31 Examples of categories.

(1) The category $\underline{\mathrm{Set}}$. The objects are all the sets; the morphisms are the functions; the identity 1_X is the identity function of the set X; and composition is the usual composition of functions.

(2) The category $\underline{\mathrm{Grp}}$ of groups. Morphisms are group homomorphisms.

(3) The category $\underline{\mathrm{Grp}}_{fin}$ of finite groups. This is a subcategory of $\underline{\mathrm{Grp}}$:

$$\mathrm{Ob}(\underline{\mathrm{Grp}}_{fin}) \subset \mathrm{Ob}(\underline{\mathrm{Grp}})$$

and the operations are the same.

(4) The category $\underline{\mathrm{Ab}}$ of abelian groups. This too is a subcategory of $\underline{\mathrm{Grp}}$.

(5) Let A be a (comm.) ring. The category $\underline{\mathrm{Mod}}\, A$ has all A-modules as its objects. The morphisms are the A-linear homomorphisms. We write

$$\mathrm{Hom}_A(M, N) := \mathrm{Hom}_{\underline{\mathrm{Mod}}(A)}(M, N).$$

(6) Let G be a group. Define a category $\underline{G}$ like this: $\mathrm{Ob}(\underline{G}) := \{\emptyset\}$, a singleton, and $\mathrm{Hom}_{\underline{G}}(\emptyset, \emptyset) := G$. The identity is $1_\emptyset := 1 \in G$, and composition is multiplication in G.

There is something special in examples (4) & (5): these are *linear categories*, as defined next.

Def. Let A be a (comm.) ring. An *A-linear category* is a category $\underline{C}$, together with an A-module structure on all the sets $\mathrm{Hom}_{\underline{C}}(X,Y)$, $X,Y \in \mathrm{Ob}(\underline{C})$. The condition is that the compositions

$$\mathrm{Hom}_{\underline{C}}(X,Y) \times \mathrm{Hom}_{\underline{C}}(Y,Z) \to \mathrm{Hom}_{\underline{C}}(X,Y)$$

are A-bilinear.

If $A = \mathbb{Z}$, then we call $\underline{C}$ a *linear category*.

We usually write $X \in \underline{C}$ instead of $X \in \mathrm{Ob}(\underline{C})$.

Exercise. Let $\underline{C}$ be a linear category. Show that $\mathrm{Hom}_{\underline{C}}(X,X)$ is a noncomm. ring, for any $X \in \underline{C}$.

Exercise Let A be a noncommutative ring. Show that $\underline{\mathrm{Mod}}\, A$ is a $Z(A)$-linear category. Here $Z(A)$ is the center of A.

33 **Def.** Let $\underline{C}$ and $\underline{D}$ be categories. A <u>functor</u> $F : \underline{C} \to \underline{D}$ consists of:

- A function $F_{ob} : Ob(\underline{C}) \longrightarrow Ob(\underline{D})$.

- A function

$$F_{X,Y} : Hom_{\underline{C}}(X,Y) \longrightarrow Hom_{\underline{D}}(F_{ob}(X), F_{ob}(Y))$$

for every $X, Y \in Ob(\underline{C})$.

The conditions are:

(i) $F_{X,X}(1\!\!1_X) = 1\!\!1_{F_{ob}(X)}$ for every $X \in Ob(\underline{C})$.

(ii) $F_{X,Z}(\psi \circ \varphi) = F_{Y,Z}(\psi) \circ F_{X,Y}(\varphi)$

for every $\varphi \in Hom_{\underline{C}}(X,Y)$ and $\psi \in Hom_{\underline{C}}(Y,Z)$.

<u>Examples</u> (1) The forgetful functor $F : \underline{Grp} \to \underline{Set}$ that forgets the group structure.

(2) Let $f : A \to B$ be a ring hom. If N is a B-module, then we can make it an A-module by $a \cdot n := f(a) \cdot n$, for $n \in N$ and $a \in A$. We get a functor $rest_f : \underline{Mod}\, B \to \underline{Mod}\, A$, called <u>restriction of scalars</u>.

34 For simplicity we write F instead of F_{Ob} & $F_{X,Y}$.

<u>Exercise</u>. Let A be a ring. For a set X define
$$F(X) := F_{fin}(X, A) \in \underline{\mathrm{Mod}}\, A.$$
Show that F can be made into a functor
$$F : \underline{\mathrm{Set}} \to \underline{\mathrm{Mod}}\, A.$$
(You have to define $F(\varphi)$ for a morphism $\varphi : X \to Y$ in $\underline{\mathrm{Set}}$.)

The functor rest_f above has a special property, which we now introduce.

<u>Def</u> Let $\underline{C}$ and $\underline{D}$ be A-linear categories, where A is some (comm.) ring. A functor $F : \underline{C} \to \underline{D}$ is called <u>A-linear</u> if for every $X, Y \in \underline{C}$ the function
$$F : \mathrm{Hom}_{\underline{C}}(X, Y) \to \mathrm{Hom}_{\underline{D}}(F(X), F(Y))$$
is A-linear.

When $A = \mathbb{Z}$, we say that F is <u>linear</u>, or <u>additive</u>.

35 <u>Examples</u> ① Let A be a (commutative) ring and $M \in \underline{\mathrm{Mod}}\, A$. For $N \in \underline{\mathrm{Mod}}\, A$ let $F(N) := M \otimes_A N$; and for $\varphi : N_0 \to N_1$ in $\underline{\mathrm{Mod}}\, A$ let

$$F(\varphi) := \mathbb{1}_M \otimes_A \varphi : M \otimes_A N_1 \to M \otimes_A N_2.$$

Using Prop. (D) on page 17, it follows that

$$F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$$

is a functor. It is easy to see that $F(\varphi + \psi) = F(\varphi) + F(\psi)$ and $F(a \cdot \varphi) = a \cdot F(\varphi)$. So F is an A-lin. functor.

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We know that if $\underline{C}$ is a linear cat, then $\mathrm{Hom}_{\underline{C}}(X,X)$ is a ring, where multiplication is composition.

<u>Prop.</u> Let $\underline{C}$ and $\underline{D}$ be linear categories, and let $F : \underline{C} \to \underline{D}$ be an additive functor. Let $X \in \underline{C}$. Then the function

$$F : \mathrm{Hom}_{\underline{C}}(X,X) \to \mathrm{Hom}_{\underline{D}}(F(X), F(X))$$

is a ring homomorphism.

<u>proof</u>. The functor axioms say that F respects $\mathbb{1}$ and composition. The axiom for additive functor says that F respects addition. $\square$