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<u>Exercise</u> Find examples of M & N s.t.

(a) M is fin. gen. (but not free)

(b) M is free (but not f.g.)

and s.t. $\chi : M^* \otimes_A N \longrightarrow \mathrm{Hom}_A(M, N)$ is <u>not</u> <u>an isomorphism</u>.

Direct Sums and Products

Let $\{Y_x\}_{x\in X}$ be a collection of sets, indexed by a set X. Recall that the cartesian product of this collection is the set

$$\prod_{x\in X} Y_x = \left\{ f : X \to \bigcup_{x\in X} Y_x \;\middle|\; f(x) \in Y_x \right\}.$$

An element $f \in \prod_{x\in X} Y_x$ is sometimes denoted by $\{y_x\}_{x\in X}$, where $y_x := f(x)$. For any $x_0 \in X$ we have the <u>projection function</u>

$$p_{x_0} : \prod_x Y_x \longrightarrow Y_{x_0}, \quad p_{x_0}(f) := f(x_0).$$

When $\{Y_x\}$ is a constant family, i.e. $Y_x = Y$ for all x, then we write

$$Y^X := \prod_{x\in X} Y_x.$$

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Fix a ring A, and consider the linear category $\underline{M} := \underline{\mathrm{Mod}}\, A$.

Def. Let $\{M_x\}_{x\in X}$ be a collection of modules $M_x \in \underline{M}$, indexed by a set X.

(1) The direct product of this collection is the module
$$\prod_{x\in X} M_x \quad \text{(the cartesian product of the sets } M_x\text{)},$$
with operations
$$\left.\begin{aligned}(f+g)(x) &:= f(x)+g(x)\\ (a.f)(x) &:= a.f(x)\end{aligned}\right) \quad \begin{aligned}&\text{for } f,g \in \textstyle\prod_x M_x\\ &a\in A\end{aligned}.$$

(2) The direct sum of this collection is the submodule
$$\bigoplus_{x\in X} M_x \subset \prod_{x\in X} M_x$$
consisting of functions with finite support. I.e. $f \in \bigoplus_x M_x$ if $f(x)=0$ for all but finitely many $x\in X$.

(cont)

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(3) For any $y \in X$ let $e_y : M_y \to \bigoplus_{x \in X} M_x$ be the hom. that sends $m \in M_y$ to the function

$$e_y(m)(x) := \begin{cases} m & \text{if } x = y \\ 0 & \text{if } x \neq y. \end{cases}$$

This is the y-th embedding.

Observe that for every $y \in X$, the function

$$p_y : \prod_{x \in X} M_x \to M_y$$

(projection on y-coordinate) is a hom. in $\underline{M}$.

For every $y \in X$ we have $p_y \circ e_y = \mathbb{1}_{M_y}$.

Of course, if X is finite then

$$\bigoplus_{x \in X} M_x = \prod_{x \in X} M_x.$$

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Thm (Univ. Prop. of $\prod$ of objects)

Let $\{M_x\}_{x\in X}$ be a collection of objects in $\underline{\text{Mod}}$ A, let $N \in \underline{\text{Mod}}$ A, and let $\{\varphi_x : N \to M_x\}_{x\in X}$ be a collection of morphisms in $\underline{\text{Mod}}$ A. There exists a unique morphism

$$\varphi : N \to \prod_{x\in X} M_x$$

s.t. for every $x' \in X$ the diagram

$$\begin{tikzcd} N \arrow[r, "\varphi"] \arrow[dr, "\varphi_{x'}"'] & \prod_{x\in X} M_x \arrow[d, "p_{x'}"] \\ & M_{x'} \end{tikzcd}$$

is commutative.

We write $\prod \varphi_x := \varphi$.

$\underline{\text{proof}}$. For any element $n \in N$ define

$$\varphi(n) := \{\varphi_x(n)\}_{x\in X} \in \prod_{x\in X} M_x.$$

This is an A-lin. hom, it satisfies $\varphi_{x'} = p_{x'} \circ \varphi$, and it is the only such hom. $\square$

Thm (Univ. Prop. of $\oplus$)

Let $\{M_x\}_{x\in X}$ be a collection of objects in $\underline{\text{Mod}}\ A$, let $N \in \underline{\text{Mod}}\ A$, and let $\{\varphi_x : M_x \to N\}_{x\in X}$ be a collection of morphisms in $\underline{\text{Mod}}\ A$. There exists a unique morphism

$$\varphi : \bigoplus_{x\in X} M_x \longrightarrow N$$

s.t. for every $x' \in X$ the diagram

$$\begin{tikzcd}
 & M_{x'} \arrow[dl, "e_{x'}"'] \arrow[dr, "\varphi_{x'}"] & \\
\bigoplus_{x\in X} M_x \arrow[rr, "\varphi"'] & & N
\end{tikzcd}$$

is commutative.

We write $\bigoplus \varphi_x := \varphi$.

Proof. For an element $\{m_x\} \in \bigoplus_{x\in X} M_x$ define

$$\varphi(\{m_x\}) := \sum_{x\in X} \varphi_x(m_x) \in N.$$

This makes sense because the collection $\{\varphi_x(m_x)\}$ has finite support. The resulting hom. $\varphi : \bigoplus M_x \to N$ is the only hom. satisfying $\varphi \circ e_{x'} = \varphi_{x'}$. $\square$

Sometimes we write $\coprod$ instead of $\bigoplus$, and then we call it a coproduct.

52) Let X be a finite set, and $\{M_x\}_{x\in X}$ a collection of objects in $\underline{\text{Mod}}\ A$. Write

$$M := \bigoplus_{x\in X} M_x = \prod_{x\in X} M_x.$$

It is easy to see that

$$\boxed{1\!\!1_M = \sum_{x\in X} e_x \circ p_x} \quad \text{and} \quad \boxed{p_x \circ e_y = \begin{cases} 1\!\!1_{M_x} & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}}$$

for all $x,y\in X$.

These equalities are in the NC ring $\text{End}_A(M) = \text{End}_{\underline{\text{Mod}}\,A}(M)$ and in the ab. grps $\text{Hom}_A(M_y, M_x)$.

Theorem Let $\{M_x\}_{x\in X}$ be a finite collection of objects in $\underline{\text{Mod}}\ A$, let $M'\in \underline{\text{Mod}}\ A$, and let

$$\{e'_x : M_x \to M'\}_{x\in X}, \quad \{p'_x : M' \to M_x\}_{x\in X}$$

be collections of morphisms in $\underline{\text{Mod}}\ A$ satisfying

$$\boxed{1\!\!1_{M'} = \sum_{x\in X} e'_x \circ p'_x} \quad \text{and} \quad \boxed{p'_x \circ e'_y = \begin{cases} 1\!\!1_{M_x} & \text{if } x=y \\ 0 & \text{if } x\neq y \end{cases}}$$

for all $x,y\in X$.

There is a unique isomorphism

$$\boxed{\varphi : M' \xrightarrow{\simeq} \bigoplus_{x\in X} M_x}$$

s.t.

$$\boxed{e_x = \varphi\circ e'_x} \quad \text{and} \quad \boxed{p'_x = p_x\circ\varphi} \quad \forall x\in X.$$

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<u>Proof.</u> Write $M := \bigoplus_{x\in X} M_x = \prod_{x\in X} M_x$.

<u>Step 1.</u>

Let $\varphi : M' \to M$ be the hom.

$$\varphi := \sum_{x\in X} e_x \circ p'_x \quad ,$$

and let $\psi : M \to M'$ be

$$\psi := \sum_{x\in X} e'_x \circ p_x .$$

For any x we have

$$\varphi \circ e'_x = \Big(\sum_{y\in X} e_y \circ p'_y\Big) \circ e'_x = \sum_y e_y \circ p'_y \circ e'_x = e_x$$

and

$$p_x \circ \varphi = p_x \circ \Big(\sum_{y\in X} e_y \circ p'_y\Big) = \sum_y p_x \circ e_y \circ p'_y = p'_x .$$

<u>Step 2.</u>

We prove that φ is bijective by showing that ψ is its inverse.

$$\psi \circ \varphi = \Big(\sum_{x\in X} e'_x \circ p_x\Big) \circ \Big(\sum_{y\in X} e_y \circ p'_y\Big)$$

$$= \sum_{x,y} e'_x \circ p_x \circ e_y \circ p'_y = \sum_x e'_x \circ p'_x = 1\!\!1_{M'}$$

and likewise $\varphi \circ \psi = 1\!\!1_M$.

Step 3. Uniqueness of φ: suppose $\varphi^\dagger : M' \to M$ is another hom. satisfying $e_x = \varphi^\dagger \circ e'_x$ and $p'_x = p_x \circ \varphi^\dagger$.

Then

$$\varphi^\dagger = \varphi^\dagger \circ 1\!\!1_{M'} = \varphi^\dagger \circ \Big(\sum_x e'_x \circ p'_x\Big) = \sum_x e_x \circ p'_x$$

$$\varphi^\dagger = 1\!\!1_M \circ \varphi^\dagger \circ 1\!\!1_{M'} = \Big(\sum_x e_x \circ p_x\Big) \circ \varphi^\dagger \circ \Big(\sum_y e'_y \circ p'_y\Big)$$

$$= \sum_{x,y} e_x \circ p_x \circ \underbrace{\varphi^\dagger \circ e'_y}_{= e_y} \circ p'_y$$

$$= \sum_x e_x \circ p'_x = \varphi. \qquad \square$$