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Derived Picard Groups of Finite-Dimensional Hereditary Algebras

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Abstract. Let A be a finite-dimensional algebra over a field k. The derived Picard group $\mathrm{DPic}_k(A)$ is the group of triangle auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ induced by two-sided tilting complexes. We study the group $\mathrm{DPic}_k(A)$ when A is hereditary and k is algebraically closed. We obtain general results on the structure of $\mathrm{DPic}_k(A)$, as well as explicit calculations for many cases, including all finite and tame representation types. Our method is to construct a representation of $\mathrm{DPic}_k(A)$ on a certain infinite quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$. This representation is faithful when the quiver $\boldsymbol{\Delta}$ of A is a tree, and then $\mathrm{DPic}_k(A)$ is discrete. Otherwise a connected linear algebraic group can occur as a factor of $\mathrm{DPic}_k(A)$. When A is hereditary, $\mathrm{DPic}_k(A)$ coincides with the full group of k-linear triangle auto-equivalences of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. Hence, we can calculate the group of such auto-equivalences for any triangulated category D equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. These include the derived categories of piecewise hereditary algebras, and of certain noncommutative spaces introduced by Kontsevich and Rosenberg.

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0 Introduction and Statement of Results

Let k be a field and A an associative unital k-algebra. We write $\mathsf{Mod}A$ for the category of left A-modules, and $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod}A)$ for the bounded derived category. Let A° be the opposite algebra and $A^{\mathrm{e}} := A \otimes_k A^{\circ}$ the enveloping algebra, so that $\mathsf{Mod}A^{\mathrm{e}}$ is the category of k-central A-A-bimodules.

A *two-sided tilting complex* a complex $T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}A^{\mathrm{e}})$ for which there exists another complex $T^{\vee} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}A^{\mathrm{e}})$ satisfying $T^{\vee} \otimes^{\mathrm{L}}_A T \cong T \otimes^{\mathrm{L}}_A T^{\vee} \cong A$. This notion is due to Rickard [Rd]. The *derived Picard group* of A (relative to k) is

$$\mathrm{DPic}_k(A) := \frac{\{\text{two-sided tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}A^{\mathrm{e}})\}}{\text{isomorphism}}$$

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with identity element A, product $(T_1, T_2) \mapsto T_1 \otimes^{\mathrm{L}}_A T_2$ and inverse $T \mapsto T^{\vee} := \mathrm{RHom}_A(T, A)$. See [Ye] for more details.

Since every invertible bimodule is a two-sided tilting complex, $\mathrm{DPic}_k(A)$ contains the (noncommutative) Picard group $\mathrm{Pic}_k(A)$ as a subgroup. It also contains a central subgroup $\langle\sigma\rangle \cong \mathbb{Z}$, where σ is the class of the two-sided tilting complex $A[1]$. In [Ye] we showed that when A is either local or commutative one has $\mathrm{DPic}_k(A) = \mathrm{Pic}_k(A) \times \langle\sigma\rangle$. This was discovered independently by Rouquier and Zimmermann [Zi], [RZ]. On the other hand, in the smallest example of a k-algebra A that is neither commutative nor local, namely the 2×2 upper triangular matrix algebra, this equality fails. These observations suggest that the group structure of $\mathrm{DPic}_k(A)$ should carry some information about the geometry of the noncommutative ring A.

This prediction is further motivated by another result in [Ye], which says that $\mathrm{DPic}_k(A)$ classifies the dualizing complexes over A. The geometric significance of dualizing complexes is well known (cf. [RD] and [YZ]).

From a broader perspective, $\mathrm{DPic}_k(A)$ is related to the geometry of noncommutative schemes on the one hand, and to mirror symmetry and deformations of (commutative) smooth projective varieties on the other hand. See [BO], [Ko], [KR] and [Or].

A good starting point for the study of the group $\mathrm{DPic}_k(A)$ is to consider *finite dimensional* k-algebras. The geometric object associated to a finite-dimensional k-algebra A is its quiver $\boldsymbol{\Delta}$, as defined by Gabriel (cf. [GR] or [ARS]). It is worthwhile to note that from the point of view of noncommutative localization theory (cf. [MR] Section 4.3) $\boldsymbol{\Delta}$ is the link graph of A. More on this in Remark 1.2.

Some calculations of the groups $\mathrm{DPic}_k(A)$ for finite-dimensional algebras have already been done. Let us mention the work of Rouquier and Zimmermann [RZ] on Brauer tree algebras, and the work of Lenzing and Meltzer [LM] on canonical algebras.

In this paper we present a systematic study the group $\mathrm{DPic}_k(A)$ when A is a finite dimensional *hereditary* algebra over an *algebraically closed* field k. We obtain general results on the structure of $\mathrm{DPic}_k(A)$, as well as explicit calculations. These results carry over to piecewise hereditary algebras, as well as to certain noncommutative schemes. The rest of the Introduction is devoted to stating our main results.

The group $\mathrm{Aut}_k(A) = \mathrm{Aut}_{\mathsf{Alg}k}(A)$ of k-algebra automorphisms is a linear algebraic group over k, via the inclusion into $\mathrm{Aut}_{\mathsf{Mod}k}(A) = \mathrm{GL}(A)$. This induces a structure of linear algebraic group on the quotient $\mathrm{Out}_k(A)$ of outer automorphisms. We denote by $\mathrm{Out}^0_k(A)$ the identity component of $\mathrm{Out}_k(A)$.

Recall that A is a *basic* k-algebra if $A/\mathfrak{r} \cong k \times \cdots \times k$, where $\mathfrak{r}$ is the Jacobson radical. For a basic algebra one has $\mathrm{Out}_k(A) = \mathrm{Pic}_k(A)$. A hereditary basic algebra A is isomorphic to the *path algebra* $k\boldsymbol{\Delta}$ of its quiver. An algebra A is *indecomposable* iff the quiver $\boldsymbol{\Delta}$ is connected.

For Morita equivalent k-algebras A and B one has $\mathrm{DPic}_k(A) \cong \mathrm{DPic}_k(B)$, and the quivers of A and B are isomorphic. According to a result of Brauer (see [Po] Section 2) one has $\mathrm{Out}^0_k(A) \cong \mathrm{Out}^0_k(B)$. If $A \cong \prod_{i=1}^n A_i$ then $\mathrm{DPic}_k(A) \cong G \ltimes \prod_{i=1}^n$

$\mathrm{DPic}_k(A_i)$, where $G \subset S_n$ is a permutation group (cf. [Ye] Lemma 2.6). Also $\boldsymbol{\Delta}(A) \cong \coprod \boldsymbol{\Delta}(A_i)$ and $\mathrm{Out}_k^0(A) \cong \prod \mathrm{Out}_k^0(A_i)$. Since the main result Theorem 0.1 is stated in terms of $\boldsymbol{\Delta}$ and $\mathrm{Out}_k^0(A)$, we allow ourselves to assume throughout that A is a basic indecomposable algebra.

Given a quiver $\boldsymbol{Q}$ we denote by $\boldsymbol{Q}_0$ its vertex set. For a pair of vertices $x, y \in \boldsymbol{Q}_0$ we write $d(x, y)$ for the arrow-multiplicity, i.e. the number of arrows $\alpha: x \to y$. Let $\mathrm{Aut}(\boldsymbol{Q}_0)$ be the permutation group of $\boldsymbol{Q}_0$, and let $\mathrm{Aut}(\boldsymbol{Q}_0; d)$ be the subgroup of arrow-multiplicity preserving permutations, namely

$$\mathrm{Aut}(\boldsymbol{Q}_0; d) = \{\pi \in \mathrm{Aut}(\boldsymbol{Q}_0) \mid d(\pi(x), \pi(y)) = d(x, y) \text{ for all } x, y \in \boldsymbol{Q}_0\}.$$

Write $\mathrm{Aut}(\boldsymbol{Q})$ for the automorphism group of the quiver $\boldsymbol{Q}$. Then $\mathrm{Aut}(\boldsymbol{Q}_0; d)$ is the image of the canonical homomorphism $\mathrm{Aut}(\boldsymbol{Q}) \to \mathrm{Aut}(\boldsymbol{Q}_0)$. The surjection $\mathrm{Aut}(\boldsymbol{Q}) \twoheadrightarrow \mathrm{Aut}(\boldsymbol{Q}_0; d)$ is split, and it is bijective iff $\boldsymbol{Q}$ has no multiple arrows.

Of particular importance to us is a certain countable quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$. This is a full subquiver of the Auslander–Reiten quiver $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A))$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ as defined by Happel [Ha]. Here $\mathsf{mod} A$ is the category of finitely generated A-modules. If A has finite representation type (i.e. $\boldsymbol{\Delta}$ is a Dynkin quiver) then $\boldsymbol{\Gamma}^{\mathrm{irr}} \cong \mathbb{Z}\boldsymbol{\Delta}$, where $\mathbb{Z}\boldsymbol{\Delta}$ is the quiver introduced by Riedtmann [Rn]. Otherwise $\boldsymbol{\Gamma}^{\mathrm{irr}} \cong \mathbb{Z} \times \mathbb{Z}\boldsymbol{\Delta}$. See Definitions 2.2 and 2.3 for the definition of the quivers $\boldsymbol{\Gamma}^{\mathrm{irr}}$ and $\mathbb{Z}\boldsymbol{\Delta}$, and see Figures 3 and 4 for illustrations. The group $\mathrm{DPic}_k(A)$ acts on $\boldsymbol{\Gamma}_0^{\mathrm{irr}}$ by arrow-multiplicity preserving permutations, giving rise to a group homomorphism $q: \mathrm{DPic}_k(A) \to \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)$.

Define the bimodule $A^* := \mathrm{Hom}_k(A, k)$. Then A^* is a two-sided tilting complex, the functor $M \mapsto A^* \otimes_A^{\mathrm{L}} M \cong \mathrm{RHom}_{\mathrm{A}}(\mathrm{M}, \mathrm{A}^*)$ is the *Serre functor* of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ in the sense of [BK], and $M \mapsto A^*[-1] \otimes_A^{\mathrm{L}} M$ is the *translation functor* in the sense of [Ha] Section I.4. We write $\tau \in \mathrm{DPic}_k(A)$ for the element represented by $A^*[-1]$. Then τ is the translation of the quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$. Let us denote by $\mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle \tau, \sigma \rangle}$ the subgroup of $\mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)$ consisting of permutations that commute with τ and σ.

Here is the main result of the paper.

THEOREM 0.1. *Let A be an indecomposable basic hereditary finite dimensional algebra over an algebraically closed field k, with quiver $\boldsymbol{\Delta}$.*

(1) *There is an exact sequence of groups*

$$1 \to \mathrm{Out}_k^0(A) \to \mathrm{DPic}_k(A) \xrightarrow{q} \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle \tau, \sigma \rangle} \to 1.$$

This sequence splits.

(2) *If A has finite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle \tau \rangle}.$$

(3) *If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \big(\mathrm{Aut}((\mathbb{Z}\boldsymbol{\Delta})_0; d)^{\langle \tau \rangle} \ltimes \mathrm{Out}_k^0(A)\big) \times \mathbb{Z}.$$

The factor $\mathbb{Z}$ of $\mathrm{DPic}_k(A)$ in part 3 is generated by σ. If $\mathbf{\Delta}$ has no multiple arrows then so does $\mathbb{Z}\mathbf{\Delta}$, and hence $\mathrm{Aut}((\mathbb{Z}\mathbf{\Delta})_0; d) = \mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})$. The proof of Theorem 0.1 is in Section 3 where it is stated again as Theorem 3.8.

Recall that a finite-dimensional k-algebra B is called *piecewise hereditary* of type $\mathbf{\Delta}$ if $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} B) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ where $A = k\mathbf{\Delta}$ for some finite quiver $\mathbf{\Delta}$ without oriented cycles. By [Rd] Corollary 3.5 one knows that $\mathrm{DPic}_k(B) \cong \mathrm{DPic}_k(A)$. The next corollary follows.

COROLLARY 0.2. *Suppose B is a piecewise hereditary k-algebra of type $\mathbf{\Delta}$. Then $\mathrm{DPic}_k(B)$ is described by Theorem 0.1 with $A = k\mathbf{\Delta}$.*

In Section 4 we work out explicit descriptions of the groups $\mathrm{Pic}_k(A)$ and $\mathrm{DPic}_k(A)$ for the Dynkin and affine quivers, as well as for some wild quivers with multiple arrows. As an example we present below the explicit description of $\mathrm{DPic}_k(A)$ for a Dynkin quiver of type A_n (which corresponds to upper triangular $n \times n$ matrices). The corollary is extracted from Theorem 4.1.

COROLLARY 0.3. *Suppose $\mathbf{\Delta}$ is a Dynkin quiver of type A_n and $A = k\mathbf{\Delta}$. Then $\mathrm{DPic}_k(A)$ is an abelian group generated by τ and σ, with one relation $\tau^{n+1} = \sigma^{-2}$.*

The relation $\tau^{n+1} = \sigma^{-2}$ was already discovered by E. Kreines (cf. [Ye] Appendix). This relation has been known also to Kontsevich, and in his terminology $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ is 'fractionally Calabi–Yau of dimension $(n-1)/(n+1)$' (see [Ko]; note that the Serre functor is $\tau\sigma$).

Suppose D is a k-linear triangulated category that's equivalent to a small category. Denote by $\mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D})$ the group of k-linear triangle auto-equivalences of D modulo functorial isomorphisms. For a finite-dimensional algebra A one has $\mathrm{DPic}_k(A) \subset \mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A))$, with equality when A is hereditary (cf. Corollary 1.9).

In [KR] Kontsevich and Rosenberg introduce the noncommutative projective space $\mathbf{NP}^n_k$, $n \geqslant 1$. They state that $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\mathbf{NP}^n_k)$ is equivalent to $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} k\mathbf{\Omega}_{n+1})$, where $\mathbf{\Omega}_{n+1}$ is the quiver in Figure 6, and $\mathsf{Coh}\mathbf{NP}^n_k$ is the category of coherent sheaves. By Beilinson's results in [Be], there is an equivalence $\mathsf{D}^{\mathrm{b}}(\mathsf{Coh}\mathbf{P}^1_k) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod} k\mathbf{\Omega}_2)$. Combining Theorem 4.3 and Corollary 1.9 we get the next corollary.

COROLLARY 0.4. *Let X be either $\mathbf{NP}^n_k$ $(n \geqslant 1)$ or $\mathbf{P}^n_k$ $(n = 1)$. Then*

$$\mathrm{Out}^{\mathrm{tr}}_k(\mathsf{D}^{\mathrm{b}}(\mathsf{Coh} X)) \cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_{n+1}(k)\big).$$

In Section 5 we look at a tree Δ with n vertices. Every orientation ω of Δ gives a quiver $\mathbf{\Delta}_\omega$. The equivalences between the various categories $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} k\mathbf{\Delta}_\omega)$ form the derived Picard groupoid $\mathrm{DPic}_k(\Delta)$. The subgroupoid generated by the two-sided tilting complexes of [APR] is called the *reflection groupoid* $\mathrm{Ref}(\Delta)$. We show that there is a surjection $\mathrm{Ref}(\Delta) \twoheadrightarrow W(\Delta)$, where $W(\Delta) \subset \mathrm{GL}_n(\mathbb{Z})$ is the Weyl group as in [BGP]. We also prove that for any orientation ω, $\mathrm{Ref}(\Delta)(\omega, \omega) = \langle \tau_\omega \rangle$ where $\tau_\omega \in \mathrm{DPic}_k(A_\omega)$ is the translation.

1. Conventions and Preliminary Results

In this section we fix notations and conventions to be used throughout the paper. This is needed since there are conflicting conventions in the literature regarding quivers and path algebras. We also prove two preliminary results.

Throughout the paper k denotes a fixed algebraically closed field. Our notation for a quiver is $\boldsymbol{Q} = (\boldsymbol{Q}_0, \boldsymbol{Q}_1)$; $\boldsymbol{Q}_0$ is the set of vertices, and $\boldsymbol{Q}_1$ is the set of arrows. For $x, y \in \boldsymbol{Q}_0$, $d(x, y)$ denotes the number of arrows $x \to y$.

In this section the letter A denotes a k-linear category that's equivalent to a small full subcategory of itself (this assumption avoids some set theoretical problems). Let us write $\mathrm{Aut}_k(\mathsf{A})$ for the class of k-linear auto-equivalences of A. Then the set

$$\mathrm{Out}_k(\mathsf{A}) = \frac{\mathrm{Aut}_k(\mathsf{A})}{\text{functorial isomorphism}} \tag{1.1}$$

is a group.

Suppose A is a k-linear additive Krull–Schmidt category (i.e. $\dim_k \mathrm{Hom}_{\mathsf{A}}(M, N) < \infty$ and all idempotents split). We define the quiver $\boldsymbol{\Gamma}(\mathsf{A})$ of A as follows: $\boldsymbol{\Gamma}_0(\mathsf{A})$ is the set of isomorphism classes of indecomposable objects of A. For two vertices x, y there are $d(x, y)$ arrows $\alpha: x \to y$, where we choose representatives $M_x \in x$, $M_y \in y$, $\mathrm{Irr}(M_x, M_y) = \mathrm{rad}(M_x, M_y)/\mathrm{rad}^2(M_x, M_y)$ is the space of irreducible morphisms and $d(x, y) := \dim_k \mathrm{Irr}(M_x, M_y)$. See [Rl] Section 2.2 for full details.

If A is a k-linear category (possibly without direct sums) we can embed it in the additive category $\mathsf{A} \times \mathbb{N}$, where a morphism $(x, m) \to (y, n)$ is an $n \times m$ matrix with entries in $\mathsf{A}(x, y) = \mathrm{Hom}_{\mathsf{A}}(x, y)$. Of course, if A is additive then $\mathsf{A} \approx \mathsf{A} \times \mathbb{N}$. If $\mathsf{A} \times \mathbb{N}$ is Krull–Schmidt then we shall write $\boldsymbol{\Gamma}(\mathsf{A})$ for the quiver $\boldsymbol{\Gamma}(\mathsf{A} \times \mathbb{N})$.

Let $\boldsymbol{Q}$ be a quiver. Assume that for every vertex $x \in \boldsymbol{Q}_0$ the number of arrows starting or ending at x is finite, and for every two vertices $x, y \in \boldsymbol{Q}_0$ there is only a finite number of oriented paths from x to y. Let $k\langle \boldsymbol{Q} \rangle$ be the *path category*, whose set of objects is $\boldsymbol{Q}_0$, the morphisms are generated by the identities and the arrows, and the only relations arise from incomposability of paths. Observe that this differs from the definition in [Rl], where the path category corresponds to $k\langle \boldsymbol{Q} \rangle \times \mathbb{N}$ in our notation. The morphism spaces of $k\langle \boldsymbol{Q} \rangle$ are $\mathbb{Z}$-graded, where the arrows have degree 1. If $I \subset k\langle \boldsymbol{Q} \rangle$ is any ideal contained in $\mathrm{rad}^2_{k\langle \boldsymbol{Q} \rangle} = \bigoplus_{n \geqslant 2} k\langle \boldsymbol{Q} \rangle_n$, and $k\langle \boldsymbol{Q}, I \rangle := k\langle \boldsymbol{Q} \rangle / I$ is the quotient category, then the additive category $k\langle \boldsymbol{Q}, I \rangle \times \mathbb{N}$ is Krull–Schmidt, and the quiver of $k\langle \boldsymbol{Q}, I \rangle$ is $\boldsymbol{\Gamma}(k\langle \boldsymbol{Q}, I \rangle) = \boldsymbol{Q}$.

Let A be a finite-dimensional k-algebra. In representation theory there are three equivalent ways to define the quiver $\boldsymbol{\Delta} = \boldsymbol{\Delta}(A)$ of A. The set $\boldsymbol{\Delta}_0$ enumerates either a complete set of primitive orthogonal idempotents $\{e_x\}_{x \in \boldsymbol{\Delta}_0}$, as in [ARS] Section III.1; or it enumerates the simple A-modules $\{S_x\}_{x \in \boldsymbol{\Delta}_0}$, as in [Rl] Section 2.1;

or it enumerates the indecomposable projective A-modules $\{P_x\}_{x\in\boldsymbol{\Delta}_0}$, as in [Rl] Section 2.4. The arrow multiplicity is in all cases

$$d(x,y) = \dim_k e_x(\mathfrak{r}/\mathfrak{r}^2)e_y = \dim_k \mathrm{Ext}^1_A(S_y, S_x) = \dim_k \mathrm{Irr}_{\mathsf{proj}A}(P_x, P_y).$$

Here $\mathfrak{r}$ is the Jacobson radical and $\mathsf{proj}A$ is the category of finitely generated projective modules, which is Krull–Schmidt. Observe that the third definition is just $\boldsymbol{\Delta}(A) = \boldsymbol{\Gamma}(\mathsf{proj}A)$.

Remark 1.2. The set $\boldsymbol{\Delta}_0$ also enumerates the prime spectrum of A, $\mathrm{Spec}A \cong \{\mathfrak{p}_x\}_{x\in\boldsymbol{\Delta}_0}$. One can show that $\mathfrak{r}/\mathfrak{r}^2 \cong \bigoplus_{x,y\in\boldsymbol{\Delta}_0}(\mathfrak{p}_x \cap \mathfrak{p}_y)/\mathfrak{p}_x\mathfrak{p}_y$ as A-A-bimodules. This implies that $d(x,y) > 0$ iff there is a second layer link $\mathfrak{p}_x \rightsquigarrow \mathfrak{p}_y$ (cf. [MR] Section 4.3.7). Thus if we ignore multiple arrows, the quiver $\boldsymbol{\Delta}$ is precisely the link graph of A.

Recall that a *translation* τ is an injective function from a subset of $\boldsymbol{Q}_0$, called the set of non-projective vertices, to $\boldsymbol{Q}_0$, such that $d(\tau(y), x) = d(x,y)$. $\boldsymbol{Q}$ is a stable translation quiver if it comes with a translation τ such that all vertices are non-projective. A *polarization* μ is an injective function defined on the set of arrows $\beta\colon x \to y$ ending in nonprojective vertices, with $\mu(\beta)\colon \tau(y) \to x$. Cf. [Rl] Section 2.2.

NOTATION 1.3. Suppose the quiver $\boldsymbol{Q}$ has a translation τ and a polarization μ. Given a nonprojective vertex $y \in \boldsymbol{Q}_0$ let $x_1, \ldots, x_m$ be some labeling, without repetition, of the set of vertices $\{x \mid \text{there is an arrow } x \to y\}$. Correspondingly, label the arrows $\beta_{i,j}\colon x_i \to y$ and $\alpha_{i,j}\colon \tau(y) \to x_i$, where $i = 1, \ldots, m$; $j = 1, \ldots, d_i$; $d_i = d(x_i, y)$; and $\alpha_{i,j} = \mu(\beta_{i,j})$. The *mesh ending at y* is the subquiver with vertices $\{\tau(y), x_i, y\}$ and arrows $\{\alpha_{i,j}, \beta_{i,j}\}$.

If $\boldsymbol{Q}$ has no multiple arrows then $d_i = 1$ and the picture of the mesh ending at y is shown in Figure 1.

The *mesh relation* at y is defined to be

$$\sum_{i=1}^{m}\sum_{j=1}^{d_i} \beta_{i,j}\alpha_{i,j} \in \mathrm{Hom}_{k\langle\boldsymbol{Q}\rangle}\big(\tau(y), y\big). \tag{1.4}$$

It is a homogeneous morphism of degree 2.

DEFINITION 1.5. Let I_{m} be the mesh ideal in the category $k\langle\boldsymbol{Q}\rangle$, i.e. the two-sided ideal generated by the mesh relations (1.4) where y runs over all nonprojective vertices. The quotient category

$$k\langle\boldsymbol{Q}, I_{\mathrm{m}}\rangle := k\langle\boldsymbol{Q}\rangle / I_{\mathrm{m}}$$

is called the *mesh category.*

Observe that in [Rn], [Rl] and [Ha] the notation for $k\langle\boldsymbol{Q}, I_{\mathrm{m}}\rangle$ is $k(\boldsymbol{Q})$.

Now let $\boldsymbol{\Delta}$ be a finite quiver without oriented cycles and $A = k\boldsymbol{\Delta}$ the path algebra. Our convention for the multiplication in A is as follows. If $x \xrightarrow{\alpha} y$ and $y \xrightarrow{\beta} z$ are

$$
\begin{tikzcd}
 & x_1 \arrow[ddr, "\beta_1"] & \\
 & \vdots & \\
\tau(y) \arrow[uur, "\alpha_1"] \arrow[r, "\alpha_i"] \arrow[ddr, "\alpha_m"'] & x_i \arrow[r, "\beta_i"] & y \\
 & \vdots & \\
 & x_m \arrow[uur, "\beta_m"'] &
\end{tikzcd}
$$

Figure 1. The mesh ending at the vertex y when $d_i = 1$.

paths in $\mathbf{\Delta}$, and if $x \xrightarrow{\gamma} z$ is the concatenated path, then $\gamma = \alpha\beta$ in A. We note that the composition rule in the path category $k\langle\mathbf{\Delta}\rangle$ is opposite to that in A, so that $\bigoplus_{x,y} \mathrm{Hom}_{k\langle\mathbf{\Delta}\rangle}(x, y) = A^{\circ}$.

For every $x \in \mathbf{\Delta}_0$ let $e_x \in A$ be the corresponding idempotent, and let $P_x = Ae_x$ be the indecomposable projective A-module. So $\{P_x\}_{x \in \mathbf{\Delta}_0}$ is a set of representatives of the isomorphism classes of indecomposable projective A-modules. Define $\mathsf{P} \subset \mathsf{mod}A$ to be the full subcategory on the objects $\{P_x\}_{x \in \mathbf{\Delta}_0}$. Then $\mathsf{P} \times \mathbb{N} \approx \mathsf{proj}A$ and $\mathbf{\Delta} \cong \mathbf{\Gamma}(\mathsf{P}) \cong \mathbf{\Gamma}(\mathsf{proj}A)$.

There is an equivalence of categories $k\langle\mathbf{\Delta}\rangle \xrightarrow{\approx} \mathsf{P}$ that sends $x \mapsto P_x$, and an arrow $\alpha: x \to y$ goes to the right multiplication $P_x = Ae_x \xrightarrow{\alpha} P_y = Ae_y$. We will identify P and $k\langle\mathbf{\Delta}\rangle$ in this way.

Recall that the automorphism group $\mathrm{Aut}_k(A)$ is a linear algebraic group. Let H be the closed subgroup

$$H := \{F \in \mathrm{Aut}_k(A) \mid F(e_x) = e_x \text{ for all } x \in \mathbf{\Delta}_0\}.$$

LEMMA 1.6. *H is connected.*

Proof. For each pair $x, y \in \mathbf{\Delta}_0$ the k-vector space $\mathsf{P}(x, y) := \mathrm{Hom}_{\mathsf{P}}(x, y) \cong e_x A e_y$ is graded. Let $\mathsf{P}(x, y)_i$ be the homogeneous component of degree i, and

$$Y := \prod_{x,y \in \mathbf{\Delta}_0} \Big(\mathrm{Aut}_k\big(\mathsf{P}(x, y)_1\big) \times \mathrm{Hom}_k\big(\mathsf{P}(x, y)_1, \mathsf{P}(x, y)_{\geqslant 2}\big)\Big).$$

This is a connected algebraic variety. Since A is generated as k-algebra by the idempotents and the arrows, and the only relations in A are the monomial relations arising from incomposability of paths, it follows that any element $F' \in Y$ extends uniquely to a k-algebra automorphism F of A that fixes the idempotents. Conversely any automorphism $F \in H$ restricts to an element F' of Y. This bijection $Y \to H$ is an isomorphism of varieties. Hence H is connected. □

The next result is partially proved in [GS] Theorem 4.8 (they assume k has characteristic 0).

PROPOSITION 1.7. *Let A be a basic hereditary finite dimensional algebra over an algebraically closed field k, with quiver $\mathbf{\Delta}$.*

(1) *There is a split exact sequence of groups*

$$1 \to \mathrm{Out}_k^0(A) \to \mathrm{Pic}_k(A) \to \mathrm{Aut}(\mathbf{\Delta}_0; d) \to 1.$$

(2) *The group $\mathrm{Out}_k^0(A)$ is trivial when $\mathbf{\Delta}$ is a tree.*

Proof. (1) Since A is basic we have $\mathrm{Out}_k(A) = \mathrm{Pic}_k(A)$. By Morita theory we have $\mathrm{Pic}_k(A) \cong \mathrm{Out}_k(\mathsf{Mod}A)$. Any auto-equivalence of the category P extends to an auto-equivalence of $\mathsf{Mod}A$ (using projective resolutions), and this induces an isomorphism of groups $\mathrm{Out}_k(\mathsf{P}) \xrightarrow{\simeq} \mathrm{Out}_k(\mathsf{Mod}A)$.

The class of auto-equivalences $\mathrm{Aut}_k(\mathsf{P})$ is actually a group here. In fact $\mathrm{Aut}_k(\mathsf{P})$ can be identified with the subgroup of $\mathrm{Aut}_k(A)$ consisting of automorphisms that permute the set of idempotents $\{e_x\} \subset A$.

Define a homomorphism of groups $q\colon \mathrm{Out}_k(A) \to \mathrm{Aut}(\mathbf{\Delta}_0; d)$ by $q(F)(x) = y$ if $FP_x \cong P_y$. Thus we get a commutative diagram

$$\begin{tikzcd} H \arrow[r] & \mathrm{Aut}_k(\mathsf{P}) \arrow[r] \arrow[d, two heads, "f"] & \mathrm{Aut}_k(A) \arrow[d, two heads, "g"] & \\ & \mathrm{Out}_k(\mathsf{P}) \arrow[r, "\cong"] & \mathrm{Out}_k(A) \arrow[r, "q"] & \mathrm{Aut}(\mathbf{\Delta}_0; d). \end{tikzcd}$$

For an element $F \in \mathrm{Aut}_k(\mathsf{P})$ we have $F|_{\{e_x\}} = qf(F)$ and, hence, $\mathrm{Ker}(q) = f(H) = g(H)$. According to Lemma 1.6, H is connected. Because g is a morphism of varieties we see that $\mathrm{Ker}(q)$ is connected. But the index of $\mathrm{Ker}(q)$ is finite, so we get $\mathrm{Ker}(q) = \mathrm{Out}_k^0(A)$.

In order to split q we choose any splitting of $\mathrm{Aut}(\mathbf{\Delta}) \twoheadrightarrow \mathrm{Aut}(\mathbf{\Delta}_0; d)$ and compose it with the homomorphism $\mathrm{Aut}(\mathbf{\Delta}) \to \mathrm{Aut}_k(\mathsf{P})$.

(2) When $\mathbf{\Delta}$ is a tree the group H is a torus: $H \cong \prod_{x,y \in \mathbf{\Delta}_0} \mathrm{Aut}_k\big(\mathsf{P}(x, y)_1\big)$. In fact H consists entirely of inner automorphisms that are conjugations by elements of the form $\sum \lambda_x e_x$ with $\lambda_x \in k^\times$. Thus $g(H) = 1$. □

The next theorem seems to be known to some experts, but we could not locate any reference in the literature. Since it is needed in the paper we have included a short proof. For a left coherent ring A (e.g. a hereditary ring) we denote by $\mathsf{mod}A$ the category of coherent A-modules. In the theorem k could be any field.

THEOREM 1.8. *Suppose A is a hereditary k-algebra. Then any k-linear triangle auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is standard.*

Proof. Let F be a k-linear triangle auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. By [Rd] Corollary 3.5 there exits a two-sided tilting complex T with $T \cong FA$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. Replacing F with $(T^\vee \otimes_A^{\mathrm{L}} -)F$ we may assume that $FA \cong A$. Hence $F(\mathsf{mod}A) \subset \mathsf{mod}A$, and $F|_{\mathsf{mod}A}$ is an equivalence. Classical Morita theory says that

$F|_{\mathsf{mod}A} \cong (P \otimes_A -)$ for some invertible bimodule P. So replacing F by $(P^\vee \otimes_A -)F$ we can assume that there is an isomorphism $\phi^0 \colon F|_{\mathsf{mod}A} \cong \mathbf{1}_{\mathsf{mod}A}$.

Now for every object $M \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ we can choose an isomorphism $M \cong \bigoplus_i M_i[-i]$ with $M_i \in \mathsf{mod}A$ (cf. [Ha] Lemma I.5.2). Define $\phi_M \colon FM \xrightarrow{\simeq} M$ to be the composition

$$FM \cong \bigoplus (FM_i)[-i] \xrightarrow{\sum \phi^0_{M_i}[-i]} \bigoplus_i M_i[-i] \cong M.$$

According to the proof of [BO] Proposition A.3, for any morphism $\alpha \colon M \to N$ one has $\phi_N F(\alpha) = \alpha \phi_M$, so $\phi \colon F \to \mathbf{1}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}$ is an isomorphism of functors. □

COROLLARY 1.9. *Suppose A is a hereditary k-algebra. Then*

$$\mathrm{DPic}_k(A) \cong \mathrm{Out}_k^{\mathrm{tr}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)).$$

Proof. The group homomorphism $\mathrm{DPic}_k(A) \to \mathrm{Out}_k^{\mathrm{tr}}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ is injective, say by [Ye] Proposition 2.2, and it is surjective by the theorem. □

2. An Equivalence of Categories

In this section we prove the technical result Theorem 2.6. It holds for any finite dimensional hereditary k-algebra A. In the special case of finite representation type, Theorem 2.6 is just [Ha] Proposition I.5.6. Our result is the derived category counterpart of [Rl] Lemma 2.3.3. For notation see Section 1 above.

We use a few facts about Auslander–Reiten triangles in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. These facts are well known to experts in representation theory, but for the benefit of other readers we have collected them in Theorems 2.1 and 2.4.

Let D be a k-linear triangulated category, which is Krull–Schmidt (as additive category). As in any Krull–Schmidt category, sink and source morphisms can be defined in D; cf. [Rl] Section 2.2. In [Ha] Section I.4, Happel defines Auslander–Reiten triangles in D, generalizing the Auslander–Reiten (or almost split) sequences in an abelian Krull–Schmidt category. A triangle $M' \xrightarrow{g} M \xrightarrow{f} M'' \to M'[1]$ in D is an Auslander–Reiten triangle if g is a source morphism, or equivalently if f is a sink morphism. As before, we denote by $M_x \in \mathsf{D}$ an indecomposable object in the isomorphism class $x \in \mathbf{\Gamma}(\mathsf{D})$.

Now let $\mathbf{\Delta}$ be a finite quiver without oriented cycles, and $A = k\mathbf{\Delta}$ the path algebra. For $M \in \mathsf{mod}k$ let $M^* := \mathrm{Hom}_k(M, k)$. Define auto-equivalences σ and τ of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ by $\sigma M := M[1]$ and $\tau M := \mathrm{RHom}_A(M, A)^*[-1] \cong A^*[-1] \otimes^{\mathrm{L}}_A M$.

THEOREM 2.1 (Happel, Ringel). *Let $A = k\mathbf{\Delta}$. Then the following hold.*

(1) *As an additive k-linear category, $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is a Krull–Schmidt category.*

(2) *The quiver* $\boldsymbol{\Gamma} := \boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ *is a stable translation quiver, and the translation* τ *satisfies* $M_{\tau(x)} \cong \tau M_x$.

(3) *The Auslander–Reiten triangles in* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ *(up to isomorphism) correspond bijectively to the meshes in* $\boldsymbol{\Gamma}$. *In the notation 1.3 with* $\boldsymbol{Q} = \boldsymbol{\Gamma}$ *these triangles are*

$$M_{\tau(y)} \xrightarrow{(g_{i,j})} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} M_{x_i} \xrightarrow{(f_{i,j})^{\mathrm{t}}} M_y \to M_{\tau(y)}[1].$$

(4) *A morphism* $(g_{i,j}) \colon M_{\tau(y)} \to \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} M_{x_i}$ *is a source morphism iff for all i,* $\{g_{i,j}\}_{j=1}^{d_i}$ *is a basis of* $\mathrm{Irr}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M_{\tau(y)}, M_{x_i})$. *Likewise a morphism* $(f_{i,j})^{\mathrm{t}} \colon \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} M_{x_i} \to M_y$ *is a sink morphism iff for all i,* $\{f_{i,j}\}_{j=1}^{d_i}$ *is basis of* $\mathrm{Irr}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M_{x_i}, M_y)$.

Proof. (1) This is implicit in [Ha] Sections I.4 and I.5. In particular [Ha] Lemma I.5.2 shows that for any indecomposable object $M \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ the ring $\mathrm{End}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M)$ is local.

(2) See [Ha] Corollary I.4.9.

(3) According to [Ha] Theorem I.4.6 and Lemma I.4.8, for each $y \in \boldsymbol{\Gamma}_0$ there exists such an Auslander–Reiten triangle. By [Ha] Proposition I.4.3 these are all the Auslander–Reiten triangles, up to isomorphism.

(4) Since source and sink morphism depend only on the structure of k-linear additive category on $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ (cf. [Ha] Section I.4.5) we may use [Rl] Lemma 2.2.3. □

The Auslander–Reiten quiver $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ contains the quiver $\boldsymbol{\Delta}$, as the full subquiver with vertices corresponding to the indecomposable projective A-modules, under the inclusion $\mathsf{mod}A \subset \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$.

DEFINITION 2.2. We call a connected component of $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ *irregular* if it is isomorphic to the connected component containing $\boldsymbol{\Delta}$, and we denote by $\boldsymbol{\Gamma}^{\mathrm{irr}}$ the disjoint union of all irregular components of $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$.

The name 'irregular' is inspired by [ARS] Section VIII.4, where regular components of $\boldsymbol{\Gamma}(\mathsf{mod}A)$ are discussed. The quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$ will be of special interest to us. It's structure is explained in Theorem 2.4 below. But first we need to recall the following definition due to Riedtmann [Rn],

DEFINITION 2.3. From the quiver $\boldsymbol{\Delta}$ one can construct another quiver, denoted by $\mathbb{Z}\boldsymbol{\Delta}$. The vertex set of $\mathbb{Z}\boldsymbol{\Delta}$ is $\mathbb{Z} \times \boldsymbol{\Delta}_0$, and for every arrow $x \xrightarrow{\alpha} y$ in $\boldsymbol{\Delta}$ there are arrows $(n, x) \xrightarrow{(n,\alpha)} (n, y)$ and $(n, y) \xrightarrow{(n,\alpha^*)} (n+1, x)$ in $\mathbb{Z}\boldsymbol{\Delta}$.

The function $\tau(n, x) = (n-1, x)$ makes $\mathbb{Z}\boldsymbol{\Delta}$ into a stable translation quiver. Observe that τ is an automorphism of the quiver $\mathbb{Z}\boldsymbol{\Delta}$, not just of the vertex set $(\mathbb{Z}\boldsymbol{\Delta})_0$. $\mathbb{Z}\boldsymbol{\Delta}$ is equipped with a polarization μ, given by $\mu(n+1, \alpha) = (n, \alpha^*)$ and $\mu(n, \alpha^*) = (n, \alpha)$. See Figures 3 and 4 in Section 4 for examples. We identify $\boldsymbol{\Delta}$ with the subquiver $\{0\} \times \boldsymbol{\Delta} \subset \mathbb{Z}\boldsymbol{\Delta}$.

Next let us define a quiver $\mathbb{Z} \times (\mathbb{Z}\mathbf{\Delta}) := \coprod_{m\in\mathbb{Z}} \mathbb{Z}\mathbf{\Delta}$; the connected components are $\{m\} \times (\mathbb{Z}\mathbf{\Delta})$, $m \in \mathbb{Z}$. Define an automorphism σ of $\mathbb{Z} \times (\mathbb{Z}\mathbf{\Delta})$ by the action $\sigma(m) = m + 1$ on the first factor. There is a translation τ and a polarization μ of $\mathbb{Z} \times (\mathbb{Z}\mathbf{\Delta})$ that extend those of $\mathbb{Z}\mathbf{\Delta} \cong \{0\} \times (\mathbb{Z}\mathbf{\Delta})$ and commute with σ.

The auto-equivalences σ and τ of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ induce commuting permutations of $\mathbf{\Gamma}_0$, which we also denote by σ and τ respectively.

THEOREM 2.4 (Happel). (1) *If A has finite representation type then there is a unique isomorphism of quivers* $\rho: \mathbb{Z}\mathbf{\Delta} \xrightarrow{\simeq} \mathbf{\Gamma}^{\mathrm{irr}}$ *which is the identity on* $\mathbf{\Delta}$ *and commutes with* τ *on vertices. Furthermore* $\mathbf{\Gamma}^{\mathrm{irr}} = \mathbf{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$.

(2) *If A has infinite representation type then there exists an isomorphism of quivers* $\rho: \mathbb{Z} \times (\mathbb{Z}\mathbf{\Delta}) \xrightarrow{\simeq} \mathbf{\Gamma}^{\mathrm{irr}}$ *which is the identity on* $\mathbf{\Delta}$ *and commutes with* τ *and* σ *on vertices. If* $\mathbf{\Delta}$ *is a tree then the isomorphism* ρ *is unique.*

Proof. This is essentially [Ha] Proposition I.5.5 and Corollary I.5.6. □

Fix once and for all for every vertex $x \in \mathbf{\Gamma}_0^{\mathrm{irr}}$ an indecomposable object $M_x \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ which represents x, and such that $M_x = P_x$ for $x \in \mathbf{\Delta}_0$. Define $\mathsf{B} \subset \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ to be the full subcategory with objects $\{M_x \mid x \in (\mathbb{Z}\mathbf{\Delta})_0\}$.

The additive category $\mathsf{B} \times \mathbb{N}$ is also Krull–Schmidt, so for $M_x, M_y \in \mathsf{B}$ the two k-modules $\mathrm{Irr}_{\mathsf{B}\times\mathbb{N}}(M_x, M_y)$ and $\mathrm{Irr}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M_x, M_y)$ could conceivably differ (cf. [Rl] Section 2.2). But this is not the case as we see in the lemma below.

LEMMA 2.5. *Suppose* $I \subset \mathbb{Z}$ *is a segment (i.e.* $I = \{i \in \mathbb{Z} \mid a \leqslant i \leqslant b\}$ *with* $a, b \in \mathbb{Z} \cup \{\pm\infty\}$*). Let* $\mathsf{B}(I) \subset \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ *be the full subcategory on the objects* M_x, $x \in I \times \mathbf{\Delta}_0 \subset \mathbf{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))_0$. *Then for any* $M_x, M_y \in \mathsf{B}(I)$ *one has*

$$\mathrm{Irr}_{\mathsf{B}(I)\times\mathbb{N}}(M_x, M_y) \cong \mathrm{Irr}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M_x, M_y).$$

Proof. Consider a sink morphism in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ ending in $M_{(n,y)}$, $(n, y) \in (\mathbb{Z}\mathbf{\Delta})_0$. By Theorem 2.1(3) and Theorem 2.4, it is of the form $(f_{i,j})^{\mathrm{t}}: \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} M_{(n-\varepsilon_i,x_i)} \to M_{(n,y)}$ with $\varepsilon_i \in \{0, 1\}$ (cf. Notation 1.3). From the definition of a sink morphism we see that this is also a sink morphism in the category $\mathsf{B} \times \mathbb{N}$.

According to [Rl] Lemma 2.2.3 (dual form), both k-modules $\mathrm{Irr}_{\mathsf{B}\times\mathbb{N}}(M_{(n-\varepsilon_i,x_i)}, M_{(n,y)})$ and $\mathrm{Irr}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)}(M_{(n-\varepsilon_i,x_i)}, M_{(n,y)})$ have the morphisms $f_{i,1}, \ldots, f_{i,d_i}$ as basis. And there are no irreducible morphisms $N \to M_{(n,y)}$ for indecomposable objects N not isomorphic to one of the $M_{(n-\varepsilon_i,x_i)}$, in either category. Thus the lemma is proved for $\mathsf{B}(I) = \mathsf{B}$.

Let $x, y \in \mathbf{\Delta}_0$ and $l, n \in \mathbb{Z}$. If $\mathrm{Hom}(M_{(l,x)}, M_{(n,y)}) \neq 0$ then necessarily $l \leqslant n$. This is clear for $l = 0$, since $M_{(0,x)}$ is a projective module, and an easy calculation shows that for $n < 0$,

$$\mathrm{H}^0(M_{(n,y)}) \cong \mathrm{H}^0(A^*[-1] \otimes_A^{\mathrm{L}} \cdots \otimes_A^{\mathrm{L}} A^*[-1] \otimes_A^{\mathrm{L}} M_{(0,y)}) = 0.$$

In general we can translate by τ^{-l}.

Now take an arbitrary segment I. The paragraph above implies that for $n, l \in I$ and $i \geqslant 0$, $\mathrm{rad}^i_{\mathsf{B}(I)\times\mathbb{N}}(M_{(l,x)}, M_{(n,y)}) = \mathrm{rad}^i_{\mathsf{B}\times\mathbb{N}}(M_{(l,x)}, M_{(n,y)})$. Hence $\mathrm{Irr}_{\mathsf{B}(I)\times\mathbb{N}}(M_{(l,x)}, M_{(n,y)}) = \mathrm{Irr}_{\mathsf{B}\times\mathbb{N}}(M_{(l,x)}, M_{(n,y)})$. □

Henceforth we shall simply write $\mathrm{Irr}(M_x, M_y)$ when $x, y \in (\mathbb{Z}\mathbf{\Delta})_0$. The lemma implies that the quiver of the category $\mathsf{B}(I)$ is the full subquiver $\mathbf{I\Delta} \subset \mathbb{Z}\mathbf{\Delta}$.

Note that for $I = \{0\}$ we get $\mathsf{B}(I) = \mathsf{P}$. Since P is canonically equivalent to $k\langle\mathbf{\Delta}\rangle$, there is a full faithful k-linear functor $G_0: k\langle\mathbf{\Delta}\rangle \to \mathsf{B}$ such that $G_0 x = M_x = P_x$ for every vertex $x \in \mathbf{\Delta}_0$, and $\{G_0(\alpha_j)\}_{j=1}^{d(x,y)}$ is a basis of $\mathrm{Irr}(M_x, M_y)$ for every pair of vertices x, y, where $\alpha_1, \ldots, \alpha_{d(x,y)}$ are the arrows $\alpha_j: x \to y$.

THEOREM 2.6. *Let $\mathbf{\Delta}$ be a finite quiver without oriented cycles, $A = k\mathbf{\Delta}$ its path algebra, $k\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$ the mesh category (Definitions 2.3 and 1.5) and $\mathsf{B} \subset \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ the full subcategory on the objects $\{M_x\}_{x\in(\mathbb{Z}\mathbf{\Delta})_0}$. Then there is a k-linear functor*

$$G: k\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}$$

such that

(i) *$Gx = M_x$ for each vertex $x \in (\mathbb{Z}\mathbf{\Delta})_0$.*
(ii) *$G|_{k\langle\mathbf{\Delta}\rangle} = G_0$.*
(iii) *G is full and faithful.*

Moreover, the functor G is unique up to isomorphism.

In other words, there is a unique equivalence G extending G_0.

Proof. Let $\boldsymbol{Q}^+ \subset \mathbb{Z}\mathbf{\Delta}$ be the full subquiver with vertex set $\{(n, y) \mid n \geqslant 0\}$. Given a vertex (n, y) in $\boldsymbol{Q}^+$, denote by $p(n, y)$ the number of its predecessors, i.e. the number of vertices (m, x) such that there is a path $(m, x) \to \cdots \to (n, y)$ in $\boldsymbol{Q}^+$. For any $p \geqslant 0$ let $\boldsymbol{Q}^+_p$ be the full subquiver with vertex set $\{(n, y) \mid n \geqslant 0,\ p(n, y) \leqslant p\}$. $\boldsymbol{Q}^+_p$ is a translation quiver with polarization, and $k\langle\boldsymbol{Q}^+_p, I_{\mathrm{m}}\rangle \subset k\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$ is a full subcategory.

By recursion on p, we will define a functor $G: k\langle\boldsymbol{Q}^+_p, I_{\mathrm{m}}\rangle \to \mathsf{B}$ satisfying conditions (i), (ii) and

(iv) *Let x, y be a pair of vertices and let $\alpha_1, \ldots, \alpha_{d(x,y)}$ be the arrows $\alpha_j: x \to y$. Then $\{G(\alpha_j)\}_{j=1}^{d(x,y)}$ is a basis of* $\mathrm{Irr}(M_x, M_y)$.

Take $p \geqslant 0$. It suffices to define $G(\alpha)$ for an arrow α in $\boldsymbol{Q}^+_p$. These arrows fall into three cases, according to their end vertex (n, y):

(a) $p(n, y) < p$, in which case any arrow α ending in (n, y) is in $\boldsymbol{Q}^+_{p-1}$, and $G(\alpha)$ is already defined.

(b) $p(n,y) = p$ and $n = 0$. Any arrow α ending in (n,y) is in $\mathbf{\Delta}$, so we define $G(\alpha) := G_0(\alpha)$. By Lemma 2.5 condition (iv) holds.

(c) $p(n,y) = p$ and $n \geqslant 1$. In this case (n,y) is a nonprojective vertex in $\boldsymbol{Q}_p^+$, and we consider the mesh ending at (n,y). The vertices with arrows to (n,y) are $(n - \varepsilon_i, x_i)$, where $i = 1, \ldots, m$; $x_i \in \mathbf{\Delta}_0$ and $\varepsilon_i = 0, 1$ (cf. Notation 1.3). Since $p(n-1,y) < p(n - \varepsilon_i, x_i) < p$ the arrows $\alpha_{i,j}$ are all in the quiver $\boldsymbol{Q}_{p-1}^+$ and, hence, $G(\alpha_{i,j})$ are defined.

According to condition (iv), Lemma 2.5 and Theorem 2.1(4) it follows that there exists an Auslander–Reiten triangle

$$M_{(n-1,y)} \xrightarrow{(G(\alpha_{i,j}))} \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{d_i} M_{(n-\varepsilon_i,x_i)} \xrightarrow{(f_{i,j})^{\mathrm{t}}} M_{(n,y)} \to M_{(n-1,y)}[1] \tag{2.7}$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$. Define

$$G(\beta_{i,j}) := f_{i,j} : M_{(n-\varepsilon_i,x_i)} \to M_{(n,y)}.$$

Note that the mesh relation $\sum \beta_{i,j}\alpha_{i,j}$ in $k\langle \boldsymbol{Q}_p^+ \rangle$ is sent by G to $\sum G(\beta_{i,j})G(\alpha_{i,j}) = 0$, so we indeed have a functor $G: k\langle \boldsymbol{Q}_p^+, I_{\mathrm{m}}\rangle \to \mathsf{B}$. Also, by Theorem 2.1(4), for any i the set $\{G(\beta_{i,j})\}_{j=1}^{d_i}$ is a basis of $\mathrm{Irr}(M_{(n-\varepsilon_i,x_i)}, M_{(n,y)})$.

Thus we obtain a functor $G: k\langle \boldsymbol{Q}^+, I_{\mathrm{m}}\rangle \to \mathsf{B}$.

By symmetry we construct a functor $G: k\langle \boldsymbol{Q}^-, I_{\mathrm{m}}\rangle \to \mathsf{B}$ for negative vertices (i.e. $n \leqslant 0$), extending G_0. Putting the two together we obtain a functor $G: k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}$ satisfying conditions (i), (ii) and (iv).

Let us prove G is fully faithful. For any $n \in \mathbb{Z}$ there is a full subquiver $\mathbb{Z}_{\geqslant n}\mathbf{\Delta} \subset \mathbb{Z}\mathbf{\Delta}$, on the vertex set $\{(i,x) \mid i \geqslant n\}$. Correspondingly there are full subcategories $k\langle \mathbb{Z}_{\geqslant n}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \subset k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$ and $\mathsf{B}(\mathbb{Z}_{\geqslant n}) \subset \mathsf{B}$. It suffices to prove that $G: k\langle \mathbb{Z}_{\geqslant n}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}(\mathbb{Z}_{\geqslant n})$ is fully faithful. By Lemma 2.5 the quiver of $\mathsf{B}(\mathbb{Z}_{\geqslant n})$ is $\mathbb{Z}_{\geqslant n}\mathbf{\Delta}$, which is pre-projective. So we can use the last two paragraphs in the proof of [Rl] Lemma 2.3.3 almost verbatim.

Finally we shall prove that G is unique up to isomorphism. Suppose $G': k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}$ is another k-linear functor satisfying conditions (i)–(iii). We will show there is an isomorphism $\phi: G \xrightarrow{\simeq} G'$ that is the identity on $k\langle \mathbf{\Delta}\rangle$.

By recursion on p we shall exhibit an isomorphism $\phi: G|_{k\langle \boldsymbol{Q}_p^+, I_{\mathrm{m}}\rangle} \xrightarrow{\simeq} G'|_{k\langle \boldsymbol{Q}_p^+, I_{\mathrm{m}}\rangle}$. It suffices to consider case (c) above, so let (n,y) be such a vertex. Then, because $G'(\alpha_{i,j}) = \phi_{(n-\varepsilon_i,x_i)} G(\alpha_{i,j}) \phi_{(n-1,y)}^{-1}$, we have

$$\sum_{i,j} G'(\beta_{i,j}) \phi_{(n-\varepsilon_i,x_i)} G(\alpha_{i,j}) = G'\left(\sum_{i,j} \beta_{i,j}\alpha_{i,j} \right) \phi_{(n-1,y)} = 0.$$

Applying $\mathrm{Hom}(-, M_{(n,y)})$ to the triangle (2.7) we obtain a morphism $a \in \mathrm{End}(M_{(n,y)})$ such that $G'(\beta_{i,j})\phi_{(n-\varepsilon_i,x_i)} = aG(\beta_{i,j})$. Because G' is faithful we see that $a \neq 0$, and since $\mathrm{End}(M_{(n,y)}) \cong k$ it follows that a is invertible. Set $\phi_{(n,y)} := a \in \mathrm{Aut}(M_{(n,y)})$. This yields the desired isomorphism $\phi: G|_{k\langle \boldsymbol{Q}_p^+, I_{\mathrm{m}}\rangle} \xrightarrow{\simeq} G'|_{k\langle \boldsymbol{Q}_p^+, I_{\mathrm{m}}\rangle}$.

By symmetry the isomorphism ϕ extends to $\boldsymbol{Q}^{-}$. $\square$

The uniqueness of G gives the next corollary.

COROLLARY 2.8. *Let F be a k-linear auto-equivalence of $k\langle\mathbb{Z}\boldsymbol{\Delta}, I_{\mathrm{m}}\rangle$ fixing all objects, and such that $F|_{k\langle\boldsymbol{\Delta}\rangle} \cong \mathbf{1}_{k\langle\boldsymbol{\Delta}\rangle}$. Then $F \cong \mathbf{1}_{k\langle\mathbb{Z}\boldsymbol{\Delta}, I_{\mathrm{m}}\rangle}$.*

Remark 2.9. Beware that if A has infinite representation type then $k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle$ is *not* equivalent to the full subcategory of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ on the objects $\{M_x\}_{x\in\boldsymbol{\Gamma}_0^{\mathrm{irr}}}$. This is because there are nonzero morphisms from the projective modules (vertices in the component $\mathbb{Z}\boldsymbol{\Delta}$) to the injective modules (vertices in $\{1\} \times \mathbb{Z}\boldsymbol{\Delta}$).

3. The Representation of $\mathrm{DPic}_k(A)$ on the Quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$

This section contains the proof of the main result of the paper, Theorem 0.1 (restated here as Theorem 3.8). It is deduced from the more technical Theorem 3.7. Throughout k is an algebraically closed field, $\boldsymbol{\Delta}$ is a connected finite quiver without oriented cycles, and $A = k\boldsymbol{\Delta}$ is the path algebra. We use the notation of previous sections.

Recall that $A^* = \mathrm{Hom}_k(A, k)$ is a tilting complex. We shall denote by τ the class of $A^*[-1]$ in $\mathrm{DPic}_k(A)$, and by σ the class of $A[1]$. We identify an element $T \in \mathrm{DPic}_k(A)$ and the induced auto-equivalence $F = T \otimes^{\mathrm{L}}_A -$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$.

LEMMA 3.1. *τ and σ are in the center of* $\mathrm{DPic}_k(A)$.

Proof. The fact that σ is in the center of $\mathrm{DPic}_k(A)$ is trivial. As for τ, this follows immediately from [Rd] Proposition 5.2 (or by [BO] Proposition 1.3, since $A^* \otimes^{\mathrm{L}}_A -$ is the Serre functor of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$). $\square$

In Definition 2.2 we introduced the quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$. Recall that for a vertex $x \in \boldsymbol{\Gamma}_0^{\mathrm{irr}}$, $M_x \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$ is the representative indecomposable object.

LEMMA 3.2. *There is a group homomorphism*

$$q\colon \mathrm{DPic}_k(A) \to \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau,\sigma\rangle}$$

such that $q(F)(x) = y$ iff $FM_x \cong M_y$.

Proof. Given an auto-equivalence F of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$, the formula $q(F)(x) = y$ iff $FM_x \cong M_y$ defines a permutation $q(F)$ of $\boldsymbol{\Gamma}_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ that preserves arrow-multiplicities. Hence it restricts to a permutation of $\boldsymbol{\Gamma}_0^{\mathrm{irr}}$. By Lemma 3.1, $q(F)$ commutes with τ and σ. $\square$

The group $\mathrm{Out}_k^0(A)$ was defined to be the identity component of $\mathrm{Out}_k(A)$.

LEMMA 3.3. $\mathrm{Ker}(q) = \mathrm{Out}_k^0(A)$.

Proof. Let $T \in \mathrm{DPic}_k(A)$. By Theorem 2.4 we know that $\boldsymbol{\Gamma}_0^{\mathrm{irr}} = \bigcup_{i,j\in\mathbb{Z}} \tau^i\sigma^j(\boldsymbol{\Delta}_0)$. Hence by Lemma 3.1, $T \in \mathrm{Ker}(q)$ iff T acts trivially on the set $\boldsymbol{\Delta}_0$. In particular, we see that $\mathrm{Ker}(q) \subset \mathrm{Pic}_k(A)$. Now use Proposition 1.7. □

LEMMA 3.4. *Suppose A has finite representation type. Then σ is in the center of the group* $\mathrm{Aut}(\boldsymbol{\Gamma}^{\mathrm{irr}})^{\langle\tau\rangle}$.

Proof. According to [Rn] Section 2, the group $\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ is abelian in all cases except D_4. But a direct calculation in this case (cf. Theorem 4.1) gives $\sigma = \tau^{-3}$. □

Before we can talk about the mesh category $k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle$ of the quiver $\boldsymbol{\Gamma}^{\mathrm{irr}}$, we have to fix a polarization μ on it. If the quiver $\boldsymbol{\Delta}$ has no multiple arrows then so does $\boldsymbol{\Gamma}^{\mathrm{irr}}$ (by Theorem 2.4), and hence there is a unique polarization on it. If $\boldsymbol{\Delta}$ isn't a tree let us choose an isomorphism $\rho: \mathbb{Z} \times (\mathbb{Z}\boldsymbol{\Delta}) \xrightarrow{\simeq} \boldsymbol{\Gamma}^{\mathrm{irr}}$ as in that theorem. This determines a polarization μ on $\boldsymbol{\Gamma}^{\mathrm{irr}}$. We also get a lifting of the permutation σ to an auto-equivalence of $k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle$.

LEMMA 3.5. *There are group homomorphisms*

$$p: \mathrm{Out}_k(k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle) \to \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau\rangle}$$

and

$$r: \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau\rangle} \to \mathrm{Out}_k(k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$$

satisfying $p(F)(x) = Fx$ for an auto-equivalence F and a vertex x; $pr = 1$; and both p and r commute with σ.

Proof. Since $\boldsymbol{\Gamma}^{\mathrm{irr}} \cong \boldsymbol{\Gamma}(k\langle\boldsymbol{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$ we get a permutation $p(F) \in \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)$. Let's prove that $p(F)$ commutes with τ in $\mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}})$. Consider a vertex $y \in \boldsymbol{\Gamma}_0^{\mathrm{irr}}$. In the Notation 1.3, there are vertices x_i and irreducible morphisms $\{F(\alpha_{i,j})\}_{j=1}^{d_i}$ and $\{F(\beta_{i,j})\}_{j=1}^{d_i}$ that form bases of $\mathrm{Irr}_{k\langle\boldsymbol{\Gamma}^{\mathrm{irr}},I_{\mathrm{m}}\rangle}(F\tau y, Fx_i)$ and $\mathrm{Irr}_{k\langle\boldsymbol{\Gamma}^{\mathrm{irr}},I_{\mathrm{m}}\rangle}(Fx_i, Fy)$ respectively. Since we have

$$\sum F(\beta_{i,j})F(\alpha_{i,j}) = 0 \in \mathrm{rad}^2_{k\langle\boldsymbol{\Gamma}^{\mathrm{irr}},I_{\mathrm{m}}\rangle}(F\tau y, Fy)/\mathrm{rad}^3_{k\langle\boldsymbol{\Gamma}^{\mathrm{irr}},I_{\mathrm{m}}\rangle}(F\tau y, Fy)$$

this must be a multiple of a mesh relation. Hence $F\tau y = \tau F y$.

Finally to define r we have to split $\mathrm{Aut}(\boldsymbol{\Gamma}^{\mathrm{irr}}) \twoheadrightarrow \mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)$ consistently with μ. It suffices to order the set of arrows $\{\alpha: x \to y\}$ for every pair of vertices $x, y \in \boldsymbol{\Gamma}_0^{\mathrm{irr}}$ consistently with μ. We only have to worry about this when A has infinite representation type. For any $x, y \in \boldsymbol{\Delta}_0$ choose some ordering of the set $\{\alpha: x \to y\}$. Using μ and σ this ordering can be transported to all of $\mathbb{Z} \times (\mathbb{Z}\boldsymbol{\Delta})$. By the isomorphism ρ of Theorem 2.4 the ordering is copied to $\boldsymbol{\Gamma}^{\mathrm{irr}}$. □

LEMMA 3.6. *There exists a group homomorphism*

$$\tilde{q}\colon \mathrm{DPic}_k(A) \to \mathrm{Out}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$$

such that $p\tilde{q} = q$.

Proof. Choose an equivalence $G\colon k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}$ as in Theorem 2.6. If A has infinite representation type then the isomorphism ρ we have chosen (as in Theorem 2.4) tells us how to extend G to an equivalence $G\colon k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle \to \coprod_{l\in\mathbb{Z}} \mathsf{B}[l]$ that commutes with σ (cf. Remark 2.9).

Let F be a triangle auto-equivalence of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. Then F induces a permutation $\pi = q(F)$ of the set $\mathbf{\Gamma}_0^{\mathrm{irr}}$ that commutes with σ. For every vertex $x \in \mathbf{\Gamma}_0^{\mathrm{irr}}$ choose an isomorphism $\phi_x\colon FM_x \xrightarrow{\simeq} M_{\pi(x)}$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. Given an arrow $\alpha\colon x \to y$ in $\mathbf{\Gamma}^{\mathrm{irr}}$, define the morphism $\tilde{q}_{\{\phi_x\}}(F)(\alpha)\colon \pi(x) \to \pi(y)$ by the condition that the diagram

$$\begin{array}{ccc}
FM_x & \xrightarrow{FG(\alpha)} & FM_y \\
{\scriptstyle\phi_x}\downarrow & & \downarrow{\scriptstyle\phi_y} \\
M_{\pi(x)} & \xrightarrow{G\tilde{q}_{\{\phi_x\}}(F)(\alpha)} & M_{\pi(y)}
\end{array}$$

commutes. Then $\tilde{q}_{\{\phi_x\}}(F) \in \mathrm{Aut}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$.

If $\{\phi'_x\}$ is another choice of isomorphisms $\phi'_x\colon FM_x \xrightarrow{\simeq} M_{\pi(x)}$ then $\{\phi'_x\phi_x^{-1}\}$ is an isomorphism of functors $\tilde{q}_{\{\phi_x\}}(F) \to \tilde{q}_{\{\phi'_x\}}(F)$, so the map $\tilde{q}\colon \mathrm{DPic}_k(A) \to \mathrm{Out}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$ is independent of these choices.

It is easy to check that $\tilde{q}$ respects composition of equivalences. □

THEOREM 3.7. *Let A be an indecomposable basic hereditary finite-dimensional k-algebra with quiver $\mathbf{\Delta}$. Then the homomorphism $\tilde{q}$ of Lemma 3.6 induces an isomorphism of groups*

$$\begin{aligned}
\mathrm{DPic}_k(A) &\cong \mathrm{Out}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)^{\langle\sigma\rangle} \\
&\cong \begin{cases} \mathrm{Out}_k\big(k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle\big) & \text{if } A \text{ has finite representation type,} \\ \mathrm{Out}_k\big(k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle\big) \times \langle\sigma\rangle & \text{otherwise.} \end{cases}
\end{aligned}$$

Proof. The proof has three parts.

(1) We show that the homomorphism

$$\tilde{q}\colon \mathrm{DPic}_k(A) \to \mathrm{Out}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)$$

of Lemma 3.6 is injective. Let T be a two-sided tilting complex such that $\tilde{q}(T) \cong \mathbf{1}_{k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle}$. Then the permutation $q(T)$ fixes the vertices of $\mathbf{\Delta} \subset \mathbf{\Gamma}^{\mathrm{irr}}$. Using the fact that $A \cong \bigoplus_{x\in\mathbf{\Delta}_0} M_x$ we see that $T \cong A$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A)$. Replacing T with H^0T we may assume T is a single bimodule. According to [Ye] Proposition 2.2, we see that T is actually an invertible bimodule. Since $k\langle\mathbf{\Delta}\rangle \to k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$ is full we get $\tilde{q}(T)|_{k\langle\mathbf{\Delta}\rangle} \cong \mathbf{1}_{k\langle\mathbf{\Delta}\rangle}$. Hence, by Morita theory, we have $T \cong A$ as bimodules.

(2) Assume A has finite representation type, so that $\mathbf{\Gamma}^{\mathrm{irr}} \cong \mathbb{Z}\mathbf{\Delta}$. We prove that

$$\tilde{q}\colon \mathrm{DPic}_k(A) \to \mathrm{Out}_k(k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle)$$

is surjective.

Consider a k-linear auto-equivalence F of $k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$. Let $\pi := p(F) \in \mathrm{Aut}((\mathbb{Z}\mathbf{\Delta})_0; d)^{\langle\tau\rangle} \cong \mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle}$ as in the proof of Lemma 3.6. According to Lemma 3.4, π commutes with σ. Define

$$M := \bigoplus_{x \in \mathbf{\Delta}_0} M_{\pi(0,x)} \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A).$$

Then for any $x, y \in \mathbf{\Delta}_0$ and integers n, i the equivalence $G\colon k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle \to \mathsf{B}$ of Theorem 2.6 produces isomorphisms

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)}(M_{\pi(0,x)}, M_{(n,y)}[i]) \\
&\quad\cong \mathrm{Hom}_{k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle}(\pi(0,x), \sigma^i(n,y)) \\
&\quad\cong \mathrm{Hom}_{k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle}((0,x), \sigma^i\pi^{-1}(n,y)) \\
&\quad\cong \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)}(M_{(0,x)}, M_{\pi^{-1}(n,y)}[i]).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)}(M, M[i]) \\
&\quad\cong \bigoplus_{x,y \in \mathbf{\Delta}_0} \mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)}(M_{(0,x)}, M_{(0,y)}[i]) \\
&\quad\cong \begin{cases} A^{\circ} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Also for any (n, y) there is some integer i and $x \in \mathbf{\Delta}_0$ such that

$$\mathrm{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)}(M_{\pi(0,x)}, M_{(n,y)}[i]) \neq 0.$$

Since any object $N \in \mathsf{D}^{\mathrm{b}}(\mathsf{mod} A)$ is a direct sum of indecomposables $M_{(n,y)}$, this implies that $\mathrm{RHom}_A(M, N) \neq 0$ if $N \neq 0$. By [Ye] Theorem 1.8 and the proof of '(ii) $\Rightarrow$ (i)' of [Ye] Theorem 1.6 there exists a two-sided tilting complex T with $T \cong M$ in $\mathsf{D}(\mathsf{Mod} A)$ (cf. [Rd] Section 3). Replacing F with $\tilde{q}(T^{\vee})F$, where $T^{\vee} := \mathrm{RHom}_A(T, A)$, we can assume that $p(F)$ is trivial.

Now that $p(F)$ is trivial, F restricts to an auto-equivalence of $k\langle \mathbf{\Delta}\rangle$, and by Proposition 1.7 we have $F|_{k\langle \mathbf{\Delta}\rangle} \cong \mathbf{1}_{k\langle \mathbf{\Delta}\rangle}$. Then Corollary 2.8 tells us $F \cong \mathbf{1}_{k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle}$.

(3) Assume A has infinite representation type. Then the quiver isomorphism ρ of Theorem 2.4 induces a group isomorphism

$$\mathrm{Out}_k(k\langle \mathbf{\Gamma}^{\mathrm{irr}}, I_{\mathrm{m}}\rangle)^{\langle\sigma\rangle} \cong \mathrm{Out}_k(k\langle \mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle) \times \langle\sigma\rangle,$$

and $\langle\sigma\rangle \cong \mathbb{Z}$. We prove that

$$\tilde{q}\colon \mathrm{DPic}_k(A) \to \mathrm{Out}_k(k\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle) \times \mathbb{Z}$$

is surjective.

Take an auto-equivalence F of $k\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle$, and write $\pi := p(F) \in \mathrm{Aut}((\mathbb{Z}\mathbf{\Delta})_0; d)^{\langle\tau\rangle}$. After replacing F with $\tau^j F$ for suitable $j \in \mathbb{Z}$, we can assume that $\pi(0, x) \in \mathbb{Z}_{\geqslant 0}\mathbf{\Delta}$ for all $x \in \mathbf{\Delta}_0$. Because $\mathbb{Z}_{\geqslant 0}\mathbf{\Delta}$ is the preprojective component of $\mathbf{\Gamma}(\mathsf{mod} A)$ (cf. [Rl]), we get

$$M := \bigoplus_{x \in \mathbf{\Delta}_0} M_{\pi(0,x)} \in \mathsf{mod} A.$$

As in part 2 above, $\mathrm{End}_A(M) = A^{\circ}$. Since M is a complete slice, [HR] Theorem 7.2 says that M is a tilting module. So M is a two-sided tilting complex over A. Replacing F by $\tilde{q}(M^{\vee})F$ we can assume $p(F)$ is trivial. Let P be an invertible bimodule such that $\tilde{q}(P)|_{k\langle\mathbf{\Delta}\rangle} \cong F|_{k\langle\mathbf{\Delta}\rangle}$. Replacing F with $\tilde{q}(P^{\vee})F$ we get $F|_{k\langle\mathbf{\Delta}\rangle} \cong \mathbf{1}_{k\langle\mathbf{\Delta}\rangle}$. Then by Corollary 2.8 we get $F \cong \mathbf{1}_{\langle\mathbb{Z}\mathbf{\Delta}, I_{\mathrm{m}}\rangle}$. □

The next theorem is Theorem 0.1 in the Introduction.

THEOREM 3.8. *Let A be an indecomposable basic hereditary finite-dimensional algebra over an algebraically closed field k, with quiver $\mathbf{\Delta}$.*

(1) *There is an exact sequence of groups*

$$1 \to \mathrm{Out}_k^0(A) \to \mathrm{DPic}_k(A) \xrightarrow{q} \mathrm{Aut}(\mathbf{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau,\sigma\rangle} \to 1.$$

This sequence splits.

(2) *If A has finite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle}.$$

(3) *If A has infinite representation type then there is an isomorphism of groups*

$$\mathrm{DPic}_k(A) \cong \big(\mathrm{Aut}((\mathbb{Z}\mathbf{\Delta})_0; d)^{\langle\tau\rangle} \ltimes \mathrm{Out}_k^0(A)\big) \times \mathbb{Z}.$$

Proof. (1) By Theorem 3.7 and Lemma 3.5 the homomorphism q is surjective. Lemma 3.3 identifies $\mathrm{Ker}(q)$.

(2) If A has finite representation type then $\mathbf{\Delta}$ is a tree, so $\mathrm{Out}_k^0(A) = 1$ by Proposition 1.7. By Theorem 2.4 and Lemma 3.4 we get

$$\mathrm{Aut}(\mathbf{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau,\sigma\rangle} \cong \mathrm{Aut}(\mathbf{\Gamma}^{\mathrm{irr}})^{\langle\tau,\sigma\rangle} \cong \mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle}.$$

(3) If A has infinite representation type then

$$\mathrm{Aut}(\boldsymbol{\Gamma}_0^{\mathrm{irr}}; d)^{\langle\tau,\sigma\rangle} \cong \mathrm{Aut}((\mathbb{Z}\boldsymbol{\Delta})_0; d)^{\langle\tau\rangle} \times \langle\sigma\rangle$$

by Theorem 2.4. We know that σ is in the center of $\mathrm{DPic}_k(A)$. □

We end the section with the following problem.

PROBLEM 3.9. The Auslander–Reiten quiver $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ is defined for any finite dimensional k-algebra A of finite global dimension. Can the action of $\mathrm{DPic}_k(A)$ on $\boldsymbol{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A))$ be used to determine the structure of $\mathrm{DPic}_k(A)$ for any such A?

4. Explicit Calculations

In this section we calculate the group structure of $\mathrm{DPic}_k(A)$ for the path algebra $A = k\boldsymbol{\Delta}$ for several types of quivers. Throughout S_m denotes the permutation group of $\{1, \ldots, m\}$.

Suppose Δ is a tree. Given an orientation ω of the edge set Δ_1, denote by $\boldsymbol{\Delta}_\omega$ the resulting quiver, and by $A_\omega := k\boldsymbol{\Delta}_\omega$. If ω and ω' are two orientations of Δ then $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A_\omega) \approx \mathsf{D}^{\mathrm{b}}(\mathsf{mod}A_{\omega'})$. This equivalence will be discussed in the next section. For now we just note that the groups $\mathrm{DPic}_k(A_\omega) \cong \mathrm{DPic}_k(A_{\omega'})$, so we are allowed to choose any orientation of Δ when computing these groups. This observation is relevant to Theorems 4.1 and 4.2 below.

THEOREM 4.1. *Let* $\boldsymbol{\Delta}$ *be a Dynkin quiver as shown in Figure 2, and let* $A := k\boldsymbol{\Delta}$ *be the path algebra. Then* $\mathrm{Pic}_k(A) \cong \mathrm{Aut}(\boldsymbol{\Delta})$ *and* $\mathrm{DPic}_k(A) \cong \mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$. *The groups* $\mathrm{Aut}(\boldsymbol{\Delta})$ *and* $\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ *are described in Table I.*

Proof. The isomorphisms are by Theorem 0.1 and Proposition 1.7. The data in the third column of Table I was calculated in [Rn] Section 4, except for the shift σ which did not appear in that paper. So we have to do a few calculations involving σ. Below are the calculations for types A_n and D_4; the rest are similar and are left to the reader as an exercise.

Table I. The group $\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ for a Dynkin quiver. The orientation of $\boldsymbol{\Delta}$ is shown in Figure 2. In types D_n and E_6, θ is the element of order 2 in $\mathrm{Aut}(\boldsymbol{\Delta})$.

Type	$\mathrm{Aut}(\boldsymbol{\Delta})$	$\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$	Relation
A_n, n even	1	$\langle\tau,\sigma\rangle \cong \mathbb{Z}$	$\tau^{n+1} = \sigma^{-2}$
A_n, n odd	1	$\langle\tau,\sigma\rangle \cong \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$	$\tau^{n+1} = \sigma^{-2}$
D_4	S_3	$\mathrm{Aut}(\boldsymbol{\Delta}) \times \langle\tau\rangle \cong S_3 \times \mathbb{Z}$	$\tau^3 = \sigma^{-1}$
D_n, $n \geqslant 5$	S_2	$\mathrm{Aut}(\boldsymbol{\Delta}) \times \langle\tau\rangle \cong S_2 \times \mathbb{Z}$	$\tau^{n-1} = \theta\sigma^{-1}$, n odd
			$\tau^{n-1} = \sigma^{-1}$, n even
E_6	S_2	$\mathrm{Aut}(\boldsymbol{\Delta}) \times \langle\tau\rangle \cong S_2 \times \mathbb{Z}$	$\tau^6 = \theta\sigma^{-1}$
E_7	1	$\langle\tau\rangle \cong \mathbb{Z}$	$\tau^9 = \sigma^{-1}$
E_8	1	$\langle\tau\rangle \cong \mathbb{Z}$	$\tau^{15} = \sigma^{-1}$

Type A_n: Choose the orientation in Figure 2. The quiver $\mathbb{Z}\mathbf{\Delta}$ looks like Figure 3. Therefore $\mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle} = \langle\tau, \eta\rangle$ where $\eta(0, 1) = (0, n)$ and $\eta(0, n) = (n-1, 1)$.

Now by [Ha] Section I.5.5 and [ARS] Sections VII.1 and VIII.5, the quiver $\mathbf{\Gamma}(\mathsf{mod}A) \subset \mathbb{Z}\mathbf{\Delta}$ is the full subquiver on the vertices in the triangle $\{(m, i) \mid m \geqslant 0,\ m+i \leqslant n\}$. The projective vertices are $(0, i)$ and the injective vertices are $(n-i, i)$, where $i \in \{1, \dots, n\}$. We see that $\sigma(0, i) = (i, n+1-i)$, and the quiver $\mathbf{\Gamma}\big((\mathsf{mod}A)[1]\big) = \sigma\big(\mathbf{\Gamma}(\mathsf{mod}A)\big)$ is the full subquiver on the vertices in the triangle $\{(m, i) \mid m \leqslant n,\ m+i \geqslant n+1\}$. Hence, $\eta = \tau\sigma$ and $\mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle} = \langle\tau, \sigma\rangle$. The relation $\tau^{-(n+1)} = \sigma^2$ is easily verified.

Type D_4: The quiver $\mathbb{Z}\mathbf{\Delta}$ is in Figure 4, and $\mathbf{\Gamma}(\mathsf{mod}A) \subset \mathbb{Z}\mathbf{\Delta}$ is a full subquiver. From the shape of $\mathbf{\Delta}$ we know that $\mathsf{mod}A$ should have 4 indecomposable projective modules, 3 having length 2 and one of them simple. From the shape of the opposite quiver $\mathbf{\Delta}^\circ$ we also know that $\mathsf{mod}A$ should have 4 indecomposable injective modules, 3 of them simple and one of length 4. Counting dimensions using Auslander–Reiten sequences we conclude that $\mathbf{\Gamma}(\mathsf{mod}A)$ is the full subquiver on the vertices $\{0, 1, 2\} \times \mathbf{\Delta}_0$. The projective vertices are $\{(0, 1), (0, i)\}$, the injective vertices are $\{(2, 1), (2, i)\}$, and the simple vertices are $\{(0, 1), (2, i)\}$, where $i \in \{2, 3, 4\}$.

For $i \in \{1, 2, 3, 4\}$ let P_i, S_i and I_i, be the projective, simple and injective modules respectively, indexed such that $P_i \twoheadrightarrow S_i \rightarrowtail I_i$, and with $P_i = M_{(0,i)}$. So $P_1 = S_1$ and $I_i = S_i$ for $i \in \{2, 3, 4\}$. By the symmetry of the quiver it follows that there is a nonzero morphism $(0, i) \to (2, i)$ in $k\langle\mathbb{Z}\mathbf{\Delta}\rangle$ for $i \in \{2, 3, 4\}$, and hence $M_{(2,i)} \cong S_i$

The rule for connecting $\mathbf{\Gamma}(\mathsf{mod}A)$ with $\mathbf{\Gamma}(\mathsf{mod}A[1])$ (see [Ha] Section I.5.5) implies that $M_{(3,1)} \cong M_{(0,1)}[1] = P_1[1]$. Therefore $M_{(3,i)} \cong P_{i'}[1]$ for $i, i' \in \{2, 3, 4\}$. Now for each such i there is an Auslander–Reiten triangle $M_{(2,i)} \to M_{(3,1)} \to M_{(3,i)} \to M_{(2,i)}[1]$. When this triangle is turned it gives an exact sequence $0 \to P_1 \to P_{i'} \to S_i \to 0$ and, hence, $i' = i$. The conclusion is that $\sigma(m, i) = (m+3, i)$ for all $(m, i) \in (\mathbb{Z}\mathbf{\Delta})_0$, so $\sigma = \tau^{-3}$. □

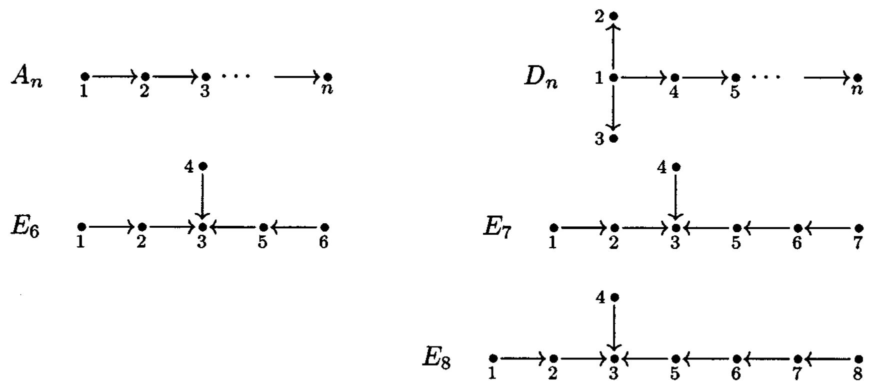

Figure 2. Orientations for the Dynkin graphs.

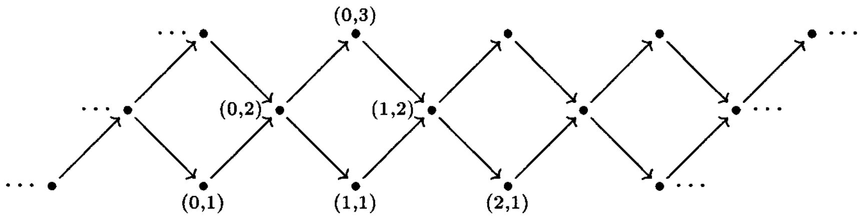

Figure 3. The quiver $\mathbb{Z}\boldsymbol{\Delta}$ for $\boldsymbol{\Delta}$ of type A_3. The vertices in mod$A$ are labeled.

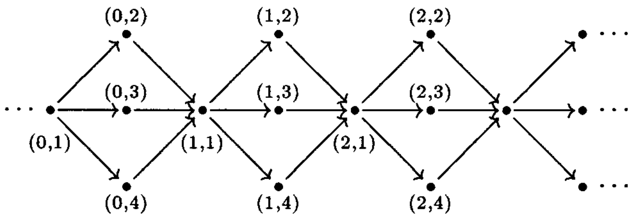

Figure 4. The quiver $\mathbb{Z}\boldsymbol{\Delta}$ for $\boldsymbol{\Delta}$ of type D_4. The vertices in mod$A$ are labeled.

THEOREM 4.2. *Let $\boldsymbol{\Delta}$ be a quiver of type $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$ or $\tilde{E}_8$, with the orientation shown in Figure 5. Then* $\mathrm{Pic}_k(A) \cong \mathrm{Aut}(\boldsymbol{\Delta})$ *and*

$$\mathrm{DPic}_k(A) \cong \mathbb{Z} \times \mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}.$$

The structure of the group $\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ *is given in Table II.*

Proof. The isomorphisms follow from Theorem 0.1 and Proposition 1.7. The structure of $\mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ is quite easy to check in all cases. In type $\tilde{D}_n$, $n \geqslant 5$ odd, the automorphism $\eta \in \mathrm{Aut}(\mathbb{Z}\boldsymbol{\Delta})^{\langle\tau\rangle}$ is

$$\eta(i,j) = \begin{cases} (i, n+2-j) & \text{if } \ j = 2, n, \\ (i - \dfrac{1-(-1)^j}{2}, n+2-j) & \text{otherwise}\ . \end{cases} \qquad \square$$

THEOREM 4.3. *For any $n \geqslant 2$ let $\boldsymbol{\Omega}_n$ be the quiver shown in Figure 6, and let $A := k\boldsymbol{\Omega}_n$ be the path algebra. Then* $\mathrm{Pic}_k(A) \cong \mathrm{PGL}_n(k)$ *and*

$$\mathrm{DPic}_k(A) \cong \mathbb{Z} \times \big(\mathbb{Z} \ltimes \mathrm{PGL}_n(k)\big).$$

In the semidirect product the action of a generator $\rho \in \mathbb{Z}$ on a matrix $F \in \mathrm{PGL}_n(k)$ is $\rho F \rho^{-1} = (F^{-1})^{\mathrm{t}}$.

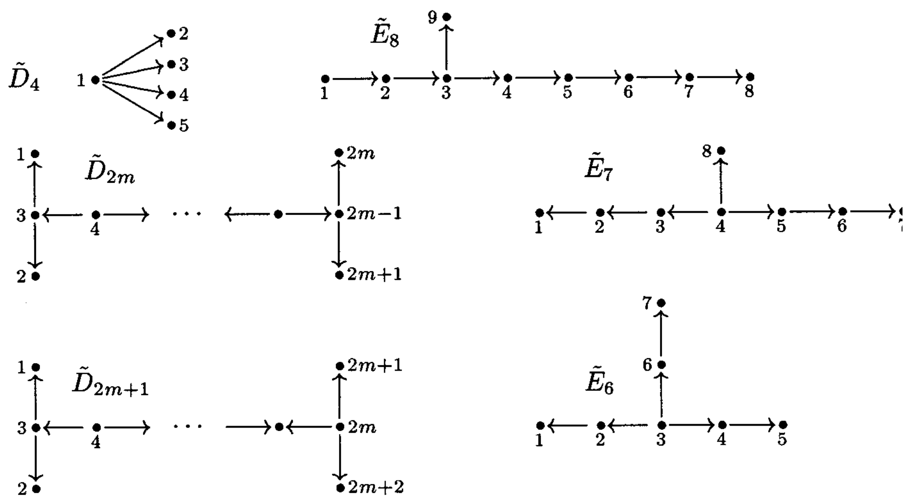

Figure 5. Orientations for the affine tree graphs.

Proof. As in the proof of Lemma 1.6 and Proposition 1.7, the group of auto-equivalences of the path category is $\mathrm{Aut}_k(k\langle\mathbf{\Omega}_n\rangle) = \mathrm{Aut}_k^0(k\langle\mathbf{\Omega}_n\rangle) \cong \mathrm{GL}_n(k)$. Hence $\mathrm{Pic}_k(A) \cong \mathrm{Out}_k(k\langle\mathbf{\Omega}_n\rangle) \cong \mathrm{PGL}_n(k)$.

Given $F \in \mathrm{Aut}_k(k\langle\mathbf{\Omega}_n\rangle)$ let $[a_{i,j}] \in \mathrm{GL}_n(k)$ be its matrix w.r.t. the basis $\{\alpha_i\}$, and let $[b_{i,j}] := ([a_{i,j}]^{-1})^{\mathrm{t}}$. Define an auto-equivalence $\tilde{F} \in \mathrm{Aut}_k^0(k\langle\mathbb{Z}\mathbf{\Omega}_n\rangle)$ with $\tilde{F}(m, \alpha_i) = \sum_j a_{i,j}(m, \alpha_j)$ and $\tilde{F}(m, \alpha_i^*) = \sum_j b_{i,j}(m, \alpha_j^*)$, $m \in \mathbb{Z}$. Then $\tilde{F}$ preserves all mesh relations, and by a linear algebra argument we see that up to scalars at each vertex, the only elements of $\mathrm{Aut}_k^0(k\langle\mathbb{Z}\mathbf{\Omega}_n\rangle)$ are of the form $\tilde{F}$.

Let $\rho \in \mathrm{Aut}(\mathbb{Z}\mathbf{\Omega}_n)$ be $\rho(m, 1) = (m, 2)$ and $\rho(m, 2) = (m+1, 1)$, with the obvious action on arrows to make it commute with the polarization μ. Then $\mathrm{Out}_k(k\langle\mathbb{Z}\mathbf{\Omega}_n, I_{\mathrm{m}}\rangle)$ is generated by $\mathrm{PGL}_n(k)$ and ρ, so $\mathrm{Out}_k(k\langle\mathbb{Z}\mathbf{\Omega}_n, I_{\mathrm{m}}\rangle) \cong \mathbb{Z} \ltimes \mathrm{PGL}_n(k)$. The formula for $\tilde{F}$ above shows that $\rho F \rho^{-1} = (F^{-1})^{\mathrm{t}}$ for $F \in \mathrm{PGL}_n(k)$.

Finally use Theorem 3.7. □

Remark 4.4. By [Be] and [BO] we see that for $n = 2$ in the theorem above, $\mathrm{DPic}_k(A) \cong \mathbb{Z} \times \mathbb{Z} \times \mathrm{PGL}_2(k)$. The apparent discrepancy is explained by the fact that $\mathbb{Z} \ltimes \mathrm{PGL}_2(k) \cong \mathbb{Z} \times \mathrm{PGL}_2(k)$ via $(m, F) \mapsto (m, H^m F)$, where $H = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$.

Table II. The groups $\mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle}$ for the affine tree quivers shown in Figure 5.

Type	$\mathrm{Aut}(\mathbf{\Delta})$	$\mathrm{Aut}(\mathbb{Z}\mathbf{\Delta})^{\langle\tau\rangle}$	Relations
$\tilde{D}_4$	S_4	$\mathrm{Aut}(\mathbf{\Delta}) \times \langle\tau\rangle \cong S_4 \times \mathbb{Z}$	
$\tilde{D}_n, n \geqslant 5$ even	$S_2 \ltimes S_2^2$	$\mathrm{Aut}(\mathbf{\Delta}) \times \langle\tau\rangle \cong (S_2 \ltimes S_2^2) \times \mathbb{Z}$	
$\tilde{D}_n, n \geqslant 5$ odd	S_2^2	$\mathrm{Aut}(\mathbf{\Delta}) \times \langle\eta\rangle \cong S_2^2 \times \mathbb{Z}$	$\eta^2 = \tau$
$\tilde{E}_6$	S_3	$\mathrm{Aut}(\mathbf{\Delta}) \times \langle\tau\rangle \cong S_3 \times \mathbb{Z}$	
$\tilde{E}_7$	S_2	$\mathrm{Aut}(\mathbf{\Delta}) \times \langle\tau\rangle \cong S_2 \times \mathbb{Z}$	
$\tilde{E}_8$	1	$\langle\tau\rangle \cong \mathbb{Z}$	

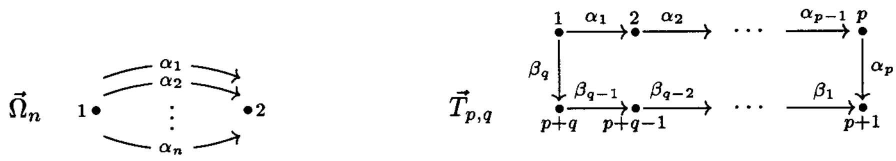

Figure 6. The quivers $\boldsymbol{\Omega}_n$ and $\boldsymbol{T}_{p,q}$.

For integers $p \geqslant q \geqslant 1$ let $\boldsymbol{T}_{p,q}$ be the quiver shown in Figure 6. Let $\boldsymbol{\Delta}$ be a quiver with underlying graph $\tilde{A}_n$. Then $\boldsymbol{\Delta}$ can be brought to one of the quivers $\boldsymbol{T}_{p,q}, p+q=n+1$, by a sequence of admissible reflections s_x^- at source vertices (see Section 6). Therefore $\mathrm{DPic}_k(k\boldsymbol{\Delta}) \cong \mathrm{DPic}_k(k\boldsymbol{T}_{p,q})$.

THEOREM 4.5. *Let A be the path algebra $k\boldsymbol{T}_{p,q}$.*

(1) *If $p=q=1$ then* $\mathrm{Pic}_k(A) \cong \mathrm{PGL}_2(k)$ *and* $\mathrm{DPic}_k(A) \cong \mathbb{Z} \times (\mathbb{Z} \ltimes \mathrm{PGL}_2(k))$.

(2) *If $p>q=1$ then* $\mathrm{Pic}_k(A) \cong \begin{bmatrix} k^\times & k \\ 0 & 1 \end{bmatrix}$ *and* $\mathrm{DPic}_k(A) \cong \mathbb{Z} \times \left(\mathbb{Z} \ltimes \begin{bmatrix} k^\times & k \\ 0 & 1 \end{bmatrix}\right)$.

(3) *If $p=q>1$ then* $\mathrm{Pic}_k(A) \cong S_2 \ltimes k^\times$ *and* $\mathrm{DPic}_k(A) \cong \mathbb{Z}^2 \times (S_2 \ltimes k^\times)$.

(4) *If $p>q>1$ then* $\mathrm{Pic}_k(A) \cong k^\times$ *and* $\mathrm{DPic}_k(A) \cong \mathbb{Z}^2 \times k^\times$.

Proof. (1) This is because $\boldsymbol{T}_{1,1} = \boldsymbol{\Omega}_2$.

(2) Here the group of auto-equivalences of $k\langle\boldsymbol{T}_{p,q}\rangle$ is, in the notation of the proof of Proposition 1.7, $\mathrm{Aut}_k(k\langle\boldsymbol{T}_{p,q}\rangle) \cong (k^\times)^{p+1} \times k$, and the group of isomorphisms is $(k^\times)^p$. Therefore $\mathrm{Out}_k(k\langle\boldsymbol{T}_{p,q}\rangle)$ is isomorphic to $k^\times \times k$ as varieties, and as matrix group $\mathrm{Out}_k(k\langle\boldsymbol{T}_{p,q}\rangle) \cong \begin{bmatrix} k^\times & k \\ 0 & 1 \end{bmatrix}$. The auto-equivalence associated to

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} \in \begin{bmatrix} k^\times & k \\ 0 & 1 \end{bmatrix}$$

is $\alpha_i \mapsto \alpha_i$ and $\beta_1 \mapsto a\beta_1 + b\alpha_p \cdots \alpha_1$.

The quiver $\mathbb{Z}\boldsymbol{T}_{p,q}$ has no multiple arrows. Let ρ be the symmetry

$$\rho(m,i) = (m,i-1) \quad \text{for} \quad i \geqslant 2, \quad \text{and} \quad \rho(m,1) = (m-1,p).$$

Then ρ generates $\mathrm{Aut}(\mathbb{Z}\boldsymbol{T}_{p,q})^{\langle\tau\rangle}$, and we can use Theorem 0.1. The action of ρ on $\mathrm{Out}_k(k\langle\boldsymbol{T}_{p,q}\rangle)$ is

$$\rho\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}\rho^{-1} = \begin{bmatrix} a & -b \\ 0 & 1 \end{bmatrix}.$$

(3) Here $\mathrm{Aut}_k^0(k\langle\boldsymbol{T}_{p,q}\rangle) \cong (k^\times)^{2p}$, and the subgroup of isomorphisms is $(k^\times)^{2p-1}$. The symmetry $\theta \in \mathrm{Aut}(\boldsymbol{T}_{p,q})$ of order 2 acts on $k^\times$ by $\theta a \theta^{-1} = a^{-1}$.

Let ρ be the symmetry

$$\rho(m,1) = (m-1,p+q), \quad \rho(m,i) = (m,i-1) \quad \text{if } 2 \leqslant i \leqslant p+1,$$

and

$$\rho(m, i) = (m-1, i-1) \quad \text{if } p+2 \leqslant i \leqslant p+q.$$

Then ρ and θ commute, and they generate $\mathrm{Aut}(\mathbb{Z}\boldsymbol{T}_{p,q})^{\langle\tau\rangle}$. The action of ρ on $\mathrm{Aut}_k(k\langle \boldsymbol{T}_{p,q}\rangle)$ is trivial.

(4) Similar to case 3. $\square$

5. The Reflection Groupoid of a Graph

In this section we interpret the reflection functors of [BGP] and the tilting modules of [APR] in the setup of derived categories.

Let Δ be a tree with n vertices. Denote by $\mathrm{Or}(\Delta)$ the set of orientations of the edge set Δ_1. For $\omega \in \mathrm{Or}(\Delta)$ let $\boldsymbol{\Delta}_\omega$ be the resulting quiver, and let A_ω be the path algebra $k\boldsymbol{\Delta}_\omega$.

Given two orientations ω, ω' let

$$\mathrm{DPic}_k(\omega, \omega') := \frac{\{\text{two-sided tilting complexes } T \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A_{\omega'} \otimes_k A_\omega^\circ))\}}{\text{isomorphism}}.$$

The *derived Picard groupoid* of Δ is the groupoid $\mathrm{DPic}_k(\Delta)$ with object set $\mathrm{Or}(\Delta)$ and morphism sets $\mathrm{DPic}_k(\omega, \omega')$. Thus when $\omega = \omega'$ we recover the derived Picard group $\mathrm{DPic}_k(A_\omega)$.

For an orientation ω and a vertex x let $P_{x,\omega} \in \mathsf{mod} A_\omega$ be the corresponding indecomposable projective module. Denote by τ_ω the translation functor of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A_\omega)$, i.e. the functor $\tau_\omega = A_\omega^*[-1] \otimes^{\mathsf{L}}_{A_\omega} -$.

Suppose $x \in (\boldsymbol{\Delta}_\omega)_0$ is a source. Define $s_x^- \omega$ to be the orientation obtained from ω by reversing the arrows starting at x. Let

$$T_{x,\omega} := \tau_\omega^{-1} P_{x,\omega} \oplus \left(\bigoplus_{y \neq x} P_{y,\omega} \right) \in \mathsf{mod} A_\omega.$$

According to [APR] Section 3, $T_{x,\omega}$ is a tilting module, with $\mathrm{End}_{A_\omega}(T_{x,\omega})^\circ \cong A_{s_x^-\omega}$. It is called an *APR tilting module*. One has isomorphisms in $\mathsf{mod} A_{s_x^-\omega}$:

$$\begin{aligned} &\mathrm{Hom}_{A_\omega}(T_{x,\omega}, P_{y,\omega}) \cong P_{y,s_x^-\omega} \quad \text{if } y \neq x, \\ &\mathrm{Hom}_{A_\omega}(T_{x,\omega}, \tau_\omega^{-1} P_{x,\omega}) \cong P_{x,s_x^-\omega}. \end{aligned} \tag{5.1}$$

Under the anti-equivalence between $\mathsf{mod} A_\omega$ and the category of finite dimensional representations of the quiver $\boldsymbol{\Delta}_\omega$, the reflection functor of [BGP] is sent to $\mathrm{Hom}_{A_\omega}(T_{x,\omega}, -) \colon \mathsf{mod} A_\omega \to \mathsf{mod} A_{s_x^-\omega}$.

DEFINITION 5.2. *The reflection groupoid* of Δ is the subgroupoid $\mathrm{Ref}(\Delta) \subset \mathrm{DPic}_k(\Delta)$ generated by the two-sided tilting complexes $T_{x,\omega} \in \mathsf{D}^{\mathrm{b}}(\mathsf{Mod}(A_\omega \otimes_k A_{s_x^-\omega}^\circ))$, as ω runs over $\mathrm{Or}(\Delta)$ and x runs over the sources in $\boldsymbol{\Delta}_\omega$.

Given an orientation ω the set $\{[P_{x,\omega}]\}_{x\in\Delta_0}$ is a basis of the Grothendieck group $\mathrm{K}_0(A_\omega) = \mathrm{K}_0(\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A_\omega))$. Let $\mathbb{Z}^{\Delta_0}$ be the free Abelian group with basis $\{e_x\}_{x\in\Delta_0}$. Then $[P_{x,\omega}] \mapsto e_x$ determines a canonical isomorphism $\mathrm{K}_0(A_\omega) \xrightarrow{\simeq} \mathbb{Z}^{\Delta_0}$. For a two-sided tilting complex $T \in \mathrm{DPic}_k(\omega, \omega')$ let $\chi_0(T)\colon \mathrm{K}_0(A_\omega) \xrightarrow{\simeq} \mathrm{K}_0(A_{\omega'})$ be $\chi_0(T)([M]) := [T \otimes^{\mathrm{L}}_{A_\omega} M]$. Using the projective bases we get a functor (when we consider a group as a groupoid with a single object)

$$\chi_0\colon \mathrm{DPic}_k(\Delta) \to \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{\Delta_0}) \cong \mathrm{GL}_n(\mathbb{Z}).$$

Recall that for a vertex $x \in \Delta_0$ one defines the reflection $s_x \in \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{\Delta_0})$ by

$$s_x e_x := -e_x + \sum_{\{x,y\}\in\Delta_1} e_y,$$
$$s_x e_y := e_y \quad \text{if } y \neq x.$$

The *Weyl group* of Δ is the subgroup $W(\Delta) \subset \mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^{\Delta_0})$ generated by the reflections s_x.

PROPOSITION 5.3. *Let x be a source in the quiver $\mathbf{\Delta}_\omega$. Then*

$$\chi_0(T_{x,\omega}) = s_x.$$

Proof. There is an Auslander–Reiten sequence

$$0 \to P_{x,\omega} \to \bigoplus_{(x\to y)\in(\mathbf{\Delta}_\omega)_1} P_{y,\omega} \to \tau_\omega^{-1} P_{x,\omega} \to 0$$

in $\mathsf{mod}A_\omega$. Applying the functor $T^\vee_{x,\omega} \otimes^{\mathrm{L}}_{A_\omega} - \cong \mathrm{RHom}_{A_\omega}(T_{x,\omega}, -)$ to this sequence, and using formula (5.1), we get a triangle

$$T^\vee_{x,\omega} \otimes^{\mathrm{L}}_{A_\omega} P_{x,\omega} \to \bigoplus_{\{x,y\}\in\Delta_1} P_{y,s_x^-\omega} \to P_{x,s_x^-\omega} \to (T^\vee_{x,\omega} \otimes^{\mathrm{L}}_{A_\omega} P_{x,\omega})[1]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}A_{s_x^-\omega})$. Hence

$$[T^\vee_{x,\omega} \otimes^{\mathrm{L}}_{A_\omega} P_{x,\omega}] = -[P_{x,s_x^-\omega}] + \sum_{\{x,y\}\in\Delta_1} [P_{y,s_x^-\omega}] \in \mathrm{K}_0(A_{s_x^-\omega}).$$

On the other hand for $y \neq x$ we have $[T^\vee_{x,\omega} \otimes^{\mathrm{L}}_{A_\omega} P_{y,\omega}] = [P_{y,s_x^-\omega}]$. This proves that $\chi_0(T^\vee_{x,\omega}) = s_x$; but $s_x = s_x^{-1}$.

An immediate consequence is: □

COROLLARY 5.4. $\chi_0(\mathrm{Ref}(\Delta)) = W(\Delta)$.

An ordering $(x_1, \ldots, x_n)$ of Δ_0 is called source-admissible for an orientation ω if x_i is a source in the quiver $\mathbf{\Delta}_{s^-_{x_{i-1}}\cdots s^-_{x_1}\omega}$ for all $1 \leqslant i \leqslant n$. Any orientation has source-admissible orderings of the vertices.

PROPOSITION 5.5. *Let* $(x_1, \ldots, x_n)$ *be a source-admissible ordering of* Δ_0 *for an orientation* ω. *Write* $\omega_i := s^-_{x_i} \cdots s^-_{x_1}\omega$, $A_i := A_{\omega_i}$ *and* $T_i := T_{x_i, \omega_{i-1}}$. *Then*

$$T_n^\vee \otimes^{\mathrm{L}}_{A_{n-1}} \cdots \otimes^{\mathrm{L}}_{A_2} T_2^\vee \otimes^{\mathrm{L}}_{A_1} T_1^\vee \cong A^*_\omega[-1]$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{Mod} A^{\mathrm{e}}_\omega)$.

Proof. For an orientation ω let $\mathbf{\Gamma}^{\mathrm{irr}}_\omega \subset \mathbf{\Gamma}(\mathsf{D}^{\mathrm{b}}(\mathsf{mod} A_\omega))$ be the quiver of Definition 2.2. As usual $(\mathbf{\Gamma}^{\mathrm{irr}}_\omega)_0$ denotes the set of vertices of $\mathbf{\Gamma}^{\mathrm{irr}}_\omega$. Let $G(\Delta)$ be the groupoid with object set $\mathrm{Or}(\Delta)$, and morphism sets $\mathrm{Iso}\big((\mathbf{\Gamma}^{\mathrm{irr}}_\omega)_0, (\mathbf{\Gamma}^{\mathrm{irr}}_{\omega'})_0\big)$ for $\omega, \omega' \in \mathrm{Or}(\Delta)$. The groupoid $G(\Delta)$ acts faithfully on the family of sets $X(\Delta) := \{(\mathbf{\Gamma}^{\mathrm{irr}}_\omega)_0\}_{\omega \in \mathrm{Or}(\Delta)}$. According to Theorem 0.1 there is an injective map of groupoids $q: \mathrm{DPic}_k(\Delta) \rightarrowtail G(\Delta)$.

Let us first assume Δ is a Dynkin graph. Then there is a canonical isomorphism of sets $X(\Delta) \cong \mathbb{Z} \times \Delta_0 \times \mathrm{Or}(\Delta)$. The action of $q(\tau_\omega)$ on $X(\Delta)$ is $q(\tau_\omega)(i, x, \omega) = (i-1, x, \omega)$. By formula (5.1), the action of $q(T^\vee_{x,\omega})$ on $X(\Delta)$ is $q(T^\vee_{x,\omega})(0, y, \omega) = (0, y, s^-_x\omega)$ if $y \neq x$, and $q(T^\vee_{x,\omega})(1, x, \omega) = (0, x, s^-_x\omega)$. Since $q(\tau_\omega)$ commutes with $q(T^\vee_{x,\omega})$ we have

$$q(T_n^\vee \otimes^{\mathrm{L}}_{A_{n-1}} \cdots \otimes^{\mathrm{L}}_{A_1} T_1^\vee)(i, x, \omega) = (i-1, x, \omega) = q(\tau_\omega)(i, x, \omega)$$

for any $x \in \Delta_0$ and $i \in \mathbb{Z}$.

If Δ is not Dynkin then $X(\Delta) \cong \mathbb{Z} \times \mathbb{Z} \times \Delta_0 \times \mathrm{Or}(\Delta)$, $q(\tau_\omega)(j, i, x, \omega) = (j, i-1, x, \omega)$, etc., and the proof is the same after these modifications.

PROPOSITION 5.6. *For any orientation* ω, $\mathrm{Ref}(\Delta)(\omega, \omega) = \langle \tau_\omega \rangle$.

Proof. We will only treat the Dynkin case; the general case is proved similarly with modifications like in the previous proof.

Let $T \in \mathrm{Ref}(\Delta)(\omega, \omega)$. From the proof above we see that $q(T)(0, x, \omega) = (i(x), x, \omega)$ for some $i(x) \in \mathbb{Z}$. A quiver map $\pi: \mathbf{\Delta}_\omega \to \mathbb{Z}\mathbf{\Delta}_\omega$ with $\pi(x) = (i(x), x)$ must have $i(x) = i$ for all x, since Δ is a tree. Therefore $q(T) = q(\tau_\omega^{-i})$. $\square$

Remark 5.7. The explicit calculations in Section 4 show that the shift $\sigma = A[1]$ is not in $\langle \tau \rangle \subset \mathrm{DPic}_k(A)$ for most algebras A. Thus $\mathrm{Ref}(\Delta) \subsetneqq \mathrm{DPic}_k(\Delta)$ for most graphs Δ.

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