DUALIZING COMPLEXES, MORITA EQUIVALENCE AND THE DERIVED PICARD GROUP OF A RING

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Abstract

Two rings A and B are said to be derived Morita equivalent if the derived categories $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A)$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} B)$ are equivalent. If A and B are derived Morita equivalent algebras over a field k, then there is a complex of bimodules T such that the functor $T \otimes^{\mathrm{L}}_{A} - : \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A) \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} B)$ is an equivalence. The complex T is called a tilting complex.

When $B = A$ the isomorphism classes of tilting complexes T form the derived Picard group $\operatorname{DPic}(A)$. This group acts naturally on the Grothendieck group $\mathrm{K}_0(A)$.

It is proved that when the algebra A is either local or commutative, then any derived Morita equivalent algebra B is actually Morita equivalent. This enables one to compute $\operatorname{DPic}(A)$ in these cases.

Assume that A is noetherian. Dualizing complexes over A are complexes of bimodules which generalize the commutative definition. It is proved that the group $\operatorname{DPic}(A)$ classifies the set of isomorphism classes of dualizing complexes. This classification is used to deduce properties of rigid dualizing complexes.

Finally finite k-algebras are considered. For the algebra A of upper triangular 2×2 matrices over k, it is proved that $t^3 = s$, where $t, s \in \operatorname{DPic}(A)$ are the classes of $A^* := \operatorname{Hom}_k(A, k)$ and $A[1]$ respectively. In the appendix, by Elena Kreines, this result is generalized to upper triangular $n \times n$ matrices, and it is shown that the relation $t^{n+1} = s^{n-1}$ holds.

0. *Introduction*

Let A and B be two rings. Recall that according to Morita theory, any equivalence between the categories of left modules $\operatorname{Mod} A \longrightarrow \operatorname{Mod} B$ is realized by a B-A-bimodule P, progenerator on both sides, as the functor $M \longmapsto P \otimes_A M$.

Happel [**5**], Cline, Parshall and Scott [**3**] and Rickard [**13**, **14**] generalized Morita theory to derived categories. Let A and B be algebras over a field k. Rickard proved that if the derived categories $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A)$ and $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} B)$ are equivalent, then there is a complex $T \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod}(B \otimes_k A^{\circ}))$ such that the functor $T \otimes^{\mathrm{L}}_{A} - :$ $\mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A) \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} B)$ is an equivalence. Here A° denotes the opposite algebra. A complex T with this property is called a *tilting complex*, and the algebras A and B are said to be *derived Morita equivalent*.

In Section 1 we recall some facts on derived categories of bimodules from [**16**]. Then we reproduce Rickard's results in the formulation needed for this paper. See Remark 1.12 regarding the generalization to an arbitrary commutative base ring k.

In Section 2 we prove that if A is either local or commutative then any derived Morita equivalent algebra B is actually Morita equivalent (in the ordinary sense). Specifically if $T \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod}(B \otimes_k A^{\circ}))$ is a tilting complex then $T \cong P[n]$ for some invertible bimodule P and some integer n (in the commutative case $\operatorname{Spec} A$ is assumed connected). See Theorems 2.3 and 2.6.

When $A = B$ the isomorphism classes of tilting complexes form a group, called the *derived Picard group* $\operatorname{DPic}(A)$. The operation is $(T_1, T_2) \longmapsto T_1 \otimes^{\mathrm{L}}_{A} T_2$, the identity is A and the inverse is $T \longmapsto T^{\vee} := \operatorname{R}\operatorname{Hom}_A(T, A)$. Let $s \in \operatorname{DPic}(A)$ be the class of the complex $A[1]$. Then the subgroup $\langle s \rangle$ is isomorphic to $\mathbb{Z}$. When A is local we show

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that $\mathrm{DPic}(A) \cong \mathbb{Z} \times \mathrm{Out}_k(A)$, where $\mathrm{Out}_k(A)$ denotes the group of outer k-algebra automorphisms (see Proposition 3.4). When A is commutative then $\mathrm{DPic}(A) \cong \mathbb{Z}^m \times \mathrm{Aut}_k(A) \ltimes \mathrm{Pic}_A(A)$, where m is the number of connected components of $\mathrm{Spec}\, A$ and $\mathrm{Pic}_A(A)$ is the usual commutative Picard group (Proposition 3.5). If A is noetherian let $\mathrm{K}_0(A) = \mathrm{K}_0(\mathrm{Mod}_{\mathrm{f}}(A))$ be the Grothendieck group. Then there is a representation $\chi_0 : \mathrm{DPic}(A) \longrightarrow \mathrm{Aut}(\mathrm{K}_0(A))$, with $\chi_0(s) = -1$.

In Section 4 we suppose A is noetherian. Then we have the notion of *dualizing complex* $R \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$, where $A^{\mathrm{e}} := A \otimes_k A^{\mathrm{o}}$ (Definition 4.1). Dualizing complexes over noncommutative algebras were introduced in [**16**], generalizing the commutative definition of [**6**]. Unlike the commutative case, where any two dualizing complexes R_1, R_2 satisfy $R_2 \cong L[n] \otimes_A R_1$ with L an invertible module and n an integer, when A is noncommutative there is no such uniqueness. The question arose of how to classify all isomorphism classes. We prove in Theorem 4.5 that given a dualizing complex R_1, any other complex R_2 is dualizing if and only if $R_2 \cong T \otimes^{\mathrm{L}}_A R_1$ for some tilting complex T. Moreover this T is unique up to isomorphism. Therefore the group $\mathrm{DPic}(A)$ classifies the isomorphism classes of dualizing complexes.

Next, in Section 5, we consider *rigid dualizing complexes*, which were defined by Van den Bergh [**15**]. One of his results was that a rigid dualizing complex is unique up to an isomorphism in $\mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}})$. We prove that this isomorphism is unique (Theorem 5.2). If A is finitely generated as k-algebra and finite over its centre then it has a rigid dualizing complex (Proposition 5.7). If A is Gorenstein and has a rigid dualizing R complex then R is also a tilting complex, and $R^{\vee} \cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, A^{\mathrm{e}})$ (Proposition 5.11). This also generalizes a result of Van den Bergh.

In Section 6 we look at a finite k-algebra A. The bimodule $A^* := \mathrm{Hom}_k(A, k)$ is the rigid dualizing complex of A. If A is a Gorenstein algebra then A^* is also a tilting complex, in which case we denote its class in $\mathrm{DPic}(A)$ by t. If moreover A has finite global dimension then $\chi_0(t) = -c$, where c is the Coxeter transformation of [**1**, Chapter VIII].

Finally in Proposition 6.5 we examine the group $\mathrm{DPic}(A)$ for the algebra

$$A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}.$$

Note that this is the smallest k-algebra which is neither commutative nor local. The ordinary noncommutative Picard group $\mathrm{Pic}(A)$ is trivial here. On the other hand, we prove that $t^3 = s$, so $\mathrm{DPic}(A) \neq \langle s \rangle$. In the appendix, by Elena Kreines, the calculation is carried out for an $n \times n$ upper triangular matrix algebra, $n \geqslant 2$. She proves that $t^{n+1} = s^{n-1}$ in this case.

1. *Morita equivalence*

Let k be a fixed base field. All k-algebras will be associative with 1. Given a k-algebra A we denote by A^{o} the opposite algebra, and by A^{e} the enveloping algebra $A \otimes A^{\mathrm{o}}$ (where $\otimes = \otimes_k$ throughout). Our modules will be by default left modules, and with this convention an A^{o}-module will mean a right A-module. Given another k-algebra B, an $(A \otimes B^{\mathrm{o}})$-module M is then just and A-B-bimodule ${}_A M_B$, central over k.

We write $\mathrm{Mod}\, A$ for the category of A-modules. Let $\mathrm{D}(\mathrm{Mod}\, A)$ be the derived category of A-modules, and for $\star = -, +, \mathrm{b}$ let $\mathrm{D}^{\star}(\mathrm{Mod}\, A)$ be the appropriate full subcategory (conventions as in [**6**]).

The forgetful functor $\operatorname{Mod}(A \otimes B^{\circ}) \longrightarrow \operatorname{Mod} A$ is exact and so induces a functor $\mathrm{D}^{\star}(\operatorname{Mod}(A \otimes B^{\circ})) \longrightarrow \mathrm{D}^{\star}(\operatorname{Mod} A)$. Now $A \otimes B^{\circ}$ is a projective A-module, so any projective (respectively flat, injective) $(A \otimes B^{\circ})$-module is projective (respectively flat, injective) over A.

Consider k-algebras A, B, C. For complexes $M \in \mathrm{D}(\operatorname{Mod}(A \otimes B^{\circ}))$ and $N \in \mathrm{D}(\operatorname{Mod}(A \otimes C^{\circ}))$, with either $M \in D^{-}$ or $N \in \mathrm{D}^{+}$, there is a derived functor

$$\mathrm{R}\operatorname{Hom}_A(M, N) \in \mathrm{D}(\operatorname{Mod}(B \otimes C^{\circ})).$$

It is calculated by replacing M with an isomorphic complex in $\mathrm{D}^{-}(\operatorname{Mod}(A \otimes B^{\circ}))$ which consists of projective modules over A, or by replacing N with an isomorphic complex in $\mathrm{D}^{+}(\operatorname{Mod}(A \otimes C^{\circ}))$ which consists of injective modules over A.

Likewise, for complexes $M \in \mathrm{D}^{-}(\operatorname{Mod}(B \otimes A^{\circ}))$ and $N \in \mathrm{D}^{-}(\operatorname{Mod}(A \otimes C^{\circ}))$ there is a derived functor

$$M \otimes^{\mathrm{L}}_A N \in \mathrm{D}(\operatorname{Mod}(B \otimes C^{\circ})).$$

It is calculated by replacing M with an isomorphic complex in $\mathrm{D}^{-}(\operatorname{Mod}(B \otimes A^{\circ}))$ which consists of flat modules over A°, or by doing the corresponding thing for N. In the case when M has finite Tor dimension over A°, that is, it is isomorphic in $\mathrm{D}(\operatorname{Mod} A^{\circ})$ to a bounded complex of flat A°-modules, then $M \otimes^{\mathrm{L}}_A N$ is defined for an unbounded N (and vice versa). For full details see [**16**].

Because the forgetful functors $\operatorname{Mod}(A \otimes B^{\circ}) \longrightarrow \operatorname{Mod} A$ and so on commute with $\mathrm{R}\operatorname{Hom}_A(-,-)$ and $- \otimes^{\mathrm{L}}_A -$ there is no need to mention them explicitly.

Recall that a complex $M \in \mathrm{D}(\operatorname{Mod} A)$ is called *perfect* if it is isomorphic to a bounded complex of finitely generated projective modules. The full subcategory $\mathrm{D}(\operatorname{Mod} A)_{\mathrm{perf}} \subset \mathrm{D}(\operatorname{Mod} A)$ consisting of perfect complexes is triangulated, and the identity functors

$$\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A) \longrightarrow \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A)_{\mathrm{perf}} \longrightarrow \mathrm{D}(\operatorname{Mod} A)_{\mathrm{perf}}$$

are equivalences, where $\operatorname{proj} A$ is the additive category of finitely generated projective A-modules.

LEMMA 1.1. (1) *Suppose that* $M \cong M_1 \oplus M_2 \in \mathrm{D}(\operatorname{Mod} A)$. *Then M is perfect if and only if both M_1 and M_2 are perfect.*

(2) *Let $M \in \mathrm{D}(\operatorname{Mod} A)$ be a perfect complex and n be an integer. If* $\mathrm{H}^p M = 0$ *for all $p > n$ then* $\mathrm{H}^n M$ *is a finitely generated module.*

Proof. (1) See [**4**, Exposé I, Proposition 4.17].

(2) Let $P \cong M$ where P is a bounded complex of finitely generated projectives. Say $P^p = 0$ for $p > m \geqslant n$. By splitting $P^n \longrightarrow \ldots \longrightarrow P^m$ we obtain a surjection $P^n \twoheadrightarrow \mathrm{H}^n M$. □

Given a complex $M \in \mathrm{D}(\operatorname{Mod} A)$, denote by $\operatorname{add} M \subset \mathrm{D}(\operatorname{Mod} A)$ the class of all direct summands of finite direct sums of M.

We say a class D_0 of objects of a triangulated category D generates it if there is no triangulated subcategory D', closed under isomorphisms, with $\mathrm{D}_0 \subset \mathrm{D}' \subsetneq \mathrm{D}$.

At this point we wish to remind the reader of the classical Morita theory.

THEOREM 1.2 (Morita theory). *Let A and B be rings. Then the following are equivalent:*

(i) *The abelian categories* $\operatorname{Mod} A$ *and* $\operatorname{Mod} B$ *are equivalent.*

(ii) *There is a B-A-bimodule P, progenerator over B; such that the canonical ring homomorphism $A^{\mathrm{o}} \longrightarrow \mathrm{End}_B(P)$ is bijective.*

(iii) *There is a B-A-bimodule P and an A-B-bimodule Q such that $P \otimes_A Q \cong B$ and $Q \otimes_B P \cong A$ as bimodules.*

If $F: \mathrm{Mod}\, A \longrightarrow \mathrm{Mod}\, B$ is the equivalence, then $P = FA$, $FM = P \otimes_A M$ and $Q = \mathrm{Hom}_B(P, B)$. In this case we say that A and B are Morita equivalent, and we call a bimodule P as above an *invertible B-A-bimodule*.

Following Rickard we make the following definition.

Definition 1.3. Let A and B be rings. If there is an equivalence of triangulated categories $F: \mathrm{D^b}(\mathrm{Mod}\, A) \longrightarrow \mathrm{D^b}(\mathrm{Mod}\, B)$ we say that A and B are *derived Morita equivalent*.

The generalization to complexes of the notion of invertible bimodule is given in the following.

Definition 1.4. Let A, B be k-algebras, and let $T \in \mathrm{D^b}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$. Suppose that

(i) $T \in \mathrm{D^b}(\mathrm{Mod}\, B)$ is a perfect complex, and add T generates $\mathrm{D^b}(\mathrm{Mod}\, B)_{\mathrm{perf}}$;

(ii) the canonical morphism $A \longrightarrow \mathrm{R\,Hom}_B(T, T)$ in $\mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}})$ is an isomorphism.

Then we call T a *tilting complex*.

In [**14**] the term ‘two-sided tilting complex’ was used.

Example 1.5. In the notation of Theorem 1.2, if P is k-central then $P \in \mathrm{Mod}(B \otimes A^{\mathrm{o}})$ is a tilting complex.

The next theorem is an immediate consequence of [**13**, Theorem 6.4] and [**14**, Theorem 3.3, Proposition 4.1]. For the convenience of the reader we have included our own proof.

Theorem 1.6 (Rickard). *The following conditions are equivalent for a complex $T \in \mathrm{D^b}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$:*

(i) *The functor*

$$T \otimes^{\mathrm{L}}_A - : \mathrm{D}^-(\mathrm{Mod}\, A) \longrightarrow \mathrm{D}^-(\mathrm{Mod}\, B)$$

is an equivalence of triangulated categories.

(i′) *The functor $T \otimes^{\mathrm{L}}_A -$ preserves bounded complexes, and induces an equivalence of triangulated categories*

$$T \otimes^{\mathrm{L}}_A - : \mathrm{D^b}(\mathrm{Mod}\, A) \longrightarrow \mathrm{D^b}(\mathrm{Mod}\, B).$$

(ii) *T is a tilting complex.*

(iii) *There exist a complex $T^{\vee} \in \mathrm{D^b}(\mathrm{Mod}\,(A \otimes B^{o}))$ and isomorphisms*

$$T^{\vee} \otimes^{\mathrm{L}}_B T \cong A \in \mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}}),$$
$$T \otimes^{\mathrm{L}}_A T^{\vee} \cong B \in \mathrm{D}(\mathrm{Mod}\, B^{\mathrm{e}}).$$

Proof. (i) $\Rightarrow$ (i′): First note that the identity functor $\mathrm{K}^-(\mathrm{Proj}\, B) \longrightarrow \mathrm{D}^-(\mathrm{Mod}\, B)$ is an equivalence, where $\mathrm{Proj}\, B$ is the additive category of projective B-modules. Now use [**13**, Proposition 6.1].

(i′) $\Rightarrow$ (ii): Let $F := T \otimes^{\mathrm{L}}_A -$. By [**13**, Propositions 6.2 and 6.3] we see that F restricts to an equivalence $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A)_{\mathrm{perf}} \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)_{\mathrm{perf}}$. Since $\mathrm{add}\, A$ generates $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A)_{\mathrm{perf}}$ it follows that $T = FA$ generates $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)_{\mathrm{perf}}$. Also F induces isomorphisms

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\, A)}(A, A[i]) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\, B)}(T, T[i]),$$

so condition (ii) of Definition 1.4 holds.

(ii) $\Rightarrow$ (i): Let

$$T^\vee := \mathrm{R\,Hom}_B(T, B) \in \mathrm{D}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}})).$$

Since $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)_{\mathrm{perf}}$ it follows that $T^\vee$ is bounded. Let

$$F^\vee := T^\vee \otimes^{\mathrm{L}}_B - : \mathrm{D}^-(\mathrm{Mod}\, B) \longrightarrow \mathrm{D}^-(\mathrm{Mod}\, A),$$

so $F^\vee \cong \mathrm{R\,Hom}_B(T, -)$. Now for any $L \in \mathrm{D}^-(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$, $M \in \mathrm{D}^-(\mathrm{Mod}\, A)$ and $N \in \mathrm{D}(\mathrm{Mod}\, B)$ one has an isomorphism

$$\mathrm{R\,Hom}_B(L \otimes^{\mathrm{L}}_A M, N) \cong \mathrm{R\,Hom}_A(M, \mathrm{R\,Hom}_B(L, N)),$$

as can be seen by taking L to be a complex of $(B \otimes A^{\mathrm{o}})$-projectives and M to be a complex of A-projectives. Therefore $F^\vee$ is a right adjoint to F, and condition (ii) of Definition 1.4 says that $1_{\mathrm{D}^-(\mathrm{Mod}\, A)} \cong F^\vee F$. Given any $M \in \mathrm{D}^-(\mathrm{Mod}\, B)$, let N be the cone on $FF^\vee M \longrightarrow M$. Because $F^\vee FF^\vee M \cong F^\vee M$ we find that $\mathrm{R\,Hom}_{\mathrm{B}}(T, N) \cong F^\vee N = 0$. Now $\mathrm{add}\, T$ generates $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)_{\mathrm{perf}}$ and $B \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)_{\mathrm{perf}}$. This implies that $N = 0$ and hence $FF^\vee \cong 1_{\mathrm{D}^-(\mathrm{Mod}\, B)}$ (cf. [**13**, Proposition 5.4]).

(iii) $\Rightarrow$ (i): The associativity of $- \otimes^{\mathrm{L}} -$ implies that $FF^\vee \cong 1_{\mathrm{D}^-(\mathrm{Mod}\, B)}$ and $F^\vee F \cong 1_{\mathrm{D}^-(\mathrm{Mod}\, A)}$.

(ii) $\Rightarrow$ (iii): Since T is a tilting complex we have

$$A \cong \mathrm{R\,Hom}_B(T, T) \cong T^\vee \otimes^{\mathrm{L}}_B T \in \mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}}).$$

By the proof of (ii) $\Rightarrow$ (i) the functor $F^\vee = T^\vee \otimes^{\mathrm{L}}_B -$ is an equivalence; hence by (i) $\Rightarrow$ (ii), $T^\vee$ is a tilting complex. Writing $T^{\vee\vee} := \mathrm{R\,Hom}_A(T^\vee, A)$, the previous arguments show that $T^{\vee\vee} \otimes^{\mathrm{L}}_A T^\vee \cong B \in \mathrm{D}(\mathrm{Mod}\, B^{\mathrm{e}})$. However $T^{\vee\vee} \cong T^{\vee\vee} \otimes^{\mathrm{L}}_A T^\vee \otimes^{\mathrm{L}}_B T \cong T$, and this completes the circle of the proof. □

Corollary 1.7. *Let A, B, C be k-algebras and let $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ and $S \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(C \otimes B^{\mathrm{o}}))$ be tilting complexes.*

(1) $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(A^{\mathrm{o}} \otimes B))$ *is a tilting complex, that is, the roles of the algebras A and B in Definition 1.4 can be exchanged.*

(2) $T^\vee \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}}))$ *from Theorem* 1.6(iii) *is a tilting complex, and it is unique up to isomorphism.*

(3) $S \otimes^{\mathrm{L}}_B T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(C \otimes A^{\mathrm{o}}))$ *is a tilting complex.*

(4) *There are equivalences of triangulated categories*

$$T \otimes^{\mathrm{L}}_{A} - : \mathrm{D}^{\star}(\mathrm{Mod}\,(A \otimes C^{\circ})) \longrightarrow \mathrm{D}^{\star}(\mathrm{Mod}\,(B \otimes C^{\circ})),$$
$$- \otimes^{\mathrm{L}}_{B} T : \mathrm{D}^{\star}(\mathrm{Mod}\,(C \otimes B^{\circ})) \longrightarrow \mathrm{D}^{\star}(\mathrm{Mod}\,(C \otimes A^{\circ}))$$

with $\star = \mathrm{b}, +, -, \varnothing$.

Proof. (1), (2), (3). These are immediate consequences of Theorem 1.6(iii).

(4) Since T is perfect over A° and over B the functors are defined on the unbounded categories D, and preserve $\mathrm{D}^{\star}$. By way-out reasons (cf. [**6**, Proposition I.7.1(iv)]) they are equivalences. □

We call the complex $T^{\vee}$ above the *inverse of* T.

The next theorem was shown to the author by V. Hinich.

THEOREM 1.8. *Let* B *be a* k*-algebra, let* $T' \in \mathrm{D}^{-}(\mathrm{Mod}\,B)$ *be a complex and let* $A := \mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,B)}(T')$. *Assume that* $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\,B)}(T', T'[i]) = 0$ *for* $i < 0$. *Then there is a complex* $T \in \mathrm{D}^{-}(\mathrm{Mod}\,(B \otimes A))$ *such that* $T \cong T'$ *in* $\mathrm{D}(\mathrm{Mod}\,B)$, *and the ring homomorphism* $A \longrightarrow \mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,B)}(T)$ *induced by the* A*-module structure of* T *is bijective.*

Proof. We shall use ideas from homotopical algebra. Suppose that C is a differential graded algebra over k, and denote by $\mathrm{DGMod}\,C$ the category of differential graded C-modules. According to [**7**, Section 3], $\mathrm{DGMod}\,C$ is a closed model category in the sense of Quillen [**12**]. The weak equivalences in $\mathrm{DGMod}\,C$ are the quasi-isomorphisms. Let $\mathrm{D}(\mathrm{DGMod}\,C) = \mathrm{Ho}(\mathrm{DGMod}\,C)$ be the homotopy category, obtained by inverting the weak equivalences. It is a triangulated category. If C is just a k-algebra (that is, $C^{i} = 0$ for $i \neq 0$) then $\mathrm{DGMod}\,C = \mathrm{C}(\mathrm{Mod}\,C)$ and $\mathrm{D}(\mathrm{DGMod}\,C) = \mathrm{D}(\mathrm{Mod}\,C)$.

According to [**7**, Theorem 3.3.1] (or [**9**, Theorem 8.2]), if $C' \longrightarrow C$ is a quasi-isomorphism of differential graded algebras, then the functor $\mathrm{D}(\mathrm{DGMod}\,C) \longrightarrow \mathrm{D}(\mathrm{DGMod}\,C')$ obtained by restriction of scalars is an equivalence.

Given our complex T', we may assume that it consists of projective B-modules. Define $A'' := \mathrm{End}_{B}(T')$, which is a differential graded algebra, and $A = \mathrm{H}^{0}A''$. Let A' be the truncation $\sigma_{\leqslant 0}A''$, that is,

$$A' := (\ldots \longrightarrow A''^{-1} \longrightarrow \mathrm{Ker}\,(A''^{0} \longrightarrow A''^{1}) \longrightarrow 0 \longrightarrow \ldots).$$

Since $A' \longrightarrow A''$ is a differential graded algebra homomorphism, we have $T' \in \mathrm{DGMod}(B \otimes A')$. On the other hand $A' \longrightarrow A$ is a quasi-isomorphism, and hence so is $B \otimes A' \longrightarrow B \otimes A$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{D}(\mathrm{Mod}\,(B \otimes A)) & \xrightarrow{G} & \mathrm{D}(\mathrm{DGMod}\,(B \otimes A)) \\ \downarrow & & \downarrow \\ \mathrm{D}(\mathrm{Mod}\,B) & \xrightarrow{=} & \mathrm{D}(\mathrm{Mod}\,B) \end{array}$$

where all the arrows are restriction of scalars. Since G is an equivalence, we can find a complex $T \in \mathrm{D}(\mathrm{Mod}\,(B \otimes A))$ such that $GT \cong T'$ in $\mathrm{D}(\mathrm{DGMod}\,(B \otimes A'))$. We may assume (by truncation) that $T \in \mathrm{D}^{-}(\mathrm{Mod}\,(B \otimes A))$, and then it has the desired properties. □

The following corollary is [**14**, Corollary 3.5]. Our proof is almost identical to Keller's in [**8**].

COROLLARY 1.9 (Rickard). *Let A and B be k-algebras, and let F: $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,A) \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,B)$ be an equivalence of triangulated categories. Then there exists a tilting complex $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ with $T \cong FA$ in* $\mathrm{D}(\mathrm{Mod}\,B)$.

Proof. According to [**13**, Propositions 6.1–6.3], F restricts to an equivalence $F: \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,A)_{\mathrm{perf}} \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,B)_{\mathrm{perf}}$. Then $T' := FA \in \mathrm{D}(\mathrm{Mod}\,B)$ is a perfect complex, add T' generates $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,B)_{\mathrm{perf}}$, $\mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,B)}(T') \cong A^{\mathrm{o}}$ and $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\,B)}(T', T'[i]) = 0$ for $i \neq 0$. Now use Theorem 1.8. □

REMARK 1.10. We did not assume that F is k-linear in the corollary. However even when F is k-linear, it is not known whether the two functors F and $T \otimes^{\mathrm{L}}_{A} -$ are necessarily isomorphic. Rickard calls an equivalence of the form $T \otimes^{\mathrm{L}}_{A} -$ *standard* (see [**13**, Section 7] and [**14**, Definition 3.4]).

To finish off this section, consider a noetherian algebra A. Then $\mathrm{Mod}_{\mathrm{f}}\,A$, the category of finitely generated modules, is abelian, and the category $\mathrm{D}_{\mathrm{f}}(\mathrm{Mod}\,A)$ of complexes with finitely generated cohomologies is triangulated.

PROPOSITION 1.11. *If A and B are both noetherian and $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ is a tilting complex then*

$$T \otimes^{\mathrm{L}}_{A} - : \mathrm{D}^{\star}_{\mathrm{f}}(\mathrm{Mod}\,A) \longrightarrow \mathrm{D}^{\star}_{\mathrm{f}}(\mathrm{Mod}\,B)$$

is an equivalence of triangulated categories for $\star = \mathrm{b}, +, -, \varnothing$. □

Proof. Since $T \otimes^{\mathrm{L}}_{A} -$ is a way-out functor in both directions and $T \otimes^{\mathrm{L}}_{A} A = T \in \mathrm{D}^{\star}_{\mathrm{f}}(\mathrm{Mod}\,B)$ the proposition follows from [**6**, Proposition I.7.3.]. □

REMARK 1.12. Throughout the paper the base ring k is a field. It is easy to see that everything in Sections 1–3 will remain valid if we let k be an arbitrary commutative ring, as long as the k-algebras A, B, C are assumed to be *projective* k-modules. With a mild modification of the proofs we can even assume these algebras are only *flat* k-modules.

For the general situation here is an approach suggested by V. Hinich and B. Keller. Consider a differential graded k-algebra $B \otimes^{\mathrm{L}}_{k} A^{\mathrm{o}} = \tilde{B} \otimes_{k} \tilde{A}^{\mathrm{o}}$, where $\tilde{A}, \tilde{B}$ are K-flat differential graded k-algebras (for example, negatively graded and flat as k-modules), and $\tilde{A} \longrightarrow A$, $\tilde{B} \longrightarrow B$ are quasi-isomorphisms. The 'derived category of bimodules' should be $\mathrm{D}(\mathrm{DGMod}\,(B \otimes^{\mathrm{L}}_{k} A^{\mathrm{o}}))$. It seems likely that all results in Sections 1–3 would still hold if we take a tilting complex to be an object of $\mathrm{D}(\mathrm{DGMod}\,(B \otimes^{\mathrm{L}}_{k} A^{\mathrm{o}}))$, satisfying the appropriate conditions. However we did not check this.

2. *Some calculations of tilting complexes*

In this section we show that derived Morita equivalence reduces to ordinary Morita equivalence when one of the algebras is local or commutative.

LEMMA 2.1. *Let $M \in \mathrm{D}^-(\mathrm{Mod}(B \otimes A^{\circ}))$ and $N \in \mathrm{D}^-(\mathrm{Mod}(A \otimes C^{\circ}))$ for k-algebras A, B, C. Then there is a convergent Künneth spectral sequence*

$$\mathrm{E}_2^{p,q} = \bigoplus_{i+j=q} \mathrm{H}^p(\mathrm{H}^i M \otimes^{\mathrm{L}}_A \mathrm{H}^j N) \Rightarrow \mathrm{H}(M \otimes^{\mathrm{L}}_A N)$$

in $\mathrm{Mod}(B \otimes C^{\circ})$. *The filtration of each* $\mathrm{H}^n(M \otimes^{\mathrm{L}}_A N)$ *is bounded. If* $i_0 \geqslant \sup\{i \mid \mathrm{H}^i M \neq 0\}$ *and* $j_0 \geqslant \sup\{j \mid \mathrm{H}^j N \neq 0\}$ *then*

$$\mathrm{H}^{i_0} M \otimes_A \mathrm{H}^{j_0} N \cong \mathrm{H}^{i_0+j_0}(M \otimes^{\mathrm{L}}_A N).$$

Proof. We can assume that M is a complex of projective $(B \otimes A^{\circ})$-modules with $M^i = 0$ for $i > i_0$, and similarly for N. Then the usual double complex calculation applies (see [**11**, Theorem XII.12.2]). In particular $\mathrm{E}_2^{p,q} = 0$ unless $p \leqslant 0$ and $q \leqslant i_0 + j_0$. □

Here is a criterion for telling when we are in the classical Morita context.

PROPOSITION 2.2. *The following conditions are equivalent for a tilting complex $T \in \mathrm{D}(\mathrm{Mod}(B \otimes A^{\circ}))$:*

(i) *$T \cong P$, where $P \in \mathrm{Mod}(B \otimes A^{\circ})$ is invertible* (*as in Theorem* 1.2).
(ii) *$\mathrm{H}^0 T$ is a projective B-module and $\mathrm{H}^p T = 0$ for $p \neq 0$.*
(iii) *$\mathrm{H}^p T = \mathrm{H}^p T^{\vee} = 0$ for $p \neq 0$, where $T^{\vee}$ is the inverse of T.*

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are trivial. As for (iii) ⇒ (i), the shape of the Künneth spectral sequence shows that

$$\begin{aligned} \mathrm{H}^0 T \otimes_A \mathrm{H}^0 T^{\vee} &\cong \mathrm{H}^0(T \otimes^{\mathrm{L}}_A T^{\vee}) \cong B \\ \mathrm{H}^0 T^{\vee} \otimes_B \mathrm{H}^0 T &\cong \mathrm{H}^0(T^{\vee} \otimes^{\mathrm{L}}_B T) \cong A. \end{aligned}$$

□

We call a ring A *local* if $A/\mathfrak{r}$ is a simple artinian ring, where $\mathfrak{r}$ is the Jacobson radical of A. (Note that the common definition of *local ring* is that $A/\mathfrak{r}$ is a division ring.)

THEOREM 2.3. *Let A and B be k-algebras, with A local, and let $T \in \mathrm{D}(\mathrm{Mod}(B \otimes A^{\circ}))$ be a tilting complex. Then $T \cong P[n]$ for some invertible bimodule P and integer n.*

Proof. Let $n := -\max\{p \mid \mathrm{H}^p T \neq 0\}$ and $m := -\max\{p \mid \mathrm{H}^p T^{\vee} \neq 0\}$. Then by Lemma 1.1, $\mathrm{H}^{-n} T$ and $\mathrm{H}^{-m} T^{\vee}$ are finitely generated nonzero modules over A° and A respectively. Since Nakayama's lemma holds for finitely generated A-modules, we have $\mathrm{H}^{-n} T \otimes_A \mathrm{H}^{-m} T^{\vee} \neq 0$. On the other hand, by Lemma 2.1 we get

$$\mathrm{H}^{-n} T \otimes_A \mathrm{H}^{-m} T^{\vee} \cong \mathrm{H}^{-(n+m)}(T \otimes^{\mathrm{L}}_A T^{\vee}).$$

Since $T \otimes^{\mathrm{L}}_A T^{\vee} \cong B$, we conclude that $m + n = 0$ and

$$\mathrm{H}^{-n} T \otimes_A \mathrm{H}^{-m} T^{\vee} \cong \mathrm{H}^0(T \otimes^{\mathrm{L}}_A T^{\vee}) \cong B.$$

Applying Lemma 2.1 again we see that

$$\mathrm{H}^{-m} T^{\vee} \otimes_B \mathrm{H}^{-n} T \cong \mathrm{H}^0(T^{\vee} \otimes^{\mathrm{L}}_B T) \cong A.$$

Therefore by (ordinary) Morita equivalence it follows that $\mathrm{H}^{-n}T$ and $\mathrm{H}^{-m}T^{\vee}$ are invertible bimodules. Just as in the proof of [**16**, Lemma 3.11] we find that $\mathrm{H}^{i}T = 0$ for $i \neq -n$. Taking $P := \mathrm{H}^{-n}T$ we have $T \cong P[n]$. $\square$

Given a complex $T \in \mathrm{D}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ there are two ring homomorphisms

$$\mathrm{Z}(B) \xrightarrow{\lambda_T} \mathrm{End}_{\mathrm{D}(\mathrm{Mod}(B\otimes A^{\mathrm{o}}))}(T) \xleftarrow{\rho_T} \mathrm{Z}(A) \tag{1}$$

from the centres of B and A, namely left and right multiplication.

Proposition 2.4. *Suppose that $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ is a tilting complex. Then the homomorphisms λ_T and ρ_T of* (1) *are both bijective.*

Proof. Applying the functor $- \otimes^{\mathrm{L}}_{A} T^{\vee}$ we get

$$\mathrm{End}_{\mathrm{D}(\mathrm{Mod}(B\otimes A^{\mathrm{o}}))}(T) \cong \mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,B^{\mathrm{e}})}(B) \cong \mathrm{End}_{\mathrm{Mod}\,B^{\mathrm{e}}}(B) = \mathrm{Z}(B).$$

Since the first isomorphism sends λ_T to λ_B we conclude that λ_T is bijective. Use the functor $T^{\vee} \otimes^{\mathrm{L}}_{B} -$ for ρ_T. $\square$

We see that $\mathrm{Z}(A) \cong \mathrm{Z}(B)$ as k-algebras (cf. also [**13**, Proposition 9.2]).

Lemma 2.5. *Suppose that A and B are k-algebras and $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$ is a tilting complex. Let $C := \mathrm{Z}(A) \cong \mathrm{Z}(B)$ as in Proposition* 2.4, *and suppose that $\tilde{C} = CS^{-1}$ is a localization of C with respect to some multiplicative set $S \subset C$. Define $\tilde{A} := \tilde{C} \otimes_C A$ and $\tilde{B} := \tilde{C} \otimes_C B$. Then*

$$\tilde{T} := \tilde{B} \otimes_B T \otimes_A \tilde{A} \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(\tilde{B} \otimes \tilde{A}^{\mathrm{o}}))$$

is a tilting complex, with inverse $\tilde{T}^{\vee} := \tilde{A} \otimes_A T^{\vee} \otimes_B \tilde{B}$.

Proof. By Proposition 2.4 the cohomology bimodules $\mathrm{H}^{p}T$ are all C-central (even though T itself need not be C-central!). From the flatness of $A \longrightarrow \tilde{A}$ and $B \longrightarrow \tilde{B}$, and using the fact that $\tilde{C} \otimes_C \tilde{C} \cong \tilde{C}$, we conclude that

$$\mathrm{H}^{p}\tilde{T} \cong \tilde{B} \otimes_B \mathrm{H}^{p}T \otimes_A \tilde{A} \cong \tilde{C} \otimes_C \mathrm{H}^{p}T.$$

Hence $\tilde{B} \otimes_B T \longrightarrow \tilde{T}$ and $T \otimes_A \tilde{A} \longrightarrow \tilde{T}$ are isomorphisms in $\mathrm{D}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$.

The functor $\mathrm{R\,Hom}_B(-, B)$ gives rise to an isomorphism

$$\mathrm{End}_{\mathrm{D}(\mathrm{Mod}(B\otimes A^{\mathrm{o}}))}(T)^{\mathrm{o}} \cong \mathrm{End}_{\mathrm{D}(\mathrm{Mod}(A\otimes B^{\mathrm{o}}))}(T^{\vee})$$

which exchanges ρ and λ. Therefore the $\mathrm{H}^{p}T^{\vee}$ are also C-central, and as above $\tilde{A} \otimes_A T^{\vee} \cong \tilde{T}^{\vee} \cong T^{\vee} \otimes_B \tilde{A}$. We see that

$$\begin{aligned} \tilde{T}^{\vee} \otimes^{\mathrm{L}}_{\tilde{B}} \tilde{T} &\cong (\tilde{A} \otimes_A T^{\vee} \otimes_B \tilde{B}) \otimes^{\mathrm{L}}_{\tilde{B}} (\tilde{B} \otimes_B T \otimes_A \tilde{A}) \\ &\cong \tilde{A} \otimes_A (T^{\vee} \otimes^{\mathrm{L}}_{B} T) \otimes_A \tilde{A} \cong \tilde{A} \end{aligned}$$

and likewise $\tilde{T} \otimes^{\mathrm{L}}_{\tilde{A}} \tilde{T}^{\vee} \cong \tilde{B}$. $\square$

In Morita equivalence (that is, Theorem 1.2), if the ring A is commutative then the isomorphism $A \cong \mathrm{Z}(B)$ makes the invertible bimodule P A-central. Since $B \cong \mathrm{End}_A(P)$ it is an Azumaya algebra over A. The next theorem says that in the commutative case, derived Morita equivalence gives nothing new.

THEOREM 2.6. *Let A and B be k-algebras, with A commutative. If A and B are derived Morita equivalent, then they are Morita equivalent.*

Proof. By Corollary 1.9 there exists a tilting complex $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}}))$. Let $T^{\vee}$ be its inverse. Write $A = A_1 \times \ldots \times A_m$ with $\operatorname{Spec} A_i$ connected. Let $B_i := A_i \otimes_A B$. Then by Lemma 2.5, $B_i \otimes_B T \otimes_A A_i$ is a tilting complex in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B_i \otimes A_i^{\mathrm{o}}))$. Thus we may assume that $\operatorname{Spec} A$ is connected.

Pick a prime ideal $\mathfrak{p} \in \operatorname{Spec} A$, let $A_{\mathfrak{p}}$ be the local ring, $B_{\mathfrak{p}} := A_{\mathfrak{p}} \otimes_A B$ and $T_{\mathfrak{p}} := B_{\mathfrak{p}} \otimes_B T \otimes_A A_{\mathfrak{p}}$. By Lemma 2.5 the complex $T_{\mathfrak{p}}$ is a tilting complex in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,(B_{\mathfrak{p}} \otimes A_{\mathfrak{p}}^{\mathrm{o}}))$, with inverse $T_{\mathfrak{p}}^{\vee} := B_{\mathfrak{p}} \otimes_B T^{\vee} \otimes_A A_{\mathfrak{p}}$. Define integers $n(\mathfrak{p})$ and $m(\mathfrak{p})$ by $n(\mathfrak{p}) := -\max\{i \,|\, \mathrm{H}^i T_{\mathfrak{p}} \neq 0\}$ and $m(\mathfrak{p}) := -\max\{i \,|\, \mathrm{H}^i T_{\mathfrak{p}}^{\vee} \neq 0\}$. As in the proof of Theorem 2.3, $\mathrm{H}^{-n(\mathfrak{p})} T_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A \mathrm{H}^{-n(\mathfrak{p})} T$ is an invertible $B_{\mathfrak{p}}$-$A_{\mathfrak{p}}$-bimodule, $\mathrm{H}^i T_{\mathfrak{p}} \cong A_{\mathfrak{p}} \otimes_A \mathrm{H}^i T = 0$ for $i \neq -n(\mathfrak{p})$, and $m(\mathfrak{p}) + n(\mathfrak{p}) = 0$.

Next consider prime ideals $\mathfrak{p} \subset \mathfrak{q}$. The previous paragraph implies that $A_{\mathfrak{p}} \otimes_A \mathrm{H}^i T = 0$ for $i \neq -n(\mathfrak{q})$, and hence $n(\mathfrak{q}) = n(\mathfrak{p})$. Because $\operatorname{Spec} A$ is connected we conclude that $n(\mathfrak{p}) = n$ is constant, and so $\mathrm{H}^i T = 0$ for $i \neq -n$. Likewise $m(\mathfrak{p}) = m = -n$ and $\mathrm{H}^i T^{\vee} = 0$ for $i \neq -m$. By Proposition 2.2 we see that the A-central bimodule $P := H^{-n} T$ is invertible. □

Here is a corollary to Theorem 2.3.

COROLLARY 2.7. *Let A and B be k-algebras, and $A \cong A_1 \times \ldots \times A_m$ with A_i local. If A and B are derived Morita equivalent, then they are Morita equivalent.*

Proof. Let $C := \mathrm{Z}(A)$, so $C = C_1 \times \ldots \times C_m$. By Lemma 2.5 every $B_i := C_i \otimes_C B$ is derived Morita equivalent to A_i. Now use Theorem 2.3. □

REMARK 2.8. R. Rouquier and A. Zimmermann have independently obtained similar results to our Theorems 2.3. and 2.6, but only in a special case: when A and B are orders over a Dedekind domain k. They also considered the derived Picard group, which they denoted by $\operatorname{TrPic}(A)$. See **[17]**.

3. *The derived Picard group*

Let us concentrate now on the case $A = B$. Recall that the k-central noncommutative *Picard group* of A is

$$\operatorname{Pic}(A) = \operatorname{Pic}_k(A) := \frac{\{\text{invertible bimodules } L \in \operatorname{Mod} A^{\mathrm{e}}\}}{\text{isomorphism}}.$$

According to Corollary 1.7 the next definition makes sense.

DEFINITION 3.1. Define the *derived Picard group* of A (relative to k) to be

$$\operatorname{DPic}(A) = \operatorname{DPic}_k(A) := \frac{\{\text{tilting complexes } T \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A^{\mathrm{e}})\}}{\text{isomorphism}}$$

with identity element A, product $(T_1, T_2) \longmapsto T_1 \otimes_A^{\mathrm{L}} T_2$ and inverse $T \longmapsto T^{\vee}$.

The group $\operatorname{DPic}(A)$ contains a copy of $\mathbb{Z}$ in its centre, as $n \longmapsto A[n]$. $\operatorname{DPic}(A)$ also contains a subgroup isomorphic to $\operatorname{Pic}(A)$, which is characterized in Proposition 2.2. Note that both $\operatorname{Pic}(A)$ and $\operatorname{DPic}(A)$ depend on k.

REMARK 3.2. If A is commutative we denote by $\mathrm{Pic}_A(A)$ the usual commutative Picard group, namely the isomorphism classes of central projective modules of rank 1. It is a subgroup of $\mathrm{Pic}(A)$ (cf. Proposition 3.5).

Let us now state some facts about invertible bimodules (which are probably well known, but we found no references). Denote by $\mathrm{Aut}(A)$ the group of k-algebra automorphisms of A. For $\sigma \in \mathrm{Aut}(A)$ let A_σ be the bimodule which is free over A and A° with basis e, and $e \cdot a = \sigma(a) \cdot e$, $a \in A$.

LEMMA 3.3. (1) *$\sigma \mapsto A_\sigma$ is a group homomorphism* $\mathrm{Aut}(A) \longrightarrow \mathrm{Pic}(A)$ *with kernel the subgroup* $\mathrm{Inn}(A)$ *of inner automorphisms.*

(2) *Suppose that L is an invertible A-bimodule which is free of rank* 1 *as left module. Then $L \cong A_\sigma$ as bimodules for some $\sigma \in \mathrm{Aut}(A)$.*

(3) *If A is local then any invertible bimodule L is free of rank* 1 *over A.*

Proof. (1) A bimodule isomorphism $A_\tau \xrightarrow{\simeq} A_\sigma$ sends the basis e of A_τ to $u \cdot e \in A_\sigma$, where u is a unit of A, and conjugation by u is $\tau\sigma^{-1}$.

(2) Choose an A-basis e of L. Then $\phi \longmapsto \phi(e)$ is a bijection $\mathrm{End}_A(L) \xrightarrow{\simeq} L$. Since right multiplication induces an isomorphism $A^{\circ} \xrightarrow{\simeq} \mathrm{End}_A(L)$ this shows that e is also a basis of L as A°-module. Conjugation by e is σ.

(3) By Nakayama's lemma it is enough to prove that $W := K \otimes_A L$ is free of rank 1 over $K := A/\mathfrak{r}$. First one checks that $W \cong K \otimes_A L \otimes_A K \cong L \otimes_A K$ as K^{e}-modules. Hence W is an invertible bimodule over K. Since $K \cong \mathrm{M}_n(D)$ for a division algebra D, by Morita equivalence $W \cong \mathrm{M}_n(V)$ as K^{e}-modules, where V is an invertible bimodule over D. It remains to prove that the free D-module V has rank 1. If $V \cong D^l$ as left modules, then $D \cong V^{\vee} \otimes_D V \cong (V^{\vee})^l$, so $l = 1$. □

The next propositions analyse $\mathrm{DPic}(A)$ in the semilocal and in the commutative cases.

PROPOSITION 3.4. *Suppose that $A \cong A_1 \times \ldots \times A_m$ where every A_i is a local k-algebra. Then*

$$\mathrm{DPic}_k(A) \cong \mathbb{Z}^m \times \mathrm{Pic}_k(A)$$
$$\mathrm{Pic}_k(A) \cong \mathrm{Out}_k(A).$$

Proof. This is an immediate consequence of Corollary 2.7, Theorem 2.3 and Lemma 3.3. □

PROPOSITION 3.5. *Suppose that A is a commutative ring. Then*

$$\mathrm{DPic}_k(A) \cong \mathbb{Z}^m \times \mathrm{Pic}_k(A)$$
$$\mathrm{Pic}_k(A) \cong \mathrm{Aut}_k(A) \ltimes \mathrm{Pic}_A(A),$$

where m is the number of connected components of $\mathrm{Spec}\, A$.

Proof. Let $A \cong A_1 \times \ldots \times A_m$ be the decomposition of A according to the connected components of $\mathrm{Spec}\, A$. Given a tilting complex $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$, Theorem

2.6 says that $T \cong P \otimes_A S$, where P is an invertible bimodule and $S = A_1[n_1] \times \ldots \times A_m[n_m]$ with $n_i \in \mathbb{Z}$. Thus $\mathrm{DPic}_k(A) \cong \mathbb{Z}^m \times \mathrm{Pic}_k(A)$. Next let $\sigma \in \mathrm{Aut}(A)$ be the automorphism determined by P (cf. Proposition 2.4). Then $L := P \otimes_A A_{\sigma^{-1}}$ is a central invertible bimodule over A. $\square$

Assume that A is noetherian. Let $\mathrm{K}_0(A) = \mathrm{K}_0(\mathrm{Mod}_{\mathrm{f}} A)$ be the Grothendieck group of A. For any $M \in \mathrm{Mod}_{\mathrm{f}} A$ let $[M]$ be its class in $\mathrm{K}_0(A)$. Then $M \longmapsto [M] := \sum (-1)^p [\mathrm{H}^p M]$ is a well-defined function $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Mod}\, A) \longrightarrow \mathrm{K}_0(A)$. Since $\mathrm{DPic}(A)$ acts on $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Mod}\, A)$ by auto-equivalences it acts also on $\mathrm{K}_0(A)$. Let $s \in \mathrm{DPic}(A)$ be the class of $A[1]$, which acts on $\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Mod}\, A)$ by a shift in degree.

Proposition 3.6. *There is a canonical group homomorphism*

$$\chi_0 : \mathrm{DPic}(A) \longrightarrow \mathrm{Aut}_{\mathbb{Z}}(\mathrm{K}_0(A))$$

with $\chi_0(s) = -1$.

Actually there are two more objects one can associate to A which are related to the representation χ_0.

The first is the noncommutative Grothendieck ring $\mathrm{K}^0(A) = \mathrm{K}^0_k(A)$, which is a rather obvious generalization of the commutative $\mathrm{K}^0(A)$. Let X be the set of isomorphism classes of A^{e}-modules T which are finitely generated projective on both sides. Define F to be the free abelian group with basis X. As abelian group, $\mathrm{K}^0(A)$ is the quotient of F by the subgroup generated by the elements $[T_0] - [T_1] + [T_2]$, for every short exact sequence $0 \longrightarrow T_0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow 0$ in $\mathrm{Mod}\, A^{\mathrm{e}}$ with $T_i \in X$. Multiplication is $[T_1] \cdot [T_2] := [T_1 \otimes_A T_2]$, and 1 is $[A]$. The Grothendieck group $\mathrm{K}_0(A)$ is a left module over $\mathrm{K}^0(A)$, by $[T] \cdot [M] := [T \otimes_A M]$ for $M \in \mathrm{Mod}\, A$, and there is a group homomorphism $\mathrm{Pic}(A) \longrightarrow \mathrm{K}^0(A)^\times$.

All the above works for complexes too. Take X to be the set of isomorphism classes of complexes $T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$ which are perfect on both sides. Define F as before, and let $\mathrm{DK}^0(A) = \mathrm{DK}^0_k(A)$ be the quotient of F by the subgroup generated by the elements $[T_0] - [T_1] + [T_2]$, for every triangle $T_0 \longrightarrow T_1 \longrightarrow T_2 \xrightarrow{+1}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$ with $T_i \in X$. Multiplication is $[T_1] \cdot [T_2] := [T_1 \otimes^{\mathrm{L}}_A T_2]$. The Grothendieck group $\mathrm{K}_0(A)$ is a left module over $\mathrm{DK}^0(A)$, by $[T] \cdot [M] := [T \otimes^{\mathrm{L}}_A M]$ for $M \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\mathrm{Mod}\, A)$. There is a ring homomorphism $\mathrm{K}^0(A) \longrightarrow \mathrm{DK}^0(A)$, and a group homomorphism $\mathrm{DPic}(A) \longrightarrow \mathrm{DK}^0(A)^\times$. To summarize, we have the following.

Proposition 3.7. $\mathrm{DK}^0(A)$ *is a ring and* $\mathrm{K}_0(A)$ *is a left* $\mathrm{DK}^0(A)$*-module. There is a group homomorphism* $\chi^0 : \mathrm{DPic}(A) \longrightarrow \mathrm{DK}^0(A)^\times$ *with* $\chi^0(s) = -1$*, and* χ_0 *factors through* χ^0.

Remark 3.8. We did not analyse the dependence of various objects, such as the group $\mathrm{DPic}(A)$, on the base field k.

4. *Classification of dualizing complexes*

In this section we assume that A is a (left and right) noetherian k-algebra. Dualizing complexes over commutative rings were introduced in [**6**]. The noncommutative version below first appeared in [**16**] (where connected graded algebras were considered).

DEFINITION 4.1. A complex $R \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$ is called *dualizing* if

(i) R has finite injective dimension over A and over A^{o};

(ii) R has finitely generated cohomology modules over A and over A^{o};

(iii) then canonical morphisms $A \longrightarrow \mathrm{R}\,\mathrm{Hom}_A(R, R)$ and $A \longrightarrow \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(R, R)$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$ are isomorphisms.

Condition (i) is equivalent to having an isomorphism $R \cong I \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$, where I is a bounded complex and each I^q is injective over A and over A^{o}. Note that this definition is left–right symmetric (that is, remains equivalent after exchanging A and A^{o}).

Given a dualizing complex R the associated duality functors are

$$D := \mathrm{R}\,\mathrm{Hom}_A(-, R) : \mathrm{D}(\mathrm{Mod}\, A)^{\mathrm{o}} \longrightarrow \mathrm{D}(\mathrm{Mod}\, A^{\mathrm{o}})$$
$$D^{\mathrm{o}} := \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(-, R) : \mathrm{D}(\mathrm{Mod}\, A^{\mathrm{o}})^{\mathrm{o}} \longrightarrow \mathrm{D}(\mathrm{Mod}\, A).$$

For a k-algebra B let $\mathrm{D}_{(\mathrm{f},\,)}(\mathrm{Mod}\,(A \otimes B))$ denote the full subcategory of $\mathrm{D}(\mathrm{Mod}\,(A \otimes B))$ whose objects are the complexes M such that for all q, $\mathrm{H}^q M$ is a finitely generated A-module. Likewise define $\mathrm{D}_{(\,,\mathrm{f})}(\mathrm{Mod}\,(A \otimes B))$ and $\mathrm{D}_{(\mathrm{f},\mathrm{f})}(\mathrm{Mod}\,(A \otimes B))$. Thus Definition 4.1(ii) says that $R \in \mathrm{D}_{(\mathrm{f},\mathrm{f})}(\mathrm{Mod}\, A^{\mathrm{e}})$.

PROPOSITION 4.2. *Let* $\mathrm{R} \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})$ *be a dualizing complex, and let B be any k-algebra.*

(1) *For any* $M \in \mathrm{D}_{(\mathrm{f},\,)}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}}))$ *one has* $DM \in \mathrm{D}_{(\,,\mathrm{f})}(\mathrm{Mod}\,(B \otimes A^{o}))$*, and there is a functorial isomorphism* $M \cong D^{\mathrm{o}}DM$*. Therefore D and* D^{o} *determine an equivalence*

$$\mathrm{D}_{(\mathrm{f},\,)}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}}))^{\mathrm{o}} \longleftrightarrow \mathrm{D}_{(\,,\mathrm{f})}(\mathrm{Mod}\,(B \otimes A^{\mathrm{o}})).$$

(2) *Let* $M \in \mathrm{D}^{-}_{(\mathrm{f},\,)}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}}))$ *and* $N \in \mathrm{D}_{(\mathrm{f},\,)}(\mathrm{Mod}\,(A \otimes B^{\mathrm{o}}))$*. Then there is a bifunctorial isomorphism*

$$\mathrm{R}\,\mathrm{Hom}_A(M, N) \cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(DN, DM)$$

in $\mathrm{D}(\mathrm{Mod}\, B)$.

Proof. (1) This is slightly stronger than [**16**, Lemma 3.5]. By adjunction we get a functorial morphism $M \longrightarrow D^{\mathrm{o}}DM$ in $\mathrm{D}(\mathrm{Mod}\,(A \otimes B))$. Now we can forget B. Since the functors D and $D^{\mathrm{o}}D$ are way-out in both directions, $DA = R \in \mathrm{D}_{\mathrm{f}}(\mathrm{Mod}\, A^{\mathrm{o}})$ and $D^{\mathrm{o}}DA \cong A$, the claim follows from [**6**, Propositions I.7.1 and I.7.3] and their opposite forms.

(2) We can assume that M is a bounded-above complex of projective $(A \otimes B^{\mathrm{o}})$-modules and R is a bounded-below complex of injective A^{e}-modules. Since $\mathrm{Hom}_A(M, R)$ is a bounded-below complex of injective A^{o}-modules, we get a morphism

$$\begin{aligned}\mathrm{R}\,\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(M, N) \longrightarrow \\ \mathrm{Hom}_{A^{\mathrm{o}}}(\mathrm{Hom}_A(N, R), \mathrm{Hom}_A(M, R)) = \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(DN, DM)\end{aligned}$$

in $\mathrm{D}(\mathrm{Mod}\, B)$, which is functorial in M, N. In order to prove it is an isomorphism we can forget B. Applying H^q we get a homomorphism $\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\, A)}(M, N[q]) \longrightarrow \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\, A^{\mathrm{o}})}(D(N[q]), DM)$, which by part (1) is bijective. $\square$

EXAMPLE 4.3. If A is a Gorenstein algebra, that is, the bimodule A has finite injective dimension over A and over A^{o}, then $R = A$ is a dualizing complex.

REMARK 4.4. One can weaken the noetherian assumption. If A is a coherent ring then the category of coherent (that is, finitely presented) A-modules is abelian, so we can work with $\mathrm{D}^{\mathrm{b}}_{\mathrm{c}}(\operatorname{Mod} A)$ and so on. Perhaps a reasonable theory can be developed for any algebra A if one works with Illusie's pseudo-coherent complexes (cf. [**4**, Exposée I]).

Let $\mathrm{D}^{-}(\operatorname{Mod} A)_{\mathrm{fpd}}$ (respectively $\mathrm{D}^{-}(\operatorname{Mod} A)_{\mathrm{fTd}}$, $\mathrm{D}^{+}(\operatorname{Mod} A)_{\mathrm{fid}}$) be the category of complexes with finite projective (respectively Tor, injective) dimension. Since A is noetherian, we have

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\operatorname{Mod} A)_{\mathrm{fTd}} = \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\operatorname{Mod} A)_{\mathrm{fpd}} = \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A)_{\mathrm{perf}} \subset \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A).$$

THEOREM 4.5. (1) *Suppose that R_1 is a dualizing complex and T is a tilting complex. Then $R_2 := R_1 \otimes^{\mathrm{L}}_A T$ is dualizing, and $T \cong \mathrm{R}\operatorname{Hom}_A(R_1, R_2)$.*

(2) *Conversely, suppose that R_1 and R_2 are dualizing complexes. Then $T := \mathrm{R}\operatorname{Hom}_A(R_1, R_2)$ is a tilting complex T, and $R_2 \cong R_1 \otimes^{\mathrm{L}}_A T$.*

(3) *Let R be a dualizing complex. Then the associated duality functors D and D^{o} induce an equivalence*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\operatorname{Mod} A)^{\mathrm{o}}_{\mathrm{fpd}} \longleftrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(\operatorname{Mod} A^{\mathrm{o}})_{\mathrm{fid}}.$$

Proof. (1) Clearly R_2 is bounded. Next let us prove that each $\mathrm{H}^n R_2$ is a finitely generated module over A. Consider the Künneth spectral sequence

$$E_2^{pq} = \bigoplus_{i+j=q} \mathrm{H}^p(\mathrm{H}^i R_1 \otimes^{\mathrm{L}}_A \mathrm{H}^j T) \Rightarrow \mathrm{H}(R_1 \otimes^{\mathrm{L}}_A T) = \mathrm{H} R_2.$$

Using a resolution of $\mathrm{H}^j T$ by finitely generated flat A-modules one easily sees that $\mathrm{H}^p(H^i R_1 \otimes^{\mathrm{L}}_A \mathrm{H}^j T)$ is finitely generated over A. Since the filtration on $\mathrm{H}^n R_2$ is bounded it follows that this too is a finitely generated A-module. Finiteness over A^{o} is proved similarly.

Given $M \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A)$ there is a natural isomorphism

$$\mathrm{R}\operatorname{Hom}_A(M, R_1 \otimes^{\mathrm{L}}_A T) \cong \mathrm{R}\operatorname{Hom}_A(M, R_1) \otimes^{\mathrm{L}}_A T.$$

If $M \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A^{\mathrm{e}})$ there is also a natural isomorphism

$$\mathrm{R}\operatorname{Hom}_A(M, R_1 \otimes^{\mathrm{L}}_A T) \cong \mathrm{R}\operatorname{Hom}_A(M \otimes^{\mathrm{L}}_A T^{\vee}, R_1)$$

where $T^{\vee} := \mathrm{R}\operatorname{Hom}_A(T, A)$. Therefore R_2 has finite injective dimension over A, $A \cong \mathrm{R}\operatorname{Hom}_A(R_2, R_2)$ and $T \cong \mathrm{R}\operatorname{Hom}_A(R_1, R_2)$. There is also a natural isomorphism

$$\mathrm{R}\operatorname{Hom}_{A^{\mathrm{o}}}(N, R_1 \otimes^{\mathrm{L}}_{\mathrm{A}} T) \cong \mathrm{R}\operatorname{Hom}_{A^{\mathrm{o}}}(N \otimes^{\mathrm{L}}_{\mathrm{A}} T^{\vee}, R_1)$$

for $N \in \mathrm{D}^{\mathrm{b}}(\operatorname{Mod} A^{\mathrm{o}})$, so R_2 has finite injective dimension over A^{o} and $A \cong \mathrm{R}\operatorname{Hom}_{A^{\mathrm{o}}}(R_2, R_2)$.

(2) By the proof of [**16**, Theorem 3.9], T is a tilting complex, and by [**16**, Lemma 3.10], $R_2 \cong R_1 \otimes^{\mathrm{L}}_A T$.

(3) Just like [**6**, Proposition IV.2.6]. □

The theorem says that $(R, T) \longmapsto R \otimes^{\mathrm{L}}_A T$ is a right action of $\operatorname{DPic}(A)$ on the set of isomorphism classes of dualizing complexes. By symmetry there is a left action $(T, R) \longmapsto T \otimes^{\mathrm{L}}_A R$. As a corollary we get the classification of isomorphism classes of dualizing complexes.

COROLLARY 4.6. *If the set*

$$\frac{\{\text{dualizing complexes } R \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, A^{\mathrm{e}})\}}{\text{isomorphism}}$$

is nonempty, then the left and right actions of the group $\mathrm{DPic}(A)$ *on it are transitive with trivial stabilizers.*

PROBLEM 4.7. In Propositions 3.5 and 3.4 we have seen that when A is commutative or local, the group $\mathrm{DPic}(A)$ consists of familiar ingredients – $\mathrm{Pic}_A(A)$, $\mathrm{Aut}(A)$ and the trivial copy of $\mathbb{Z}$ (cf. also Section 6). On the other hand $\mathrm{DPic}(A)$ classifies the isomorphism classes of dualizing complexes. Now it is known in commutative algebraic geometry that dualizing complexes are in close relation to localization. For instance, a ring with a dualizing complex is catenary; a dualizing complex can be represented by a residual complex, which is a sum of local cohomology modules (see [**6**]). This leads us to ask whether some obstructions to localization can be found in $\mathrm{DPic}(A)$ when A is noncommutative? More specifically, is there a relation between the group structure of $\mathrm{DPic}(A)$ and the link graph of maximal ideals in $\mathrm{Spec}\, A$?

5. *Rigid dualizing complexes*

In this section we use the action of the group $\mathrm{DPic}(A)$ on the set of isomorphism classes of dualizing complexes to study certain properties of dualizing complexes. In particular we shall be interested in rigid dualizing complexes, which were recently introduced by M. Van den Bergh. As in Section 4, A is a noetherian k-algebra.

First we need some notational conventions on modules with multiple actions. For an element $a \in A$ we denote by $a^{\mathrm{o}} \in A^{\mathrm{o}}$ the same element. Thus for $a_1, a_2 \in A$, $a_1^{\mathrm{o}} \cdot a_2^{\mathrm{o}} = (a_2 \cdot a_1)^{\mathrm{o}} \in A^{\mathrm{o}}$. With this notation if M is a right A-module then the left A^{o} action is $a^{\mathrm{o}} \cdot m = m \cdot a$, $m \in M$. The algebra A^{e} has an involution $A^{\mathrm{e}} \longrightarrow (A^{\mathrm{e}})^{\mathrm{o}}$, $a_1 \otimes a_2^{\mathrm{o}} \longmapsto a_2 \otimes a_1^{\mathrm{o}}$ which allows us to regard every left A^{e}-module M as a right A^{e}-module in a consistent way:

$$(a_1 \otimes a_2^{\mathrm{o}}) \cdot m = (a_2 \otimes a_1^{\mathrm{o}})^{\mathrm{o}} \cdot m = m \cdot (a_2 \otimes a_1^{\mathrm{o}}) = a_1 \cdot m \cdot a_2.$$

Given an $(A \otimes B^{\mathrm{o}})$-module M and a $(B \otimes A^{\mathrm{o}})$-module N we define a mixed action of $A^{\mathrm{e}} \otimes B^{\mathrm{e}}$ on the tensor product $M \otimes N$ as follows. A^{e} acts on $M \otimes N$ by the outside action

$$(a_1 \otimes a_2^{\mathrm{o}}) \cdot (m \otimes n) := (a_1 \cdot m) \otimes (n \cdot a_2),$$

whereas B^{e} acts on $M \otimes N$ by the inside action

$$(b_1 \otimes b_2^{\mathrm{o}}) \cdot (m \otimes n) := (m \cdot b_2) \otimes (b_1 \cdot n).$$

By default we regard the outside action as a left action and the inside action as a right action. If $A = B$ and $M = N$ then the two actions by A^{e} on $M \otimes M$ are interchanged via the involution $m_1 \otimes m_2 \longmapsto m_2 \otimes m_1$. However for the sake of definiteness in this case, given an A^{e}-module L, $\mathrm{Hom}_{A^{\mathrm{e}}}(L, M \otimes M)$ shall refer to homomorphisms $L \longrightarrow M \otimes M$ which are A^{e}-linear with respect to the outside action.

The next definition is due to Van den Bergh [**15**].

DEFINITION 5.1. A *rigid dualizing complex* over A is a pair (R, ρ) where R is a dualizing complex and

$$\rho : R \xrightarrow{\simeq} \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R \otimes R)$$

is an isomorphism in $\mathrm{D}(\mathrm{Mod}\,(A^{\mathrm{e}}))$.

Van den Bergh proved that any two rigid dualizing complexes are isomorphic. We improve on this slightly in the following.

THEOREM 5.2. *Suppose that* (R_1, ρ_1) *and* (R_2, ρ_2) *are two rigid dualizing complexes. Then there is a unique isomorphism* $\phi : R_1 \xrightarrow{\simeq} R_2$ *in* $\mathrm{D}(\mathrm{Mod}\,(A^{\mathrm{e}}))$ *making the diagram*

$$\begin{tikzcd} R_1 \arrow[r, "\rho_1"] \arrow[d, "\phi"'] & \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R_1 \otimes R_1) \arrow[d, "\phi\otimes\phi"'] \\ R_2 \arrow[r, "\rho_2"] & \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R_2 \otimes R_2) \end{tikzcd}$$

commute.

First we need the following lemma.

LEMMA 5.3. *Let R be a dualizing complex. Then the two ring homomorphisms* $\lambda_R, \rho_R : \mathrm{Z}(A) \longrightarrow \mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})}(R)$*, namely left and right multiplication, are bijective.*

Proof. The proof is similar to Proposition 2.4. Define functors $D := \mathrm{R}\,\mathrm{Hom}_A(-, R)$ and $D^{\mathrm{o}} := \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(-, R)$. Since $A \cong D^{\mathrm{o}}DA \cong D^{\mathrm{o}}R$ in $\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})$ it follows (by applying D^{o}) that

$$\mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})}(R, R) \cong \mathrm{Hom}_{\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})}(A, A)^{\mathrm{o}}.$$

This sends the left action λ_R of $\mathrm{Z}(A)$ on R to the right action ρ_A of $\mathrm{Z}(A)$ on A. However

$$\mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})}(A) = \mathrm{End}_{\mathrm{Mod}\,A^{\mathrm{e}}}(A) = \mathrm{Z}(A)$$

(via $\lambda_A = \rho_A$). Hence λ_R is bijective. Do the same for ρ_R. □

Proof of Theorem 5.2. Suppose we are given some isomorphism $\phi' : R_1 \xrightarrow{\simeq} R_2$. Let $\psi \in \mathrm{Aut}(R_2)$ satisfy

$$(\phi' \otimes \phi')\rho_1 = \rho_2 \psi \phi',$$

and define $\phi := \psi^{-1}\phi'$. By Lemma 5.3 there are elements $a, b \in \mathrm{Z}(A)^{\times}$ such that

$$\psi^{-1} = a \otimes 1 = 1 \otimes b^{\mathrm{o}} \in \mathrm{End}_{\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})}(R_2).$$

Hence $\phi = (a \otimes 1)\phi' = (1 \otimes b^{\mathrm{o}})\phi'$. Because ρ_2 and ψ are A^{e}-linear we get

$$\begin{aligned}(\phi \otimes \phi)\rho_1 &= (a \otimes b^{\mathrm{o}})(\phi' \otimes \phi')\rho_1 \\ &= (a \otimes b^{\mathrm{o}})\rho_2 \psi \phi' \\ &= \rho_2 (a \otimes b^{\mathrm{o}})\psi \phi' \\ &= \rho_2((a \otimes 1)\psi)(1 \otimes b^{\mathrm{o}})\phi' \\ &= \rho_2 \phi.\end{aligned}$$

In other words diagram (2) is commutative. If ϕ'' also makes (2) commutative, then writing $\phi'' = (c \otimes 1)\phi$ with $c \in \mathrm{Z}(A)^{\times}$, the same computation shows that $c = 1$.

It remains to produce ϕ'. Consider the complexes $T := \mathrm{R}\,\mathrm{Hom}_A(R_1, R_2)$ and $T^{\mathrm{o}} := \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{o}}}(R_1, R_2)$. Then by Theorem 4.5

$$R_2 \cong T^{\mathrm{o}} \otimes^{\mathrm{L}}_A R_1 \cong R_1 \otimes^{\mathrm{L}}_A T.$$

Now using ρ_1 and ρ_2 we obtain isomorphisms in $\mathrm{D}(\mathrm{Mod}(A^{\mathrm{e}}))$:

$$\begin{aligned} R_2 &\cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R_2 \otimes R_2) \\ &\cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, (R_1 \otimes^{\mathrm{L}}_A T) \otimes (T^{\mathrm{o}} \otimes^{\mathrm{L}}_A R_1)) \\ &\cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R_1 \otimes R_1) \otimes^{\mathrm{L}}_{A^{\mathrm{e}}} (T \otimes T^{\mathrm{o}}) \\ &\cong R_1 \otimes^{\mathrm{L}}_{A^{\mathrm{e}}} (T \otimes T^{\mathrm{o}}) \\ &\cong T^{\mathrm{o}} \otimes^{\mathrm{L}}_A R_1 \otimes^{\mathrm{L}}_A T \\ &\cong R_2 \otimes^{\mathrm{L}}_A T \end{aligned}$$

so again by Theorem 4.5, $T \cong A$. □

Usually we will leave the isomorphism ρ implicit, and just speak of a rigid dualizing complex R.

LEMMA 5.4. *Suppose that A is commutative, integral of dimension n and smooth over k. Then $\Omega^n_{A/k}[n]$ is a rigid dualizing complex.*

Proof. There is a natural isomorphism $\Omega^n_{A/k} \otimes \Omega^n_{A/k} \cong \Omega^{2n}_{A^{\mathrm{e}}/k}$ by wedge product. By [**6**, Proposition III.8.4] we get a natural isomorphism

$$\rho : \Omega^n_{A/k}[n] \xrightarrow{\simeq} \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, \Omega^{2n}_{A^{\mathrm{e}}/k}[2n]).$$

□

REMARK 5.5. Observe that this ρ is actually the fundamental class of the diagonal $X \hookrightarrow X \times X$, $X = \mathrm{Spec}\,A$. Locally there are generators $a_1, \ldots, a_n$ for $\mathrm{Ker}(A^{\mathrm{e}} \longrightarrow A)$, and then ρ is given by the generalized fraction

$$\begin{bmatrix} \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_n \\ a_1 \ldots a_n \end{bmatrix}.$$

REMARK 5.6. J. Lipman (in unpublished notes) studied the canonical isomorphism

$$f^{!}\mathcal{O}_Y \cong \mathrm{R}\,\mathrm{Hom}_{X \times_Y X}(\mathcal{O}_X, f^{!}\mathcal{O}_Y \boxtimes f^{!}\mathcal{O}_Y) \tag{3}$$

where $f : X \longrightarrow Y$ is a flat morphism of schemes, in connection with the relative fundamental class of f. When Y is a Gorenstein scheme, $\mathscr{R} := f^{!}\mathcal{O}_Y$ is a dualizing complex on X. This generalizes Lemma 5.4.

A ring homomorphism $A \longrightarrow B$ is called *finite* if B is a finitely generated left and right A-module.

PROPOSITION 5.7. *Suppose that A is finite over its centre and finitely generated as k-algebra. Then A has a rigid dualizing complex.*

Proof. Choose a finite centralizing homomorphism $C \longrightarrow A$, with $C = k[t_1, \ldots, t_n]$ a commutative polynomial algebra. Let $R_C := \Omega^n_{C/k}[n]$, with ρ_C as in Lemma 5.4. Define $R_A := \mathrm{R}\,\mathrm{Hom}_C(A, R_C)$, which by [**16**, Proposition 5.2] is a dualizing complex over A. One has

$$\begin{aligned} R_A \otimes R_A &= \mathrm{R}\,\mathrm{Hom}_C(A, R_C) \otimes \mathrm{R}\,\mathrm{Hom}_C(A, R_C) \\ &= \mathrm{R}\,\mathrm{Hom}_{C^{\mathrm{e}}}(A^{\mathrm{e}}, R_C \otimes R_C). \end{aligned}$$

Next using ρ_C we obtain an isomorphism

$$\begin{aligned} \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, R_A \otimes R_A) &\cong \mathrm{R}\,\mathrm{Hom}_{A^{\mathrm{e}}}(A, \mathrm{R}\,\mathrm{Hom}_{C^{\mathrm{e}}}(A^{\mathrm{e}}, R_C \otimes R_C)) \\ &\cong \mathrm{R}\,\mathrm{Hom}_{C^{\mathrm{e}}}(A, R_C \otimes R_C) \\ &\cong \mathrm{R}\,\mathrm{Hom}_C(A, \mathrm{R}\,\mathrm{Hom}_{C^{\mathrm{e}}}(C, R_C \otimes R_C)) \\ &\cong \mathrm{R}\,\mathrm{Hom}_C(A, R_C) \\ &\cong R_A \end{aligned}$$

which we label ρ_A. □

PROPOSITION 5.8. *Let $A \longrightarrow B$ be a finite homomorphism of k-algebras, and suppose that (R_A, ρ_A) and (R_B, ρ_B) are rigid dualizing complexes. Assume that for some commutative finitely generated k-algebra C there exists a homomorphism $C \longrightarrow A$, which makes A and B finite C-algebras. Then there is a canonical morphism $\mathrm{Tr}_{B/A} : R_B \longrightarrow R_A$ in $\mathrm{D}(\mathrm{Mod}(A^{\mathrm{e}}))$.*

Proof. Choose such a homomorphism $C \longrightarrow A$, and pick a rigid dualizing complex (R_C, ρ_C). By Proposition 5.7 and Theorem 5.2 there are unique isomorphisms $R_A \cong \mathrm{R}\,\mathrm{Hom}_C(A, R_C)$ and $R_B \cong \mathrm{R}\,\mathrm{Hom}_C(B, R_C)$. We obtain $\mathrm{Tr}_{B/A}$ by applying $\mathrm{R}\,\mathrm{Hom}_C(-, R_C)$ to the morphism $A \longrightarrow B$ in $\mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}})$. This is independent of C by Theorem 5.2. □

REMARK 5.9. These results are interesting even for A commutative. For instance, if A, B are integral of dimension n and smooth over k, and if $A \longrightarrow B$ is a finite homomorphism, then we obtain $\mathrm{Tr}_{B/A} : \Omega^n_{B/k}[n] \longrightarrow \Omega^n_{A/k}[n]$. This trace coincides with the trace of [**6**]. If $A \longrightarrow B$ is also étale then $\Omega^n_{B/k} \cong B \otimes_A \Omega^n_{A/k}$, and $\mathrm{Tr}_{B/A}$ is induced from $B \longrightarrow \mathrm{End}_A(B) \xrightarrow{\mathrm{Tr}} A$.

Derived equivalent algebras have 'the same' dualizing complexes, as shown by the following.

PROPOSITION 5.10. *Let A and B be noetherian k-algebras, $R \in \mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}})$ be a dualizing complex, and $T \in \mathrm{D}(\mathrm{Mod}(B \otimes A^{\mathrm{o}}))$ be a tilting complex. Then*

$$R^T := T \otimes^{\mathrm{L}}_A R \otimes^{\mathrm{L}}_A T^{\vee} \in \mathrm{D}(\mathrm{Mod}\, B^{\mathrm{e}})$$

is a dualizing complex. If in addition (R, ρ) is a rigid dualizing complex, then (R^T, ρ^T) is rigid, where ρ^T is induced naturally by ρ.

Proof. Since for any $M \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\, B)$ we have

$$\mathrm{R}\,\mathrm{Hom}_B(M, R^T) \cong \mathrm{R}\,\mathrm{Hom}_A(T^{\vee} \otimes^{\mathrm{L}}_B M, R) \otimes^{\mathrm{L}}_A T^{\vee}$$

and so on, it follows that R^T is dualizing.

In the rigid situation, first note that $R^T \cong (T \otimes T^\vee) \otimes^{\mathrm{L}}_{A^e} R$, and $T \otimes T^\vee \in \mathrm{D}(\mathrm{Mod}\,(B^e \otimes (A^e)^o))$ is a tilting complex. Using the isomorphism ρ we obtain

$$\begin{aligned}
\mathrm{R}\,\mathrm{Hom}_{B^e}(B, R^T \otimes R^T) &\cong \mathrm{R}\,\mathrm{Hom}_{B^e}(B, (T \otimes^{\mathrm{L}}_A R) \otimes (R \otimes^{\mathrm{L}}_A T^\vee)) \otimes^{\mathrm{L}}_{A^e} (T^\vee \otimes T) \\
&\cong \mathrm{R}\,\mathrm{Hom}_{B^e}(B, (T \otimes T^\vee) \otimes^{\mathrm{L}}_{A^e} (R \otimes R)) \otimes^{\mathrm{L}}_{A^e} (T^\vee \otimes T) \\
&\cong \mathrm{R}\,\mathrm{Hom}_{A^e}(A, R \otimes R) \otimes^{\mathrm{L}}_{A^e} (T^\vee \otimes T) \\
&\cong R \otimes^{\mathrm{L}}_{A^e} (T^\vee \otimes T) \\
&\cong R^T.
\end{aligned}$$

This determines ρ^T. □

The next proposition generalizes [**15**, Proposition 8.4], which gives a formula for the rigid dualizing complex R when A is a Gorenstein algebra and $R \cong L[n]$ for an invertible bimodule L.

Proposition 5.11. *Suppose that A is a Gorenstein algebra and R is a rigid dualizing complex. Then R is a tilting complex and*

$$R^\vee = \mathrm{R}\,\mathrm{Hom}_A(R, A) \cong \mathrm{R}\,\mathrm{Hom}_{A^e}(A, A^e) \in \mathrm{D}(\mathrm{Mod}\,A^e).$$

Proof. R is tilting by Theorem 4.5. Then it is a straightforward calculation:

$$\begin{aligned}
R &\cong \mathrm{R}\,\mathrm{Hom}_{A^e}(A, R \otimes R) \\
&\cong \mathrm{R}\,\mathrm{Hom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_{A^e} (R \otimes R) \\
&\cong R \otimes^{\mathrm{L}}_A \mathrm{R}\,\mathrm{Hom}_{A^e}(A, A^e) \otimes^{\mathrm{L}}_A R
\end{aligned}$$

so applying $R^\vee \otimes^{\mathrm{L}}_{A^e} -$ and then $- \otimes^{\mathrm{L}}_{A^e} R^\vee$ we get what we want. □

6. *Finite k-algebras*

In this section A is a finite k-algebra. We write $M^* = DM := \mathrm{Hom}_k(M, k)$ for an A-module M. The bimodule A^* is then injective on both sides, and $M^* \cong \mathrm{Hom}_A(M, A^*)$ for any $M \in \mathrm{D}(\mathrm{Mod}\,A)$.

Proposition 6.1. (1) *A^* is a rigid dualizing complex over A.*
(2) *$T \in \mathrm{D}^{\mathrm{b}}(\mathrm{Mod}\,A^e)$ is a tilting complex if and only if T^* is a dualizing complex.*
(3) *A is a Gorenstein algebra if and only if A^* is a tilting complex. In this case,*

$$A^* \otimes^{\mathrm{L}}_A M \cong \mathrm{R}\,\mathrm{Hom}_A(M, A)^*$$

for any $M \in \mathrm{D}^-_{\mathrm{f}}(\mathrm{Mod}\,A)$.

Proof. (1) By the proof of Proposition 5.7.
(2) Use the duality D (cf. Proposition 4.2).
(3) Since A is a Gorenstein algebra if and only if $R = A$ is a dualizing complex, this is a consequence of part (2). Using a projective resolution of M we get a functorial morphism

$$A^* \otimes^{\mathrm{L}}_A M \longrightarrow \mathrm{Hom}_{A^o}(\mathrm{R}\,\mathrm{Hom}_A(M, A), A^*).$$

By way-out arguments it suffices to check that this is an isomorphism for $M = A$, which is clear. □

REMARK 6.2. When the dualizing complex R is a single bimodule in degree 0, it is called a *cotilting module* in the literature. The name is justified by Proposition 6.1(2) (and cf. Theorem 4.5).

REMARK 6.3. The derived functor $A^* \otimes^{\mathrm{L}}_A -$ is discussed in [**5**] and in [**14**, Section 5]. If A is a hereditary algebra then by Proposition 6.1(3) we have

$$\mathrm{H}^{-1}(A^* \otimes^{\mathrm{L}}_A M) \cong \mathrm{Ext}^1_A(M, A)^* \cong D\,\mathrm{Tr}\,M$$

for every $M \in \mathrm{Mod}_{\mathrm{f}}\,A$. Here $D\,\mathrm{Tr}$ is the 'dual of the transpose' functor of [**1**, Chapter IV], which induces the translation function in the *Auslander–Reiten quiver* of A.

Now assume that A has finite global dimension. Let $S_1, \dots, S_n$ be a complete set of nonisomorphic simple A-modules, and let $P_1, \dots, P_n$ (respectively $I_1, \dots, I_n$) be the corresponding indecomposable projective (respectively injective) modules. Then the Grothendieck group $\mathrm{K}_0(A) = \mathrm{K}_0(\mathrm{Mod}_{\mathrm{f}}\,A)$ is a free $\mathbb{Z}$-module with basis either of the sets $\{[S_i]\}_{i=1}^n$, $\{[P_i]\}$ or $\{[I_i]\}$. The *Coxeter transformation* $c \in \mathrm{Aut}(\mathrm{K}_0(A))$ is defined by $c([P_i]) := -[I_i]$ (see [**1**, Section VIII.2]).

In Proposition 3.6 we defined the representation $\chi_0 : \mathrm{DPic}(A) \longrightarrow \mathrm{Aut}(\mathrm{K}_0(A))$. Denote by t the class of A^* in $\mathrm{DPic}(A)$.

PROPOSITION 6.4. $\chi_0(t) = -c$.

Proof. This follows from Proposition 6.1(3) and [**1**, Proposition VII.2.2(a)]. □

For the remainder of the section we shall examine the algebra

$$A = \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}.$$

(This was suggested by T. Stafford.) Observe that A is the smallest k-algebra which is neither commutative nor local, so Propositions 3.5 and 3.4 do not apply. In the classification by Dynkin quivers (diagrams), the algebra A corresponds to the quiver $\Delta = A_2 = (\bullet \longrightarrow \bullet)$. That is, $A \cong k\Delta$, the path algebra of Δ.

Let P_1, P_2 (respectively S_1, S_2) be the projective (respectively simple) A-modules

$$P_1 = S_1 := \begin{bmatrix} k \\ 0 \end{bmatrix}; \quad P_2 := \begin{bmatrix} k \\ k \end{bmatrix}; \quad S_2 := \begin{bmatrix} 0 \\ k \end{bmatrix},$$

so that $A = P_1 \oplus P_2$ as A-modules.

PROPOSITION 6.5. (1) $\mathrm{Pic}(A) = 1$.

(2) *There are isomorphisms in* $\mathrm{D}(\mathrm{Mod}\,A)$*:*

$$\begin{aligned} A^* \otimes^{\mathrm{L}}_A S_1 &\cong P_2 \\ A^* \otimes^{\mathrm{L}}_A P_2 &\cong S_2 \\ A^* \otimes^{\mathrm{L}}_A S_2 &\cong S_1[1]. \end{aligned}$$

(3) *There is an isomorphism in* $\mathrm{D}(\mathrm{Mod}\,A^{\mathrm{e}})$*:*

$$A^* \otimes^{\mathrm{L}}_A A^* \otimes^{\mathrm{L}}_A A^* \cong A[1].$$

Proof. (1) First note that the indecomposable projective modules P_1 and P_2 have different lengths. Hence if L is an invertible bimodule we must have $L \otimes_A P_1 \cong P_1$ and $L \otimes_A P_2 \cong P_2$. Therefore $L \cong A$ as A-modules. According to Lemma 3.3(2) we get $L \cong A_\sigma$ as bimodules, for some $\sigma \in \mathrm{Aut}(A)$. However one sees that any such σ is conjugation by a matrix

$$\begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix},$$

so $A_\sigma \cong A$ as bimodules and $\mathrm{Pic}(A) \cong \mathrm{Out}(A) = 1$.

(2) A straightforward calculation using the isomorphism of A^{e}-modules

$$A^* \cong \begin{bmatrix} k & 0 \\ k & k \end{bmatrix} = \begin{bmatrix} k & k \\ k & k \end{bmatrix} \Big/ \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix}$$

induced by the trace pairing on

$$M_2(k) = \begin{bmatrix} k & k \\ k & k \end{bmatrix}.$$

(3) By part (2) we obtain this isomorphism in $\mathrm{D}(\mathrm{Mod}\, A)$. Now apply Proposition 2.2, Lemma 3.3 and part (1) above. □

As before denote by s the class of $A[1]$ in $\mathrm{DPic}(A)$. The action of s on $\mathrm{D}(\mathrm{Mod}\, A)$ is by a shift in degree, and the subgroup $\langle s \rangle$ is then isomorphic to $\mathbb{Z}$. Proposition 6.5(3) gives the following remarkable fact.

Corollary 6.6. $t^3 = s$.

In terms of the representation χ_0 and the basis $\{[S_1], [S_2]\}$ of $\mathrm{K}_0(A)$ we get

$$\chi_0(s) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \chi_0(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}.$$

Remark 6.7. These results were extended by E. Kreines to upper triangular $n \times n$ matrix rings, $n \geqslant 2$ (see the appendix). In particular she showed that $t^{n+1} = s^{n-1}$. This is in agreement with the fact that the order of the Coxeter transformation c is $n+1$ (cf. [**1**, p. 289]).

Problem 6.8. Let A be an indecomposable, elementary, hereditary k-algebra of finite representation type. What is the structure of the group $\mathrm{DPic}(A)$? (This problem is solved in an upcoming paper by J. Miyachi and the author.) Is it true that $\mathrm{DPic}(A) \cong \mathbb{Z}$ with generator t? What is the structure of the rings $\mathrm{K}^0(A)$ and $\mathrm{DK}^0(A)$? How do $\mathrm{DPic}(A)$ and $\mathrm{DK}^0(A)$ fit in with other invariants of A?

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Appendix A. The algebra of $n \times n$ upper triangular matrices (by Elena Kreines)

Let us consider the upper triangular $n \times n$ matrix algebra A over a field k, where $n \geqslant 2$. Let $A^* := \mathrm{Hom}_k(A, k)$, which is known to be a tilting complex, and define the functor $F : \mathrm{D}(\mathrm{Mod}\, A) \longrightarrow \mathrm{D}(\mathrm{Mod}\, A)$, $FM := A^* \otimes^{\mathrm{L}}_A M$.

Theorem A.1. *There is an isomorphism*

$$F^{n+1}A = \underbrace{A^* \otimes^{\mathrm{L}}_A \ldots \otimes^{\mathrm{L}}_A A^*}_{n+1} \otimes^{\mathrm{L}}_A A \cong A[n-1]$$

in $\mathrm{D}(\mathrm{Mod}\, A)$.

The proof of the theorem appears at the end of the appendix.

Corollary A.2. *We get an isomorphism*

$$\underbrace{A^* \otimes^{\mathrm{L}}_A \ldots \otimes^{\mathrm{L}}_A A^*}_{n+1} \cong A[n-1]$$

in $\mathrm{D}(\mathrm{Mod}\, A^{\mathrm{e}})$. *Hence* $t^{n+1} = s^{n-1}$ *in* $\mathrm{DPic}(A)$.

Proof. By [**2**], $\mathrm{Aut}(A) = \mathrm{Inn}(A)$, and thus we can use the proof of Proposition 6.5(3). $\square$

Let $\mathrm{M}_n(k)$ denote the full matrix algebra, and let $\mathfrak{r} \subset A$ be the ideal of strictly upper triangular matrices. Then the trace pairing on $\mathrm{M}_n(k)$ identifies $A^* \cong \mathrm{M}_n(k)/\mathfrak{r}$ as A-bimodules.

For $1 \leqslant i \leqslant j \leqslant n$ let I^i_j be the A-module represented as a column

$$I^i_j := \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ k \\ \vdots \\ k \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \longleftarrow i \\ \\ \longleftarrow j \\ \\ \\ \\ \end{array}$$

The left action of A on I^i_j is as follows. For $i = 1$ this is the usual matrix multiplication, and for $i > 1$ we have $I^i_j \cong I^1_j / I^1_{i-1}$.

We see that $P_j := I^1_j$ is a projective module, and $A = \bigoplus_{j=1}^n P_j$. Also $I_i := I^i_n$ is an injective module, and $A^* = \bigoplus_{i=1}^n I_i$. The module $S_i := I^i_i$ is simple.

For the proof of the theorem we need two lemmas.

Lemma A.3. *For* $i = 1, \ldots, n$ *we have* $FP_i \cong I^i_n$.

Proof. Since the module P_i is projective we have $A^* \otimes^{\mathrm{L}}_A P_i = A^* \otimes_A P_i$. By tensoring the short exact sequence

$$0 \longrightarrow \bigoplus_{\substack{j=1,\dots,n \\ j \neq i}} P_j \longrightarrow A \longrightarrow P_i \longrightarrow 0$$

with the module A^*, and noting that $A^* \cdot P_j = I^j_n \subset A^*$ (where we view $P_j \subset A$ as a left ideal), we obtain

$$FP_i = A^* \otimes_A P_i \cong \frac{A^*}{A^* \cdot (\bigoplus_{j \neq i} P_j)} = \frac{A^*}{\bigoplus_{j \neq i} I^j_n} \cong I^i_n.$$

□

LEMMA A.4. *If $i > 1$ then $FI^i_j \cong I^{i-1}_{j-1}[1]$.*

Proof. The module I^i_j is not projective. A projective resolution for this module is the short exact sequence

$$0 \longrightarrow P_{i-1} \longrightarrow P_j \longrightarrow I^i_j \longrightarrow 0.$$

By tensoring this sequence with the module A^* and using Lemma A.3 we obtain the exact sequence

$$I^{i-1}_n \xrightarrow{\phi} I^j_n \longrightarrow A^* \otimes_A I^i_j \longrightarrow 0.$$

Let us denote by $M_1 \subset A$ the set of matrices whose only nonzero entries are in the first row. It is easy to see that $M_1 \cdot I^i_j = 0$ (since $i > 1$) and that $A^* \cdot M_1 = A^*$. This implies that $\operatorname{Coker}(\phi) = A^* \otimes_A I^i_j = 0$. Since $\operatorname{Ker}(\phi)$ is a submodule of I^{i-1}_n of length $j-i+1$ we must have $\operatorname{Ker}(\phi) = I^{i-1}_{j-1}$. □

Proof of Theorem A.1. By the two lemmas

$$\begin{aligned}
FP_i &\cong I^i_n \\
F^2P_i &\cong I^{i-1}_{n-1}[1] \\
&\vdots \\
F^iP_i &\cong I^1_{n+1-i}[i-1] = P_{n+1-i}[i-1] \\
&\vdots \\
F^{n+1}P_i &\cong P_i[n-1].
\end{aligned}$$

However $A = \bigoplus_{i=1}^n P_i$ as A-modules, and hence $F^{n+1}A \cong A[n-1]$ as claimed. □

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