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The Residue Complex of a Noncommutative Graded Algebra

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0. INTRODUCTION

Suppose A is a finitely generated commutative algebra over a field k. According to Grothendieck duality theory, there is a canonical complex $\mathscr{K}_A^{\cdot}$ of A-modules, called the *residue complex*. It is characterized as the Cousin complex of the twisted inverse image $\pi^! k$, where $\pi : X = \operatorname{Spec} A \to k$ is the structural morphism. $\mathscr{K}_A^{\cdot}$ has the decomposition

$$\mathscr{K}_A^{-q} = \bigoplus_{x \in X_q / X_{q-1}} \mathscr{K}_A(x) \tag{0.1}$$

where $X_q / X_{q-1} \subseteq X$ is the set of points of dimension q (the q-skeleton) and $\mathscr{K}_A(x)$ is an injective hull of the residue field $k(x)$. The coboundary operator $\delta: \mathscr{K}_A(x) \to \mathscr{K}_A(y)$ is nonzero precisely when y is an immediate specialization of x. For a discussion of the commutative theory see [RD] and [Ye2].

In this paper we propose a definition of the residue complex $R^{\cdot}$ of a noncommutative Noetherian graded k-algebra $A = k \oplus A_1 \oplus A_2 \oplus \cdots$.

We begin, in Section 1, with the *generalized Auslander–Gorenstein* (A-G) condition. This condition can be checked whenever A has a dualizing complex; if A is Gorenstein (i.e., has finite injective dimension) it reduces

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to the usual A-G condition. The generalized A-G condition is necessary for the existence of a residue complex (see below) and seems to be a reasonable requirement if A is expected to have any geometry associated to it. We generalize a result of Bjork and Levasseur to the effect that the *canonical dimension* Cdim $:= -j$, where $j(M)$ is the grade of the module M, is a finitely partitive exact dimension function (Theorem 1.3). We also extend results of [ATV2] regarding normalization of Cohen–Macaulay modules of dimension 1 (Theorem 1.9).

In Section 2 we define a *strong residue complex* over A (Definition 2.3). This is a refinement of the notion of *balanced dualizing complex* which appeared in [Ye1]. The strong residue complex $R^{\cdot}$ is unique, up to an isomorphism of complexes of graded bimodules (Theorem 2.4). So when it exists, $R^{\cdot}$ is a new invariant of A. The algebraic structure of $R^{\cdot}$ should carry some "geometric information" about A, in analogy to the commutative case. Existence is proved in two general circumstances: (i) A is finite over its center; and (ii) A is the twisted homogeneous coordinate ring of a triple $(X, \sigma, \mathscr{L})$ (Propositions 2.11, 2.8). In Section 3 we prove existence for a three-dimensional Sklyanin algebra (see below).

There is evidence that many important algebras, including some four-dimensional A-S (Artin–Schelter) regular algebras, do not have strong residue complexes [ASZ]. Guided by this evidence we devised the definition of *weak residue complex* (Def. 2.14). However, we do not have a single example of an algebra which admits a weak residue complex but not a strong one. We show that the existence of a weak residue complex implies the generalized A-G condition (Theorem 2.18).

Section 3 is devoted to proving that a three-dimensional Sklyanin algebra (see [ST, ATV1]) has a strong residue complex. Let $(E, \sigma, \mathscr{L})$ be the triple defining A; so E is an elliptic curve, and the automorphism σ is a translation. We show that A is localizable at every σ-orbit on E (Proposition 3.5). This fact is used to show that the minimal left graded-injective resolution $I^{\cdot}$ of A is also the minimal right resolution. According to [Aj3] the modules I^q have the correct GK dimensions. Therefore by tensoring with the dualizing bimodule ω we obtain the residue complex $R^{\cdot} = \omega \otimes_A I^{\cdot}$ (Theorem 3.13, Corollary 3.14).

1. THE GENERALIZED AUSLANDER–GORENSTEIN CONDITION

In [Ye1] some ideas of Grothendieck duality theory were extended to noncommutative rings, and we shall briefly review them here. Suppose $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ is a Noetherian graded algebra over a field k. It follows that A is a finitely generated algebra. By default an A-module will mean a graded left module. Let $\mathsf{GrMod}(A)$ be the abelian category of graded left A-modules with degree 0 homomorphisms, and let $\mathsf{GrMod}_{\mathrm{f}}(A)$ be the subcategory of finite (that is, finitely generated) modules. We write $\mathrm{Hom}^{\mathrm{gr}}_A(M, N)_i$ for the group of degree i homomorphisms between graded left A-modules, so

$$\mathrm{Hom}^{\mathrm{gr}}_A(M, N)_i = \mathrm{Hom}_{\mathsf{GrMod}(A)}(M, N(i)),$$

where $N(i)$ is the shifted module. Define

$$\mathrm{Hom}^{\mathrm{gr}}_A(M, N) := \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}^{\mathrm{gr}}_A(M, N)_i \in \mathsf{GrMod}(k).$$

Note that if M is finite then $\mathrm{Hom}^{\mathrm{gr}}_A(M, N) = \mathrm{Hom}_A(M, N)$.

We denote by A° the opposite ring, and $A^{\mathrm{e}} := A \otimes_k A^{\circ}$. A right module (resp. a bimodule) is regarded as a left A° (resp. A^{e}) module.

Remark 1.1. Most definitions, operations, and conditions in this paper have a left–right symmetry, expressible by interchanging A and A°. For instance, if $M, N \in \mathsf{GrMod}(A^{\circ})$ we get $\mathrm{Hom}^{\mathrm{gr}}_{A^{\circ}}(M, N) \in \mathsf{GrMod}(k)$.

Denote by $\mathsf{D}(\mathsf{GrMod}(A))$ the derived category of the abelian category $\mathsf{GrMod}(A)$. Let $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{GrMod}(A))$ be the subcategory of bounded complexes with finite cohomologies. Recall that a complex $R^{\cdot} \in \mathsf{D}^{+}(\mathsf{GrMod}(A^{\mathrm{e}}))$ is called *dualizing* if $R^{\cdot}$ has finite injective dimension over A and A°; each $\mathrm{H}^q R^{\cdot}$ is finite over A and A°; and the natural morphisms $A \to \mathrm{RHom}^{\mathrm{gr}}_A(R^{\cdot}, R^{\cdot})$ and $A \to \mathrm{RHom}^{\mathrm{gr}}_{A^{\circ}}(R^{\cdot}, R^{\cdot})$ are isomorphisms in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$. Then the functors $\mathrm{RHom}^{\mathrm{gr}}_A(-, R^{\cdot})$ and $\mathrm{RHom}^{\mathrm{gr}}_{A^{\circ}}(-, R^{\cdot})$ are anti-equivalences between $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{GrMod}(A))$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{GrMod}(A^{\circ}))$. The dualizing complex $R^{\cdot}$ is unique in the following sense: any other dualizing complex is isomorphic in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$ to $R^{\cdot} \otimes_A L[n]$, for some invertible bimodule L and integer n (see [Ye1, Theorem 3.9]).

Let $\mathfrak{m}$ be the augmentation ideal of A. Write $\Gamma_{\mathfrak{m}}$ (resp., $\Gamma_{\mathfrak{m}^{\circ}}$) for the functor of left (resp. right) $\mathfrak{m}$-torsion. A dualizing complex $R^{\cdot}$ is called *balanced* if there are isomorphisms $\mathrm{R}\Gamma_{\mathfrak{m}} R^{\cdot} \cong \mathrm{R}\Gamma_{\mathfrak{m}^{\circ}} R^{\cdot} \cong A^*$ in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$. Here $A^* := \mathrm{Hom}^{\mathrm{gr}}_k(A, k)$, the graded-injective hull of the trivial module k. The balanced dualizing complex $R^{\cdot}$ is unique up to isomorphism in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$. For example, a Noetherian Artin–Schelter regular algebra A of dimension n has an invertible bimodule ω s.t. $\omega[n]$ is a balanced dualizing complex (see [Ye1, Cor. 4.14]).

Suppose $R^{\cdot}$ is a dualizing complex over A. Given a finite graded A-module M, its *grade number* w.r.t. $R^{\cdot}$ is defined to be

$$j_{A;R^{\cdot}}(M) := \inf\{q \mid \mathrm{Ext}^q_A(M, R^{\cdot}) \neq 0\} \in \mathbb{Z} \cup \{\infty\}.$$

Note that if A is Gorenstein (i.e. it has finite injective dimension) and $R^{\cdot} = A$ we recover the usual grade number.

DEFINITION 1.2. We say A satisfies the *generalized Auslander–Gorenstein* (A-G) condition if for every $M \in \mathsf{GrMod}_{\mathrm{f}}(A)$, integer q and graded submodule $N \subseteq \mathrm{Ext}^q_A(M, R^{\cdot})$, one has $j_{A^{\circ}; R^{\cdot}}(N) \geq q$, and if the same holds with A, A° interchanged.

It is easily seen that this definition does not depend on the particular dualizing complex $R^{\cdot}$. Indeed, if we take any other complex $\tilde{R}^{\cdot}$, then it is isomorphic in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$ to $R^{\cdot} \otimes_A L[n]$, and these twists will cancel out. The condition is clearly left–right symmetric (cf. Remark 1.1). In Section 2 we will relate the generalized A-G condition with residue complexes.

The next theorem generalizes results of Bjork [Bj] and Levasseur [Le].

THEOREM 1.3. *Suppose A satisfies the generalized Auslander–Gorenstein condition. Then $M \mapsto -j_{A; R^{\cdot}}(M)$ is a finitely partitive exact dimension function on* $\mathsf{GrMod}_{\mathrm{f}}(A)$ (*see* [MR, *Sects*. 6.8, 8.3]).

Proof. According to [Ye1, Prop. 2.4], we can assume $R^{\cdot}$ is a bounded complex of bimodules and each R^q is graded-injective over A and A°. Then the adjunction homomorphism $M \to H^{\cdot}$, where

$$H^{\cdot} := \mathrm{Hom}^{\mathrm{gr}}_{A^{\circ}}\big(\mathrm{Hom}^{\mathrm{gr}}_A(M, R^{\cdot}), R^{\cdot}\big)$$

is a quasi-isomorphism. Pick a positive integer d large enough so that $R^q \neq 0$ only if $|q| \leq d$. Consider the decreasing filtration on $H^{\cdot}$ given by the subcomplexes

$$F^p H^{\cdot} := \mathrm{Hom}^{\mathrm{gr}}_{A^{\circ}}\big(\mathrm{Hom}^{\mathrm{gr}}_A(M, R^{\cdot}), R^{\geq p}\big).$$

Then F is an exhaustive filtration, and there is a convergent spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^{\mathrm{gr},\, p}_{A^{\circ}}\big(\mathrm{Ext}^{\mathrm{gr},\, -q}_A(M, R^{\cdot}), R^{\cdot}\big) \Rightarrow M. \tag{1.1}$$

The corresponding decreasing filtration

$$M = F^{-d}M \supset F^{-d+1}M \supset \cdots \supset F^{d+1}M = 0$$

is called the b-filtration in [Le].

The generalized A-G condition tells us that $E_2^{p,q} = 0$ if $p < -q$. So the spectral sequence lives in a bounded region of the (p, q) plane: $p \geq -q$ and $|q|, |p| \leq d$. We conclude from formula (1.1) that for every $|p| \leq d$

there is an exact sequence of graded A-modules

$$0 \to \frac{F^p M}{F^{p+1} M} \to E_2^{p,-p} \to Q^p \to 0$$

with Q^p a subquotient of $\bigoplus_i E_2^{p+1+i,-p-i}$. Therefore $j_{A;R^\cdot}(F^p M / F^{p+1} M) \geq p$ (cf. [Bj, Thm. 1.3] and [Le, Thm. 2.2]).

From here the proof continues just like in [Bj, Propositions 1.6, 1.8] and [Le, Sects. 2–4]. ∎

From here to the end of this section we will assume A satisfies the generalized A-G condition, and also that it has some balanced dualizing complex $R^\cdot$. The uniqueness of $R^\cdot$ in $\mathsf{D}(\mathsf{GrMod}(A^e))$ justifies the following definition.

DEFINITION 1.4. The *canonical dimension* of a finite graded A-module M is

$$\operatorname{CDim} M := -j_{A;R^\cdot}(M) \in \mathbb{Z} \cup \{-\infty\}.$$

COROLLARY 1.5. *Any finite A-module M has a critical composition series w.r.t.* CDim.

Proof. See [Le, (4.6.4)] or [MR, Lemma 6.2.10 and Prop. 6.2.20]. ∎

PROPOSITION 1.6. Let M be a finite graded A-module.

1. One has

$$\operatorname{CDim} M \in \{-\infty, 0, 1, \ldots, \operatorname{CDim} A\},$$

CDim $M \leq 0$ iff M is $\mathfrak{m}$-torsion, and CDim $M = -\infty$ iff $M = 0$.

2. If $\operatorname{Ext}_A^q(M, R^\cdot) \neq 0$ then $-\operatorname{CDim} M \leq q \leq 0$.

Proof. (1) Suppose M has finite length. Since $R^\cdot$ is balanced, $\operatorname{RHom}_A^{\mathrm{gr}}(M, R^\cdot) \cong M^*$, so $\operatorname{CDim} M \in \{-\infty, 0\}$. Now suppose M is a critical module. Then either $M \cong k$, or M has a nonzero finite length quotient $\overline{M}$, in which case $\operatorname{CDim} M > \operatorname{CDim} \overline{M} = 0$. But any module M has a critical composition series.

(2) The inequality $q \geq -\operatorname{CDim} M$ is trivial. By the generalized A-G condition and part 1 we have $-q \geq \operatorname{CDim} \operatorname{Ext}_A^q(M, R^\cdot) \geq 0$. ∎

Let us finish off this section with an application, due to Artin. It is a generalization of [ATV2, Propositions 6.3 and 6.6].

DEFINITION 1.7. We say a finite graded A-module M is *Cohen–Macaulay* (C-M) if $\operatorname{RHom}_A^{\mathrm{gr}}(M, R^\cdot) \cong M^\vee[n]$ for some A°-module $M^\vee$ and integer n.

The A°-module $M^{\vee}$ is called the dual module of M, and it is also C-M: $(M^{\vee})^{\vee} = M$. Of course, $n = \mathrm{CDim}\, M = \mathrm{CDim}\, M^{\vee}$.

We shall abbreviate the dualizing functors as follows: $D := \mathrm{RHom}^{\mathrm{gr}}_{A}(-, R^{\cdot})$ and $D^{\circ} := \mathrm{RHom}^{\mathrm{gr}}_{A^{\circ}}(-, R^{\cdot})$. Fix for the remainder of the section an isomorphism $\mathrm{R}\Gamma_{\mathfrak{m}} R^{\cdot} \cong A^*$ in $\mathsf{D}(\mathsf{GrMod}(A^{\mathrm{e}}))$ (a rigidification of $R^{\cdot}$). This determines an isomorphism $\mathrm{R}\Gamma_{\mathfrak{m}^{\circ}} R^{\cdot} \cong A^*$ such that $D^{\circ}Dk \cong k \cong (k^*)^*$ (see [Ye1, Remark 5.7]).

Proposition 1.8. *Suppose A satisfies the generalized A-G condition.*

1. *Let M be a finite graded A-module with* $\mathrm{CDim}\, M = 1$. *Then M is C-M iff it is $\mathfrak{m}$-torsion free.*

2. *Suppose $\phi : M' \to M$ is a homomorphism between C-M modules of dimension* 1, *which is an isomorphism modulo $\mathfrak{m}$-torsion. Then $\phi^{\vee} : M^{\vee} \to (M')^{\vee}$ is also an isomorphism modulo $\mathfrak{m}$-torsion. To be precise, there is a natural exact sequence of A°-modules*

$$0 \to M^{\vee} \xrightarrow{\phi^{\vee}} (M')^{\vee} \to \mathrm{Coker}(\phi)^* \to 0.$$

Proof. 1. First assume M is $\mathfrak{m}$-torsion free. Set $N^{-1} := \mathrm{H}^{-1}DM$ and $N^0 := \mathrm{H}^0 DM$. Let $\sigma_{\leq q}$ and $\sigma_{>q}$ be the truncation functors of [RD, Chap. 1, Sect. 7]. Since $\sigma_{\leq -1} DM \cong N^{-1}[1]$ and $\sigma_{\leq 0}\sigma_{>-1} DM \cong N^0$ we get a triangle

$$N^{-1}[1] \to DM \to N^0 \to N^{-1}[2] \tag{1.2}$$

in $\mathsf{D}^{\mathrm{b}}_{\mathrm{f}}(\mathsf{GrMod}(A^{\circ}))$. By the generalized A-G condition the module N^0 has finite length, so $D^{\circ}N^0 = (N^0)^*$. Because $\mathrm{CDim}\, N^{-1} \leq 1$ it follows that $\mathrm{H}^q D^{\circ} N^{-1} \neq 0$ only for $q = -1, 0$. Therefore $\mathrm{H}^0 D(N^{-1}[2]) = 0$. Applying $\mathrm{H}^0 D^{\circ}$ to the triangle (1.2) we get $0 \to (N^0)^* \to M$. The conclusion is that $N^0 = 0$, so M is C-M with dual $M^{\vee} = N^{-1}$.

Conversely, suppose M is C-M, so $DM = M^{\vee}[1]$. Let $T := \Gamma_{\mathfrak{m}} M$, $\overline{M} := M/T$. The triangle $T \to M \to \overline{M} \to T[1]$ gives an exact sequence

$$\mathrm{H}^0 DM \to \mathrm{H}^0 DT \to \mathrm{H}^1 D\overline{M}.$$

Since M is C-M we have $\mathrm{H}^0 DM = 0$. By Proposition 1.6, $\mathrm{H}^1 D\overline{M} = 0$. Therefore $T^* = \mathrm{H}^0 DT = 0$, so M is $\mathfrak{m}$-torsion free.

2. Let $N := \mathrm{Coker}(\phi)$. Since M' is $\mathfrak{m}$-torsion free, it follows that $\mathrm{Ker}(\phi) = 0$, so there is a triangle $M' \to M \to N \to M'[1]$. Apply $\mathrm{H}^0 D$ to this triangle, and use the fact that $DN \cong N^*$. ∎

Theorem 1.9. *Suppose A has a balanced dualizing complex and satisfies the generalized Auslander–Gorenstein condition. Let M be a Cohen–Macaulay A-module with* $\mathrm{CDim}\, M = 1$. *Then there is an A-module* Norm M, *which is* *functorial in M. There is a natural exact sequence of A-modules*

$$0 \to M \to \operatorname{Norm} M \to (M^{\vee})^* \to 0. \tag{1.3}$$

If $M \to \tilde{M}$ is an isomorphism modulo $\mathfrak{m}$-torsion then $\operatorname{Norm} M \to \operatorname{Norm} \tilde{M}$ *is an isomorphism. The module* $\operatorname{Norm} M$ *is $\mathfrak{m}$-torsion free. There is a natural isomorphism* $(\operatorname{Norm} M)^* \cong \operatorname{Norm}(M^{\vee})$.

Proof. For $n \geq 0$ define A-modules $M'_n := M_{\geq n} \subseteq M$ and $M''_n := M/M'_n$. So M'_n is a C-M module and M''_n is of finite length. The triangle

$$DM''_n \to DM \to DM'_n \to (DM''_n)[1]$$

gives an exact sequence

$$0 \to M^{\vee} \to (M'_n)^{\vee} \to (M''_n)^* \to 0. \tag{1.4}$$

Taking k-linear duals we obtain an inverse system

$$0 \to M''_n \to \left((M'_n)^{\vee}\right)^* \to (M^{\vee})^* \to 0 \tag{1.5}$$

and in the limit we get the sequence (1.3), where $\operatorname{Norm} M := \lim_{\leftarrow n}((M'_n)^{\vee})^*$. Clearly this construction is functorial for A-linear homomorphisms $\phi : M \to \tilde{M}$ between C-M modules. If $\operatorname{Ker}(\phi)$ and $\operatorname{Coker}(\phi)$ are $\mathfrak{m}$-torsion then $\phi : M'_n \to \tilde{M}'_n$ is bijective for $n \gg 0$, so $\operatorname{Norm}(\phi)$ is also bijective.

Next we shall prove that $N := \operatorname{Norm} M$ is $\mathfrak{m}$-torsion free. For any integer m consider the A°-submodule $(M^{\vee})'_m := (M^{\vee})_{\geq m} \subseteq M^{\vee}$ and the quotient $(M^{\vee})''_m := M^{\vee}/(M^{\vee})'_m$. Set $\tilde{M}_{-m} := ((M^{\vee})'_m)^{\vee}$. The exact sequence

$$0 \to (M^{\vee})'_m \to M^{\vee} \to (M^{\vee})''_m \to 0,$$

when dualized, gives, according to Proposition 1.8, an exact sequence

$$0 \to M \to \tilde{M}_{-m} \to \left((M^{\vee})''_m\right)^* \to 0. \tag{1.6}$$

Since $N \cong \operatorname{Norm} \tilde{M}_{-m}$ we get injections $M \to \tilde{M}_{-m} \to N$. On comparing the size of cokernels in formulas (1.3) and (1.6) we conclude that $\lim_{m \to} \tilde{M}_{-m} \cong N$. Therefore N is $\mathfrak{m}$-torsion free.

Now consider the C-M A°-module $M^{\vee}$. In the construction of $\operatorname{Norm}(M^{\vee})$, the sequence corresponding to (1.4) is (1.6), so

$$\left(\operatorname{Norm}(M^{\vee})\right)^* = \lim_{m \to} \tilde{M}_{-m} \cong N. \qquad \blacksquare$$

2. RESIDUE COMPLEXES—DEFINITIONS AND PROPERTIES

Let $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ be a Noetherian graded algebra over a field k. Suppose dim is an exact dimension function for A-modules (in the sense of [MR, Sect. 6.8]). Really we need two such functions, $\dim_A : \mathsf{GrMod}_{\mathrm{f}}(A) \to \mathbb{N} \cup \{-\infty\}$ and $\dim_{A^\circ} : \mathsf{GrMod}_{\mathrm{f}}(A^\circ) \to \mathbb{N} \cup \{-\infty\}$, but we will try to keep this fact invisible, when possible.

DEFINITION 2.1. Let M be a (left graded) A-module and q an integer. Define $\Gamma_{\mathsf{M}_q} M$ to be the sum of all finite submodules $M' \subseteq M$ with $\dim M' \leq q$. Let $\mathsf{M}_q \subseteq \mathsf{GrMod}(A)$ be the subcategory whose objects are the modules M satisfying $\Gamma_{\mathsf{M}_q} M = M$. For a right module N we write $\Gamma_{\mathsf{M}^\circ_q} N \subseteq N$ and the corresponding category is $\mathsf{M}^\circ_q \subseteq \mathsf{GrMod}(A^\circ)$.

One should think of $\Gamma_{\mathsf{M}_q} M$ as the submodule of elements "supported on M_q," in analogy to commutative algebraic geometry. For any module M there is a filtration

$$0 = \Gamma_{\mathsf{M}_{-1}} M \subseteq \Gamma_{\mathsf{M}_0} M \subseteq \cdots \subseteq \Gamma_{\mathsf{M}_d} M = M$$

where $d = \dim_A A$.

The subquotients are

$$\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} M := \Gamma_{\mathsf{M}_q} M / \Gamma_{\mathsf{M}_{q-1}} M. \tag{2.1}$$

We get additive functors Γ_{M_q} and $\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}}$ on the category of graded left modules. If M is a bimodule then for any $a \in A$, right multiplication by a preserves $\Gamma_{\mathsf{M}_q} M$. Hence the functors Γ_{M_q} and $\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}}$ send bimodules to bimodules.

DEFINITION 2.2. 1. A nonzero (graded left) A-module M is said to be *pure* of dimension q (w.r.t. dim) if $\Gamma_{\mathsf{M}_q} M = M$ and $\Gamma_{\mathsf{M}_{q-1}} M = 0$.

2. An A-module M is said to be *essentially pure* of dimension q if there is an essential submodule $M' \subseteq M$ which is pure of dimension q.

3. The algebra A is called *pure* if every essentially pure graded A-module or A°-module is pure.

DEFINITION 2.3. A *strong residue complex* over A w.r.t. dim is a complex of bimodules $R^{\cdot}$ satisfying:

(i) Each bimodule R^q is a graded-injective module over A and A°.

(ii) Each bimodule R^q is pure of dimension $-q$ over A and A°.

(iii) $R^{\cdot}$ is a balanced dualizing complex.

It is immediate to see that the complex $R^{\cdot}$ is bounded; in fact, $R^q \neq 0$ only for $-d \leq q \leq 0$, where $d = \min\{\dim_A A, \dim_{A^\circ} A\}$.

THEOREM 2.4. *A strong residue complex is unique. Specifically, if $R^{\cdot}$ and $\tilde{R}^{\cdot}$ are two strong residue complexes, then there is an isomorphism of complexes of graded bimodules $\phi : R^{\cdot} \to \tilde{R}^{\cdot}$, and ϕ is unique up to a constant in k^*.*

The proof is given after some preparatory results.

LEMMA 2.5. *The functors Γ_{M_q} and $\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}}$ have derived functors*

$$\mathrm{R}\Gamma_{\mathsf{M}_q}, \mathrm{R}\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} : \mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}})) \to \mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}})).$$

If $I^{\cdot} \in \mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}}))$ is a complex with each I^p a graded-injective A-module, then $\mathrm{R}\Gamma_{\mathsf{M}_q} I^{\cdot} = \Gamma_{\mathsf{M}_q} I^{\cdot}$ and $\mathrm{R}\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} I^{\cdot} = \Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} I^{\cdot}$.

Proof. The proof is based on that of [Ye1, Theorem 1.2], which in turn relies on [RD, Chap. I, Theorem 5.1]. Any complex $M^{\cdot} \in \mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}}))$ is quasi-isomorphic to some complex $I^{\cdot}$ as above (see [Ye1, Lemma 1.1]). Thus it suffices to prove that if $I^{\cdot}$ is such a complex which is acyclic, then the complexes $\Gamma_{\mathsf{M}_q} I^{\cdot}$ and $\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} I^{\cdot}$ are also acyclic.

Denote by δ the coboundary operator of $I^{\cdot}$. Suppose $x \in \Gamma_{\mathsf{M}_q} I^p$, $\delta x = 0$. Let $L \subseteq A$ be the annihilator of x, so $\dim A/L \leq q$. Since the complex $\mathrm{Hom}^{\mathrm{gr}}_A(A/L, I^{\cdot})$ is acyclic, there is some $y \in \Gamma_{\mathsf{M}_q} I^{p-1}$ with $\delta y = x$. This proves the acyclicity of $\Gamma_{\mathsf{M}_q} I^{\cdot}$. From the exact sequence of complexes

$$0 \to \Gamma_{\mathsf{M}_{q-1}} I^{\cdot} \to \Gamma_{\mathsf{M}_q} I^{\cdot} \to \Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} I^{\cdot} \to 0$$

we see that $\Gamma_{\mathsf{M}_q/\mathsf{M}_{q-1}} I^{\cdot}$ is also acyclic. ∎

LEMMA 2.6. *Suppose $R^{\cdot}$ is a strong residue complex w.r.t.* dim. *Then the generalized A-G condition holds and* dim = Cdim (*for A and A°*).

Proof. If $\dim M < -q$ then $\mathrm{Hom}_A(M, R^q) = 0$, and therefore $\mathrm{Ext}^q_A(M, R^{\cdot}) = 0$. This means that $\mathrm{Cdim}\, M \leq \dim M$.

Take any surjection $\bigoplus_{i=1}^m A(n_i) \twoheadrightarrow M$ in $\mathsf{GrMod}(A)$. Then the A°-module $\mathrm{Ext}^q_A(M, R^{\cdot})$ is a subquotient of $\bigoplus R^q(-n_i)$, and hence $\dim \mathrm{Ext}^q_A(M, R^{\cdot}) \leq -q$. At this point we have proved the generalized A-G condition. Next, the convergence of the spectral sequence (1.1) implies that $\dim M \leq \max\{\dim E_2^{p,q}\}$. But $\dim E_2^{p,q} \leq -p$, and $E_2^{p,q} \neq 0$ implies $-p \leq q \leq \mathrm{Cdim}\, M$. ∎

Proof of Theorem 2.4. The proof is an adaptation of ideas found in [RD, Chap. IV]. First observe that by Lemma 2.6, both $R^{\cdot}$ and $\tilde{R}^{\cdot}$ are strong residue complexes w.r.t. Cdim. We define Γ_{M_q} using this dimension function. Let $M^{\cdot}$ be any complex in $\mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}}))$. Replace $M^{\cdot}$ by a quasi-isomorphic complex $I^{\cdot}$ as in Lemma 2.5. Define a decreasing filtration on $I^{\cdot}$ by $F^p I^{\cdot} := \Gamma_{\mathsf{M}_{-p}} I^{\cdot}$. This filtration gives the usual spectral sequence of a filtered complex, and after identifying terms we obtain

$$E_1^{p,q} = \mathrm{H}^{p+q}\big(F^p I^{\cdot}/F^{p+1} I^{\cdot}\big) = \mathrm{H}^{p+q} R\Gamma_{\mathsf{M}_{-p}/\mathsf{M}_{-p-1}} M^{\cdot} \Rightarrow \mathrm{H}^{p+q} M^{\cdot}$$

(see [ML, Chap. XI, Sect. 8]). Define the (left) Cousin complex of $M^{\cdot}$ to be the complex $(EM^{\cdot})^p := E_1^{p,0}$ with operator $d_1^{p,0} : E_1^{p,0} \to E_1^{p+1,0}$. The result is a functor $E : \mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}})) \to \mathsf{C}(\mathsf{GrMod}(A^{\mathrm{e}}))$, where the latter is the (abelian) category of complexes of graded bimodules.

If $R^{\cdot}$ is a strong residue complex, then $\Gamma_{\mathsf{M}_{-p}/\mathsf{M}_{-p-1}} R^q = R^q$ if $q = p$ and 0 otherwise. Therefore $ER^{\cdot} \cong R^{\cdot}$ as complexes.

Now according to [Ye1, Sect. 4], balanced dualizing complexes are unique up to isomorphism in $\mathsf{D}^+(\mathsf{GrMod}(A^{\mathrm{e}}))$. Choose such an isomorphism $\psi : R^{\cdot} \to \tilde{R}^{\cdot}$, which is known to be unique up to a constant. Then $\phi = E(\psi) : R^{\cdot} \to \tilde{R}^{\cdot}$ is the desired isomorphism. ▮

The next proposition is a generalization of [Aj3, Theorem 3.14].

PROPOSITION 2.7. *If A has a strong residue complex then it is a pure algebra.*

Proof. Let M be a finite A-module and $M' \subseteq M$ an essential submodule, pure of dimension q. It will suffice to produce an injection $M' \to (R^{-q})^i$ for some i. Suppose $N \subseteq M'$ is critical. By the generalized A-G condition there is a nonzero homomorphism $\phi : N \to R^{-q}$, which by purity must be injective. Since every nonzero A-module has a critical submodule (cf. Corollary 1.5) it follows that there is an essential submodule $N_1 \oplus \cdots \oplus N_i \subseteq M'$ with all N_i critical. Choose injective homomorphisms $\phi_i : N_i \to R^{-q}$ and let $\psi : M' \to (R^{-q})^i$ be any extension of $\bigoplus \phi_i$. Then ψ is necessarily injective. ▮

When we can associate with A a sufficiently rich geometry, e.g., when the projective spectrum Proj A is a classical projective scheme (in the terminology of [AZ]), one would expect that A would have a strong residue complex. The propositions below justify this expectation. First consider the twisted homogeneous coordinate ring of a triple $(X, \sigma, \mathscr{L})$, where X is a proper scheme, σ is an automorphism, and $\mathscr{L}$ is a σ-ample invertible sheaf (cf. [AV]).

PROPOSITION 2.8. *Suppose A is a twisted homogeneous coordinate ring. Then A has a strong residue complex, w.r.t. to* dim = Kdim (*Krull dimension*).

Proof. A balanced dualizing complex $R^{\cdot}$ exists by [Ye1, Theorem 7.3]. It is the cone over the natural homomorphism of complexes $\Gamma_* \mathscr{K}_X^{\cdot} \to A^*$

arising from Grothendieck duality. Here $\mathscr{K}_X^{\cdot}$ is the residue complex of X. For each q, R^q is a graded-injective module over A and A°.

Since $R^0 \cong A^*$, it has Kdim $= 0$. For $q < 0$ we have $R^q \cong \Gamma_* \mathscr{K}_X^{q+1}$. Because of the equivalence of categories between $\mathsf{GrMod}(A)$ modulo $\mathfrak{m}$-torsion and quasi-coherent $\mathscr{O}_X$-modules, it follows that for any nonzero coherent sheaf $\mathscr{M}$, Kdim $\Gamma_* \mathscr{M} = \dim \operatorname{Supp} \mathscr{M} + 1$. It is known that the quasi-coherent sheaf $\mathscr{K}_X^{q+1}$ is pure of dimension $-q-1$ (by this we mean that each nonzero coherent subsheaf $\mathscr{M} \subseteq \mathscr{K}_X^{q+1}$ has $\dim \operatorname{Supp} \mathscr{M} = -q-1$). Hence R_A^q is pure of Kdim $= -q$. All this works for right modules too. ∎

Remark 2.9. One can show that if some positive power of $\mathscr{L}^{\sigma} \otimes \mathscr{L}^{-1}$ is in the identity component $\operatorname{Pic}^0 X$ of the Picard scheme of X, then for each graded A-module M one has the equality GKdim $M =$ Kdim M. On the other hand, in [AV, Example 5.18] we see a twisted homogeneous coordinate ring A with GKdim $A = 5$ and Kdim $A = 3$.

The decomposition $\mathscr{K}_X^{\cdot} = \bigoplus_{x \in X} \mathscr{K}_X(x)$ (cf. formula (0.1)) induces a bimodule decomposition $R^{\cdot} = (\bigoplus_T R(T)) \oplus A^*$, where T runs through the σ-orbits in X and $R(T) := \bigoplus_{x \in T} \Gamma_* \mathscr{K}_X(x)$. It is known that $\mathscr{K}_X(x)$ is an indecomposable injective in $\mathsf{QCoh}(X)$, so $\Gamma_* \mathscr{K}_X(x)$ is indecomposable in $\mathsf{GrMod}(A)$.

PROBLEM 2.10. Is $R(T)$ an indecomposable bimodule?

The second general situation to consider is an algebra finite over its center.

PROPOSITION 2.11. *If A is finite over its center then it has a strong residue complex*, *w.r.t* dim = Kdim = GKdim.

Proof. There is a finite centralizing homomorphism $C \to A$, where $C = k[t_1, \dots, t_d]$ is a (commutative) polynomial ring, and the variables t_i all have degree $e \geq 1$. The algebra C has a residue complex $R_C^{\cdot}$. If $e = 1$ use Prop. 2.8 with $X = \mathbf{P}_k^{d-1}$; if $e > 1$ simply take the same complex as for $e = 1$ and change the grading. Let $R_A^{\cdot} := \operatorname{Hom}_C^{\mathrm{gr}}(A, R_C^{\cdot})$. According to [Ye1, Theorem 5.4] this is a balanced dualizing complex over A. Each R_A^q is graded-injective on both sides. Since as a C-module R_A^q embeds into a finite direct sum of twists of R_C^q, it is pure of GK dimension $-q$. ∎

Here again commutative geometry says there is a bimodule decomposition $R^{\cdot} = \bigoplus_{\mathfrak{p}} R(\mathfrak{p})$, where $\mathfrak{p}$ runs over the graded primes of the center of A.

PROBLEM 2.12. Is $R(\mathfrak{p})$ an indecomposable bimodule?

Remark 2.13. Let A_q be the multiparameter quantum deformation of the polynomial ring $A = k[t_1, \dots, t_d]$, depending on a $d \times d$ matrix $q =$

$[q_{ij}]$ (see [Ye3]). We do not know whether, for all q, A_q admits a strong residue complex. The problem is that localization destroys the $\mathbb{Z}^d$-grading which is used to deform A-modules into A_q-modules, so the residue complex $R_A^{\cdot}$ cannot be deformed.

In Section 3 we shall prove that a three-dimensional Sklyanin algebra has a strong residue complex. Recent work of Ajitabh *et al.* [ASZ] shows that some four-dimensional Artin–Schelter regular algebras do not admit strong residue complexes. They actually find an algebra A such that in the minimal graded-injective resolution $0 \to A \to I^{-4} \to I^{-3} \to \cdots$, each I^q is essentially pure of dimension $-q$ (w.r.t. Cdim = GKdim), but I^{-1} is not pure. Influenced by this result we make the next definition, even though we have no example (so far) of an algebra with a weak residue complex but no strong residue complex.

DEFINITION 2.14. A weak residue complex w.r.t. dim is a complex of bimodules $R^{\cdot}$ satisfying:

(i) Each bimodule R^q is a graded-injective module over A and A°.

(ii) Each bimodule R^q is essentially pure of dimension $-q$ over A and A°, and there is equality $\Gamma_{\mathsf{M}_q} R^{\cdot} = \Gamma_{\mathsf{M}^{\circ}_q} R^{\cdot} \subseteq R^{\cdot}$.

(iii) $R^{\cdot}$ is a balanced dualizing complex.

Let $J^{\cdot}$ be a complex of graded-injective A-modules. We say $J^{\cdot}$ is a minimal injective complex if for every q, $\operatorname{Ker}(\delta : J^q \to J^{q+1}) \subseteq J^q$ is an essential submodule. Any complex $M^{\cdot} \in \mathsf{D}^+(\mathsf{GrMod}(A))$ admits a quasi-isomorphism to a minimal injective complex $J^{\cdot}$, and one can easily check that this $J^{\cdot}$ is unique up to isomorphism (cf. [Ye1, Lemma 4.2]). Observe that minimality has nothing to do with a dimension function, nor is $M^{\cdot} \mapsto J^{\cdot}$ functorial.

LEMMA 2.15. *Suppose $J^{\cdot}$ is a complex of graded-injective A-modules with J^q essentially pure of dimension $-q$. Then $J^{\cdot}$ is minimal.*

Proof. Pick an integer q. Let $M := \operatorname{Ker}(\delta : J^q \to J^{q+1})$ and let I be a graded-injective hull of M. So $J^q \cong I \oplus I'$ and $\delta : I' \to J^{q+1}$ is an injection. By the purity assumption we get $I' = 0$. ∎

We conclude:

PROPOSITION 2.16. *If $R^{\cdot}$ and $\tilde{R}^{\cdot}$ are weak residue complexes, then they are isomorphic as complexes of A-modules and as complexes of A°-modules. In particular, if one is a strong residue complex then so is the other.*

PROBLEM 2.17. Is it possible for an algebra A to admit two weak residue complexes $R^{\cdot}$ and $\tilde{R}^{\cdot}$ which are not isomorphic as complexes of graded bimodules? (Of course A cannot be pure.)

At this point we wish to relate residue complexes to the generalized A-G condition.

THEOREM 2.18. *Let* dim *be an exact dimension function for* A. *Suppose that either condition holds*:

(i) *A admits a strong residue complex.*

(ii) *A admits a weak residue complex, and every finite left or right graded A-module has a* dim *critical composition series.*

Then A satisfies the generalized A-G condition, and $\dim = \mathrm{Cdim}$.

LEMMA 2.19. *Say $R^{\cdot}$ is the residue complex in condition* (ii) *of the theorem. Let M be a critical finite module with* $\dim M = d$. *Then for every $q > -d$ there is a finite module $\overline{M}$ with* $\dim \overline{M} < d$, *and a homomorphism $M \to \overline{M}$, s.t.* $\mathrm{Ext}^q_A(\overline{M}, R) \to \mathrm{Ext}^q_A(M, R^{\cdot})$ *is surjective.*

Proof. Write $E(M) := \mathrm{Ext}^q_A(M, R^{\cdot})$. Say $[\phi] \in E(M)$ is represented by $\phi : M \to R^q$. Because M is critical (and therefore pure of dimension d) and R^q is essentially pure of dimension $-q$, ϕ cannot be injective. So $\overline{M}_\phi := \mathrm{Im}(\phi)$ has $\dim \overline{M}_\phi < d$ and $[\phi] \in \mathrm{Im}(E(\overline{M}_\phi) \to E(M))$. Now choose $[\phi_1], \dots, [\phi_m]$ which generate $E(M)$ over A°. Then $\overline{M} := \bigoplus \overline{M}_{\phi_i}$ has the required properties. ▮

LEMMA 2.20. *Let M be a finite A-module. Assume* $\dim M = d$. *Then in the situation of condition* (ii) *of the theorem*:

1. $\dim \mathrm{Ext}^q_A(M, R^{\cdot}) \leq d$ *for all* q.
2. $\dim \mathrm{Ext}^q_A(M, R^{\cdot}) < d$ *for all* $q > -d$.
3. $\mathrm{Ext}^q_A(M, R^{\cdot}) = 0$ *for all* $q < -d$.

Proof. Say $\bigoplus_{i=1}^m A(n_i) \twoheadrightarrow M$ is a presentation of M. Then

$$\mathrm{Hom}_A(M, R^q) \subseteq \Gamma_{\mathsf{M}_d}\Big(\bigoplus R^q(-n_i)\Big) = \Gamma_{\mathsf{M}^\circ_d}\Big(\bigoplus R^q(-n_i)\Big).$$

Since $\mathrm{Ext}^q_A(M, R^{\cdot})$ is a subquotient of $\mathrm{Hom}_A(M, R^q)$ this implies part 1. If moreover $d < -q$ then $\Gamma_{\mathsf{M}_d} R^q = 0$, giving part 3.

Let us prove part 2. We may assume M is critical. Then the assertion is a consequence of Lemma 2.19 and part 1 applied to $\overline{M}$. ▮

Note that the two lemmas work also for right modules (exchange A and A°).

Proof of Theorem 2.18. We need only consider condition (ii) of the theorem (cf. Lemma 2.6). Say $\dim M = d$. By part 3 of Lemma 2.20 we have Cdim $M \leq d$. Suppose Cdim $M < d$. Then by parts 1 and 2 of the lemma all the terms in the spectral sequence (1.1) have dim $< d$, which is impossible. The conclusion is Cdim $M = \dim M$.

To prove the generalized A-G condition it suffices to check that $\dim \operatorname{Ext}^q_A(M, R^\cdot) \leq -q$. We will do so by induction on $d = \dim M$. For $d \leq -q$ this is part 1 of Lemma 2.20. For $d > -q$ and M critical, the module $\overline{M}$ of Lemma 2.19 has dim $\overline{M} < d$ so we can use induction. For other modules this is true by looking at a critical composition series. ∎

PROBLEM 2.21. Is it true that every algebra which satisfies the generalized A-G condition admits a weak residue complex? It was proved in [Le] and [TV] that Sklyanin algebras of all dimensions satisfy the A-G condition, yet it is not known even whether every four-dimensional Sklyanin algebra admits a weak residue complex.

Let us finish this section with the Cohen–Macaulay case.

COROLLARY 2.22. *Assume the hypotheses of Theorem* 2.18. *Furthermore, assume* $R^\cdot \cong \omega[d]$ *in* $\mathsf{D}(\mathsf{GrMod}(A^e))$ *for some bimodule* ω *and some integer* d. *Then* $d =$ Cdim A, *and*

$$0 \to \omega \to R^{-d} \to \cdots \to R^0 \to 0 \tag{2.2}$$

is a minimal graded-injective resolution of ω, *both as left and right module*.

Proof. The isomorphism $\operatorname{RHom}^{\mathrm{gr}}_A(A, R^\cdot) = R^\cdot \cong \omega[d]$ means that A is a C-M A-module with Cdim $A = d$ and dual module $A^\vee = \omega$. Hence $R^{-d-1} = 0$ and we deduce the exact sequence (2.2). By Lemma 2.19 it is a minimal resolution. ∎

3. THE RESIDUE COMPLEX OF A THREE-DIMENSIONAL SKLYANIN ALGEBRA

In this section k is an algebraically closed field. We assume A is a three-dimensional Sklyanin algebra (see [ST]), which is the same as a type A three-dimensional regular algebra with three generators (in the classification of [ATV1]). The triple $(E, \sigma, \mathscr{L})$ consists of a smooth elliptic curve $E \subseteq \mathbf{P}^2_k$, an invertible sheaf $\mathscr{L} = \mathscr{O}_E(1)$, and a translation σ by some point of $E(k)$. We shall prove that A is localizable at any σ-orbit $T \subseteq E(k)$. Such a result was obtained in [Aj2] for twisted homogeneous coordinate rings of $\mathbf{P}^1_k$, by another method.

Let B be the twisted homogeneous coordinate ring of the triple $(E, \sigma, \mathscr{L})$. Then $B \cong A/(g)$ where g is a central element of A of degree

3. An $\mathscr{O}_E$-module $\mathscr{M}$ defines a left graded B-module

$$\Gamma_* \mathscr{M} := \bigoplus_{n \in \mathbb{Z}} \Gamma(E, \mathscr{L}^{(1-\sigma^n)/(1-\sigma)} \otimes \mathscr{M}^{\sigma^n}),$$

where the exponents are in the integral group ring $\mathbb{Z}\langle \sigma \rangle$ and $\mathscr{M}^\sigma := \sigma^* \mathscr{M}$. If $\mathscr{M}$ is equivariant w.r.t. σ then $\Gamma_* \mathscr{M}$ is actually a B-B-bimodule, and if $\mathscr{A}$ is an equivariant $\mathscr{O}_E$-algebra, then $\Gamma_* \mathscr{A}$ is a graded k-algebra with an algebra homomorphism $B \to \Gamma_* \mathscr{A}$ (cf. [AV] and [Ye1]).

Given a point $p \in E(k)$ let $\mathscr{I}(p) := k(E)/\mathscr{O}_{E,p}$, considered as a quasi-coherent sheaf. So $\mathscr{I}(p)$ is an injective hull of the residue field $k(p)$, and there is an exact sequence

$$0 \to \mathscr{O}_E \to k(E) \to \bigoplus_{p \in E(k)} \mathscr{I}(p) \to 0. \tag{3.1}$$

Let $B_E := \Gamma_* k(E)$, a graded k-algebra, and $I_B(p) := \Gamma_* \mathscr{I}(p)$, a graded left B-module. Recall that the point module N_p is the module $(\Gamma_* k(p))_{\geq 0}$.

LEMMA 3.1. *$B_E \cong \operatorname{Frac}^{\mathrm{gr}} B$, the graded total ring of fractions. $I_B(p)$ is a left graded-injective hull of N_p. Applying Γ_* to the sequence* (3.1) *we get an exact sequence of graded left B-modules*

$$0 \to B \to B_E \to \bigoplus_{p \in E(k)} I_B(p).$$

It is the beginning of a minimal graded-injective resolution, and the only missing term is $B^* = \operatorname{Hom}_k^{\mathrm{gr}}(B, k)$.

Proof. By [Ye1, Theorem 7.3], plus the fact that $\mathscr{O}_E \cong \omega_E$ (noncanonically). ∎

Fix a σ-orbit $T \subseteq E(k)$. Then $\bigoplus_{p \in T} k(p)$ is an equivariant sheaf, and hence $\bigoplus_{p \in T} N_p$ is a B–B-bimodule. Let

$$I_B(T) := \bigoplus_{p \in T} I_B(p) \cong \Gamma_* \left(\bigoplus_{p \in T} \mathscr{I}(p) \right).$$

This too is a bimodule, and is also a graded-injective hull of $\bigoplus_{p \in T} N_p$ on both sides. Define the $\mathscr{O}_E$-subalgebra $\mathscr{O}_{E,T} \subseteq k(E)$ by

$$\Gamma(U, \mathscr{O}_{E,T}) := \bigcap_{p \in T \cap U} \mathscr{O}_{E,p}$$

for $U \subseteq E$ open. Then we get a σ-equivariant exact sequence

$$0 \to \mathscr{O}_{E,T} \to k(E) \to \bigoplus_{p \in T} \mathscr{I}(p) \to 0, \tag{3.2}$$

from which we see that $\mathcal{O}_{E,T}$ is a σ-equivariant quasi-coherent sheaf. Let $B_T := \Gamma_* \mathcal{O}_{E,T}$, a graded subalgebra of B_E. Define a multiplicative set

$$S_T := B \cap \{\text{homogeneous units of } B_T\}. \tag{3.3}$$

PROPOSITION 3.2. *The sequence of B_T–B_T-bimodules*

$$0 \to B_T \to B_E \xrightarrow{\delta_{E,T}} I_B(T) \to 0, \tag{3.4}$$

gotten by applying Γ_* to (3.2), *is exact. S_T is a left and right denominator set in B, and $B_T = S_T^{-1}B = BS_T^{-1}$.*

To prove the proposition we first need two lemmas.

LEMMA 3.3. *Given $p \in E(k) - T$ there is some $b \in B_1$, s.t. $b(p) = 0$ but $b(q) \neq 0$ for every $q \in T$.*

Proof. Say σ is translation by $r \in E(k)$ and $T = q_0 + \langle r \rangle$ in the group structure of $E(k)$. Given any nonzero $b \in B_1 = \Gamma(E, \mathcal{L})$ (which is the same as a line $\{b = 0\}$ in $\mathbf{P}_k^2$) its divisor of zeroes is $\{p_1, p_2, p_3\}$, and these points satisfy $p_1 + p_2 + p_3 = 0$. Consider a line through $p_1 = p$; then $p_2 \in T$ iff p_3 is in the σ-orbit $T' := -p - q_0 + \langle r \rangle$. Now $E(k)$ being a divisible group, the cyclic subgroup $\langle r \rangle$ has infinite index. Hence in $E(k)$ there are infinitely many σ-orbits, and so there are infinitely many lines through p which do not intersect T at all. ∎

LEMMA 3.4. *Consider the left B-module $BS_T^{-1} \subseteq B_T$. Then $BS_T^{-1} = \lim_{s \to} Bs^{-1}$, the limit over $s \in S_T$.*

Proof. We have to prove that given $s_1, s_2 \in S_T$ there is some $s \in S_T$ s.t. $Bs_1^{-1} + Bs_2^{-1} \subseteq Bs^{-1}$. For any nonzero $s \in B$ let $\mathcal{R}(s) \subseteq \mathcal{O}_{E,T}$ be the sheaf associated to the free module Bs^{-1}; so $Bs^{-1} \cong \Gamma_* \mathcal{R}(s)$. It therefore suffices to prove that for some s, $\mathcal{R}(s_1) + \mathcal{R}(s_2) \subseteq \mathcal{R}(s)$.

Now $\mathcal{R}(s_i) = \mathcal{O}_E(D_i)$ for some effective divisors D_i supported on $E - T$. Let $D := D_1 + D_2$, so $\mathcal{R}(s_i) \subseteq \mathcal{O}_E(D)$. Say $D = \sum_{j=1}^n p_j$ (with repetition). By Lemma 3.3 we can find $b_j \in B_1$ s.t. $b_j(\sigma^{j-1}(p_j)) = 0$ but for all $q \in T$, $b_j(q) \neq 0$. Then taking $s := b_1 \cdots b_n \in S_T$ we get $\mathcal{O}_E(D) \subseteq \mathcal{R}(s)$. ∎

Proof of Proposition 3.2. First observe that (3.2) is a σ-equivariant sequence of $\mathcal{O}_{E,T}$-modules, so (3.4) is a sequence of graded B_T–B_T-bimodules.

Choose any affine open set $U \subseteq E$ containing T. This is possible since $T \neq E(k)$ (cf. Lemma 3.3) and we can take $U = E - \{p'\}$ for some $p' \notin T$. Since $\mathrm{H}^1(U, \mathcal{O}_E) = 0$, it follows that $k(E) \to \Gamma(U, \bigoplus_{p \in E(k)} \mathcal{J}(p))$ is surjective. But $I_B(T)$ is a direct summand of $\Gamma(U, \bigoplus_{p \in E(k)} \mathcal{J}(p))$. This proves that $\delta_{E,T}$ is surjective in degree 0. For other degrees just twist everything.

To prove the second assertion it suffices, by Lemma 3.4, to prove that $B_T = BS_T^{-1}$ (cf. [MR, Chap. 3.1]). Now BS_T^{-1} is a graded left B-module. Let $\mathscr{R}$ be the sheaf on E associated to BS_T^{-1}, so $\mathscr{O}_E \subseteq \mathscr{R} \subseteq \mathscr{O}_{E,T}$. If $p \in T$ then the stalk $\mathscr{O}_{E,p} = (\mathscr{O}_{E,T})_p$, so a fortiori $\mathscr{R}_p = (\mathscr{O}_{E,T})_p$. If $p \notin T$ then by Lemma 3.3 we may find some $a_i, b_i \in B_1 = \Gamma(E, \mathscr{L})$, $i \geq 1$, s.t. $a_i(\sigma^{i-1}(p)) \neq 0$, $b_i(\sigma^{i-1}(p)) = 0$ and for all $q \in T$, $b_i(q) \neq 0$. Since $b_i \in S_T$ we get

$$c_n = a_1 \cdots a_n b_n^{-1} \cdots b_1^{-1} \in \left(BS_T^{-1}\right)_0 \subseteq \Gamma(E, \mathscr{R}).$$

So $c_n \in \mathscr{R}_p \subseteq k(E)$ has a pole of order at least n. This implies that $\mathscr{R}_p = k(E) = (\mathscr{O}_{E,T})_p$.

We have shown that $\mathscr{R} = \mathscr{O}_{E,T}$. Since $BS_T^{-1} = \lim_{s \to} Bs^{-1}$ and $Bs^{-1} \cong \Gamma_* \mathscr{R}(s)$ it follows that $BS_T^{-1} \cong \Gamma_* \mathscr{R} = B_T$.

Finally, by the left–right symmetry of Γ_* for equivariant sheaves (cf. [Ye1, Prop. 6.17]) we also get $S_T^{-1}B = B_T$. ∎

Define

$$\tilde{S}_T := \{s \in A \mid s \text{ is homogeneous and } s + (g) \in S_T \subseteq B\}$$

which is clearly a multiplicative set. Let $Q := \operatorname{Frac}^{\mathrm{gr}} A$, the graded total ring of fractions.

PROPOSITION 3.5. *$\tilde{S}_T$ is a left and right denominator set, with ring of fractions $A_T := A\tilde{S}_T^{-1} \subseteq Q$.*

Proof. Copy the proof of [Aj1, Chap. III, Prop. 3.6]. ∎

From here to Corollary 3.14 we will assume the automorphism σ has infinite order.

Consider a minimal graded-injective resolution of A as a left module:

$$0 \to A \to I^{-3} \xrightarrow{\delta} I^{-2} \xrightarrow{\delta} I^{-1} \xrightarrow{\delta} I^0 \to 0 \tag{3.5}$$

(with unusual numbering). Inside Q there are the two subrings $\Lambda := A[g^{-1}]$ and

$$A_E = A_{(g)} := \{as^{-1} \mid s \text{ is homogeneous, } s \notin (g)\}$$

(cf. [Aj3]). Define an A_E–A_E-bimodule $I_A(E) := Q/A_E$. For every $p \in E(k)$ let $I_A(p)$ be a graded-injective hull of N_p (as an A-module). In [Aj3] Ajitabh proves the following:

THEOREM 3.6. *The left A-module I^{-q} is pure of GK dimensions q. Moreover,*

$$I^{-3} \cong Q$$

$$I^{-2} \cong \frac{Q}{\Lambda} \oplus I_A(E)$$

$$I^{-1} \cong \bigoplus_{p \in E(k)} I_A(p)$$

$$I^0 \cong A^*(3)$$

as graded left A-modules. The homomorphism $\delta : I^{-3} \to I^{-2}$ is the sum of the two projections $Q \twoheadrightarrow Q/\Lambda$ and $Q \twoheadrightarrow I_A(E)$.

The algebra Q is filtered by the "fractional ideals" $A_E g^n$, $n \in \mathbb{Z}$ (the "(g)-adic valuation") and we denote by $\mathrm{gr}^{(g)}\ Q$ the resulting graded algebra. Note that this algebra carries two gradings. Since g is a central regular element, we see that $\mathrm{gr}^{(g)}Q = B_E[\bar{g}, \bar{g}^{-1}]$, where $\bar{g}$ is the symbol of g, and this algebra is isomorphic to a Laurent polynomial algebra over B_E in the central indeterminate $\bar{g}$. Similarly, we have $\mathrm{gr}^{(g)}A = B[\bar{g}]$.

Suppose M is a (g)-torsion left A-module. Then we write

$$M_{-n} := \mathrm{Hom}_A\big(A/(g^{n+1}), M\big) \subseteq M.$$

This defines a decreasing exhaustive filtration on M, with $M_1 = 0$. Denote by $\mathrm{gr}^{(g)}M$ the associated graded module.

LEMMA 3.7. *The left A-modules $I_A(E)$, I^{-1}, and I^0 are (g)-torsion. The modules $\mathrm{gr}^{(g)}I_A(E)$, $\mathrm{gr}^{(g)}I^{-1}$, and $\mathrm{gr}^{(g)}I^0$ are $B[\bar{g}^{-1}]$-modules; in fact, writing M for either of these modules we get a bijection*

$$B\big[\bar{g}^{-1}\big] \otimes_B M_0 \xrightarrow{\cong} \mathrm{gr}^{(g)}M.$$

Proof. According to Theorem 3.6, I^{-1} has GK dimension 1. Since no power of σ fixes the class of $[\mathscr{L}]$ in Pic E it follows that I^{-1} is (g)-torsion (see [ATV2, Prop. 7.8]). The other two modules are trivially (g)-torsion. Almost by definition multiplication by $\bar{g}$ is injective on $\mathrm{gr}^{(g)}_{-n}M$, $n > 0$. Since M is a graded-injective A-module it is g-divisible, and so $\mathrm{gr}^{(g)}M$ is uniquely $\bar{g}$-divisible. ∎

The class $\bar{g}^{-1}$ of g^{-1} in $I_A(E) = Q/A_E$ is killed by (g). Thus it induces a degree 0 B-module homomorphism $B(3) \xrightarrow{\cdot \bar{g}^{-1}} I_A(E)_0$.

LEMMA 3.8. 1. *The sequence*

$$0 \to B(3) \xrightarrow{\cdot \bar{g}^{-1}} I_A(E)_0 \xrightarrow{\delta} I_0^{-1} \xrightarrow{\delta} I_0^0 \to 0$$

is a minimal left graded-injective resolution of $B(3)$ as a B-module.

2. *The sequence*

$$0 \to B[\bar{g}^{-1}](3) \xrightarrow{\cdot \bar{g}^{-1}} \mathrm{gr}^{(g)} I_A(E) \xrightarrow{\mathrm{gr}^{(g)}(\delta)} \mathrm{gr}^{(g)} I^{-1} \xrightarrow{\mathrm{gr}^{(g)}(\delta)} \mathrm{gr}^{(g)} I^{0} \to 0$$

of $B[\bar{g}^{-1}]$*-modules is exact.*

Proof. 1. Since Q/Λ has no (g)-torsion, it follows that

$$\mathrm{Hom}_A(B, I^{\cdot}) = \left(0 \to I_A(E)_0 \xrightarrow{\delta} I_0^{-1} \xrightarrow{\delta} I_0^0 \to 0\right).$$

But $\mathrm{Ext}_A^q(B, A) = 0$, unless $q = 1$, in which case it is isomorphic to $B(3)$. Hence the sequence is exact. Clearly each $\mathrm{Hom}_A(B, I^q)$ is a graded-injective B-module. By Theorem 3.6 and Lemma 2.20 we see that the resolution is minimal.

2. Use Lemma 3.7. ∎

Now fix a σ-orbit $T \subseteq E(k)$. Set

$$I_A(T) := \bigoplus_{p \in T} I_A(p) \subseteq I^{-1}$$

and let $\delta_{E,T} : I_A(E) \to I_A(T)$ be the homomorphism

$$\delta_{E,T} : I_A(E) \hookrightarrow I^{-2} \xrightarrow{\delta} I^{-1} \twoheadrightarrow I_A(T).$$

PROPOSITION 3.9. $\delta_{E,T}$ *is surjective.*

Proof. We shall prove by induction on n that $(\delta_{E,T})_{-n} : I_A(E)_{-n} \to I_A(T)_{-n}$ is surjective. For $n = 0$ this is done in Prop. 3.2 (in view of Lemma 3.8 part 1, and Lemma 3.1). Therefore by Lemma 3.8, part 2, $\mathrm{gr}^{(g)}(\delta_{E,T})$ is surjective. Now suppose $x \in I_A(T)_{-(n+1)} - I_A(T)_{-n}$, with symbol $[x] \in \mathrm{gr}^{(g)}_{-(n+1)} I_A(T)$. Then $[x] = \mathrm{gr}^{(g)}(\delta_{E,T})([y])$ for some $y \in I_A(E)_{-(n+1)}$. But then $x - \delta_{E,T}(y) \in I_A(T)_{-n}$, and we can use the induction hypothesis. ∎

LEMMA 3.10. $I_A(T)$ *is a left* A_T*-module, and* $\delta_{E,T} : I_A(E) \to I_A(T)$ *is* A_T*-linear.*

Proof. It suffices to prove that for all $n \geq 0$, $A_T \otimes_A I_A(T)_{-n} \cong I_A(T)_{-n}$. For $n = 0$ this is done in Proposition 3.2 (B_T is the image of A_T under the projection $A_E \to B_E$). To prove the claim for $n > 0$ it is enough to show that every $s \in \tilde{S}_T$ acts invertibly on $I_A(T)_{-n}$. Look at the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_A(T)_0 & \longrightarrow & I_A(T)_{-(n+1)} & \xrightarrow{g\cdot} & I_A(T)_{-n} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle s\cdot} & & \downarrow{\scriptstyle s\cdot} & & \downarrow{\scriptstyle s\cdot} & & \\ 0 & \longrightarrow & I_A(T)_0 & \longrightarrow & I_A(T)_{-(n+1)} & \xrightarrow{g\cdot} & I_A(T)_{-n} & \longrightarrow & 0. \end{array}$$

Now by induction the two extreme vertical arrows are bijective. Therefore so is the middle one. ▮

PROPOSITION 3.11. *The kernel of* $\delta_{E,T}$ *is*

$$\bigcup_{n \geq 1} A_T g^{-n} = \frac{A_T[g^{-1}] + A_E}{A_E} \subseteq \frac{Q}{A_E} = I_A(E).$$

In particular, it is a sub-A_T*–*A_T*-bimodule of* $I_A(E)$.

Proof. We shall prove by induction on n that $\operatorname{Ker}((\delta_{E,T})_{-n}) : I_A(E)_{-n} \to I_A(T)_{-n}$ is the submodule $A_T g^{-n}$. For $n = 0$ this is done in Proposition 3.2. Now for any n, $\delta_{E,T}(g^{-n}) = 0$, since we can start with $g^{-n} \in Q = I^{-3}$, and then corresponding to the sequence

$$Q \xrightarrow{\delta} \frac{Q}{\Lambda} \oplus I_A(E) \xrightarrow{\delta} I^{-1} = \bigoplus_{T'} I_A(T')$$

(sum on all orbits T') we get

$$g^{-n} \mapsto (g^{-n}, g^{-n}) = (0, g^{-n}) \mapsto 0 = \sum_{T'} \delta_{E,T'}(g^{-n}).$$

By by Lemma 3.10, $\operatorname{Ker}(\delta_{E,T})$ is an A_T-module, so $A_T g^{-n} \subseteq \operatorname{Ker}((\delta_{E,T})_{-n})$.

Now let $x \in I_A(E)_{-(n+1)} - I_A(E)_{-n}$, $\delta_{E,T}(x) = 0$. Since $\operatorname{Ker}(\operatorname{gr}^{(g)}(\delta_{E,T})) = B_T[\bar{g}^{-1}]$ (cf. Lemma 3.7), there is some $a \in A_T$ s.t. $x - ag^{-(n+1)} \in I_A(E)_{-n}$ and $\delta_{E,T}(x - ag^{-(n+1)}) = 0$. Here we can use induction. ▮

Remark 3.12. Observe the similarity to the proof of [Ye2, Theorem 4.3.13], the main step in constructing the residue complex on a scheme. In both instances the surjection from a generic component to a special component is used to parametrize the special component.

THEOREM 3.13. *Suppose* A *is a three-dimensional Sklyanin algebra over an algebraically closed field, and the automorphism* σ *has infinite order. Then there is an exact sequence of graded* A*–*A*-bimodules*

$$0 \to A \to I^{-3} \xrightarrow{\delta} I^{-2} \xrightarrow{\delta} I^{-1} \xrightarrow{\delta} I^0 \to 0$$

which is a minimal graded-injective resolution of A, both as a left and right module. Moreover, I^{-q} is pure of GK dimension q, both as left and right module.

Proof. By Theorem 3.6, I^{-3}, I^{-2} are bimodules, which are graded-injective modules on both sides, and $\delta : I^{-3} \to I^{-2}$ is a bimodule map. According to Prop. 3.11, for every orbit T the kernel $\mathrm{Ker}(\delta_{E,T}) \subseteq I_A(E)$ is a sub-bimodule. Furthermore this same kernel occurs in the minimal right resolution of A. Since $\delta_{E,T}$ is surjective (Prop. 3.9) this endows $I_A(T)$ with a bimodule structure. Now $I^{-1} = \bigoplus_T I_A(T)$. We conclude that $0 \to A \to I^{-3} \to I^{-2} \to I^{-1}$ is a bimodule complex which is at the same time the beginning of a left and the beginning of a right minimal resolution. Since the sequence (3.5) is exact we see that $\mathrm{Coker}(\delta : I^{-2} \to I^{-1}) \cong I^0$. This puts a bimodule structure on I^0, and necessarily $I^0 \cong A^*(3)$ as right modules. ∎

Finally, we have

COROLLARY 3.14. *Let A be a three-dimensional Sklyanin algebra. Then A has a strong residue complex w.r.t.* GKdim.

Proof. If σ has finite order then A is finite over its center, so we can apply Proposition 2.11. Otherwise let ω be the dualizing bimodule of A, namely the bimodule s.t. $\omega[3]$ is a balanced dualizing complex. (Actually in this special case ω is just $A(-3)$.) Taking the complex $I^{\cdot}$ of the theorem we see that $R^{\cdot} := \omega \otimes_A I^{\cdot}$ is a strong residue complex. ∎

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