<u>Theorem</u> (Noether Normalization)
Let $\mathbb{K}$ be a field, and let A be a nonzero finitely generated $\mathbb{K}$-ring. There exist elements $a_1, \dots, a_n \in A$ s.t. the following conditions hold:

(i) The sequence $(a_1, \dots, a_n)$ is algebraically independent over $\mathbb{K}$.

(ii) A is a finitely generated module over the subring $\mathbb{K}[a_1, \dots, a_n]$.

In other words, the theorem says that the ring hom.

$$\mathbb{K}[t_1, \dots, t_n] \to A, \quad t_i \mapsto a_i$$

from the polynomial ring is <u>injective</u> and <u>finite</u>.

<u>Proof</u> Since A is fin. gen. as $\mathbb{K}$-ring, there exist <u>finite</u> ring homs

$$f : \mathbb{K}[t_1, \dots, t_n] \to A$$

(e.g. surjections)

for sufficiently large n. Let us choose such a hom. f with n <u>minimal</u>. We will prove that f is injective. Define $B := \mathbb{K}[t_1, \dots, t_n]$.

⇒ (cont)

112 (cont)

Let $\mathbb{b} := \mathrm{Ker}(f)$, and assume that $\mathbb{b} \neq 0$.
Take any element $b \in \mathbb{b}$, $b \neq 0$.

We claim that the polynomial $b = b(\underline{t})$ is nonconstant. Otherwise $b \in \mathbb{K}^\times$, so $\mathbb{b} = \mathbb{K}[\underline{t}]$ and $A = 0$.

Now we use the lemma. There are $b_1, \ldots, b_{n-1} \in B$, s.t. letting $b_n := b$, the ring hom.

$$g : \mathbb{K}[s_1, \ldots, s_n] \to B = \mathbb{K}[\underline{t}], \quad s_i \mapsto b_i,$$

is finite. Consider the hom.

$$f \circ g : \mathbb{K}[s_1, \ldots, s_n] \to A .$$

It is a finite ring homom, and $(f \circ g)(s_n) = f(b_n) = 0$.
Write $a_i := (f \circ g)(s_i) \in A$; so $a_n = 0$.

Let $h : \mathbb{K}[s_1, \ldots, s_{n-1}] \to A$ be the restriction of $f \circ g$ to this subring. Then

$$\mathrm{Im}(h) = \mathbb{K}[a_1, \ldots, a_{n-1}] = \mathbb{K}[a_1, \ldots, a_{n-1}, a_n] = \mathrm{Im}(f \circ g).$$

We see that h is a finite hom.

This contradicts the minimality of n $\square$.

not done in class

Lemma Let $\mathbb{K}$ be a field, and let A be a finite $\mathbb{K}$-ring which is an integral domain. Then A is a field.

pf. Let $a \in A$ be nonzero. Since A is finite over $\mathbb{K}$, the element a is integral over $\mathbb{K}$. Say $p(t)$ is a monic poly in $\mathbb{K}[t]$ s.t. $p(a)=0$. We can factor $p(t)$ in $\mathbb{K}[t]$ into

$$p(t) = t^m \cdot q(t), \quad q(t) = t^n + \sum_{i=0}^{n-1} \lambda_i t^i, \quad \lambda_0 \neq 0 .$$

Since A is an integ. dom. & $a \neq 0$, get $q(a)=0$.

I.e. $-\lambda_0 = a\left(1 + \sum_{i=1}^{n-1} \lambda_i a^i\right)$

so $1 = a \cdot \left(\frac{1}{-\lambda_0} \cdot \left(1 + \sum_{i=1}^{n-1} \lambda_i a^i\right)\right)$ in A. $\square$

Lemma Let $\mathbb{K}$ be a field, $\mathbb{K}[\underline{t}] := \mathbb{K}[t_1, \dots, t_n]$ the poly. ring, and $\lambda_1, \dots, \lambda_n \in \mathbb{K}$. Then the ideal $\mathfrak{m} := (t_1 - \lambda_1, \dots, t_n - \lambda_n)$ in $\mathbb{K}[\underline{t}]$ is maximal, and $\mathbb{K} \to \mathbb{K}[\underline{t}]/\mathfrak{m}$ is bijective.

pf. Exercise.

If $\mathbb{K}$ is not alg. closed, then $\mathbb{K}[\underline{t}]$ has many other max. ideals...

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Hilbert NSZ

Theorem. Let $\mathbb{K}$ be an algebraically closed field, let $\mathbb{K}[\underline{t}] := \mathbb{K}[t_1, \dots, t_n]$ be the polynomial ring in n variables, and let $\mathfrak{m}$ be a maximal ideal in $\mathbb{K}[\underline{t}]$.

Then $\mathfrak{m} = (t_1 - \lambda_1, \dots, t_n - \lambda_n)$ for $\lambda_1, \dots, \lambda_n \in \mathbb{K}$.

proof. Consider the field $L := \mathbb{K}[\underline{t}]/\mathfrak{m}$. It is a fin. gen. $\mathbb{K}$-ring. By Noether Normalization, there are $b_1, \dots, b_m \in L$, algebraically independent over $\mathbb{K}$,

s.t. L is finite over the subring $\mathbb{K}[b_1, \dots, b_m] =: B$.

We claim that $m = 0$. If not, then the ideal $\mathfrak{b} := (b_1, \dots, b_m)$ in B is nonzero. But then $\mathfrak{b} \cdot L$ is a nonzero L-submod. of L, so $L = \mathfrak{b} \cdot L$. Applying pre-Nakayama (page 88) to the fin. gen. B-module L, we see that for some $b \in \mathfrak{b}$ we have $(1 + b) \cdot L = 0$. Hence $1 + b = 0$ in $B \subset L$. Therefore $-1 = b \in \mathfrak{b}$. But $\mathfrak{b}$ is a max. ideal of B (since $B \cong \mathbb{K}[s_1, \dots, s_m]$). Contradiction

We conclude that $m = 0$, and so L is finite over $\mathbb{K}$. But $\mathbb{K}$ is alg closed, so $\mathbb{K} \to L = \mathbb{K}[\underline{t}]/\mathfrak{m}$ is an isom. For each i there is $\lambda_i \in \mathbb{K}$ s.t. $t_i \equiv \lambda_i \mod \mathfrak{m}$. So $t_i - \lambda_i \in \mathfrak{m}$. By Lemma, $\mathfrak{m} = (t_1 - \lambda_1, \dots, t_n - \lambda_n)$. Δ

<u>Cor.</u> Let A be a fin. gen. ring over an alg. closed field $\mathbb{K}$. Let $\mathfrak{m} \subset A$ be a max. ideal. Then the can. ring hom. $\mathbb{K} \to A/\mathfrak{m}$ is bijective.

<u>proof</u> <u>Exercise</u>.

<u>Exercise</u>. Prove this version of NSZ: $\mathbb{K}$ is any field, A is a fin. gen. $\mathbb{K}$-ring, $\mathfrak{m} \subset A$ max. Then $\mathbb{K} \to A/\mathfrak{m}$ is finite.

Geometric Significance of NSZ

Let $\mathbb{K}$ be an alg. closed field, and A a finite type $\mathbb{K}$-ring. Assume A is an integral domain. The topological space Spec A (with Zar. top) will be denoted by X.

By NSZ, any max. ideal $\mathfrak{m}$ of A all has residue field $\mathbb{k}(\mathfrak{m}) = \mathbb{K}$.

Let us denote by $X(\mathbb{K})$ the set of max ideals of $X = \operatorname{Spec} A$, with the induced subspace top. Any $f \in A$ is now a function

$$f : X(\mathbb{K}) \to \mathbb{K}, \quad f(\mathfrak{m}) := \lambda, \text{ where}$$

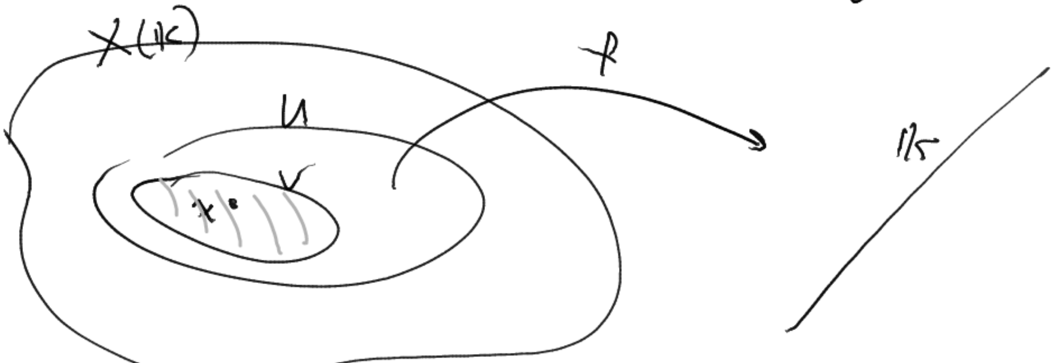
$\lambda \in \mathbb{K}$ is the unique elt s.t. $f - \lambda \in \mathfrak{m}$.

The principal closed sets are

$$Z(f) = \{x \in X(\mathbb{K}) \mid f(x) = 0\}.$$

15 We call $X(\mathbb{k})$ an <u>affine algebraic variety</u>.

Let $U \subset X(\mathbb{k})$ be an open set. A function $f : U \to \mathbb{k}$ is called regular if



each $x \in U$ has open nbhd V, s.t.

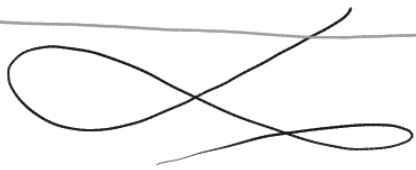
$$f|_V = \frac{g}{h}\Big|_V, \text{ where } g, h \in A \text{ and } h(y) \neq 0 \ \forall\, y \in V.$$

We denote by $\mathcal{O}(U)$ the set of reg. func's on U. It is a fin. gen. $\mathbb{k}$-ring. A can be recovered as

$$\boxed{A = \mathcal{O}(X(\mathbb{k})).}$$

A choice of surj. $\mathbb{k}[t_1, \dots, t_n] \to A$ gives an embedding

$$\boxed{\begin{gathered} X(\mathbb{k}) \subset \mathbb{k}^n, \\ x \mapsto (t_1(x), \dots, t_n(x)) \end{gathered}}$$



11-6

Transcendece Degree

(no proofs)

Let $K \subset L$ be a field extension.

L is algebraic over K if every element $b \in L$ is algebraic over K; i.e. $p(b) = 0$ for some nonzero poly. $p(t) \in K[t]$.

It is easy to see that L is algebraic over K iff $L = \bigcup_i L_i$, where each L_i is a finite field ext. of K.

A transcendece basis of L over K is a collection $\{b_i\}_{i\in I}$ of elts. of L, s.t.

- The collection is algebraically independent over K.
- L is algebraic over the subfield $K(\{b_i\}_{i\in I})$.

Thm. (1) There exists a trans. basis $\{b_i\}_{i\in I}$ of L over K.

(2) If $\{c_j\}_{j\in J}$ is another trans. basis of L over K, then $|J| = |I|$.

The cardinality $|I|$ is called the trans. degree.

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<u>Theorem</u>. Let $\mathbb{k}$ be a field, and let A be a finitely generated $\mathbb{k}$-ring that is an integral domain, with $L := \mathrm{Frac}(A)$. Suppose $\mathbb{k}[t_1, \ldots, t_n] \to A$ is a finite injective ring hom. (as guaranteed by Noether Normalization).
Then n is the transcendence degree of L over $\mathbb{k}$.

<u>Proof</u>. <u>Exercise</u>. (Hint: use lemma on top of page 112 - not done in class.)

to here 23/12