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<u>Theorem</u> Let $F: \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, B$ be an additive functor, and let $\{M_x\}_{x \in X}$ be a <u>finite</u> collection in $\underline{\mathrm{Mod}}\, A$.

For any $y \in X$ consider the hom's

$$F(p_y): F\Big(\bigoplus_x M_x\Big) \to F(M_y)$$

and

$$e_y : F(M_y) \to \bigoplus_{x \in X} F(M_x).$$

Then

$$\boxed{\sum_{x \in X} e_x \circ F(p_x) \; : \; F\Big(\bigoplus_{x \in X} M_x\Big) \longrightarrow \bigoplus_{x \in X} F(M_x)}$$

is an isomorphism in $\underline{\mathrm{Mod}}\, B$.

We summarize this by "an additive functor respects finite $\oplus$".

<u>Proof</u>. Let's write $M := \bigoplus_{x \in X} M_x$ and $N := \bigoplus_{x \in X} F(M_x)$.

Also let $\varphi := \sum_x e_x \circ F(p_x) : F(M) \to N$

$\psi : \sum_x F(e_x) \circ p_x : N \to M$.

cont.

(56) cont.

Then

$$\varphi\circ\psi=\sum_{x,y}F(e_x)\circ p_x\circ e_y\circ F(p_x)$$

$$=\sum_x F(e_x)\circ F(p_x)=F\Big(\sum_x e_x\circ p_x\Big)=F(1\!\!1_N)=1\!\!1_{F(N)}$$

and

$$\psi\circ\varphi=\sum_{x,y}e_x\circ F(p_x)\circ F(e_y)\circ p_y=$$

$$=\sum_{x,y}e_x\circ F(p_x\circ e_y)\circ p_y=\sum_x e_x\circ F(1\!\!1_{N_x})\circ p_x$$

$$=\sum_x e_x\circ 1\!\!1_{F(N_x)}\circ p_x=\sum_x e_x\circ p_x=1\!\!1_N.$$

Thus φ is an isom. $\square$

__Exercise__. Find an additive functor

$$F:\underline{\mathrm{Mod}}\ A\to\underline{\mathrm{Mod}}\ A$$

s.t. (a) does not respect $\bigoplus$.

(b) " " " $\prod$.

You first need to find the can. homs. $\varphi: F(\bigoplus N_x)\to\bigoplus F(N_x)$

$\varphi:\prod F(N_x)\to F(\prod N_x)$.

Give specific counterexamples !

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<u>Prop.</u> Let $\{\varphi_x : M_x \to N_x\}$ be a collection of hom's in <u>Mod</u> A. Write $M := \bigoplus_x M_x$, $N := \bigoplus_x N_x$ and $\varphi := \bigoplus \varphi_x$.
Also write $\hat{M} := \prod_x M_x$, $\hat{N} := \prod_x N_x$ and $\hat{\varphi} := \prod \varphi_x$.

So $\varphi : M \to N$ and $\hat{\varphi} : \hat{M} \to \hat{N}$ are hom's in <u>Mod</u> A.

TFAE:

(i) All the hom's φ_x are injective (resp. surjective, resp. bijective).

(ii) The hom φ is injective (resp. surjective, resp. bijective).

(iii) The hom $\hat{\varphi}$ is injective (resp. surjective, resp. bijective).

<u>Pf.</u> <u>Exercise.</u>

Lemma. Let

$$(\varphi) \qquad 0 \to M_0 \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_2 \to 0$$

be an exact seq. in $\mathrm{Mod}\ A$.

TFAE: (i) There is $\sigma_1 : M_1 \to M_0$ s.t. $\sigma_1 \circ \varphi_1 = \mathbb{1}_{M_0}$.

(ii) There is $\sigma_2 : M_2 \to M_1$ s.t. $\varphi_2 \circ \sigma_2 = \mathbb{1}_{M_2}$.

(iii) There is an isomorphism $\sigma_1 : M_1 \xrightarrow{\cong} M_0 \oplus M_2$ s.t. the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_0 \arrow[r, "\varphi_0"] \arrow[d, "\mathbb{1}_{M_0}"'] & M_1 \arrow[r, "\varphi_1"] \arrow[d, "\sigma"'] & M_2 \arrow[r] \arrow[d, "\mathbb{1}_{M_2}"] & 0 \\
0 \arrow[r] & M_0 \arrow[r, "e_0"] & M_0 \oplus M_2 \arrow[r, "p_2"] & M_2 \arrow[r] & 0
\end{tikzcd}$$

is commutative.

pf. Exercise.

Def An exact seq. (φ) that satisfies the equivalent conditions of the lemma is called split

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