Prop. Let A and B be rings.

(1) Let $M \in \underline{\mathrm{Mod}}\, A$. M is the zero module iff $\mathrm{Hom}_A(M,M)$ is the zero ring.

(2) Let $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B$ be an additive functor. Then $F(\{0\}) = \{0\}$; ie. the zero A-module is sent by F to the zero B-module.

proof (1). M is the zero module means $M = \{0\}$. This happens iff the identity hom. $\mathbb{1}_M$ equals the zero hom. 0_M. Now $\mathbb{1}_M, 0_M \in \mathrm{Hom}_A(M,M)$; and this ring is trivial iff $\mathbb{1}_M = 0_M$.

(2) Consider the ring hom.
$$F : \mathrm{Hom}_A(M,M) \to \mathrm{Hom}_B(F(M), F(M)).$$
If the first ring is zero, then $\mathbb{1}_M = 0_M$ (by part 1).
Now $\mathbb{1}_{F(M)} = F(\mathbb{1}_M) = F(0_M) = 0_{F(M)}$,
so again by part 1 we have $F(M) = \{0\}$. $\square$

Exercise (1) Is there a nonzero additive functor
$$F : \underline{\mathrm{Mod}}\, \mathbb{F}_5 \to \underline{\mathrm{Mod}}\, \mathbb{F}_7 \ ?$$
(2) Is there a nonzero add. functor
$$F : \underline{\mathrm{Mod}}\, \mathbb{Q} \to \underline{\mathrm{Mod}}\, \mathbb{F}_7 \ ?$$
Why?

37) Let $f: A \to B$ be a ring hom. We saw the A-linear functor

$$\mathrm{rest}_f : \underline{\mathrm{Mod}}\, B \to \underline{\mathrm{Mod}}\, A.$$

Is there some functor in the opposite direction, ie.

$$\underline{\mathrm{Mod}}\, A \longrightarrow \underline{\mathrm{Mod}}\, B \quad ?$$

Yes - and we know it! This is the <u>base change</u> (or induction) functor

$$\mathrm{ind}_f\,(M) := B \otimes_A M$$

and

$$\mathrm{ind}_f\,(\varphi : M_1 \to M_2) := \left(1_B \otimes \varphi : B \otimes_A M_1 \to B \otimes_A M_2\right).$$

This two is an A-lin. functor.

The functors rest_f and ind_f are <u>adjoints</u>. This property will be defined and explained later. ~~but the meaning is carried by the next proposition~~

<u>Notation</u> For a category $\underline{C}$ and $X \in \mathrm{Ob}(\underline{C})$ we write

$$\mathrm{End}_{\underline{C}}\,(X) := \mathrm{Hom}_{\underline{C}}\,(X, X).$$

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38. Prop. Let $f : A \to B$ be a ring hom. For any $M \in \underline{\mathrm{Mod}}\, A$ and $N \in \underline{\mathrm{Mod}}\, B$, the function

$$\boxed{\eta_{M,N} : \mathrm{Hom}_B(\mathrm{ind}_f(M), N) \to \mathrm{Hom}_A(M, \mathrm{rest}_f(N))},$$

that sends $\psi : B \otimes_A M \to N$ to $\varphi : M \to N$, $\varphi(m) := \psi(1 \otimes m)$, is bijective.

Pf. Given $\varphi : M \to N$ in $\underline{\mathrm{Mod}}\, A$, consider the A-bilin function $\beta : B \times M \to N$, $\beta(b, m) := b \cdot \varphi(m)$. This gives rise to an A-lin. hom. $\psi : B \otimes_A M \to N$. A little calculation shows that ψ is B-linear. We get a function $\zeta_{M,N}(\varphi) := \psi$,

$$\zeta_{M,N} : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_B(B \otimes_A M, N).$$

Another calc. shows that $\zeta_{M,N}$ is inverse to $\eta_{M,N}$. $\square$

hard

Exercise Let $K := \mathbb{Q}[\sqrt{2}]$ and $L := \mathbb{Q}[\sqrt{-1}]$.

(1) Show there are no ring homs. $K \to L$ and $L \to K$.

(2) Find a nonzero additive function
$F : \underline{\mathrm{Mod}}\, K \to \underline{\mathrm{Mod}}\, L$. (Hint: use the ring homs $K \to \mathbb{C}$ and $L \to \mathbb{C}$.)

<u>Def</u>. Let A and B be rings, and let $F : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, B$ be an additive functor. We say that F is <u>exact</u> (resp. <u>left exact</u>, resp. <u>right exact</u>) if for any exact sequence

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0 \qquad (\heartsuit)$$

(resp.
$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'',$$
resp.
$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0\ )$$

in $\underline{\text{Mod}}\, A$, the sequence

$$0 \to F(M') \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(M'') \to 0$$

(resp.
$$0 \to F(M') \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(M''),$$
resp.
$$F(M') \xrightarrow{F(\varphi)} F(M) \xrightarrow{F(\psi)} F(M'') \to 0\ )$$

is exact in $\underline{\text{Mod}}\, B$.

($\heartsuit$) By a prop. (page 36) we know that $F(\{0\}) = \{0\}$.

<u>Exercise</u> Prove that TFAE for an additive functor $F : \underline{\text{Mod}}\, A \to \underline{\text{Mod}}\, B$:

(i) F is exact.

(ii) F is right exact and preserves injections (i.e. φ is injec. $\Rightarrow$ $F(\varphi)$ is injec.)

(iii) F is left exact and preserves surjections.

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Example. Consider a ring hom. $f: A \to B$.
The functor rest_f is exact: if
$$0 \to N' \xrightarrow{\varphi} N \xrightarrow{\psi} N'' \to 0$$
is an exact seq. in $\underline{\mathrm{Mod}\, B}$, then applying rest_f to this seq., it becomes an exact seq. of A-mods.
This is because kernels & cokernels are "encoded" in the abelian group structures.

This is false for ind_f.

Example Take $A := \mathbb{Z}$ and $B := \mathbb{Z}/(6)$. Consider the ex. seq.
$$\begin{array}{ccccccccc} 0 \to & M_0 & \xrightarrow{\varphi_1} & M_1 & \xrightarrow{\varphi_2} & M_2 & \to 0 \\ & \| & & \| & & \| & \\ & \mathbb{Z} & \underset{1 \mapsto 6}{\to} & \mathbb{Z} & \underset{1 \mapsto 1}{\to} & \mathbb{Z}/(6) & \end{array}$$
in $\underline{\mathrm{Mod}}\, \mathbb{Z}$ [$= \underline{\mathrm{Ab}}$]. Applying $\mathrm{ind}_f = B \otimes_A -$

[here $f: \mathbb{Z} \to \mathbb{Z}/(6)$ is the obvious ring hom.]

we get
$$0 \to \mathbb{Z}/(6) \xrightarrow[1 \mapsto 0]{\overset{0}{\mathrm{ind}_f(\varphi_2)}} \mathbb{Z}/(6) \xrightarrow[1 \mapsto 1]{\overset{\cong}{\mathrm{ind}_f(\varphi_2)}} \mathbb{Z}/(6) \to 0$$

no exactness here!
$\mathrm{Im}(\,) = 0 \quad \mathrm{Ker}(\,) = \mathbb{Z}/(6)$

Yet - in this example we did have exactness on the right! This is no accident...

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