Homotopies

(A is a commutative ring)

Def.

Let M, N be complexes (of A-modules) and let

$$\varphi_0, \varphi_1 : M \to N$$

be homomorphisms in $\underline{C}(\underline{\text{Mod}}\, A)$.

A homotopy $\gamma : \varphi_0 \to \varphi_1$ is an A-lin. hom. $\gamma : M \to N$ of degree -1, such that

$$\boxed{d_N \circ \gamma + \gamma \circ d_M = \varphi_1 - \varphi_0}$$

as hom's $M \to N$.

In the def. I used the "graded" notation.

Note that φ_i satisfy $d_N \circ \varphi_i = \varphi_i \circ d_M$.

Exercise Translate this def. to the "collection" notation, i.e. $\varphi_i = \{\varphi_i^j\}_{j \in \mathbb{Z}}$ etc.

Def. Let $\varphi_0, \varphi_1 : M \to N$ be homs in $\underline{C}(\underline{\text{Mod}}\, A)$. We say that φ_1 is homotopic to φ_0 if there exists a homotopy $\gamma : \varphi_0 \to \varphi_1$. This is denoted by $\varphi_0 \sim_h \varphi_1$.

Prop. Homotopy is an equivalence relation on the set $\mathrm{Hom}_{\underline{C}(\underline{\mathrm{Mod}}\,A)}(M, N)$.

pf. Exercise.

Prop. Let $\varphi_0, \varphi_1 : M \to N$ and $\psi_0, \psi_1 : N \to P$ be hom's in $\underline{C}(\underline{\mathrm{Mod}}\,A)$. If $\varphi_0 \sim_h \varphi_1$ and $\psi_0 \sim_h \psi_1$, then $\psi_0 \circ \varphi_0 \sim_h \psi_1 \circ \varphi_1$.

pf Exercise

Prop Let $\varphi_0, \varphi_1 : M \to N$ be hom's in $\underline{C}(\underline{\mathrm{Mod}}\,A)$. If φ_0 & φ_1 are homotopic, then

$$H^i(\varphi_0) = H^i(\varphi_1),$$

as hom's

$$H^i(M) \to H^i(N),$$

for all i.

proof. Say $\gamma : \varphi_0 \to \varphi_1$ is a homotopy. Take $[m] \in H^i(M)$. Then

$$H^i(\varphi_j)([m]) = [\varphi_j(m)], \quad j = 0, 1.$$

But

$$\varphi_1(m) - \varphi_0(m) = (\varphi_1 - \varphi_0)(m) = (d_N \circ \gamma + \gamma \circ d_M)(m)$$

$d(m)=0$

$$= d_N(\gamma(m)),$$

so $[\varphi_1(m)] = [\varphi_0(m)]$. $\square$

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Def. A hom. $\varphi: M \to N$ in $\underline{C}(\underline{Mod}\, A)$ is called a <u>homotopy equivalence</u> if there is a hom. $\psi: N \to M$ in $\underline{C}(\underline{Mod}\, A)$, such that $\psi \circ \varphi \sim_h 1_M$ and $\varphi \circ \psi \sim_h 1_N$.

Exercise. Show that homotopy equiv. is and equiv. relation on the set $Ob(\underline{C}(\underline{Mod}\, A))$.

Def. A hom. $\varphi: M \to N$ in $\underline{C}(\underline{Mod}\, A)$ is called a <u>quasi-isomorphism</u> if $H(\varphi): H(M) \to H(N)$ is an isom. of graded A-mod's.

Above $H(M) := \bigoplus_{i \in \mathbb{Z}} H^i(M)$ and $H(\varphi) := \bigoplus_i H^i(\varphi)$.

Prop If $\varphi: M \to N$ is a homotopy equivalence, then it is a quasi-isomorphism

proof Let $\psi: N \to M$ be a homotopy-inverse of φ; i.e. $\psi \circ \varphi \sim_h 1_M$ and $\varphi \circ \psi \sim_h 1_N$.

Then
$$H(\psi) \circ H(\varphi) \underset{\text{funct. of } H}{=} H(\psi \circ \varphi) \underset{\psi\circ\varphi \sim_h 1_M}{=} H(1_M) \underset{\text{identity}}{=} 1_{H(M)}$$

150 Likewise $H(\varphi) \circ H(\psi) = \mathbb{1}_{H(N)}$.

This shows that $H(\varphi)$ is an isom of graded A-modules. $\square$

Digression Let $M, N \in \underline{C}(\underline{\mathrm{Mod}}\, A)$. Then

$$(M \text{ isom. to } N) \Rightarrow (M \text{ homotopy equ. to } N) \Rightarrow$$
$$(M \text{ qu. isom. to } N).$$

This is analogous to $\underline{\mathrm{Top}} \ni X, Y$:

$$(X \text{ homeomorphic to } Y) \Rightarrow (X \text{ hom. equiv. to } Y)$$
$$\Rightarrow \left(\text{there is } f : X \to Y \text{ s.t. } H_i(f) : H_i(X; \mathbb{Z}) \to H_i(Y; \mathbb{Z}) \text{ are } \cong\right).$$

Functors & Complexes

Def Suppose $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$ is an A-linear functor (*)

(1) Given $(M, d_M) \in \underline{C}(\underline{\mathrm{Mod}}\, A)$, let $(F(M), d_{F(M)})$ be the complex $F(M)^i := F(M^i)$ and $d^i_{F(M)} := F(d^i_M)$.

$$F(M) = \left(\cdots \to F(M^i) \xrightarrow{F(d^i_M)} F(M^{i+1}) \to \cdots \right)$$

(2) Given $\varphi : M \to N$ in $\underline{C}(\underline{\mathrm{Mod}}\, A)$, let $F(\varphi) : F(M) \to F(N)$ be the hom. in $\underline{C}(\underline{\mathrm{Mod}}\, A)$ with $F(\varphi)^i := F(\varphi^i)$.

Prop $F : \underline{C}(\underline{\mathrm{Mod}}\, A) \to \underline{C}(\underline{\mathrm{Mod}}\, A)$ is an A-lin functor.

pf Exercise.

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One might ask if F preserves quasi-isoms, i.e. if for any quasi-isom $\varphi : M \to N$, the hom $F(\varphi) : F(M) \to F(N)$ is a qu.-isom. The answer is: No.

(*) More generally can look at $\mathbb{K}$-algebras A & B (non-comm), and at a $\mathbb{K}$-linear functor $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B$.