BEN GURION UNIVERSITY OF THE NEGEV

אוניברסיטת בן גוריון בנגב

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פרופ׳ אמנון יקותיאלי
המחלקה למתמטיקה
אוניברסיטת בן גוריון
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Commutative Algebra and Homological Algebra 201-2-2011
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Final Assignment

Instructions

(1) The assignment is due on 02/03/2014. It can be written in Hebrew or English. Typed work is preferred; if handwritten then please write clearly in pen.
(2) You are required to work alone. Do not consult with others. You may ask me questions by email (including asking for hints), and I will try to help, within reason.
(3) Quoting results from the course notes is allowed and even encouraged. If you need to quote results from the literature (e.g. a textbook) then please give precise reference.
(4) **You should answer 7 questions from among questions numbers 1 - 13**. The choice is up to you. Please do not answer more than 7. List the numbers of the questions you chose to answer at the beginning.
(5) Note that Questions 12 and 13 are harder, so answering them correctly will give you extra credit (for instance to compensate for mistakes you may make in other questions).
(6) You are allowed to use questions that you did not answer, when answering later questions. But be sure to mention which questions or results you are using.

In this assignment A is a nonzero commutative ring. The category of A-modules is denoted by $\mathsf{Mod}\, A$. It is an A-linear category. We use the graded notation for complexes; i.e. a complex is a pair (M, d_M), where $M = \bigoplus_{i \in \mathbb{Z}} M^i$ is a graded A-module, and d_M is a degree 1 endomorphism of M satisfying $\mathrm{d}_M \circ \mathrm{d}_M = 0$.

Let M be an A-module. Let us recall how the functors $\mathrm{Tor}_i^A(M, -)$ were defined. These are the left derived functors $\mathrm{L}_i F_M$ of the functor $F_M := M \otimes_A -$. Thus for every $N \in \mathsf{Mod}\, A$ we choose a projective resolution $\tilde{\eta}_N : P_N \to N$, and we define

$$\mathrm{Tor}_i^A(M, N) := \mathrm{H}^{-i}(M \otimes_A P_N) \in \mathsf{Mod}\, A.$$

For every homomorphism $\phi : N \to N'$ in $\mathsf{Mod}\, A$ we choose a lift $\tilde{\phi} : P_N \to P_{N'}$, namely a homomorphism $\tilde{\phi}$ in $\mathsf{C}(\mathsf{Mod}\, A)$ satisfying $\tilde{\eta}_{N'} \circ \tilde{\phi} = \phi \circ \tilde{\eta}_N$. Then we define

$$\mathrm{Tor}_i^A(M, \phi) := \mathrm{H}^{-i}(\mathbf{1}_M \otimes \tilde{\phi}) : \mathrm{Tor}_i^A(M, N) \to \mathrm{Tor}_i^A(M, N').$$

Now suppose M and N are complexes of A-modules. Define a graded A-module $M \otimes_A N$ as follows:

$$(M \otimes_A N)^i := \bigoplus_{j \in \mathbb{Z}} M^j \otimes_A N^{i-j}$$

and

$$M \otimes_A N := \bigoplus_{i \in \mathbb{Z}} (M \otimes_A N)^i.$$

Question 1. Show that $M \otimes_A N$ can be made into a complex of A-modules, with differential d satisfying

$$\mathrm{d}(m \otimes n) = \mathrm{d}_M(m) \otimes n + (-1)^j m \otimes \mathrm{d}_N(n)$$

for $m \in M^j$ and $n \in N^k$.

Question 2. Show that the complex $M \otimes_A N$ is functorial in M and N. Namely given homomorphisms $\phi : M \to M'$ and $\psi : N \to N'$ in $\mathsf{C}(\mathsf{Mod}\, A)$, there is an induced homomorphism

$$\phi \otimes \psi : M \otimes_A N \to M' \otimes_A N'$$

in $\mathsf{C}(\mathsf{Mod}\, A)$; and these induced homomorphisms satisfy the conditions of a functor.

A complex $M = \bigoplus_i M^i$ is called *nonpositive* if $M^i = 0$ for all $i > 0$.
We want to prove the next theorem.

Theorem 1. *Let $\phi : M \to N$ be a quasi-isomorphism $\mathsf{C}(\mathsf{Mod}\, A)$, and let P be a complex of flat A-modules. Assume that P, M and N are nonpositive complexes. Then the homomorphism*

$$\mathbf{1}_P \otimes \phi : P \otimes_A M \to P \otimes_A N$$

is a quasi-isomorphism.

The proof goes in several steps.

Question 3. Let P be a flat A-module and let $\phi : M \to N$ be a quasi-isomorphism in $\mathsf{C}(\mathsf{Mod}\, A)$. Prove that

$$\mathbf{1}_P \otimes \phi : P \otimes_A M \to P \otimes_A N$$

is a quasi-isomorphism.

Question 4. Let P be a complex, and let i_0 be an integer. Define $P' \subset P$ to be the graded submodule $P' := \bigoplus_{i > i_0} P^i$. And define $P'' := P/P'$. Show that P' and P'' are complexes, and there is an exact sequence of complexes

$$0 \to P' \xrightarrow{\alpha} P \xrightarrow{\beta} P'' \to 0.$$

Question 5. Let $\phi : M \to N$ and P be as in Theorem 1. Let i_0 be an integer, and define P' and P'' as in the previous question. Prove that there is a commutative diagram of complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' \otimes_A M & \xrightarrow{\alpha \otimes \mathbf{1}_M} & P \otimes_A M & \xrightarrow{\beta \otimes \mathbf{1}_M} & P'' \otimes_A M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathbf{1}_{P'} \otimes \phi} & & \downarrow{\scriptstyle \mathbf{1}_{P} \otimes \phi} & & \downarrow{\scriptstyle \mathbf{1}_{P''} \otimes \phi} & & \\
0 & \longrightarrow & P' \otimes_A N & \xrightarrow{\alpha \otimes \mathbf{1}_N} & P \otimes_A N & \xrightarrow{\beta \otimes \mathbf{1}_N} & P'' \otimes_A N & \longrightarrow & 0
\end{array}$$

and that the rows in this diagram are exact sequences.

Question 6. Let $\phi : M \to N$ and P be as in Theorem 1. Assume that there is a nonpositive integer i_0 such that $P^i = 0$ for all $i < i_0$; so that $P = \bigoplus_{i_0 \leq i \leq 0} P^i$. Prove Theorem 1 under this assumption. (Hint: use questions 3 and 5, and induction on i_0.)

Question 7. Prove Theorem 1. (Hint: show that in order to prove that

$$\mathrm{H}^j(\mathbf{1}_P \otimes \phi) : \mathrm{H}^j(P \otimes_A M) \to \mathrm{H}^j(P \otimes_A N)$$

is an isomorphism for a specific integer j, we can assume that $P^k = 0$ for all $k < j - 1$.)

Here are two important corollaries.

Corollary 2. *Let M and N be A-modules. For every i there is an A-linear isomorphism*

$$\mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^A(N, M).$$

Actually more is true: this isomorphism is natural in both M and N, etc. But we will not prove that.

Question 8. Prove Corollary 2. (Hint: use Theorem 1 to concluce that the homomorphisms of complexes

$$\tilde{\eta}_M \otimes \mathbf{1}_{P_N} : P_M \otimes_A P_N \to M \otimes_A P_N$$

and

$$\mathbf{1}_{P_M} \otimes \tilde{\eta}_N : P_M \otimes_A P_N \to P_M \otimes_A N$$

are quasi-isomorphisms. Also show that $P_M \otimes_A N \cong N \otimes_A P_M$ as complexes.)

A *flat resolution* of an A-module N is a quasi-isomorphism $Q \to N$, where Q is a nonpositive complex of flat A-modules. The next corollary says that Tor can be calculated using flat resolutions.

Corollary 3. *Let M and N be A-modules, and let $Q \to N$ be a flat resolution of N. Then for every i there is an A-linear isomorphism*

$$\mathrm{Tor}_i^A(M, N) \cong \mathrm{H}^{-i}(M \otimes_A Q).$$

Question 9. Prove Corollary 3. (Hint: like proof of Corollary 2.)

Here is a characterization of flat modules via Tor.

Theorem 4. *The following three conditions are equivalent for an A-module M.*

(i) *M is flat.*
(ii) *$\mathrm{Tor}_i^A(M, -) = 0$ for all $i > 0$.*
(iii) *$\mathrm{Tor}_1^A(M, -) = 0$.*

Question 10. Prove Theorem 4. (Hint: for (iii) $\Rightarrow$ (i) use the long exact sequence of Tor. For for (i) $\Rightarrow$ (ii) use Corollaries 2 and 3.)

Here is an application of the last result.

Theorem 5. *Let A be a noetherian local ring and let M be a finitely generated A-module. The following two conditions are equivalent.*

(i) *M is free.*
(ii) *M is flat.*

Question 11. Prove Theorem 5. (Hint: for (ii) $\Rightarrow$ (i) show that the proof of the Theorem on page 100 of the notes works here with a slight modification, using Theorem 4.)

The last theorem we consider is this:

Theorem 6. *Let A be a noetherian ring and let M be a finitely generated A-module. The following two conditions are equivalent.*

(i) *M is projective.*
(ii) *M is flat.*

Question 12. (harder!) Take any prime ideal $\mathfrak{p}$ in A. Show that for any $M, N \in \mathsf{Mod}\, A$ there is a homomorphism

$$\chi_{M,N} : \mathrm{Hom}_A(M, N) \otimes_A A_{\mathfrak{p}} \to \mathrm{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}),$$

which is a natural transformation in both M and N. Moreover, if M is finitely generated, then $\chi_{M,N}$ is an isomorphism.

Question 13. (harder!) Prove Theorem 6. Hint: Consider a surjection $\phi : N \to N'$ in $\mathsf{Mod}\, A$, and the exact sequence

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N') \to L \to 0.$$