

(5.9) The \otimes functor is special:

Thm. Let $M \in \text{Mod } A$. The functor

$$F: \text{Mod } A \rightarrow \text{Mod } A, \quad F := M \otimes_A -$$

respects all direct sums. Namely, for any collection $\{N_x\}_{x \in X}$ in $\text{Mod } A$, the canonical hom.

$$\sum_{x \in X} F(e_x) \circ p_x : \bigoplus_{x \in X} F(N_x) \rightarrow F\left(\bigoplus_{x \in X} N_x\right)$$

is bijective.

Proof. Note that $\sum_x F(e_x) \circ p_x$ is the hom. ψ in the proof of the previous thm. We want to construct its inverse.

$$\text{Let } \beta: M \times \left(\bigoplus_x N_x\right) \rightarrow \bigoplus_x (M \otimes_A N_x)$$

$$\text{be } \beta(m, \{n_x\}) := \{m \otimes n_x\}_{x \in X},$$

for any fin. supp. collection $\{n_x\}_{x \in X}$. This is a bilin. funct, so get here.

$$\psi: M \otimes_A \left(\bigoplus_x N_x\right) \rightarrow \bigoplus_x (M \otimes_A N_x).$$

↘ (cont)

(60)

↘ (cont.)

The module $(M \otimes_A \bigoplus_x N_x)$ is gen. by the \bigvee ^{elements} $m \otimes e_x(m_x)$, for $m \in M$, $x \in X$ and $m_x \in N_x$. The mod. $\bigoplus_x (M \otimes_A N_x)$ is gen. by elts. $e_x(m \otimes n_x)$.
But $\psi(m \otimes e_x(m_x)) = e_x(m \otimes n_x)$, and vice versa for φ . So ψ is an inverse of φ . \square .



Once we introduce the concept of natural isomorphism between functors, we will prove:

Thm. The following are equivalent for an additive functor $F: \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A$.

- (i) F is right exact and commutes with direct sums.
- (ii) F is isomorphic to the functor $M \otimes_A -$, where $M := F(A) \in \underline{\text{Mod}} A$.



(61)

Cor. Let $\{M_x\}_{x \in X}$ be a collection of A -mods.
TFAE:

- (i) All the modules M_x are flat.
- (ii) The module $\bigoplus_{x \in X} M_x$ is flat.

pf. Take any injection $\psi: N_0 \rightarrow N_1$. There is a comm diagram

$$\begin{array}{ccc} \bigoplus_x (N_0 \otimes_A M_x) & \longrightarrow & N_0 \otimes_A \left(\bigoplus_x M_x \right) \\ \bigoplus (\psi \otimes 1_{M_x}) \downarrow & & \downarrow \psi \otimes \left(\bigoplus 1_{M_x} \right) \\ \text{prev. } \bigoplus_x (N_1 \otimes_A M_x) & \longrightarrow & N_1 \otimes_A \left(\bigoplus_x M_x \right) \end{array}$$

By the theorem, the horizontal arrows are bijective.

The lemma on p. 58 says that all the maps $\psi \otimes 1_{M_x}$ are injective iff $\bigoplus_x (\psi \otimes 1_{M_x})$ is inj.
We see that this holds iff $\psi \otimes \left(\bigoplus_x 1_{M_x} \right)$ is inj. \square



62

The Functor Hom and Projectives

Fix a ring A (comm.) and a module M .
 For any module N there is a module
 $F(N) := \text{Hom}_A(M, N)$.

For any hom. $\varphi: N_0 \rightarrow N_1$, there is a hom.

$$F(\varphi) = \text{Hom}(\mathbb{1}_M, \varphi): F(N_0) \rightarrow F(N_1)$$

defined by

$$\text{Hom}(\mathbb{1}_M, \varphi)(\psi) := \varphi \circ \psi = \varphi \circ \psi \circ \mathbb{1}_M.$$

We get an A -linear functor (it's easy to check)

$$F = \text{Hom}_A(M, -): \underline{\text{Mod}} A \rightarrow \underline{\text{Mod}} A.$$

Prop. The functor $\text{Hom}_A(M, -)$ is left exact.

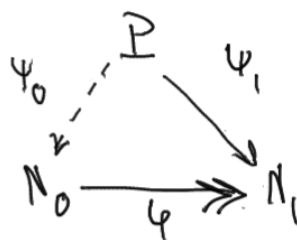
Pr. Exercise (Hint: it is quite easy!)

Def. An A -module P is called a projective
module if the functor $\text{Hom}_A(P, -)$ is
exact.

(63) Theorem (Characterization of Proj. Mods.)

TFAE for an A -module P :

- (i) P is projective.
- (ii) The functor $\text{Hom}_A(P, -)$ respects surjections.
- (iii) Let $\varphi: N_0 \rightarrow N_1$ be a surjection in $\text{Mod } A$. Given any hom. $\psi_1: P \rightarrow N_1$, there exists a hom. $\psi_0: P \rightarrow N_0$ s.t. $\psi_1 = \psi_0 \circ \varphi$.



- (iv) P is a direct summand of a free A -module.

Proof.

(i) \Leftrightarrow (ii): We know that $\text{Hom}_A(P, -)$ is always left exact (Prop. p 61). So $\text{Hom}_A(P, -)$ is exact iff it respects surjections.

(ii) \Leftrightarrow (iii): consider the surjection $\varphi: N_0 \rightarrow N_1$ and the hom.

$$\text{Hom}(H_P, \varphi): \text{Hom}_A(P, N_0) \rightarrow \text{Hom}_A(P, N_1)$$

It sends:

$$\psi_0 \longmapsto \psi_0 \circ \varphi$$

So condition (iii) is precisely the surjectivity of $\text{Hom}(H_P, \varphi)$.

(iii) \Rightarrow (iv): Choose a surjection $\varphi: F \rightarrow P$ from a free module F .

\Downarrow cont

(64) \searrow cont

By cond. (iii) there is $\sigma: P \rightarrow F$ st. $\psi \circ \sigma = 1_P$.
 Consider the exact sequence

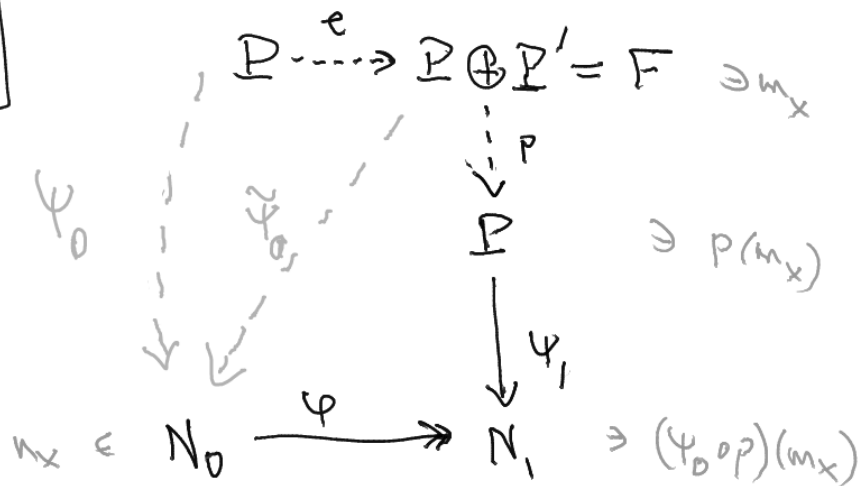
$$(*) \quad 0 \rightarrow P' \rightarrow F \xrightarrow{\psi} P \rightarrow 0$$

where $P' := \text{Ker}(\psi)$. The hom. σ splits $(*)$, so
 by Lemma (p. 58) we get $F \cong P \oplus P'$.

(iv) \Rightarrow (iii): We are given the solid part of
 the diagram below:

i.e.

$$\begin{array}{l} \psi_1: P \rightarrow N_1 \\ \psi: N_0 \rightarrow N_1 \text{ surj.} \end{array}$$



Let P' be a module st. $F = P \oplus P'$ is free, with
 basis $\{m_x\}_{x \in X}$. For every $x \in X$ choose $n_x \in N_0$
 s.t.

$$\psi(n_x) = (\psi_0 \circ p)(m_x).$$

This can be done since ψ is surjective. By the univ.
 prop. of a basis, there is $\tilde{\psi}_0: F \rightarrow N_0$ s.t. $\tilde{\psi}_0(m_x) = n_x$.

So
$$\psi \circ \tilde{\psi}_0 = \psi_1 \circ p: F \rightarrow N_1.$$

Let
$$\psi_0 := \tilde{\psi}_0 \circ e: P \rightarrow N_0.$$

\searrow (cont)

(cont) Then

65

$$\psi \circ \psi_0 = \psi \circ \tilde{\psi}_0 \circ e = \psi_1 \circ \rho \circ e = \psi_1 \circ \mathbb{1}_P = \psi_1. \quad \square$$



Corollary. If P is projective then it is flat.

Pr. The A -module A is flat since $A \otimes_A M \cong M$ canonically ($1 \otimes m \mapsto m$).

Take F free, with basis $\{m_x\}_{x \in X}$. Then $F \cong \bigoplus_{x \in X} A$.

By Cor. p. 61, imp. (i) \Rightarrow (ii), get that F is flat.

Finally choose P' s.t. $F := P \oplus P'$ is free.

By the implic. (ii) \Rightarrow (i) of Cor. (p. 61) conclude that P is flat. \square



Exercise Prove that \mathbb{Q} is not a projective \mathbb{Z} -module.
(we will see soon that \mathbb{Q} is flat over \mathbb{A} .)



to have
20/11