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Prop. Any A-module is a quotient of a free A-mod.

pf. Let M be an A-module. Choose any family $\{m_x\}_{x\in X}$ of elements of M that generates it. Such exist; e.g. we can take $X := M$ and $m_x := x$. Now we can either use the universal property of the free module $F_{fin}(X,A) =: F$ and its basis $\{\delta_x\}$ to see that there is a hom. $\varphi : F \to M$ s.t. $\varphi(\delta_x) = m_x$; or we can just write the formula for φ explicitly:

$$\varphi(\{a_x\}) = \sum_x a_x \cdot m_x \in M.$$

Since $\{m_x\}$ generates, this hom. is surjective. $\square$

In this prop. we tacitly assume that A is nontriv.; we shall always do so when talking about free modules.

A homomorphism between free modules

$$\varphi : F_{fin}(X,A) \longrightarrow F_{fin}(Y,A)$$

is described by an $X \times Y$ matrix with entries in A:

$$\Phi : X \times Y \to A \text{, or } \Phi = \{a_{x,y}\}_{x,y\in X}$$

satisfying

$$\varphi(\delta_x) = \sum_{y\in Y} a_{x,y} \cdot \delta_y$$

Of course the matrices that appear are "finite in the y-direction".

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Exact Sequences

This is "homological algebra".

Let A be a ring. A sequence of A-modules is the data

$$\cdots M_{-1} \xrightarrow{\varphi_{-1}} M_0 \xrightarrow{\varphi_1} M_1 \to \cdots \xrightarrow{\varphi_i} M_i \cdots$$

consisting of modules M_i and homomorphisms φ_i. The sequence could be finite or infinite on either side.

Def Let

$$(*) \qquad \cdots M_{i-1} \xrightarrow{\varphi_i} M_i \xrightarrow{\varphi_{i+1}} M_{i+1} \cdots$$

be a sequence of A-modules. We say that $(*)$ is exact at M_i if

$$\mathrm{Im}(\varphi_i) = \mathrm{Ker}(\varphi_{i+1}).$$

We say $(*)$ if exact if it is exact at all modules M_j of it (that are neither first nor last).

Def. A short exact sequence is an exact seq. with five modules, the first and last being 0. I.e.

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0.$$

Let's see what this means. Exactness on M' means

20 that $\mathrm{Ker}(\varphi) = \mathrm{Im}(0 \to M') = 0 \subseteq M'$. I.e. φ is injective.⊙ Likewise ψ is surjective. Exactness at M says that $\mathrm{Ker}(\psi) \cong M'$. We see that

$$\boxed{\frac{M}{M'} \cong M''}$$

⊙ So we can identify M' with a submod. of M, using φ.

Exercise. $0 \to M_0 \to M_1 \to M_2 \to 0$ ex. seq. of fin. rank free mods. Then $\sum_i (-1)^i \cdot \mathrm{rank}(M_i) = 0$.

Presentation of Modules

Let M be an A-module. A presentation of M is an exact sequence

$$(\dagger) \qquad F_1 \xrightarrow{\psi} F_0 \xrightarrow{\varphi} M \to 0$$

where F_0 and F_1 are free modules.

Every module M has a presentation: choose a surjection $\varphi : F_0 \to M$ from a free module F_0 and then a surjection $\psi : F_1 \to \mathrm{Ker}(\varphi)$ from a free mod. F_1.

Some presentations are better. A finite presentation of M is a pres. $(\dagger)$ in which F_0 & F_1 have finite ranks. If M has a fin. presentation then we call it finitely presented.

21) Example. Let K be a field, and let A be a polynomial ring over K in fin. many variables. We will prove later that any fin. gen. A-module is fin. pres.

Now assume $A = K[t_1, t_2, \ldots]$, polynomials in countably many variables. Let $\mathfrak{m}$ be the ideal gen. by the variables. Then $M := A/\mathfrak{m}$ is a fin. gen. A-module (it is cyclic), by it is not fin. presented! (Not easy to prove.)

Exercise. Let M be a free A-module (A nontrivial). Show that M is finitely generated iff it has finite rank.

In a presentation (P), a basis of F_0 is referred to as "generators of M"; and a basis of F_1 is "relations".

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Tensor Products

This is an extremely important operation, but quite tricky!

Fix a ring A. Let M, N, L be A-modules. A <u>bilinear</u> function

$$\beta : M \times N \longrightarrow L$$

is a function satisfying:

- $\beta(m_1 + m_2, n) = \beta(m_1, n) + \beta(m_2, n)$
- $\beta(a \cdot m, n) = a \cdot \beta(m, n)$

lin. in 1^{st} arg

- $\beta(m, n_1 + n_2) = \beta(m, n_1) + \beta(m, n_2)$
- $\beta(m, a \cdot n) = a \cdot \beta(m, n)$

lin. in 2^{nd} arg

Notice that if $\beta_1, \beta_2 : M \times L \to L$ are bilinear func's, then so are $\beta_1 + \beta_2$ and $a \cdot \beta_1$.

<u>Def</u>. Let M and N be A-modules. A <u>tensor product</u> of M and N is a pair (β, L), consisting of an a module L and a bilin. func.

$$\beta : M \times N \to L,$$

having this universal property:

(♡) Let L' be any A-mod, and let $\beta' : M \times N \to L'$ be an A-bilin function. Then there exists a unique A-lin hom

$$\varphi : L \longrightarrow L'$$

s.t. $\quad \beta' = \varphi \circ \beta.$

23 We can express this condition as a commutative diagram:

$$\begin{tikzcd} M \times N \arrow[r, "\beta"] \arrow[dr, "\beta'"'] & L \arrow[d, dashed, "\varphi"] \\ & L' \end{tikzcd}$$

Prop. Let M & N be A-modules. Suppose that (L, β) and (L', β') are both tensor products of M & N. Then there is a unique A-linear isomorphism

$$\varphi : L \xrightarrow{\simeq} L'$$

s.t.

$$\beta' = \varphi \circ \beta.$$

proof. Exercise.

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Thm. Let M & N be A-modules. Then a tensor product (L, β) of M & N exists.

proof. We will construct L using gen's & rel's. Consider the set $M \times N$ and the free module $F := F_{fin}(M \times N, A)$.

Inside F we consider the submodule K generated by all elements of these kinds:

(cont)