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Theorem. Let A be a ring and $M \in \underline{\mathrm{Mod}}\, A$. The functor
$$F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A,$$
$$F(N) := M \otimes_A N, \quad F(\varphi) := 1\!\!1_M \otimes \varphi,$$
is right exact.

proof. Consider an exact sequence
$$N_0 \xrightarrow{\varphi_1} N_1 \xrightarrow{\varphi_2} N_2 \to 0$$
in $\underline{\mathrm{Mod}}\, A$. We have to prove that the sequence
$$M \otimes_A N_0 \xrightarrow{1\!\!1_M \otimes \varphi_1} M \otimes_A N_1 \xrightarrow{1\!\!1_M \otimes \varphi_2} M \otimes_A N_2 \to 0$$
is exact.

(cont. ↘)

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Exactness at $M\otimes_A N_2$, i.e. surjectivity of $1_M\otimes\varphi_2$, is easy. It suffices to prove that any pure tensor $m\otimes n_2$ is in the image. Since φ_2 is surjective, we can find $n_1\in N_1$ s.t. $\varphi_2(n_1)=n_2$. Then $m\otimes n_2=(1_M\otimes\varphi_2)(m\otimes n_1)$.

Exactness at $M\otimes_A N_1$ is harder, and we will this *indirectly*.

Define $$L:=\frac{M\otimes_A N_1}{\mathrm{Im}(1_M\otimes\varphi_1)}\in\underline{\mathrm{Mod}}\,A.$$

We are going to construct an *isomorphism* $\psi: L\to M\otimes_A N_2$ s.t. the diagram

(†)
$$\begin{tikzcd}
M\otimes_A N_1 \arrow[r, "1_M\otimes\varphi_2"] \arrow[d, "\pi"'] & M\otimes_A N_2 \\
L \arrow[ur, "\psi\ \boxed{\cong ?}"'] &
\end{tikzcd}$$

is *commutative*. Here π is the canon. surjection. This will show that
$$\mathrm{Ker}(1_M\otimes\varphi_2)=\mathrm{Ker}(\psi\circ\pi)=\mathrm{Ker}(\pi)=\mathrm{Im}(1_M\otimes\varphi_1),$$
i.e. exactness at $M\otimes_A N_1$.

(cont)

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The construction of ψ is easy. Since $\varphi_2 \circ \varphi_1 = 0$, it follows that $(1_M \otimes \varphi_2) \circ (1_M \otimes \varphi_1) = 0$, so there is a unique ψ making the diagram $(*)$ commute.

To prove that ψ is bijective we are going to to construct its inverse. Consider the function

$$\beta : M \times N_2 \to L,$$

$$\beta(m, n_2) := \pi(m \otimes n_1),$$

where n_1 is any element of N_1 s.t. $\varphi_2(n_1) = n_2$.

We can find such n_1 because φ_2 is surjective. But why is β independent of choice of n_1? If n_1' is any other choice, then $\varphi_2(n_1' - n_1) = 0$, so $n_1' = n_1 + \varphi_1(n_0)$ for some $n_0 \in N_0$, by exactness at N_1.

Thus

$$\pi(m \otimes n_1') = \pi(m \otimes n_1 + m \otimes \varphi_1(n_0)) = \pi(m \otimes n_1).$$

(Cont)

We proved that β is well defined. It is easy to check that β is bilinear. Hence it induces a hom.

$$\chi : M \otimes_A N_2 \longrightarrow L$$

in $\underline{\text{Mod}}\ A$ s.t. the diagram

$$(\dagger\dagger) \qquad \begin{tikzcd} M \otimes_A N_1 \arrow[r, "1_M \otimes \varphi_2"] \arrow[d, "\pi"'] & M \otimes_A N_2 \arrow[dl, "\chi"] \\ L & \end{tikzcd}$$

is commutative. Because π and $1_M \otimes \varphi_2$ are surjective, and commutativity of diagrams $(\dagger)$ and $(\dagger\dagger)$, it follows that $\chi \circ \psi = 1_L$ and

$$\psi \circ \chi = 1_{M \otimes_A N_2}.$$

Hence ψ is bijective.

$\square$

<u>Def</u>. An A-module M is called <u>flat</u> if the functor $F : \underline{\text{Mod}}\ A \to \underline{\text{Mod}}\ A$, $F(N) := M \otimes_A N$, $F(\varphi) := 1_M \otimes \varphi$, is <u>exact</u>.

<u>Exercise</u> Let M be a fin. gen. $\mathbb{Z}$-module. Prove that M is <u>free</u> iff it is <u>flat</u>.

 — We will return to flatness later!

45 So far we did not see any example of tensors that are not pure. The next exercises will do just that.

<u>Exercise</u>. Here A is a nonzero ring. For an A-module M let $M^* := \mathrm{Hom}_A(M, A)$. This is the <u>dual module</u>.

(1) Let $M, N \in \underline{\mathrm{Mod}}\, A$.
Show that there is a unique hom.

$$\boxed{\chi : M^* \otimes_A N \longrightarrow \mathrm{Hom}_A(M, N)}$$

in $\underline{\mathrm{Mod}}\, A$ s.t.

$$\chi(\mu \otimes n)(m) = \mu(m) \cdot n$$

for any $\mu \in M^*$, $n \in N$ and $m \in M$.

(2) Assume M is free of finite rank, with basis $(m_1, \dots, m_r)$. Show that M^* is also free, with basis $(\mu_1, \dots, \mu_r)$ satisfying

$$\mu_i(m_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

This is the <u>dual basis</u> to $(m_1, \dots, m_r)$.

(3) Again assume M is free of finite rank. Prove that χ is bijective.
(from part 1)

(4) Assume M & N are free, with bases $\underline{m} = (m_1, \dots, m_r)$ and $(n_1, \dots, n_s)$. Write the matrix of the hom. $\chi(\mu_i \otimes n_j) : M \to N$ w.r.t $\underline{m}$ & $\underline{n}$.

(46) <u>Exercise</u>. Here K is a field, and $V, W \in \underline{\mathrm{Mod}}_{\mathrm{fin}} K$ (i.e. fin. dim. vector spaces).

(1) For $\varphi \in \mathrm{Hom}_K(V, W)$ define

$$\mathrm{rank}(\varphi) := \mathrm{rank}(\mathrm{Im}(\varphi))$$

Here $\mathrm{Im}(\varphi)$ is the K-submodule of W, and it has finite rank. Prove that $\mathrm{rank}(\varphi) = 1$ iff φ is a nonzero pure tensor, i.e.

$$\varphi = \chi(\mu \otimes w) \text{ for some nonzero}$$

$\mu \in V^*$ and $w \in W$, where χ is the can. isom. from the prev. exercise.

(2) Find an elt. of $V^* \otimes W$ which is not a pure tensor. (Must assume V & W have ranks ≥ 2.)

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