Strategic decision: revert to commutative rings. This simplifies things!

Example Let A be a comm. ring and $N \in \underline{\mathrm{Mod}}\, A$. Then

$$\mathrm{Hom}_A(-, N) : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$$

is a contravariant A-linear functor.

Theorem Let $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$ be an A-linear contravariant functor (A is commutative.)

There are A-linear contravariant functors

$$R^i F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A \quad , \quad i \in \mathbb{N},$$

with these properties.

(1) There is a nat. trans. $\eta : F \to R^0 F$, which is an isomorphism if F is left exact.

(2) If F is exact, then $R^i F = 0$ for all $i \geq 1$.

(3) Suppose $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is a short exact seq. in $\underline{\mathrm{Mod}}\, A$. Then there are homs. $\partial^i : R^{i-1}(M') \to R^i(M'')$ s.t. the sequence

$$R^{i-1}(M') \xrightarrow{\partial^i} R^i(M'') \xrightarrow{R^i(\psi)} R^i(M) \xrightarrow{R^i(\varphi)} R^i(M') \xrightarrow{\partial^{i+1}} R^{i+1}(M'')$$

is exact.

<u>Example</u>. $N \in \underline{\mathrm{Mod}}\, A$. $F := \mathrm{Hom}_A(-, N)$ is left exact contravariant: if

$$M' \to M \to M'' \to 0$$

is exact, then so is

$$0 \to F(M'') \to F(M) \to F(M') .$$

Here $\quad \mathrm{Ext}^i_A(-, N) := R^i F \quad .$

I suggest reading (later) about the deeper theory of <u>derived categories</u>. There is a link on the course web page to notes on that.

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Complexes of Modules

Fix a commutative ring A. (Almost everything here works for left modules over a NC ring...!)

<u>Def.</u> A <u>complex</u> of A-modules (or complex in <u>Mod</u> A) is a collection $M = \{M^i\}_{i \in \mathbb{Z}}$ of A-modules, and a collection $d = \{d^i\}_{i \in \mathbb{Z}}$ of A-linear homomorphisms

$$d^i : M^i \to M^{i+1}$$

The condition is that $d^{i+1} \circ d^i = 0$.

We often write (M, d) like this:

$$(M, d) = \left(\cdots \to M^i \xrightarrow{d^i} M^{i+1} \to \cdots \right)$$

d is called the <u>differential</u> of M, or the <u>coboundary operator</u>. Sometimes we write $d_M := d$.

Sometimes it is convenient to view M as a <u>graded A-module</u>: $M = \bigoplus_{i \in \mathbb{Z}} M^i$.

Then $d : M \to M$ is an A-lin. hom. of degree 1, and $d \circ d = 0$.

<u>Examples</u> (1) Let $M^0 \in \underline{\mathrm{Mod}}\, A$. We can make it into a complex by taking $M^i := 0$ for $i \neq 0$, and $d^i := 0$.

(2) Let X be a topological space. Then the chain complex of X with integer coefficients is a complex of $\mathbb{Z}$-modules.

There is a little problem: upper vs. lower indices (chain vs. cochain). It is dealt with by letting $M_i := M^{-i}$, and $d_i : M_i \to M_{i-1}$ is $d^i : M^{-i} \to M^{-i+1}$.

<u>Def</u> Let (M, d_M) and (N, d_N) be complexes of A-modules. A <u>homomorphism</u> of complexes

$$\varphi : (M, d_M) \to (N, d_N)$$

is a collection $\varphi = \{\varphi^i\}_{i \in \mathbb{Z}}$ of A-linear homs. $\varphi^i : M^i \to N^i$, s.t.

$$d_N^i \circ \varphi^i = \varphi^{i+1} \circ d_M^i$$

for all i.

In the "graded" notation $\varphi : M = \bigoplus_i M^i \to N = \bigoplus_i N^i$ is a degree 0 A-lin. hom, satisfying

$$d_N \circ \varphi = \varphi \circ d_M .$$

(137) The complexes and their hom's form a category, that we denote by $\underline{C}(\underline{\text{Mod}}\, A)$. This is an A-lin. cat.

We often leave the differential d_M implicit, and talk about a complex of modules M.

Def. Let M be a complex in Mod A.

(1) Let $\boxed{Z^i(M) := \mathrm{Ker}(d^i : M^i \to M^{i+1})}$ and $\boxed{B^i(M) := \mathrm{Im}(d^{i-1} : M^{i-1} \to M^i)}$.
These are the i-cocycles and i-coboundaries of M.

(2) The i-th cohomology of M is the A-module

$$\boxed{H^i(M) := Z^i(M) / B^i(A).}$$

This makes sense: $d^i \circ d^{i-1} = 0$, so $B^i(M) \subset Z^i(M) \subset M^i$.

If $\varphi : M \to N$ is a hom. in $\underline{C}(\underline{\text{Mod}}\, A)$, then $\varphi(Z^i(M)) \subset Z^i(N)$ and $\varphi(B^i(M)) \subset B^i(N)$.

Def. Given a hom. $\varphi : M \to N$ in $\underline{C}(\underline{\text{Mod}}\, A)$, let

$$H^i(\varphi) : H^i(M) \to H^i(N)$$

be the A-lin. hom. induced by φ^i.

For $m \in Z^i(M)$ let us write $[m] := m + B^i(M) \in H^i(M)$.
Then $H^i(\varphi)([m]) = [\varphi^i(m)]$.

Prop. The assignment $M \mapsto H^i(M)$ and $\varphi \mapsto H^i(\varphi)$ is an A-linear functor $H^i : \underline{C}(\underline{\text{Mod}}\, A) \to \underline{\text{Mod}}\, A$.

Proof. Exercise.

Def. A short exact sequence in $\underline{C}(\underline{\mathrm{Mod}}\, A)$ is a diagram

$$(*) \qquad 0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

in $\underline{C}(\underline{\mathrm{Mod}}\, A)$, such for every i the seq.

$$0 \to L^i \xrightarrow{\varphi^i} M^i \xrightarrow{\psi^i} N^i \to 0$$

in $\underline{\mathrm{Mod}}\, A$ is exact.

Prop. Consider the diagram $(*)$. It is exact iff

$$0 \to \bigoplus_{i \in \mathbb{Z}} L^i \xrightarrow{\oplus \varphi^i} \bigoplus_i M^i \xrightarrow{\oplus \psi^i} \bigoplus_i N^i \to 0$$

is an exact seq in $\underline{\mathrm{Mod}}\, A$.

pf. Exercise.

<u>Theorem</u>. (Exact sequence of complexes)

Let $\left(0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0\right) = \underline{S}$

be an exact sequence in $\underline{C}(\underline{\mathrm{Mod}}\, A)$. Then there are A-lin homs

$$\partial^i_{\underline{S}} : H^i(N) \to H^{i+1}(L) \quad , \ i \in \mathbb{Z},$$

called <u>connecting homomorphisms</u>.

The collection $\{\partial^i_{\underline{S}}\}$ have these properties:

i) (Long exact seq. in cohomology)

The sequence of A-modules

$$\dots \to H^{i-1}(N) \xrightarrow{\partial^{i-1}_{\underline{S}}} H^i(L) \xrightarrow{H^i(\varphi)} H^i(M) \xrightarrow{H^i(\psi)} H^i(N) \xrightarrow{\partial^i_{\underline{S}}} H^{i+1}(L) \to \dots$$

is exact.

ii) (Functoriality) Suppose we have a second ex. seq.

$$\left(0 \to \tilde{L} \xrightarrow{\tilde{\varphi}} \tilde{M} \xrightarrow{\tilde{\psi}} \tilde{N} \to 0\right) = \tilde{\underline{S}},$$

in $\underline{C}(\underline{\mathrm{Mod}}\, A)$, and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & L & \xrightarrow{\varphi} & M & \xrightarrow{\psi} & N & \to & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & \tilde{L} & \xrightarrow[\tilde{\varphi}]{} & \tilde{M} & \xrightarrow[\tilde{\psi}]{} & \tilde{N} & \to & 0 \end{array}$$

in $\underline{C}(\underline{\mathrm{Mod}}\, A)$.

(140) Then for every i the diagram

$$
\begin{tikzcd}
H^i(N) \arrow[r, "\partial^i_{\underline{\varepsilon}}"] \arrow[d, "H^i(\gamma)"'] & H^{i+1}(L) \arrow[d, "H^{i+1}(\alpha)"] \\
H^i(\tilde{N}) \arrow[r, "\partial^i_{\tilde{\underline{\varepsilon}}}"] & H^{i+1}(\tilde{L})
\end{tikzcd}
$$

is commutative.

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