(141) <u>Exercise</u> (not related to thm.)

Let $M = (\cdots \to M^i \xrightarrow{d^i} M^{i+1} \to \cdots)$ be a complex of A-mods.
Show that $M_\cdot$ is an exact sequence iff $H^i(M) = 0$ for all i. In this case M is called an <u>acyclic complex</u>.

<u>Proof</u> (of theorem) By "diagram chasing":

<u>Step 1.</u> Take a cohomology class $[n] \in H^i(N)$, where $n \in Z^i(N)$.
We have to find a suitable cohom. class $\partial_S^i([n]) \in H^{i+1}(L)$.

Since $\psi^i : M^i \to N^i$ is surj., there is $m \in M^i$ s.t. $\psi^i(m) = n$.

$$\begin{tikzcd}
l \arrow[r, mapsto] & d(m) \arrow[r, mapsto] & 0 \\[-20pt]
\rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} & \\[-20pt]
L^{i+1} \arrow[r, "\varphi"] & M^{i+1} \arrow[r, two heads, "\psi"] & N^{i+1} \\
 & M^i \arrow[u, "d"] \arrow[r, two heads, "\psi"] & N^i \arrow[u, "d"'] \\[-20pt]
 & \rotatebox{90}{$\in$}\ m & \rotatebox{90}{$\ni$}\ n
\end{tikzcd}$$

Now $\psi(d(m)) = d(\psi(m)) = d(n) = 0$,

so by exactness at M^{i+1} there is $l \in L^{i+1}$ s.t. $\varphi(l) = d(m)$.

Next,
$\varphi(d(l)) = d(\varphi(d(m))) =$
$\varphi((d \circ d)(m)) = 0$. Since φ is $\hookrightarrow$,

$$\begin{tikzcd}
d(l) & L^{i+2} \arrow[r, hook] & M^{i+2} & 0 \\
l \in \arrow[u, mapsto] & L^{i+1} \arrow[u, "d"] \arrow[r, "\varphi"] & M^{i+1} \arrow[u] \ni d(m) \arrow[ur, mapsto] &
\end{tikzcd}$$

(142) conclude that $d(l)=0$. So $l \in Z^{i+1}(L)$.
We define

$$\boxed{\partial([n]) := [l] \in H^{i+1}(L).}$$

Step 2. Here we prove independence of choices.

We compare the process $n \rightsquigarrow m \rightsquigarrow l$ above, to an alternative process $n' \rightsquigarrow m' \rightsquigarrow l'$, where $n' \in Z^i(N)$, $[n'] = [n]$, $\psi(m') = n$, $\varphi(l') = d(m)$.

We must show that $[l'] = [l]$.

Since $[n'] = [n]$, there's $z \in N^{i-1}$ s.t. $n' - n = d(z)$. Choose $y \in M^{i-1}$ s.t. $\psi(y) = z$, and consider $z := m' - m - d(y) \in M^i$.
We have

$$\psi(z) = \psi(m') - \psi(m) - \psi(d(y)) = n' - n - d(\psi(y)) =$$
$$= n' - n - d(z) = 0.$$

By exact. $\exists$ $x \in L^i$ s.t. $\varphi(x) = z$.
Now

$$\varphi(l' - l - d(x)) = d(m') - d(m') - \underbrace{\varphi(d(x))}_{\substack{\| \\ d(\varphi(x)) \\ \| \\ d(z) \\ \| \\ d(m') - d(m)}} = 0.$$

Since φ is inj, $\Rightarrow$
$l' - l = d(x)$.
Thus $[l'] = [l]$.

Step 3 $\partial^i_{\underline{S}}$ is A-linear

$\boxed{\text{Exercise.}}$

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Step 4. Now we prove exactness at $H^i(N)$

$$H^i(M) \xrightarrow{H^i(\psi)} H^i(N) \xrightarrow{\partial_0^i} H^{i+1}(L)$$

Take $[n] \in H^i(N)$. Assume $[n] = H^i(\psi)([m])$ for some $m \in Z^i(M)$. Then $d(m) = 0$, and $l = 0$ in $Z^{i+1}(L)$ lifts $d(m)$. Thus

$$\partial^i([n]) = [0] = 0.$$

Conversely, say $\partial^i([n]) = 0$. This means that in the process of construction $n \rightsquigarrow m \rightsquigarrow l$, we have $l \in B^{i+1}(L)$. Thus $l = d(x)$ for some $x \in L^i$. Now

$$d(m) = \varphi(l) = \varphi(d(x)) = d(\varphi(x)).$$

Let
$$y := m - \varphi(x) \in M^i.$$

Then
$$d(y) = d(m) - d(\varphi(x)) = 0,$$

so $y \in Z^i(M)$. And

$$\psi(y) = \underbrace{\psi(m)}_{=n} - \underbrace{\psi(\varphi(x))}_{=0} = n, \quad \text{so}$$

$$H^i(\psi)([y]) = [\psi(y)] = [n].$$

<u>Step 5</u>. Exactness at $H^i(M)$.

$$H^i(L) \xrightarrow{H^i(\varphi)} H^i(M) \xrightarrow{H^i(\psi)} H^i(N)$$

Since $\psi \circ \varphi = 0$ and H^i is an additive functor, we get $H^i(\psi) \circ H^i(\varphi) = 0$.

Now let $[m] \in H^i(M)$ be s.t. $H^i(\psi)([m]) = 0$. So $\psi(m) = d(z)$ for some $z \in N^{i-1}$.

Let $y \in M^{i-1}$ be s.t. $\psi(y) = z$.

So $\psi(d(y)) = d(\psi(y)) = d(z) = \psi(m)$.

$$\begin{tikzcd}
\ell & m-d(y) \arrow[r, mapsto] & 0 & \\
L^i \arrow[r, "\varphi"] & M^i \arrow[r, "\psi"] & N^i & \ni \psi(m) \\
 & M^{i-1} \arrow[u, "d"] \arrow[r, two heads, "\psi"] & N^{i-1} \arrow[u, "d"] & \ni z \arrow[u] \\
 & y \arrow[rr, mapsto] & & z
\end{tikzcd}$$

We get $\psi(m - d(y)) = 0$, so $\exists\ \ell \in L^i$ s.t. $\varphi(\ell) = m - d(y)$.

Now

$$\varphi(d(l)) = d(\varphi(l)) = d(m - d(y))$$
$$= d(m) - d(d(y)) = 0 - 0 = 0.$$

Since φ is inj. $\Rightarrow$ $d(l) = 0$; so $l \in Z^i(L)$. And

$$H^i(\varphi)([l]) = [\varphi(l)] = [m - d(y)] = [m].$$

Step 6. Exactness at $H^i(L)$.

Exercise. (Similar to step 4)

to here 6/1

Step 7. Functoriality. Given comm. diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & M \arrow[r, "\psi"] \arrow[d, "\beta"'] & N \arrow[r] \arrow[d, "\gamma"] & 0 & \Sigma \\
0 \arrow[r] & L_\dagger \arrow[r, "\varphi_\dagger"] & M_\dagger \arrow[r, "\psi_\dagger"] & N_\dagger \arrow[r] & 0 & \Sigma_\dagger
\end{tikzcd}$$

in $\mathsf{C}(\mathrm{Mod}\, A)$, rows ex. seq. Take $[n] \in H^i(N)$.

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$$\left(H^{i+1}(\alpha)\circ\partial^i_{\underline{s}}\right)([n]) = \left(\partial^i_{\underline{s}_t}\circ H^i(\gamma)\right)([n]).$$

Say $n \rightsquigarrow m \rightsquigarrow l$ is a choice process for $\partial^i_{\underline{s}}([n])$. So $\psi(m)=n$ and $\varphi(l)=d(m)$.

Let $n_t := \gamma(n) \in N^i_t$, $m_t := \beta(m)$ and $l_t := \alpha(l)$.

Then $d(n_t)=d(\gamma(n))=\gamma(d(n))=0$, so $n_t \in Z^i(N_t)$ and $\boxed{H^i(\gamma)([n])=[n_t].}$ Now

$$\psi_t(m_t)=\psi_t(\gamma(m))=\gamma(\psi(m))=\gamma(n)=n_t$$

and $$\varphi_t(l_t)=\varphi_t(\alpha(l))=\beta(\varphi(l))$$
$$=\beta(m)=m_t.$$

We see that $n_t \rightsquigarrow m_t \rightsquigarrow l_t$ is a choice proc. for $\boxed{\partial^i_{\underline{s}_t}([n_t])=[l_t]}$.

Finally

$$\boxed{H^{i+1}(\varphi)([l])=[\varphi(l)]=[l_t].}$$

We get

$$H^{i+1}(\varphi)\left(\partial^i_{\underline{s}}([n])\right)=H^{i+1}(\varphi)([l])$$
$$=[l_t]=\partial^i_{\underline{s}_t}([n_t])=\partial^i_{\underline{s}}\left(H^i(\gamma)([n])\right).$$

$\square$