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Theorem. Let  $A$  be a ring and  $M \in \text{Mod } A$ . The functor

$$F : \text{Mod } A \rightarrow \text{Mod } A;$$

$$F(N) := M \otimes_A N, \quad F(\varphi) := 1_M \otimes \varphi,$$

is right exact.

proof. Consider an exact sequence

$$N_0 \xrightarrow{\varphi_1} N_1 \xrightarrow{\varphi_2} N_2 \rightarrow 0$$

in  $\text{Mod } A$ . We have to prove that the sequence

$$M \otimes_A N_0 \xrightarrow{1_M \otimes \varphi_1} M \otimes_A N_1 \xrightarrow{1_M \otimes \varphi_2} M \otimes_A N_2 \rightarrow 0$$

is exact.

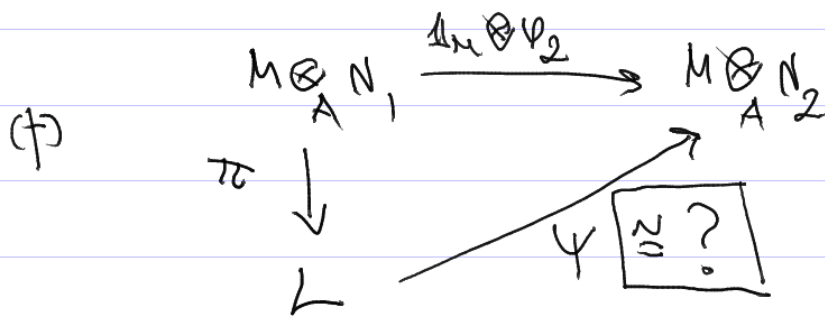
(cont. ~~11~~)

Exactness at  $M \otimes_A N_2$ , i.e. surjectivity of  $1_M \otimes \varphi_2$ , is easy. It suffices to prove that any pure tensor  $m \otimes n_2$  is in the image. Since  $\varphi_2$  is surjective, we can find  $n_1 \in N_1$  s.t.  $\varphi_2(n_1) = n_2$ . Then  $m \otimes n_2 = (1_M \otimes \varphi_2)(m \otimes n_1)$ .

Exactness at  $M \otimes_A N_1$  is harder, and we will do this indirectly.

Define  $L := \frac{M \otimes_A N_1}{\text{Im}(1_M \otimes \varphi_1)} \in \text{Mod } A$ .

We are going to construct an isomorphism  $\psi: L \rightarrow M \otimes_A N_2$  s.t. the diagram



is commutative. Here  $\pi$  is the canon. surjection. This will show that

$$\text{Ker}(1_M \otimes \varphi_2) = \text{Ker}(\psi \circ \pi) = \text{Ker}(\pi) = \text{Im}(1_M \otimes \varphi_1),$$

i.e. exactness at  $M \otimes_A N_1$ .

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The construction of  $\psi$  is easy. Since  $\psi_2 \circ \psi_1 = 0$ , it follows that  $(1_M \otimes \psi_2) \circ (1_M \otimes \psi_1) = 0$ , so there is a unique  $\psi$  making the diagram (†) commute.

To prove that  $\psi$  is bijective we are going to construct its inverse. Consider the function

$$\beta: M \times N_2 \rightarrow Z,$$

$$\beta(m, n_2) := \pi(m \otimes n_1),$$

where  $n_1$  is any element of  $N_1$  s.t.  $\psi_2(n_1) = n_2$ .

We can find such  $n_1$  because  $\psi_2$  is surjective. But why is  $\beta$  independent of choice of  $n_1$ ? If  $n_1'$  is any other choice, then  $\psi_2(n_1' - n_1) = 0$ , so  $n_1' = n_1 + \psi_1(n_0)$  for some  $n_0 \in N_0$ , by exactness at  $N_1$ .

Thus

$$\pi(m \otimes n_1') = \pi(m \otimes n_1 + m \otimes \psi_1(n_0)) = \pi(m \otimes n_1).$$

(cont) ↘

We proved that  $\beta$  is well defined. It is easy to check that  $\beta$  is bilinear. Hence it induces a hom.

$$\gamma: M \otimes_A N_2 \rightarrow L$$

in  $\text{Mod } A$  s.t. the diagram

$$(f) \quad \begin{array}{ccc} M \otimes_A N_1 & \xrightarrow{1_M \otimes \psi_2} & M \otimes_A N_2 \\ \pi \downarrow & & \searrow \gamma \\ L & & \end{array}$$

is commutative. Because  $\pi$  and  $1_M \otimes \psi_2$  are surjective, and commutativity of diagrams (f) and (ft), it follows that  $\gamma \circ \psi = 1_L$  and

$$\psi \circ \gamma = 1_{M \otimes_A N_2}.$$

Hence  $\psi$  is bijective.  $\square$



Def. An  $A$ -module  $M$  is called flat if the functor  $F: \text{Mod } A \rightarrow \text{Mod } A$ ,  $F(N) := M \otimes_A N$ ,  $F(\varphi) := 1_M \otimes \varphi$ , is exact.

Exercise Let  $M$  be a fin. gen.  $\mathbb{Z}$ -module. Prove that  $M$  is free iff it is flat.  $\infty$

page 4 — We will return to flatness later!

So far we did not see any example of tensors that are not pure. The next exercises will do just that.

Exercise. Here  $A$  is a nonzero ring. For an  $A$ -module  $M$  let  $M^* := \text{Hom}_A(M, A)$ . This is the dual module.

(1) Let  $M, N \in \text{Mod } A$ .

Show that there is a unique hom.

$$\chi: M^* \otimes_A N \rightarrow \text{Hom}_A(M, N)$$

in  $\text{Mod } A$  s.t.

$$\chi(\mu \otimes n)(m) = \mu(m) \cdot n$$

for any  $\mu \in M^*$ ,  $n \in N$  and  $m \in M$ .

(2) Assume  $M$  is free of finite rank, with basis  $(m_1, \dots, m_r)$ . Show that  $M^*$  is also free, with basis  $(\mu_1, \dots, \mu_r)$  satisfying

$$\mu_i(m_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$$

This is the dual basis to  $(m_1, \dots, m_r)$ .

(3) Again assume  $M$  is free of finite rank. Prove that  $\chi$  is bijective.  
(from part 1)

(1) Assume  $M$  &  $N$  are free, with bases  $(m_1, \dots, m_r)$  and  $(n_1, \dots, n_s)$ . Write the matrix of the hom.  $\chi(\mu_i \otimes \nu_j): M \rightarrow N$  w.r.t  $m$  &  $n$ .

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Exercise. Here  $K$  is a field, and  $V, W \in \text{Mod}_{\text{fin}} K$  (i.e. fin. dim. vector spaces).

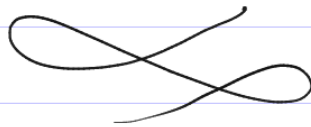
(1) For  $\varphi \in \text{Hom}_K(V, W)$  define

$$\text{rank}(\varphi) := \text{rank}(\text{Im}(\varphi))$$

Here  $\text{Im}(\varphi)$  is the  $K$ -submodule of  $W$ , and it has finite rank. Prove that  $\text{rank}(\varphi) = 1$  iff  $\varphi$  is a nonzero pure tensor, i.e.

$\varphi = \chi(\mu \otimes w)$  for some nonzero  $\mu \in V^*$  and  $w \in W$ , where  $\chi$  is the can. isom. from the prev. exercise.

(2) Find an elt. of  $V^* \otimes W$  which is not a pure tensor. (Must assume  $V$  &  $W$  have ranks  $\geq 2$ .)



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