol{U}, \mathcal{M})\big)$ is the usual Čech resolution of the sheaf $\mathcal{M}$ with respect to the covering $\boldsymbol{U}$ (cf. equation 3.2)). So it suffices to prove that for all q and $\boldsymbol{i} \in \boldsymbol{\Delta}_q^{m,\mathrm{nd}}$ the sheaves $g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}$ are $\Gamma(X, -)$-acyclic.

First let's assume $\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$-module. Let $\mathfrak{U}_{\boldsymbol{i}}$ be the open formal subscheme of $\mathfrak{X}$ supported on $U_{\boldsymbol{i}}$. Then $g_{\boldsymbol{i}}^{-1}\mathcal{M}$ is a coherent $\mathcal{O}_{\mathfrak{U}_{\boldsymbol{i}}}$-module, and both $g_{\boldsymbol{i}} : \mathfrak{U}_{\boldsymbol{i}} \to \mathfrak{X}$ and $\mathfrak{U}_{\boldsymbol{i}} \to \operatorname{Spec}\mathbb{K}$ are affine morphisms. By [EGA I, Theorem 10.10.2] it follows that $g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M} = \mathrm{R}g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}$, and also

$$\Gamma(U_{\boldsymbol{i}}, g_{\boldsymbol{i}}^{-1}\mathcal{M}) = \mathrm{R}\Gamma(U_{\boldsymbol{i}}, g_{\boldsymbol{i}}^{-1}\mathcal{M}) \cong \mathrm{R}\Gamma(X, \mathrm{R}g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}) \cong \mathrm{R}\Gamma(X, g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}).$$

We conclude that $\mathrm{H}^j(X, g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}) = 0$ for all $j > 0$.

In the general case when $\mathcal{M}$ is a direct limit of coherent $\mathcal{O}_{\mathfrak{X}}$-modules we still get $\mathrm{H}^j(X, g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1}\mathcal{M}) = 0$ for all $j > 0$.

(2) By part (2) of the theorem we get a functorial isomorphism $\mathrm{R}\Gamma(X, \mathcal{M}) \cong \mathrm{R}\Gamma\big(X, \hat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \mathcal{M})\big)$. Now use part (1) above. □

4. Mixed Resolutions

In this section $\mathbb{K}$ is s field of characteristic 0 and X is a finite type $\mathbb{K}$-scheme.

Let us begin be recalling the definition of the sheaf of principal parts $\mathcal{P}_X$ from [EGA IV]. Let $\Delta : X \to X^2 = X \times_{\mathbb{K}} X$ be the diagonal embedding. By completing X^2 along $\Delta(X)$ we obtain a noetherian formal scheme $\mathfrak{X}$, and $\mathcal{P}_X := \mathcal{O}_{\mathfrak{X}}$. The two projections $\mathrm{p}_i : X^2 \to X$ give rise to two ring homomorphisms $\mathrm{p}_i^* : \mathcal{O}_X \to \mathcal{P}_X$. We view $\mathcal{P}_X$ as a left (resp. right) $\mathcal{O}_X$-module via p_1^* (resp. p_2^*).

Recall that a connection ∇ on an $\mathcal{O}_X$-module $\mathcal{M}$ is a $\mathbb{K}$-linear sheaf homomorphism $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ satisfying the Leibniz rule $\nabla(fm) = \mathrm{d}(f) \otimes m + f\nabla(m)$ for local sections $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$.

Definition 4.1. Consider the de Rham differential $\mathrm{d}_{X^2/X} : \mathcal{O}_{X^2} \to \Omega^1_{X^2/X}$ relative to the morphism $\mathrm{p}_2 : X^2 \to X$. Since $\Omega^1_{X^2/X} \cong \mathrm{p}_1^* \, \Omega^1_X = \mathrm{p}_1^{-1} \Omega^1_X \otimes_{\mathrm{p}_1^{-1}\mathcal{O}_X} \mathcal{O}_{X^2}$ we obtain a $\mathbb{K}$-linear homomorphism $\mathrm{d}_{X^2/X} : \mathcal{O}_{X^2} \to \mathrm{p}_1^* \, \Omega^1_X$. Passing to the completion along the diagonal $\Delta(X)$ we get a connection of $\mathcal{O}_X$-modules

$$\nabla_{\mathcal{P}} : \mathcal{P}_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \tag{4.2}$$

called the *Grothendieck connection.*

Note that the connection $\nabla_{\mathcal{P}}$ is $\mathrm{p}_2^{-1}\mathcal{O}_X$ -linear. It will be useful to describe $\nabla_{\mathcal{P}}$ on the level of rings. Let $U = \operatorname{Spec} C \subset X$ be an affine open set. Then

$$\Gamma(U, \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega^1_C \otimes_C (\widehat{C \otimes C}) \cong \widehat{\Omega^1_C \otimes C},$$

the I-adic completion, where $I := \operatorname{Ker}(C \otimes C \to C)$. And $\nabla_{\mathcal{P}} : \widehat{C \otimes C} \to \widehat{\Omega^1_C \otimes C}$ is the completion of $\mathrm{d} \otimes \mathbf{1} : C \otimes C \to \Omega^1_C \otimes C$.

As usual the connection $\nabla_{\mathcal{P}}$ of (4.2) induces differential operators of left $\mathcal{O}_X$-modules

$$\nabla_{\mathcal{P}} : \Omega^i_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \to \Omega^{i+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X$$

for all $i \geq 0$, by the rule

$$\nabla_{\mathcal{P}}(\alpha \otimes b) = \mathrm{d}(\alpha) \otimes b + (-1)^i \alpha \wedge \nabla_{\mathcal{P}}(b). \tag{4.3}$$

Theorem 4.4. *Assume X is a smooth n-dimensional $\mathbb{K}$-scheme. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Then the sequence of sheaves on X*

$$\begin{aligned} 0 \to \mathcal{M} \xrightarrow{m \mapsto 1 \otimes m} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \xrightarrow{\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \\ \cdots \xrightarrow{\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}} \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to 0 \end{aligned} \tag{4.5}$$

is exact.

Proof. The proof is similar to that of [Ye1, Theorem 4.5]. We may restrict to an affine open set $U = \operatorname{Spec} B \subset X$ that admits an étale coordinate system $\boldsymbol{s} = (s_1, \ldots, s_n)$, i.e. $\mathbb{K}[\boldsymbol{s}] \to B$ is an étale ring homomorphism. It will be convenient to have another copy of B, which we call C; so that $\Gamma(U, \mathcal{P}_X) = \widehat{B \otimes C}$, the I-adic completion, where $I := \operatorname{Ker}(B \otimes C \to B)$. We shall identify B and C with their images inside $B \otimes C$, and denote the copy of the element s_i in C by r_i. Letting $t_i := r_i - s_i \in B \otimes C$ we then have $t_i = \tilde{s}_i = 1 \otimes s_i - s_i \otimes 1$ in our earlier notation. Note that $\Omega_{\mathbb{K}[\boldsymbol{s}]} \subset \Omega_B$ is a sub DG algebra, and $B \otimes_{\mathbb{K}[\boldsymbol{s}]} \Omega_{\mathbb{K}[\boldsymbol{s}]} \to \Omega_B$ is a bijection.

By definition

$$\Gamma(U, \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X) \cong \Omega_B \otimes_B (\widehat{B \otimes C}) \cong \widehat{\Omega_B \otimes C}. \tag{4.6}$$

The differential $\nabla_{\mathcal{P}}$ on the left goes to the differential $\mathrm{d}_B \otimes \mathbf{1}_C$ on the right. Consider the sub DG algebra $\Omega_{\mathbb{K}[\boldsymbol{s}]} \otimes C \subset \Omega_B \otimes C$. We know that $\mathbb{K} \to \Omega_{\mathbb{K}[\boldsymbol{s}]}$ is a quasi-isomorphism; therefore so is $C \to \Omega_{\mathbb{K}[\boldsymbol{s}]} \otimes C$.

Because $t_i + s_i = r_i \in C$ we see that $C[\boldsymbol{s}] = C[\boldsymbol{t}] \subset B \otimes C$. Therefore we obtain C-linear isomorphisms

$$\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes C \cong \Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{s}] = \Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}].$$

So there is a commutative diagram

$$(4.7)\qquad \begin{array}{ccccccccccc} 0 & \longrightarrow & C & \longrightarrow & C[\boldsymbol{t}] & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}] & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots\ \Omega^n_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & B \otimes C & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_B \otimes C & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots\ \Omega^n_B \otimes C & \longrightarrow & 0 \end{array}$$

of C-modules. The top row is exact, and the vertical arrow are inclusions. Let us introduce a new grading on $\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[\boldsymbol{t}]$ as follows: $\deg(s_i) := 1$, $\deg(t_i) := 1$, $\deg(\mathrm{d}(s_i)) := 1$ and $\deg(c) := 0$ for every nonzero $c \in C$. Since $\nabla_{\mathcal{P}}(t_i) = -\mathrm{d}(s_i)$ we see that $\nabla_{\mathcal{P}}$ is homogeneous of degree 0, and thus the top row in (4.7) is an exact sequence in the category $\mathsf{GrMod}\, C$ of graded C-modules. Now each term in this sequence is a free graded C-module, and therefore this sequence is split in $\mathsf{GrMod}\, C$.

The $\boldsymbol{t}$-adic inv structure on $C[\boldsymbol{t}]$ can be recovered from the grading, and this inv structure is the same as the I-adic inv structure on $B \otimes C$. Therefore the completion is $\Omega^p_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] \cong \widehat{\Omega^p_B \otimes C}$. Thus the diagram (4.7) is transformed to the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C & \longrightarrow & C[[\boldsymbol{t}]] & \xrightarrow{\nabla_{\mathcal{P}}} & \Omega^1_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots\ \Omega^n_{\mathbb{K}[\boldsymbol{s}]} \otimes_{\mathbb{K}[\boldsymbol{s}]} C[[\boldsymbol{t}]] & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C & \longrightarrow & \widehat{B \otimes C} & \xrightarrow{\nabla_{\mathcal{P}}} & \widehat{\Omega^1_B \otimes C} & \xrightarrow{\nabla_{\mathcal{P}}} & \cdots\ \widehat{\Omega^n_B \otimes C} & \longrightarrow & 0 \end{array}$$

in which the top tow is continuously C-linearly split, and the vertical arrows are bijections. Hence the bottom row is split exact. Comparing this to (4.6) we conclude that the sequence of right $\mathcal{O}_U$-modules

$$0 \to \mathcal{O}_U \xrightarrow{\mathrm{p}_2^*} \mathcal{P}_X|_U \xrightarrow{\nabla_{\mathcal{P}}} \big(\Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X\big)|_U \xrightarrow{\nabla_{\mathcal{P}}} \cdots \big(\Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{P}_X\big)|_U \to 0$$

is split exact.

Therefore it follows that for any $\mathcal{O}_X$-module $\mathcal{M}$ the sequence (4.5), when restricted to U, is split exact. □

Let us now fix an affine open covering $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ of X.

Let $\mathcal{I}_X = \operatorname{Ker}(\mathcal{P}_X \to \mathcal{O}_X)$. This is a defining ideal of the noetherian formal scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) := (X, \mathcal{P}_X)$. So $\mathcal{P}_X$ is an inv module over itself with the $\mathcal{I}_X$-adic inv structure. Given quasi-coherent $\mathcal{O}_X$-modules $\mathcal{M}$ and $\mathcal{N}$, the tensor product $\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a dir-coherent $\mathcal{P}_X$-module, and so it has the $\mathcal{I}_X$-adic dir-inv structure. See Example 1.4. In particular

$$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} = \bigoplus\nolimits_{p \geq 0} \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

becomes a dir-inv $\mathbb{K}_X$-module.

Lemma 4.8. *$\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$ is a DG Ω_X-module in $\mathsf{Dir\,Inv\,Mod}\,\mathbb{K}_X$, with differential $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$.*

Proof. Because $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$ is a differential operator of $\mathcal{P}_X$-modules, it is continuous for the $\mathcal{I}_X$-adic dir-inv structure. See [Ye2, Proposition 2.3]. □

Henceforth we will write $\nabla_{\mathcal{P}}$ instead of $\nabla_{\mathcal{P}} \otimes \mathbf{1}_{\mathcal{M}}$.

Definition 4.9. Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. For any $p, q \in \mathbb{N}$ define

$$\mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) := \widehat{\tilde{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

The Grothendieck connection

$$\nabla_{\mathcal{P}} : \Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{p+1}_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$$

induces a homomorphism of sheaves

$$\nabla_{\mathcal{P}} : \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}^{p+1,q}_{\boldsymbol{U}}(\mathcal{M}).$$

We also have $\partial : \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}^{p,q+1}_{\boldsymbol{U}}(\mathcal{M})$. Define

$$\mathrm{Mix}^{i}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus\nolimits_{p+q=i} \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}),$$

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus\nolimits_{i} \mathrm{Mix}^{i}_{\boldsymbol{U}}(\mathcal{M})$$

and

$$\mathrm{d}_{\mathrm{mix}} := \partial + (-1)^q \nabla_{\mathcal{P}} : \mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}^{p+1,q}_{\boldsymbol{U}} \oplus \mathrm{Mix}^{p,q+1}_{\boldsymbol{U}}(\mathcal{M}). \tag{4.10}$$

The complex $\big(\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}), \mathrm{d}_{\mathrm{mix}}\big)$ is called the *mixed resolution of* $\mathcal{M}$.

There are functorial embeddings of sheaves

$$\mathcal{M} \subset \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M} \subset \widehat{\tilde{\mathrm{N}}}^0 \mathrm{C}(\boldsymbol{U}, \Omega^0_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \mathrm{Mix}^{0,0}_{\boldsymbol{U}}(\mathcal{M}) \tag{4.11}$$

and

$$\mathrm{Mix}^{p,q}_{\boldsymbol{U}}(\mathcal{M}) \subset \prod_{l \in \mathbb{N}} \prod_{\boldsymbol{i} \in \boldsymbol{\Delta}^m_l} g_{\boldsymbol{i}*}\, g_{\boldsymbol{i}}^{-1} \left(\Omega^q(\boldsymbol{\Delta}^l_{\mathbb{K}}) \,\widehat{\otimes}\, (\Omega^p_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})\right); \tag{4.12}$$

see Lemma 3.1.

Proposition 4.13. (1) *$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$ is a sheaf of super-commutative associative unital DG $\mathbb{K}$-algebras. There are two $\mathbb{K}$-algebra homomorphisms $\mathrm{p}_1^*, \mathrm{p}_2^* : \mathcal{O}_X \to \mathrm{Mix}^0_{\boldsymbol{U}}(\mathcal{O}_X)$.*

(2) *Let $\mathcal{M}$ be a quasi-coherent $\mathcal{O}_X$-module. Then $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is a left DG $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-module.*

(3) *If $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of finite rank then the multiplication map*

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$$

is an isomorphism.

Proof. By by Lemmas 3.1 and 3.7. □

Note that $\mathrm{d}_{\mathrm{mix}} \circ \mathrm{p}_2^* : \mathcal{O}_X \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$ is zero, but $\mathrm{d}_{\mathrm{mix}} \circ \mathrm{p}_1^* \neq 0$.

Proposition 4.14. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules. Suppose*

$$\phi : \prod_{i=1}^{r} (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

is a continuous Ω_X-multilinear sheaf morphism of degree d. Then there is a unique $\mathbb{K}$-multilinear sheaf morphism of degree d

$$\widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \phi) : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_1) \times \cdots \times \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_r) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$$

which is compatible with ϕ via the embedding (4.12).

Proof. This is an immediate consequence of Lemma 3.7. □

Suppose we are given $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\, \mathcal{O}_X)$. Define

$$\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})^i := \bigoplus_{p+q=i} \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p)$$

with differential

$$\mathrm{d}_{\mathrm{mix}} + (-1)^q \mathrm{d}_{\mathcal{M}} : \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^p) \to \mathrm{Mix}^{q+1}_{\boldsymbol{U}}(\mathcal{M}^p) \oplus \mathrm{Mix}^q_{\boldsymbol{U}}(\mathcal{M}^{p+1}).$$

Theorem 4.15. *Let X be a smooth separated $\mathbb{K}$-scheme, and let $\boldsymbol{U} = \{U_{(0)}, \ldots, U_{(m)}\}$ be an affine open covering of X.*

1. *There is a functorial quasi-isomorphism $\mathcal{M} \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ for $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\, \mathcal{O}_X)$.*
2. *Given $\mathcal{M} \in \mathsf{C}^+(\mathsf{QCoh}\, \mathcal{O}_X)$, the canonical morphism $\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \to \mathrm{R}\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big)$ in $\mathsf{D}(\mathsf{Mod}\, \mathbb{K})$ is an isomorphism.*
3. *The quasi-isomorphism in part (1) induces a functorial isomorphism $\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \cong \mathrm{R}\Gamma(X, \mathcal{M})$ in $\mathsf{D}(\mathsf{Mod}\, \mathbb{K})$.*

Proof. (1) Write $\mathcal{N} := \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}$. A filtration argument and Theorem 4.4 show that the inclusion $\mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism. Next we view $\mathcal{N}$ as a bounded below complex in $\mathsf{Dir\,Inv\,Mod}\, \mathbb{K}_X$. By Theorem 3.8(2) we have a quasi-isomorphism $\mathcal{N} \to \widehat{\tilde{\mathrm{N}}}^q\mathrm{C}(\boldsymbol{U}, \mathcal{N}) = \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$.

(2) This is due to Corollary 3.10(1), applied to the formal scheme $(X, \mathcal{P}_X)$ and the complex $\mathcal{N}$ of dir-coherent $\mathcal{P}_X$-modules defined above.

(3) This assertion is an immediate consequence of parts (1) and (2). □

Corollary 4.16. *In the situation of the theorem, suppose $\mathcal{M}, \mathcal{N} \in \mathsf{C}^+(\mathsf{QCoh}\, \mathcal{O}_X)$ and $\phi : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$ is a $\mathbb{K}$-linear quasi-isomorphism. Then*

$$\Gamma(X, \phi) : \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) \to \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})\big)$$

is a quasi-isomorphism.

Proof. Consider the commutative diagram

$$\begin{array}{ccc} \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) & \xrightarrow{\Gamma(X,\phi)} & \Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})\big) \\ \downarrow & & \downarrow \\ \mathrm{R}\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})\big) & \xrightarrow{\mathrm{R}\Gamma(X,\phi)} & \mathrm{R}\Gamma\big(X, \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})\big) \end{array}$$

in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K})$. By part (2) of the theorem the vertical arrows are isomorphisms. Since ϕ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,\mathbb{K}_X)$ it follows that the bottom arrow is an isomorphism. $\square$

Given a quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ and an integer i define

$$\mathrm{G}^i \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \bigoplus_{q \geq i} \operatorname{Mix}^q_{\boldsymbol{U}}(\mathcal{M}).$$

Then $\{\mathrm{G}^i \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})\}_{i\in\mathbb{Z}}$ is a descending filtration of $\operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ by subcomplexes, satisfying $\mathrm{G}^i \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) = \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M})$ for $i \ll 0$ and $\bigcap_i \mathrm{G}^i \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) = 0$. For any i define

$$\operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) := \mathrm{G}^i \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}) \,/\, \mathrm{G}^{i+1} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{M}).$$

The functor

$$\operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}} : \mathsf{QCoh}\,\mathcal{O}_X \to \mathsf{Mod}\,\mathbb{K}_X$$

is additive, but we do not know whether it is exact. The next theorem asserts this in a very special case.

Consider the sheaves of DG Lie algebras $\mathcal{T}_{\mathrm{poly},X}$ and $\mathcal{D}_{\mathrm{poly},X}$ as complexes of quasi-coherent $\mathcal{O}_X$-modules (cf. [Ye3, Proposition 3.18]). According to [Ye1, Theorem 0.4] there is a quasi-isomorphism

$$\mathcal{U}_1 : \mathcal{T}_{\mathrm{poly},X} \to \mathcal{D}_{\mathrm{poly},X}.$$

Theorem 4.17. *For any i the homomorphism of complexes*

$$\operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{U}_1) : \operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{T}_{\mathrm{poly},X}) \to \operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{D}_{\mathrm{poly},X})$$

is a quasi-isomorphism.

Proof. Given a point $x \in X$ choose an affine open neighborhood V of x which admits an étale morphism $V \to \mathbf{A}^n_{\mathbb{K}}$. By [Ye2, Theorem 4.11] the map of complexes

$$\mathcal{U}_1|_V : \mathcal{T}_{\mathrm{poly},X}|_V \to \mathcal{D}_{\mathrm{poly},X}|_V$$

is a homotopy equivalence in $\mathsf{C}^+(\mathsf{QCoh}\,\mathcal{O}_V)$. Since $\operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}$ is an additive functor we see that $\operatorname{gr}^i_{\mathrm{G}} \operatorname{Mix}_{\boldsymbol{U}}(\mathcal{U}_1)|_V$ is a quasi-isomorphism. $\square$

Remark 4.18. We know very little about the structure of the sheaves $\widehat{\widetilde{\mathrm{N}}}^q \mathrm{C}(\boldsymbol{U}, \mathcal{M})$, even when $\mathcal{M} = \mathcal{O}_X$. Cf. [HS].

5. Simplicial Sections

Let X be a $\mathbb{K}$-scheme, and let $X = \bigcup_{i=0}^m U_{(i)}$ be an open covering, with inclusions $g_{(i)} : U_{(i)} \to X$. We denote this covering by $\boldsymbol{U}$. For any multi-index $\boldsymbol{i} = (i_0, \ldots, i_q) \in \boldsymbol{\Delta}^m_q$ we write $U_{\boldsymbol{i}} := \bigcap_{j=0}^q U_{(i_j)}$, and we define the scheme $U_q := \coprod_{\boldsymbol{i}\in\boldsymbol{\Delta}^m_q} U_{\boldsymbol{i}}$. Given $\alpha \in \boldsymbol{\Delta}^q_p$ and $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$ there is an inclusion of open sets $\alpha_* : U_{\boldsymbol{i}} \to U_{\alpha_*(\boldsymbol{i})}$. These patch to a morphism of schemes $\alpha_* : U_q \to U_p$, making $\{U_q\}_{q\in\mathbb{N}}$ into a simplicial scheme. The inclusions $g_{(i)} : U_{(i)} \to X$ induce inclusions $g_{\boldsymbol{i}} : U_{\boldsymbol{i}} \to X$ and morphisms $g_q : U_q \to X$; and one has the relations $g_p \circ \alpha_* = g_q$ for any $\alpha \in \boldsymbol{\Delta}^q_p$.

Definition 5.1. Let $\pi : Z \to X$ be a morphism of $\mathbb{K}$-schemes. A *simplicial section of* π based on the covering $\boldsymbol{U}$ is a sequence of morphisms

$$\boldsymbol{\sigma} = \{\sigma_q : \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q \to Z\}_{q\in\mathbb{N}}$$

satisfying the following conditions.

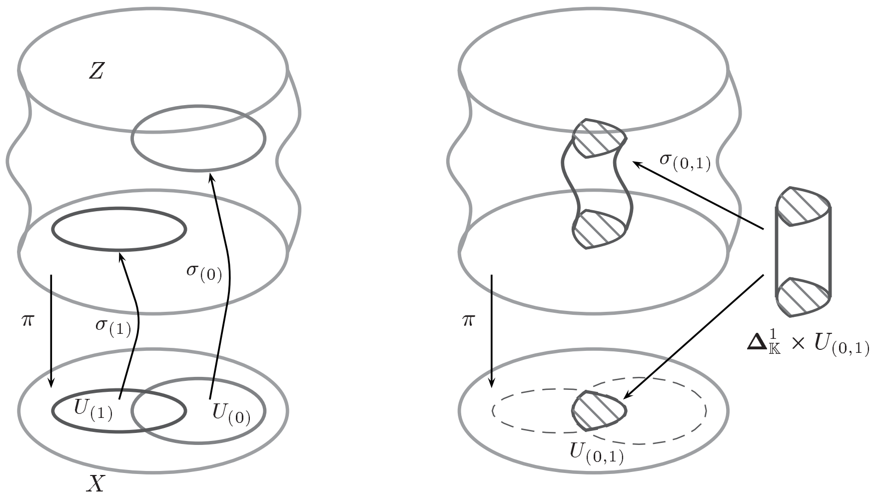

FIGURE 1. An illustration of a simplicial section $\boldsymbol{\sigma}$ based on an open covering $\boldsymbol{U} = \{U_{(i)}\}$. On the left we see two components of $\boldsymbol{\sigma}$ in dimension $q = 0$; and on the right we see one component in dimension $q = 1$.

(i) For any q the diagram

$$\begin{array}{ccc} \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q & \xrightarrow{\sigma_q} & Z \\ {\scriptstyle \mathrm{p}_2} \downarrow & & \downarrow {\scriptstyle \pi} \\ U_q & \xrightarrow{g_q} & X \end{array}$$

is commutative.

(ii) For any $\alpha \in \boldsymbol{\Delta}^q_p$ the diagram

$$\begin{array}{ccccc} & & \boldsymbol{\Delta}^p_{\mathbb{K}} \times U_p & & \\ & \overset{\mathbf{1} \times \alpha^*}{\nearrow} & & \overset{\sigma_p}{\searrow} & \\ \boldsymbol{\Delta}^p_{\mathbb{K}} \times U_q & & & & Z \\ & \underset{\alpha_* \times \mathbf{1}}{\searrow} & & \underset{\sigma_q}{\nearrow} & \\ & & \boldsymbol{\Delta}^q_{\mathbb{K}} \times U_q & & \end{array}$$

is commutative.

Given a multi-index $\boldsymbol{i} \in \boldsymbol{\Delta}^m_q$ we denote by $\sigma_{\boldsymbol{i}}$ the restriction of σ_q to $\boldsymbol{\Delta}^q_{\mathbb{K}} \times U_{\boldsymbol{i}}$. See Figure 1 for an illustration.

As explained in the introduction, simplicial sections arise naturally in several contexts, including deformation quantization.

Let A be an associative unital super-commutative DG $\mathbb{K}$-algebra. Consider homogeneous A-multilinear functions $\phi : M_1 \times \cdots \times M_r \to N$, where $M_1, \ldots, M_r, N$ are DG A-modules. There is an operation of composition for such functions: given functions $\psi_i : \prod_j L_{i,j} \to M_i$ the composition is $\phi \circ (\psi_1 \times \cdots \times \psi_r) : \prod_{i,j} L_{i,j} \to N$. There is also a summation operation: if $\phi_j : \prod_i M_i \to N$ are homogeneous of equal degree then so is their sum $\sum_j \phi_j$. Finally, let $\mathrm{d} : \prod_i M_i \to \prod_i M_i$ be the function

$$\mathrm{d}(m_1, \ldots, m_r) := \sum_{i=1}^{r} \pm\big(m_1, \ldots, \mathrm{d}(m_i), \ldots, m_r\big)$$

with Koszul signs. All the above can of course be sheafified, i.e. $\mathcal{A}$ is a sheaf of DG algebras on a scheme Z etc.

As before let $\pi : Z \to X$ be a morphism if $\mathbb{K}$-schemes, and let $\boldsymbol{U} = \{U_{(i)}\}$ be an open covering of X. Suppose $\boldsymbol{\sigma}$ is a simplicial section of π based on $\boldsymbol{U}$. We consider Ω^p_X as a discrete inv $\mathbb{K}_X$-module, and $\Omega_X = \bigoplus_{p \geq 0} \Omega^p_X$ has the $\bigoplus$ dir-inv structure. Likewise for $\Omega_Z = \bigoplus_{p \geq 0} \Omega^p_Z$.

Suppose $\mathcal{M}$ is a quasi-coherent $\mathcal{O}_X$-module. Then, as explained in Section 4, $\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})$ is a DG Ω_Z-module on Z, with the Grothendieck connection $\nabla_{\mathcal{P}}$. And $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M})$ is a DG $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-module on X, with differential $\mathrm{d}_{\mathrm{mix}}$.

Theorem 5.2. *Let $\pi : Z \to X$ be a morphism of schemes, and suppose $\boldsymbol{\sigma}$ is a simplicial section of π based on an open covering $\boldsymbol{U}$ of X. Let $\mathcal{M}_1, \ldots, \mathcal{M}_r, \mathcal{N}$ be quasi-coherent $\mathcal{O}_X$-modules, and let*

$$\phi : \prod_{i=1}^{r} \big(\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)\big) \to \Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

be a continuous Ω_Z-multilinear sheaf morphism on Z of degree k. Then there is an induced $\mathrm{Mix}_{\boldsymbol{U}}(\mathcal{O}_X)$-multilinear sheaf morphism of degree k

$$\boldsymbol{\sigma}^*(\phi) : \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_1) \times \cdots \times \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{M}_r) \to \mathrm{Mix}_{\boldsymbol{U}}(\mathcal{N})$$

on X with the following properties:

(i) *The assignment $\phi \mapsto \boldsymbol{\sigma}^*(\phi)$ respects the operations of composition and summation.*

(ii) *If $\phi = \widehat{\pi^*}(\phi_0)$ for some continuous Ω_X-multilinear morphism*

$$\phi_0 : \prod_{i=1}^{r} \big(\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i\big) \to \Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}$$

then $\boldsymbol{\sigma}^*(\phi) = \widehat{\tilde{\mathrm{N}}}\mathrm{C}(\boldsymbol{U}, \phi_0)$.

(iii) *Assume that*

$$\nabla_{\mathcal{P}} \circ \phi - (-1)^k \phi \circ \nabla_{\mathcal{P}} = \psi$$

for some continuous Ω_Z-multilinear sheaf morphism

$$\psi : \prod_{i=1}^{r} \big(\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_i)\big) \to \Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})$$

of degree $k+1$. Then

$$\mathrm{d}_{\mathrm{mix}} \circ \boldsymbol{\sigma}^*(\phi) - (-1)^k \boldsymbol{\sigma}^*(\phi) \circ \mathrm{d}_{\mathrm{mix}} = \boldsymbol{\sigma}^*(\psi).$$

Before the proof we need an auxiliary result.

Lemma 5.3. *Let A and B be complete DG algebras in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K}$, and let $f^* : A \to B$ be a continuous DG algebra homomorphism. To any DG A-module M in $\mathsf{Dir}\,\mathsf{Inv}\,\mathsf{Mod}\,\mathbb{K}$ we assign the DG B-module $f^* M := B \mathbin{\widehat{\otimes}}_A M$. Then to any continuous A-multilinear function $\phi : \prod_i M_i \to N$ we can assign a continuous B-multilinear function $f^*(\phi) : \prod_i f^*(M_i) \to f^*(N)$. This assignment is functorial in f^*, and respects the operations of composition and summation. If ϕ and ψ are such continuous A-multilinear functions, homogeneous of degrees k and $k+1$ respectively and satisfying*

$$\mathrm{d} \circ \phi - (-1)^k \phi \circ \mathrm{d} = \psi,$$

then

$$\mathrm{d} \circ f^*(\phi) - (-1)^k f^*(\phi) \circ \mathrm{d} = f^*(\psi).$$

Proof. This is all straightforward, except perhaps the last assertion. For that we make the calculations. By continuity and multilinearity it suffices to show that

$$\big(\mathrm{d} \circ f^*(\phi)\big)(\beta) - (-1)^k \big(f^*(\phi) \circ \mathrm{d}\big)(\beta) = f^*(\psi)(\beta)$$

for $\beta = (\beta_1, \ldots, \beta_r)$, with $\beta_i = b_i \otimes m_i$, $b_i \in B^{p_i}$ and $m_i \in M^{q_i}$. Then

$$\begin{aligned} \big(\mathrm{d} \circ f^*(\phi)\big)(\beta) &= \mathrm{d}\big(\pm b_1 \ldots b_r \cdot \phi(m_1, \ldots, m_r)\big) \\ &= \pm \mathrm{d}(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot \mathrm{d}\big(\phi(m_1, \ldots, m_r)\big) \end{aligned}$$

with Koszul signs. Since

$$\mathrm{d}(\beta_i) = \mathrm{d}(b_i) \otimes m_i \pm b_i \otimes \mathrm{d}(m_i)$$

we also have

$$\begin{aligned} \big(f^*(\phi) \circ \mathrm{d}\big)(\beta) &= \sum_i \pm f^*(\phi)\big(\beta_1, \ldots, \mathrm{d}(\beta_i), \ldots, \beta_r\big) \\ &= \sum_i \Big(\pm b_1 \cdots \mathrm{d}(b_i) \cdots b_r \cdot \phi(m_1, \ldots, m_r) \\ &\qquad \pm b_1 \ldots b_r \cdot \phi\big(m_1, \ldots, \mathrm{d}(m_i) \cdots m_r\big) \Big) \\ &= \pm \mathrm{d}(b_1 \cdots b_r) \cdot \phi(m_1, \ldots, m_r) \pm b_1 \cdots b_r \cdot \phi\big(\mathrm{d}(m_1, \ldots, m_r)\big). \end{aligned}$$

Finally

$$f^*(\psi)(\beta) = \pm b_1 \cdots b_r \cdot \psi(m_1, \ldots, m_r),$$

and the signs all match up. □

Proof of the theorem. For a sequence of indices $\boldsymbol{i} = (i_0, \ldots, i_l) \in \boldsymbol{\Delta}_l^m$ let us introduce the abbreviation $Y_{\boldsymbol{i}} := \boldsymbol{\Delta}_{\mathbb{K}}^l \times U_{\boldsymbol{i}}$, and let $\mathrm{p}_2 : Y_{\boldsymbol{i}} \to U_{\boldsymbol{i}}$ be the projection. The simplicial section $\boldsymbol{\sigma}$ restricts to a morphism $\sigma_{\boldsymbol{i}} : Y_{\boldsymbol{i}} \to Z$.

By Lemma 5.3, applied with respect to the DG algebra homomorphism $\sigma_{\boldsymbol{i}}^* : \sigma_{\boldsymbol{i}}^{-1} \Omega_Z \to \Omega_{Y_{\boldsymbol{i}}}$, there is an induced continuous $\Omega_{Y_{\boldsymbol{i}}}$-multilinear morphism

$$\begin{aligned} \sigma_{\boldsymbol{i}}^*(\phi) : \prod_{j=1}^r \Big(\Omega_{Y_{\boldsymbol{i}}} \mathbin{\widehat{\otimes}}_{\sigma_{\boldsymbol{i}}^{-1} \Omega_Z} \sigma_{\boldsymbol{i}}^{-1} \big(\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} \, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_j) \big) \Big) \\ \to \ \Omega_{Y_{\boldsymbol{i}}} \mathbin{\widehat{\otimes}}_{\sigma_{\boldsymbol{i}}^{-1} \Omega_Z} \sigma_{\boldsymbol{i}}^{-1} \big(\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} \, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N}) \big) \end{aligned}$$

Now for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{M}$ we have an isomorphism of dir-inv DG $\Omega_{Y_{\boldsymbol{i}}}$-modules

$$\Omega_{Y_{\boldsymbol{i}}} \mathbin{\widehat{\otimes}}_{\sigma_{\boldsymbol{i}}^{-1} \Omega_Z} \sigma_{\boldsymbol{i}}^{-1} \big(\Omega_Z \mathbin{\widehat{\otimes}}_{\mathcal{O}_Z} \widehat{\pi^*} \, (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}) \big) \cong \Omega_{Y_{\boldsymbol{i}}} \mathbin{\widehat{\otimes}}_{\mathcal{O}_{Y_{\boldsymbol{i}}}} \mathrm{p}_2^* (\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}).$$

Under the DG algebra isomorphism $\mathrm{p}_{2*}\Omega_{Y_{\boldsymbol{i}}} \cong \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \otimes \Omega_{U_{\boldsymbol{i}}}$ there is a dir-inv DG module isomorphism

$$\mathrm{p}_{2*}\big(\Omega_{Y_{\boldsymbol{i}}} \widehat{\otimes}_{\mathcal{O}_{Y_{\boldsymbol{i}}}} \mathrm{p}_2^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})\big) \cong \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \,\widehat{\otimes}\, (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M})|_{U_{\boldsymbol{i}}}.$$

Thus we obtain a family of morphisms

$$\begin{aligned} \sigma^*_{\boldsymbol{i}}(\phi) : \prod_{j=1}^{r} \Big(\Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \,\widehat{\otimes}\, (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{M}_j)|_{U_{\boldsymbol{i}}} \Big) \\ \to \ \Omega(\boldsymbol{\Delta}^l_{\mathbb{K}}) \,\widehat{\otimes}\, (\Omega_X \otimes_{\mathcal{O}_X} \mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{N})|_{U_{\boldsymbol{i}}}\big) \end{aligned}$$

indexed by $\boldsymbol{i}$ and satisfying the simplicial relations. Now use Lemma 3.6 to obtain $\boldsymbol{\sigma}^*(\phi)$. Properties (i-iii) follow from Lemma 5.3. □

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