(11-8) Dimension Theorem

Let A be an integral domain fin. gen. as ring over a field K. A chain of prime ideals in A is a seq.

$$\mathbb{P}_0 \subsetneq \mathbb{P}_1 \subsetneq \cdots \subsetneq \mathbb{P}_n$$

of prime ideals.

The length of the chain is n.

Def The Krull dimension of A is the supremum of lengths of chains of primes in A.

Theorem Let $\mathbb{K}[t_1, \dots, t_n] \to A$ be a fin. injective ring hom. (as in Noether Normaliz.)

(1) The Krull dimension of A is n.

(2) The trans. deg. of $\mathrm{Frac}(A)$ over K is n.

Natural Transformations

In this part of the course we discuss general categories

Def. Let $\underline{C}$ and $\underline{D}$ be categories, and let $F, G : \underline{C} \to \underline{D}$ be functors. A *natural transformation* $\eta : F \to G$ is the data of a morphism

$$\eta_C : F(C) \to G(C)$$

in the category $\underline{D}$ for every object $C \in \underline{C}$.
The condition is this: for every morphism $f : C_1 \to C_2$ in $\underline{C}$, the diagram

$$\begin{array}{ccc} F(C_1) & \xrightarrow{F(f)} & F(C_2) \\ {\scriptstyle \eta_{C_1}} \downarrow & & \downarrow {\scriptstyle \eta_{C_2}} \\ G(C_1) & \xrightarrow[G(f)]{} & G(C_2) \end{array}$$

in $\underline{D}$ is commutative.

Example. Let A be a comm. ring, let $M, N \in \underline{\text{Mod}}\, A$, and let $\psi : M \to N$ be an A-lin. hom. Define functors $F, G : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, A$, $F(L) := M \otimes_A L$ and $G(L) := N \otimes_A L$.
Let $\eta_L : M \otimes_A L \to N \otimes_A L$ be $\eta_L := \psi \otimes \mathbb{1}_L$.

We will show that $\eta : F \to G$ is a nat. trans. Take a hom. $\varphi : L_1 \to L_2$. Then we have

$$\begin{tikzcd}
m\otimes l \in & M \otimes_A L_1 \arrow[r, "F(\varphi)", "1_M \otimes \varphi"'] \arrow[d, "\eta_{L_1}"', "\psi \otimes 1_{L_1}"] & M \otimes_A L_2 \arrow[d, "\eta_{L_2}"', "\psi \otimes 1_{L_2}"] & m\otimes\varphi(l) \\
\psi(m)\otimes l & N \otimes_A L_1 \arrow[r, "G(\varphi)", "1_N \otimes \varphi"'] & N \otimes_A L_2 & \psi(m)\otimes\varphi(l) \\
 & & \psi(m)\otimes\varphi(l) &
\end{tikzcd}$$

a commutative diagram

Def. A morphism $f : C \to D$ in a category $\underline{C}$ is called an isomorphism if there is a morphism $g : D \to C$ s.t. $g \circ f = 1_C$ and $f \circ g = 1_D$.

Exercise. Prove that if $f : C \to D$ is an isomorphism in a category $\underline{C}$, then its inverse $g : D \to C$ is unique.

Def. Let $F, G : \underline{C} \to \underline{D}$ be functors.

A natural transformation $\eta : F \to G$ is called a natural isomorphism if $\eta_C : F(C) \to G(C)$ is an isom., for every $C \in \underline{C}$.

121 **Prop.** Let $\eta : F \to G$ be a natural isomorphism. For any $C \in \underline{C}$ let $\zeta_C : G(C) \to F(C)$ be the inverse of η_C. Then $\zeta : G \to F$ is a natural isomorphism.

Pf. Exercise.

Example. Consider $F = M \otimes_A -$, $G = N \otimes_A -$ from the last example. We will show that $\eta : F \to G$ is a nat. isom. iff $\varphi : M \to N$ is an isom. First assume φ is an isom. in $\underline{\text{Mod}}\, A$. Let $\chi := \varphi^{-1} : N \to M$. Define a nat. trans $\zeta : G \to F$ by

$$\zeta_L := \chi \otimes 1\!\!1_L : N \otimes_A L \to M \otimes_A L.$$

This is an isom $\forall L$, so a nat. isom.

Conversely, assume $\eta : F \to G$ is a nat. isom. Then $\eta_A : F(A) \to G(A)$ is $\cong$.

But the comm. diag:

$$\begin{array}{ccccc} F(A) & = & M \otimes_A A & \xrightarrow{\cong} & M \\ \eta_A \downarrow \cong & & \downarrow \varphi \otimes 1\!\!1_A & & \downarrow \varphi \\ G(A) & = & N \otimes_A A & \xrightarrow{\cong} & N \end{array}$$

shows that φ is $\cong$.

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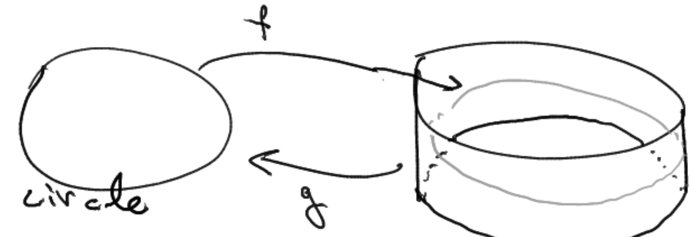
Equivalence of Categories

An equivalence of categories is like a homotopy equiv. of topological spaces:



$g \circ f = 1\!\!1$ cylinder $f \circ g \underset{\text{homotopy}}{\sim} 1\!\!1$

Def. Let $F : \underline{C} \to \underline{D}$ be a functor. We say that F is an equivalence if there is a functor $G : \underline{D} \to \underline{C}$, and natural isom's

$$\eta : G \circ F \xrightarrow{\cong} 1\!\!1_{\underline{C}}$$

$$\xi : F \circ G \xrightarrow{\cong} 1\!\!1_{\underline{D}}.$$

Example. Let $\underline{Set}_f$ be the category of finite sets. For $n \in \mathbb{N}$ let

$$X_n := \{1, 2, \ldots, n\} \subset \mathbb{N}.$$

Let $\underline{D}$ be the full subcat. of $\underline{Set}_f$ on the set of objects $\{X_n\}_{n \in \mathbb{N}}$. So

$$\mathrm{Hom}_{\underline{D}}(X_m, X_n) = \mathrm{Hom}_{\underline{Set}}(X_m, X_n).$$

Consider the inclusion functor $F : \underline{D} \to \underline{Set}_f$. The next prop. shows that F is an equiv.

(123) First a general theorem that I want prove.

Thm Let $F : \underline{C} \to \underline{D}$ be a functor. TFAE:

(i) F is an equivalence.

(ii) F is fully faithful and essentially surjective on objects.

Explanation: • fully faithfull means that $\forall\, C_0, C_1 \in Ob(\underline{C})$ the function $F : Hom_{\underline{C}}(C_0, C_1) \to Hom_{\underline{D}}(F(C_0), F(C_1))$ is bijective.

• F is essent. surj. on objects if $\forall\, D \in ob(\underline{D})$ $\exists$ $C \in ob(\underline{C})$ and an isom $D \xrightarrow{\simeq} F(C)$.

I will prove the next weaker prop.

Prop. Let $\underline{D}$ be a category, let $\underline{C}$ be a full subcategory of $\underline{D}$ and let $F : \underline{C} \to \underline{D}$ be the inclusion functor. If every object of $\underline{D}$ is isomorphic to some object of $\underline{C}$, then F is an equivalence.

Proof. We define a functor $G : \underline{D} \to \underline{C}$ like this. Take $D \in Ob(\underline{D})$. If $D \in Ob(\underline{C})$ then define $G(D) := D$ and $\eta_D := \mathbb{1}_D : D \to G(D)$. If $D \notin Ob(\underline{C})$, then choose some object $G(D) \in Ob(\underline{C})$, with an isomorphism $\eta_D : D \xrightarrow{\simeq} G(D)$.

(124) By assumption this is possible. (We secretly use the axiom of choice.)
So we have a function $G : Ob(\underline{D}) \to Ob(\underline{C})$.

Now let $f : D_0 \to D_1$ be a morphism in $\underline{D}$. define
$$G(f) := \eta_{D_1} \circ f \circ \eta_{D_0}^{-1} : G(D_0) \to G(D_1).$$

$$\begin{tikzcd} D_0 \arrow[r, "f"] \arrow[d, "\eta_{D_0}"', "\cong"] & D_1 \arrow[d, "\eta_{D_1}", "\cong"'] \\ G(D_0) \arrow[r, dashed, "G(f)"'] & G(D_1) \end{tikzcd}$$

This is a functor: given $D_0 \xrightarrow{f_1} D_1 \xrightarrow{f_2} D_2$ in $\underline{D}$,

$$\begin{tikzcd} D_0 \arrow[r, "f_1"] \arrow[d, "\eta_{D_0}"'] & D_1 \arrow[r, "f_2"] \arrow[d, "\eta_{D_1}"] & D_2 \arrow[d, "\eta_{D_2}"] \\ G(D_0) \arrow[r, "G(f_1)"] & G(D_1) \arrow[r, "G(f_2)"] & G(D_2) \end{tikzcd}$$

$$G(f_2 \circ f_1) = \eta_{D_2} \circ f_2 \circ f_1 \circ \eta_{D_0}^{-1} =$$
$$= \underbrace{\eta_{D_2} \circ f_2 \circ \eta_{D_1}^{-1}} \circ \underbrace{\eta_{D_1} \circ f_1 \circ \eta_{D_0}^{-1}} = G(f_2) \circ G(f_1).$$

And $G(1\!\!1_D) = \eta_D \circ \eta_D^{-1} = 1\!\!1_{G(D)}$. $\boxed{\text{for } C \in Ob(\underline{C})}$

Note that $(G \circ F)(C) = C$, and for $f : C \to C$ we have $(G \circ F)(f) = G(f) = \eta_C \circ f \circ \eta_C^{-1} = f$.

So $G \circ F = 1\!\!1_{\underline{C}}$.

Finally we have to find a nat. isom $\eta : 1_{\underline{D}} \xrightarrow{\simeq} F \circ G$. Take any D.
Then $(F\circ G)(D) = G(D)$, and we already chose an isom. $\eta_D : D \xrightarrow{\simeq} G(D)$.
Let $\eta := \{\eta_D\}$. It remain to show that $\eta : 1_{\underline{D}} \to F\circ G$ is a nat. trans. Take $f : D_0 \to D_1$.
So $(F\circ G)(f) = G(f) = \eta_{D_1} \circ f \circ \eta_{D_0}^{-1}$.
Thus the diag.

$$\begin{tikzcd}
D_0 \arrow[r, "f"] \arrow[d, "\eta_{D_0}"'] & D_1 \arrow[d, "\eta_{D_1}"] \\
(F\circ G)(D_0) \arrow[r, "G(f)"'] & (F\circ G)(D_1)
\end{tikzcd}$$

is comm. $\square$