

47

Exercise Find examples of  $M$  &  $N$  st.

(a)  $M$  is fin. gen. (but not free)

(b)  $M$  is free (but not f.g.)

and s.t.  $\chi: M^* \otimes_A N \rightarrow \text{Hom}_A(M, N)$  is not  
an isomorphism.



## Direct Sums and Products

Let  $\{Y_x\}_{x \in X}$  be a collection of sets, indexed by a set  $X$ . Recall that the cartesian product of this collection is the set

$$\prod_{x \in X} Y_x = \left\{ f: X \rightarrow \bigcup_{x \in X} Y_x \mid f(x) \in Y_x \right\}.$$

sometimes

An element  $f \in \prod_{x \in X} Y_x$  is denoted by  $\{y_x\}_{x \in X}$ , where  $y_x = f(x)$ . For any  $x_0 \in X$  we have the projection function

$$p_{x_0}: \prod_x Y_x \rightarrow Y_{x_0}, \quad p_{x_0}(*) := f(x_0).$$

When  $\{Y_x\}$  is a constant family, i.e.  $Y_x = Y$  for all  $x$ , then we write

$$Y^X := \prod_{x \in X} Y_x.$$

Fix a ring  $A$ , and consider the linear category  
 $\underline{M} := \underline{\text{Mod}} A$ .

Def. Let  $\{M_x\}_{x \in X}$  be a collection of modules  
 $M_x \in \underline{M}$ , indexed by a set  $X$ .

(1) The direct product of this collection is the  
 module

$$\prod_{x \in X} M_x \quad (\text{the cartesian product of the sets } M_x),$$

with operations

$$\left. \begin{aligned} (f+g)(x) &:= f(x) + g(x) \\ (a \cdot f)(x) &:= a \cdot f(x) \end{aligned} \right\} \begin{array}{l} \text{for } f, g \in \prod_x M_x \\ a \in A \end{array}$$

(2) The direct sum of this collection is  
 the submodule

$$\bigoplus_{x \in X} M_x \subset \prod_{x \in X} M_x$$

consisting of functions with finite support. I.e.  
 $f \in \bigoplus_x M_x$  if  $f(x) = 0$  for all but finitely many  $x \in X$ .

(cont)

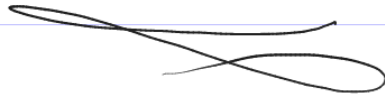
49

(cont)

(3) For any  $y \in X$  let  $e_y: M_y \rightarrow \bigoplus_{x \in X} M_x$  be the hom. that sends  $m \in M_y$  to the function

$$e_y(m)(x) := \begin{cases} m & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

This is the  $y$ -th embedding.



Observe that for every  $y \in X$ , the function

$$p_y: \prod_{x \in X} M_x \rightarrow M_y$$

(projection on  $y$ -coordinate) is a hom. in  $\underline{M}$ .

For every  $y \in X$  we have  $p_y \circ e_y = \text{id}_{M_y}$ .

Of course, if  $X$  is finite then

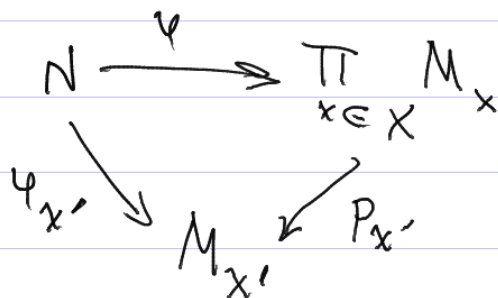
$$\bigoplus_{x \in X} M_x = \prod_{x \in X} M_x.$$

Thm (Univ. Prop. of  $\Pi$ ) <sup>of objects</sup>

Let  $\{M_x\}_{x \in X}$  be a collection in  $\text{Mod } A$ , let  $N \in \text{Mod } A$ , and let  $\{\varphi_x: N \rightarrow M_x\}_{x \in X}$  be a collection of morphisms in  $\text{Mod } A$ . There exists a unique morphism

$$\varphi: N \rightarrow \prod_{x \in X} M_x$$

s.t. for every  $x' \in X$  the diagram



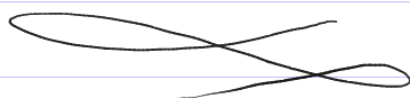
is commutative.



We write  $\prod \varphi_x := \varphi$ .

proof. For any element  $n \in N$  define  $\varphi(n) := \{\varphi_x(n)\}_{x \in X} \in \prod_{x \in X} M_x$ .

This is an  $A$ -lin. hom, it satisfies  $\varphi_{x'} = P_{x'} \circ \varphi$ , and it is the only such hom.  $\square$



(51)

Thm (Univ. Prop. of  $\oplus$ ) <sup>of objects</sup>

Let  $\{M_x\}_{x \in X}$  be a collection in  $\text{Mod } A$ , let  $N \in \text{Mod } A$ , and let  $\{\varphi_x: M_x \rightarrow N\}_{x \in X}$  be a collection of morphisms in  $\text{Mod } A$ . There exists a unique morphism

$$\varphi: \bigoplus_{x \in X} M_x \rightarrow N$$

s.t. for every  $x' \in X$  the diagram

$$\begin{array}{ccc} & M_{x'} & \\ e_{x'} \swarrow & & \searrow \varphi_{x'} \\ \bigoplus_{x \in X} M_x & \xrightarrow{\varphi} & N \end{array}$$

is commutative.



We write  $\bigoplus \varphi_x := \varphi$ .

Proof. For an element  $\{m_x\} \in \bigoplus_{x \in X} M_x$  define

$$\varphi(\{m_x\}) := \sum_{x \in X} \varphi_x(m_x) \in N.$$

This makes sense because the collection  $\{\varphi_x(m_x)\}$  has finite support. The resulting hom.  $\varphi: \bigoplus M_x \rightarrow N$  is the only hom. satisfying  $\varphi \circ e_{x'} = \varphi_{x'}$ .  $\square$



Sometimes we write  $\coprod$  instead of  $\bigoplus$ , and then we call it a coproduct.

page 5

52

Let  $X$  be a finite set, and  $\{M_x\}_{x \in X}$  a collection of objects in  $\text{Mod } A$ . Write

$$M := \bigoplus_{x \in X} M_x = \prod_{x \in X} M_x.$$

It is easy to see that

$$\mathbb{1}_M = \sum_{x \in X} e_x \circ p_x$$

and

$$p_x \circ e_y = \begin{cases} \mathbb{1}_{M_x} & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases}$$

for all  $x, y \in X$ .

These equalities are in the  $\mathbb{K}$ -ring  $\text{End}_A(M) = \text{End}_{\text{Mod } A}(M)$  and  $\text{Hom}_A(M_y, M_x)$  the ab. grops

Theorem. Let  $\{M_x\}_{x \in X}$  be a finite collection <sup>of objects</sup> in  $\text{Mod } A$ , let  $M' \in \text{Mod } A$ , and let  $\{e'_x : M_x \rightarrow M'\}_{x \in X}$ ,  $\{p'_x : M' \rightarrow M_x\}_{x \in X}$  be collections of morphisms in  $\text{Mod } A$  satisfying

$$\mathbb{1}_{M'} = \sum_{x \in X} e'_x \circ p'_x$$

and

$$p'_x \circ e'_y = \begin{cases} \mathbb{1}_{M_x} & \text{if } x=y \\ 0 & \text{if } x \neq y \end{cases}$$

for all  $x, y \in X$ .

There is a unique isomorphism

$$\psi: M' \xrightarrow{\cong} \bigoplus_{x \in X} M_x$$

s.t.

$$e_x = \psi \circ e'_x$$

and

$$p'_x = p_x \circ \psi$$

$$\forall x \in X.$$

Proof. Write  $M := \bigoplus_{x \in X} M_x = \prod_{x \in X} M_x$ .

Step 1.

Let  $\varphi: M' \rightarrow M$  be the hom.

$$\varphi := \sum_{x \in X} e_x \circ p'_x,$$

and let  $\psi: M \rightarrow M'$  be

$$\psi := \sum_{x \in X} e'_x \circ p_x.$$

For any  $x$  we have

$$\varphi \circ \psi = \left( \sum_{y \in X} e_y \circ p'_y \right) \circ e'_x = \sum_y e_y \circ p'_y \circ e'_x = e_x$$

and

$$p_x \circ \varphi = p_x \circ \left( \sum_{y \in X} e_y \circ p'_y \right) = \sum_y p_x \circ e_y \circ p'_y = p'_x.$$

Step 2.

We prove that  $\varphi$  is bijective by showing that  $\psi$  is its inverse.

$$\begin{aligned} \varphi \circ \psi &= \left( \sum_{x \in X} e'_x \circ p_x \right) \circ \left( \sum_{y \in X} e_y \circ p'_y \right) \\ &= \sum_{x, y} e'_x \circ p_x \circ e_y \circ p'_y = \sum_x e'_x \circ p'_x = \mathbb{1}_{M'} \end{aligned}$$

and likewise  $\psi \circ \varphi = \mathbb{1}_M$ .



54



Step 3. Uniqueness of  $\psi$ : suppose  $\psi^t: M' \rightarrow N$  is another hom. satisfying  $e_x = \psi^t \circ e'_x$  and  $p'_x = p_x \circ \psi^t$ .

Then

$$\psi^t = \psi^t \circ \mathbb{1}_{M'} = \psi^t \circ \left( \sum_x e'_x \circ p'_x \right) = \sum_x e_x \circ p'_x$$

$$\psi^t = \mathbb{1}_N \circ \psi^t \circ \mathbb{1}_{M'} = \left( \sum_x e_x \circ p_x \right) \circ \psi^t \circ \left( \sum_y e'_y \circ p'_y \right)$$

$$= \sum_{x,y} e_x \circ p_x \circ \underbrace{\psi^t \circ e'_y}_{= e_y} \circ p'_y$$

$$= \sum_x e_x \circ p'_x = \psi.$$

□