Definition. Let A be a commutative ring, and let $F: \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$ be an A-linear functor.

(1) For any $M \in \underline{\mathrm{Mod}}\, A$ choose a projective resolution $\boxed{\tilde{\eta}_M : P_M \to M}$, and define the A-modules

(†) $$\boxed{L_i F(M) := H^{-i}(P_M)} \quad , \quad i \in \mathbb{Z}.$$

(2) For any hom. $\varphi : M \to N$ in $\underline{\mathrm{Mod}}\, A$ choose a hom. $\boxed{\tilde{\varphi} : P_M \to P_N}$ in $\underline{C}(\underline{\mathrm{Mod}}\, A)$ s.t.

$$\varphi \circ \tilde{\eta}_M = \tilde{\eta}_N \circ \tilde{\varphi} \quad ,$$

and define A-lin. homs.

$$\boxed{L_i F(\varphi) : L_i F(M) \to L_i F(N)}$$

by the formula

(††) $$\boxed{L_i F(\varphi) := H^{-i}(\tilde{\varphi})}$$

(3) For any $M \in \underline{\mathrm{Mod}}\, A$ let $\boxed{\eta_M : L_0 F(M) \to F(M)}$ be the hom.

(†††) $$\boxed{\eta_M := H^0(F(\tilde{\eta}_M)) : H^0(F(P_M)) \to F(M).}$$

Note that all choices in this definition are possible.

<u>Theorem</u>. Let A be a comm. ring, and let $F : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, A$ be an A-linear functor.

(1) Take $i \in \mathbb{Z}$. The assignments $M \mapsto L_i F(M)$ and $\varphi \mapsto L_i F(\varphi)$, for objects and morphisms in $\underline{\text{Mod}}\, A$, are an A-linear functor

$$L_i F : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, A.$$

(2) The homomorphism $\eta_M : L_0 F(M) \to F(M)$, for $M \in \underline{\text{Mod}}\, A$, are a natural transformation

$$\eta : L_0 F \to F.$$

If F is right exact, then η is a natural isomorphism

(3) $L_i F = 0$ for $i < 0$.

162 Proof. (1) First we prove that L_iF is a functor. If $\varphi = 1\!\!1_M$ (and $N=M$), then our lift $\tilde{\varphi}: P_M \to P_M$ of $1\!\!1_M$ is homotopic to $1\!\!1_{P_M}: P_M \to P_M$, by Theorem p. 156. Hence, by Prop. p.148 we have

$$H^{-i}(F(\tilde{\varphi})) = H^{-i}(F(1\!\!1_{P_M})): H^{-i}(F(P_M)) \to H^{-i}(F(P_M)).$$

But $H^{-i}(F(1\!\!1_{P_M})) = 1\!\!1_{H^{-i}(F(P_M))}$.

Thus
$$L_iF(1\!\!1_M) = 1\!\!1_{L_iF(M)}.$$

Now consider
$$\begin{tikzcd} L \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & M \arrow[r, "\psi"] & N \end{tikzcd}$$

and the liftings $\tilde{\varphi}$, $\tilde{\psi}$ and $\widetilde{\psi\circ\varphi}$.

Since both $\tilde{\psi}\circ\tilde{\varphi}$ and $\widetilde{\psi\circ\varphi}$ lift $\psi\circ\varphi$, they are homotopic. Thus

$$L_i(F)(\psi)\circ L_iF(\varphi) \overset{\text{def}}{=} H^{-i}(F(\tilde{\psi}))\circ H^{-i}(F(\tilde{\varphi}))$$
$$\overset{H^{-i}, F \text{ functors}}{=} H^{-i}(F(\tilde{\psi}\circ\tilde{\varphi}))$$
$$\overset{\text{homotopy}}{=} H^{-i}(F(\widetilde{\psi\circ\varphi})) \overset{\text{def}}{=} L_iF(\psi\circ\varphi).$$

So L_iF is a functor. The fact that it is A-linear is left as an Exercise.

(2) We have to prove that for any hom. $\varphi : M \to N$ in $\underline{\text{Mod}}\,A$, the diag.

(†)

$$\begin{tikzcd} L_0F(M) \arrow[r, "L_0F(\varphi)"] \arrow[d, "\eta_M"'] & L_0F(N) \arrow[d, "\eta_N"] \\ F(M) \arrow[r, "F(\varphi)"] & F(N) \end{tikzcd}$$

is comm. But this follows from comm. of diag

(‡)

$$\begin{tikzcd} \underline{P}^{-1}_M \arrow[r, "\tilde{\varphi}^{-1}"] \arrow[d, "d"'] & \underline{P}^{-1}_N \arrow[d, "d"] \\ \underline{P}^0_M \arrow[r, "\tilde{\varphi}^0"] \arrow[d, "\tilde{\eta}_M"'] & \underline{P}^0_N \arrow[d, "\tilde{\eta}_N"] \\ M \arrow[r, "\varphi"] \arrow[d] & N \arrow[d] \\ 0 & 0 \end{tikzcd}$$

,

which implies comm. of diag

(‡‡)

$$\begin{tikzcd} F(\underline{P}^{-1}_N) \arrow[r, "F(d)"] \arrow[d, "F(\tilde{\varphi}^{-1})"] & F(\underline{P}^0_M) \arrow[r, "F(\tilde{\eta}_M)"] \arrow[d, "F(\tilde{\varphi}^0)"] & F(M) \arrow[r] \arrow[d, "F(\varphi)"] & 0 \\ F(\underline{P}^{-1}_N) \arrow[r, "F(d)"'] & F(\underline{P}^0_N) \arrow[r, "F(\tilde{\eta}_N)"'] & F(N) \arrow[r] & 0 \end{tikzcd}$$

which in turn implies the commutativity of

$$(\diamondsuit)\qquad \begin{tikzcd} H^0(F(\underline{P}_M)) \arrow[r, "H^0(F(\tilde{\eta}_M))"] \arrow[d, "H^0(F(\tilde{\varphi}))"] & F(M) \arrow[d, "F(\varphi)"] \\ H^0(F(\underline{P}_N)) \arrow[r, "H^0(F(\tilde{\eta}_N))"] & F(M) \end{tikzcd}$$

But $(\diamondsuit)$ is $(\spadesuit)$ by definition.

If F is right exact, then rows in $(\heartsuit\heartsuit)$ are exact sequences, and therefore rows in $(\diamondsuit)$ are $\cong$.

(3) Trivial. $\triangleleft$

Def. The functor $L_i F$ is called the <u>i-th left derived functor</u> of F.

(165) <u>Theorem</u> (Independence)

The functors $L_i F$ are independent of the choices made in Def. (p. 160) in the following sense. Suppose for each module M we choose another proj. res. $\tilde{\eta}'_M : \underline{P}'_M \to M$, and for every hom $\varphi : M \to N$ we choose some lifting $\tilde{\varphi}' : \underline{P}'_M \to \underline{P}'_N$. Let $L'_i F$ be the corresponding functors, i.e.

$$L'_i F(M) := H^{-i}(F(\underline{P}'_M)) \quad \text{and} \quad L'_i F(\varphi) := H^{-i} F(\tilde{\varphi}').$$

Let $\eta' : L'_0 F \to F$ be the nat. trans.

$$\eta'_M := H^0(F(\tilde{\eta}'_M)) : L'_0 F(M) \to F(M).$$

Then: (1) For every i there is a natural isomorphism $\boxed{\chi_i : L'_i F \xrightarrow{\simeq} L_i F.}$

(2) The diagram of nat. trans.

$$\begin{tikzcd} L'_0 F \arrow[rr, "\chi_0"] \arrow[dr, "\eta'"'] & & L_0 F \arrow[dl, "\eta"] \\ & F(M) & \end{tikzcd}$$

is commutative.

~~proof~~. (Sketch). For any M choose a hom of complexes $\tilde{\chi}_M : \underline{P}'_M \to \underline{P}_M$ lifting $\mathbb{1}_M : M \to M$.

The hom. $\tilde{\chi}_M$ is a homotopy equivalence — use Thm (p. 156). Then $F(\tilde{\chi}_M) : F(\underline{P}'_M) \to F(\underline{P}_M)$ is qu-isom. For each i let

$$\chi_{i,M} := H^{-i}(F(\tilde{\chi}_M)) : L'_i F(M) \xrightarrow{\cong} L_i F(M).$$

For any $\varphi : M \to N$ we have a diagram

$$\begin{tikzcd} \underline{P}'_M \arrow[r, "\tilde{\varphi}'"] \arrow[d, "\tilde{\chi}_M"'] & \underline{P}'_N \arrow[d, "\tilde{\chi}_N"] \\ \underline{P}_M \arrow[r, "\tilde{\varphi}"] & \underline{P}_N \end{tikzcd}$$

which is commutative up to homotopy — i.e. $\tilde{\chi}_N \circ \tilde{\varphi}' \sim_h \tilde{\varphi} \circ \tilde{\chi}_M$. This is because both lift φ.

Applying $H^i \circ F$ we get a comm. diag.

$$\begin{tikzcd} L'_i F(M) \cong H^i(F(\underline{P}'_M)) \arrow[r, "L'_i F(\varphi)"] \arrow[d, "\chi_{M,i}"'] & H^i(F(\underline{P}'_N)) \cong L'_i F(N) \arrow[d, "\chi_{N,i}"] \\ L_i F(M) \cong H^i(F(\underline{P}_M)) \arrow[r, "L_i F(\varphi)"] & H^i(F(\underline{P}_N)) \cong L_i F(N) \end{tikzcd}$$

This is (1). For (2): have comm. diagram ......

$$\begin{tikzcd} \underline{P}'^{0}_M \arrow[rr, "\tilde{\chi}^0_M"] \arrow[dr] & & \underline{P}^0_M \arrow[dl] \\ & M & \end{tikzcd}$$

$\square$

16.7

Exact Sequences

<u>Lemma</u>. Let $(*)$ $0 \to L \to M \to N \to 0$ be an exact seq. in <u>Mod</u> A. Then there is a comm. diagram

$$\begin{tikzcd} 0 \arrow[r] & P_L \arrow[r] \arrow[d, "\eta_L"'] & P_M \arrow[r] \arrow[d, "\eta_M"'] & P_N \arrow[r] \arrow[d, "\eta_N"] & 0 \\ 0 \arrow[r] & L \arrow[r] & M \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

in $\underline{C}$(<u>Mod</u> A), such that the vertical arrows are projective resolutions, and the first row is exact.

<u>pf</u>. <u>Step 1</u>. Given $(*)$, we find proj. mods P_L^0, P_M^0, P_N^0, and a comm. diag

$$(**)\quad \begin{tikzcd} 0 \arrow[r] & P_L^0 \arrow[r, "\tilde\varphi^0"] \arrow[d, "\eta_L^0"'] & P_M^0 \arrow[r, "\tilde\psi^0"] \arrow[d, "\eta_M^0"'] & P_N^0 \arrow[r] \arrow[d, "\eta_N^0"] & 0 \\ 0 \arrow[r] & L \arrow[r, "\varphi"] & M \arrow[r, "\psi"] & N \arrow[r] & 0 \end{tikzcd}$$

s.t. first row exact. Here's how: choose any surjections $P_L^0 \twoheadrightarrow L$ & $P_N^0 \twoheadrightarrow N$. Let $P_M^0 := P_L^0 \oplus P_N^0$ ... also proj.

Choose $\alpha : P_N^0 \to M$ lifting ψ. Let $\eta_M^0 := \varphi \circ \eta_L^0 \oplus \alpha$. Get comm. diag. $(**)$

Where P_L^0 & P_N^0 are proj. mods.

(168) Clearly first row is exact; but

Why is η^0_M surjective? We view columns of $(**)$ as complexes, with $P^0_L, \dots$ in deg 0. So $(**)$ is an exact seq. of complexes. In H^{-1} get seq. of maps

$$\begin{array}{ccccccc} 0 \to & \mathrm{Coker}(\eta^0_L) & \to & \mathrm{Coker}(\eta^0_M) & \to & \mathrm{Coker}(\eta^0_N) & \to \cdots \\ & \| & & \| & & \| & \\ & 0 & & ? & & 0 & \end{array}$$

Since η^0_L & η^0_N surj $\Rightarrow$ η^0_M surj.

<u>Step 2.</u> Say we have, for $i \geq 0$, comm. diag. of mods,

$$\begin{array}{ccccccccc} 0 & \to & P^{-i}_L & \to & P^{-i}_M & \to & P^{-i}_N & \to & 0 \\ & & \downarrow d^{-i}_L & & \downarrow d^{-i}_M & & \downarrow d^{-i}_N & & \\ & & \vdots & & \vdots & & \vdots & & \\ 0 & \to & P^0_L & \to & P^0_M & \to & P^0_N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L & \to & M & \to & N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \\ & & D_L & & D_M & & D_N & & \end{array}$$

P^j_L, P^j_M, P^j_N proj, exact rows; and columns exact in deg $> -i$.

so $0 \to D_L \to D_M \to D_N \to 0$ ex. seq. of complexes.

Consider ex. seq. in cohomology:

$$\begin{array}{ccccccccc} & & \mathrm{Ker}(d^{-i}_L) & \to & \mathrm{Ker}(d^{-i}_M) & \to & \mathrm{Ker}(d^{-i}_N) & & \\ & & \| & & \| & & \| & & \\ H^{-i-1}(D_N) & \to & H^{-i}(D_L) & \to & H^{-i}(D_M) & \to & H^{-i}(D_N) & \to & H^{-i+1}(D_L) \\ \| & & & & & & & & \| \\ 0 & & & & & & & & 0 \end{array}$$

since $D^{-i-1}_N = 0$

since column exact at $P^{-i+1}_L = D^{-i+1}_L$

By Step 1 can produce new exact row of projectives:

$$\begin{array}{ccccccc} 0 \to & P^{-i-1}_L & \to & P^{-i-1}_M & \to & P^{-i-1}_N & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \end{array}$$

$\square$

169 **Theorem**. Given a short exact sequence

$$\underline{s} = (0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0)$$

in $\underline{\mathrm{Mod}}\ A$, there are connecting homomorphisms

$$\partial_i^{\underline{s}} : L_i F(N) \to L_{i-1} F(L)$$

in $\underline{\mathrm{Mod}}\ A$, such that the sequence

$$\cdots \to L_{i+1} F(N) \xrightarrow{\partial_{i+1}^{\underline{s}}} L_i F(L) \xrightarrow{L_i F(\varphi)} L_i F(M) \xrightarrow{L_i F(\psi)} L_i F(N) \xrightarrow{\partial_i^{\underline{s}}} L_{i-1} F(L) \to \cdots$$

is <u>exact</u>.

<u>Pf</u>. By Thm (p. 165) can use any proj. resolutions. We take those provided by previous lemma. The <u>rows</u> are <u>split exact</u> - since $\underline{P}_N^{-i}$ are proj. Hence for every i we have an ex. seq

$$0 \to F(\underline{P}_L^{-i}) \to F(\underline{P}_M^{-i}) \to F(\underline{P}_N^{-i}) \to 0.$$

So the seq. of complexes

$$0 \to F(\underline{P}_L) \xrightarrow{F(\tilde{\varphi})} F(\underline{P}_M) \xrightarrow{F(\tilde{\psi})} F(\underline{P}_N) \to 0$$

is exact. Acc. to Thm. (p. 139) we get long exact seq. in cohomology. $\square$

170

The Functors Tor

<u>Def.</u> Let A be a commutative ring, and let $M \in \underline{\text{Mod}}\, A$. Let $F : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, A$ be the functor

$$\boxed{F := M \otimes_A - .}$$

The left derived functor $L_i F$ are denoted by

$$\boxed{\mathrm{Tor}_i^A(M, -) := L_i F.}$$

We know that

$$\mathrm{Tor}_0^A(M, -) \cong M \otimes_A -$$

as functors.

What is significance of Tor_i^A, for $i \geq 1$?

Is it true that

$$\mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^A(N, M) \quad ?$$

Answers in final assignment!

~ End ~