(12b)

Morita Equivalence

I will do a special case of this important theory.

We work with non-commutative rings. Let A be a ring. A left A-module is an abelian group M, with a ring hom. $\rho : A \to \mathrm{End}_{\mathbb{Z}}(M)$. For $a \in A$ and $m \in M$ we write $a \cdot m := \rho(a)(m) \in M$.

We denote by $\underline{\mathrm{Mod}}\, A$ the category of left A-modules, and A-linear homs.
It is a $\mathbb{Z}$-linear category. (Even a $Z(A)$-lin. cat.)

Let $n \geq 1$.

The set of $n \times n$ matrices $M_n(A)$ is a ring - the usual matrix operations make sense.

Let $\underline{0}$ and $\underline{1}$ be the zero and one matrices.
There is an injective ring hom $A \to M_n(A)$,
$a \mapsto a \cdot \underline{1} = \begin{bmatrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{bmatrix}$. The center of $M_n(A)$ is $Z(A)$. So $\underline{\mathrm{Mod}}\, M_n(A)$ is a $Z(A)$-linear cat.

<u>Theorem</u>. There is a $Z(A)$-linear equivalence of categories

$$F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, M_n(A).$$

Proof. Let M be an A-module. Define $F(M) := M^{\oplus n}$.
We want . make $M^{\oplus n}$ into an $M_n(A)$-module by viewing $M^{\oplus n}$ as a column

$$M^{\oplus n} = \begin{bmatrix} M \\ \vdots \\ M \end{bmatrix} = \left\{ \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \,\middle|\, m_1, \dots, m_n \in M \right\}$$

Then $M_n(A)$ acts on $M^{\oplus n}$ by matrix multip. from the left.

If $\varphi : M \to M'$ is an A-lin. hom., then

$$F(\varphi) := \begin{bmatrix} \varphi \\ \vdots \\ \varphi \end{bmatrix} : \begin{bmatrix} M \\ \vdots \\ M \end{bmatrix} \longrightarrow \begin{bmatrix} M' \\ \vdots \\ M' \end{bmatrix} \quad \text{is } M_n(A)\text{-linear.}$$

It is easy to check that F is a $Z(A)$-lin. functor.

Before we define a quasi-inverse G, we have to know more about the ring $M_n(A)$.

For $1 \le i, j \le n$ let e_{ij} be the matrix

$$e_{ij} := \begin{array}{cc} & j \\ i & \begin{bmatrix} 0 & \cdots & & 0 \\ \vdots & & 1 & \vdots \\ 0 & & \cdots & 0 \end{bmatrix} \end{array} \in M_n(A).$$

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It is easy to check that

$$(\dagger) \qquad \underline{e}_{i,j} \cdot \underline{e}_{j',k} = \begin{cases} \underline{e}_{i,k} & \text{if } j = j' \\ \underline{0} & \text{if } j \neq j' \end{cases},$$

$$(\dagger\dagger) \qquad \sum_{i=1}^{n} \underline{e}_{i,i} = \underline{1},$$

$$(\dagger\dagger\dagger) \quad \underline{e}_{i,j} \cdot (a \cdot \underline{1}) = (a \cdot \underline{1}) \cdot \underline{e}_{i,j} = a \cdot \underline{e}_{i,j} \quad \text{for } a \in A.$$

Let $L \in \underline{\mathrm{Mod}}\ M_n(A)$. Then by $(\dagger\dagger\dagger)$ the abelian subgroup $\underline{e}_{i,i} \cdot L =: L_i$ is an A-submod. of L; and by $(\dagger\dagger)$ we have an A-mod. isom.

$$\boxed{(*) \qquad L = \bigoplus_{i=1}^{n} L_i \,.}$$

← "row decomposition"

Consider the operation $\underline{e}_{i,j} \cdot -$ on L. We have $\underline{e}_{i,j} \cdot L_{j'} = 0$ for $j' \neq j$, and $\underline{e}_{i,j} \cdot L_j = \underline{e}_{i,i} \cdot \underline{e}_{i,j} \cdot L_j \subset L_i$. [using $(\dagger)$].
Thus we have A-mod. hom.

$$(**) \qquad \underline{e}_{i,j} \cdot - : L_j \to L_i \,.$$

Since $\underline{e}_{j,i} \cdot \underline{e}_{i,j} = \underline{e}_{j,j}$, we see that $(**)$ is $\cong$.

Define $G(L) := \underline{e}_{1,1} \cdot L = L_1$.

Suppose $\psi : L \to L'$ is a hom. in $\underline{\mathrm{Mod}}\ M_n(A)$.

Then

$$\Psi(L_1)=\Psi(\underline{e}_{1,1}\cdot L)\subset \underline{e}_{1,1}\cdot L'_1 ,$$

so get hom of A-modules

$$G(\Psi):=\Psi|_{G(L)} : G(L)\longrightarrow G(L').$$

Thus get a $Z(A)$-lin. functor

$$G : \underline{\mathrm{Mod}}\ M_n(A)\longrightarrow \underline{\mathrm{Mod}}\ A.$$

The next step is to find a natural isom

$$\eta : \mathbb{1}_{\underline{\mathrm{Mod}}\,A}\xrightarrow{\simeq} G\circ F.$$

Take an A-mod. M. Then $F(M)=\begin{bmatrix} M\\ \vdots\\ M\end{bmatrix}$ and $G(M)=\underline{e}_{1,1}\cdot F(M)=\begin{bmatrix} M\\ 0\\ \vdots\\ 0\end{bmatrix}$. There is an obvious A-lin. isom.

$$\eta_M : M\xrightarrow{\simeq}\begin{bmatrix} M\\ 0\\ \vdots\\ 0\end{bmatrix},$$

and it is natural.

Finally we need a nat. isom.

$$\zeta : \mathbb{1}_{\underline{\mathrm{Mod}}\,M_n(A)}\xrightarrow{\simeq} F\circ G.$$

Take $L\in \underline{\mathrm{Mod}}\ M_n(A)$. So $G(L)=L_1=\underline{e}_{1,1}\cdot L$ and $(F\circ G)(L)=\begin{bmatrix} L_1\\ \vdots\\ L_1\end{bmatrix}=L_1\oplus\cdots\oplus L_1$. On the other hand, by $(*)$ we have $L=L_1\oplus\cdots\oplus L_n$. Define

A-mod. isoms

$\Rightarrow$

$$\zeta_L : L \longrightarrow (F\circ G)(L),$$

$$\zeta_L|_{L_i} : L_i \xrightarrow{\cong} L_1, \quad \zeta_L|_{L_i} := \underline{e}_{1,i}$$

This is an $\underline{M}_n(A)$-lin. isom. And it is natural. (check this!)

$\square$

Exercise

- There is an A-mod. isom. $\boxed{\underline{M}_n(A) \cong A \otimes_{\mathbb{Z}} \underline{M}_n(\mathbb{Z})}$ (*)

- The mult. in $\underline{M}_n(A)$ satisfies (using can. isom. (*))

$$(a \otimes \underline{b}) \cdot (a' \otimes \underline{b}') = (aa') \otimes (\underline{b} \cdot \underline{b}')$$

for $a, a' \in A$; $\underline{b}, \underline{b}' \in \underline{M}_n(\mathbb{Z})$.

(131) Derived Functors

Let $\mathbb{K}$ be a comm. ring. A $\mathbb{K}$-algebra is a ring A, together with a ring hom. $\mathbb{K} \to Z(A)$.
A hom. of $\mathbb{K}$-algebras is a ring hom $A \to B$ that commutes with the homs. from $\mathbb{K}$.

Example Any ring is a $\mathbb{Z}$-algebra.

If A is a $\mathbb{K}$-algebra, then $\underline{\mathrm{Mod}}\, A$ is a $\mathbb{K}$-lin. category.

Suppose A & B are $\mathbb{K}$-algebras, and

$$F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B \text{ is a } \mathbb{K}\text{-lin. functor}$$

We will learn about the left derived functors $L_i F$, and the right derived functors $R^i F$.

We will prove:

Theorem. There are additive functors

$$L_i F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B \quad , \quad i \in \mathbb{N},$$

with these properties.

(1) There is a nat. trans. $\eta : L_0 F \to F$, which is an isomorphism if F is right exact.

(2) If F is exact, then $L_i F = 0$ for all $i \geq 1$.

(3) Suppose $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is a short exact seq. in $\underline{\text{Mod}}\, A$. Then there are homs.

$$\partial_i : L_{i+1}(M'') \to L_i(M')$$

s.t. the sequence

$$L_{i+1}(M'') \xrightarrow{\partial_i} L_i(M') \xrightarrow{L_i(\varphi)} L_i(M) \xrightarrow{L_i(\psi)} L_i(M'') \xrightarrow{\partial_{i-1}} L_{i-1}(M')$$

is exact.

<u>Example</u> Say N is a right A-module (or A is commutative). Let $F(M) := N \otimes_A M$. Then $L_i F =: \mathrm{Tor}_i^A(N, -)$.

For right derived: $F \xrightarrow{\sim} R^0 F$; $R^i F = 0$ for $i \geq 1$ if F exact; $\partial^i : R^i(M'') \to R^{i+1}(M')$; long ex. seq. [not important for us]

The <u>opposite</u> of a category $\underline{C}$ is the cat. $\underline{C}^{op}$, with same objects, but $\mathrm{Hom}_{\underline{C}^{op}}(C, D) := \mathrm{Hom}_{\underline{C}}(D, C)$, and composition is reversed.

A <u>contravariant functor</u> $F : \underline{C} \to \underline{D}$ is a functor $F : \underline{C}^{op} \to \underline{D}$.

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