(5.9) The $\otimes$ functor is special:

<u>Thm.</u> Let $M \in \underline{\mathrm{Mod}}\, A$. The functor

$$F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A, \quad F := M \otimes_A -$$

respects <u>all</u> direct sums. Namely, for any collection $\{N_x\}_{x \in X}$ in $\underline{\mathrm{Mod}}\, A$, the canonical hom.

$$\boxed{\sum_{x \in X} F(e_x) \circ p_x \; : \; \bigoplus_{x \in X} F(N_x) \to F\Big(\bigoplus_{x \in X} N_x\Big)}$$

is bijective.

<u>Proof.</u> Note that $\sum_x F(e_x) \circ p_x$ is the hom. ψ in the proof of the previous thm. We want to construct its inverse.

Let $\beta : M \times \big(\bigoplus_x N_x\big) \to \bigoplus_x \big(M \otimes_A N_x\big)$ be

$$\beta(m, \{n_x\}) := \{m \otimes n_x\}_{x \in X},$$

for any fin. supp. collection $\{n_x\}_{x \in X}$. This is a bilin. funct, so get hom.

$$\varphi : M \otimes_A \big(\bigoplus_x N_x\big) \to \bigoplus_x \big(M \otimes_A N_x\big).$$

(cont)

The module $\left(M \otimes_A \bigoplus_x N_x\right)$ is gen. by the elements $m \otimes e_x(m_x)$, for $m \in M$, $x \in X$ and $m_x \in M_x$. The mod. $\bigoplus_x \left(M \otimes_A N_x\right)$ is gen. by elts. $e_x(m \otimes n_x)$. But $\varphi(m \otimes e_x(m_x)) = e_x(m \otimes n_x)$, and vice versa for ψ. So ψ is an inverse of φ. $\square$

Once we introduce the concept of <u>natural</u> <u>isomorphism</u> <u>between</u> <u>functors</u>, we will prove:

<u>Thm</u>. The following are equivalent for an additive functor $F : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A$.

(i) F is right exact and commutes with direct sums.

(ii) F is isomorphic to the functor $M \otimes_A -$, where $M := F(A) \in \underline{\mathrm{Mod}}\, A$.

Cor. Let $\{M_x\}_{x\in X}$ be a collection of A-mod's. TFAE:

(i) All the modules M_x are flat.

(ii) The module $\bigoplus_{x\in X} M_x$ is flat.

pf. Take any injection $\varphi: N_0 \to N_1$. There is a comm diagram

$$\begin{array}{ccc}
\bigoplus_x (N_0 \otimes_A M_x) & \longrightarrow & N_0 \otimes_A (\bigoplus_x M_x) \\
{\scriptstyle \oplus(\varphi\otimes 1_{M_x})} \downarrow & & \downarrow {\scriptstyle \varphi\otimes(\oplus 1_{M_x})} \\
\bigoplus_x (N_1 \otimes_A M_x) & \longrightarrow & N_1 \otimes_A (\bigoplus_x M_x)
\end{array}$$

By the prev. theorem, the horizontal arrows are bijective.

The lemma on p. 58 says that all the hom's $\varphi \otimes 1_{M_x}$ are injective iff $\bigoplus_x (\varphi \otimes 1_{M_x})$ is inj. We see that this holds iff $\varphi \otimes (\bigoplus_x 1_{M_x})$ is inj. $\square$

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The Functor Hom and Projectives

Fix a ring A (comm.) and a module M.
For any module N there is a module

$$F(N) := \mathrm{Hom}_A(M, N).$$

For any hom. $\varphi : N_0 \to N_1$ there is a hom.

$$F(\varphi) = \mathrm{Hom}(1_M, \varphi) : F(N_0) \to F(N_1)$$

defined by

$$\mathrm{Hom}(1_M, \varphi)(\psi) := \varphi \circ \psi = \varphi \circ \psi \circ 1_M.$$

We get an A-linear functor (it's easy to check)

$$\boxed{F = \mathrm{Hom}_A(M, -) : \underline{\mathrm{Mod}}\, A \to \underline{\mathrm{Mod}}\, A.}$$

<u>Prop.</u> The functor $\mathrm{Hom}_A(M,-)$ is <u>left</u> exact.

<u>Pf.</u> <u>Exercise</u> (Hint: it is quite easy!)

<u>Def.</u> An A-module P is called a <u>projective module</u> if the functor $\mathrm{Hom}_A(P,-)$ is <u>exact</u>.

Theorem (Characterization of Proj. Mods.)

TFAE for an A-module P:

(i) P is projective.

(ii) The functor $\mathrm{Hom}_A(P,-)$ respects surjections.

(iii) Let $\varphi : N_0 \to N_1$ be a surjection in $\mathrm{Mod}\,A$. Given any hom. $\psi_1 : P \to N_1$, there exists a hom. $\psi_0 : P \to N_0$ s.t.
$$\psi_1 = \varphi \circ \psi_0.$$

$$\begin{tikzcd} & P \arrow[dl, dashed, "\psi_0"'] \arrow[dr, "\psi_1"] & \\ N_0 \arrow[rr, two heads, "\varphi"'] & & N_1 \end{tikzcd}$$

(iv) P is a direct summand of a free A-module.

Proof.

(i) $\Leftrightarrow$ (ii): We know that $\mathrm{Hom}_A(P,-)$ is always left exact (Prop. p 61). So $\mathrm{Hom}_A(P,-)$ is exact iff it respects surjections

(ii) $\Leftrightarrow$ (iii): Consider the surjection $\varphi : N_0 \to N_1$ and the hom.
$$\mathrm{Hom}(1\!\!1_P, \varphi) : \mathrm{Hom}_A(P, N_0) \longrightarrow \mathrm{Hom}_A(P, N_1)$$
It sends:
$$\psi_0 \longmapsto \varphi \circ \psi_0$$
So condition (iii) is precisely the surjectivity of $\mathrm{Hom}(1\!\!1_P, \varphi)$.

(iii) $\Rightarrow$ (iv): Choose a surjection $\varphi : F \to P$ from a free module F.

cont

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By cond. (iii) there is $\sigma : P \to F$ s.t. $\varphi \circ \sigma = \mathbb{1}_P$.
Consider the exact sequence

$$(*) \quad 0 \longrightarrow P' \to F \xrightarrow{\varphi} P \to 0$$

where $P' := \mathrm{Ker}(\varphi)$. The hom. σ splits $(*)$, so by Lemma (p. 58) we get $F \cong P \oplus P'$.

<u>(iv) $\Rightarrow$ (iii):</u> We are given the solid part of the diagram below:

i.e.
$\psi_1 : P \to N_1$
$\varphi : N_0 \to N_1$ surj.

$$\begin{tikzcd}
 & P \arrow[r, dashed, "e"] \arrow[ddl, dashed, "\psi_0"'] & P \oplus P' = F \ni m_x \arrow[d, dashed, "p"] \arrow[ddll, dashed, "\tilde{\psi}_0"] \\
 & & P \ni p(m_x) \arrow[d, "\psi_1"] \\
n_x \in N_0 \arrow[rr, "\varphi"] & & N_1 \ni (\psi_0 \circ p)(m_x)
\end{tikzcd}$$

Let P' be a module s.t. $F := P \oplus P'$ is free, with basis $\{m_x\}_{x \in X}$. For every $x \in X$ choose $n_x \in N_0$ s.t.

$$\varphi(n_x) = (\psi_1 \circ p)(m_x).$$

This can be done since φ is surjective. By the univ. prop. of a basis, there is $\tilde{\psi}_0 : F \to N_0$ s.t. $\tilde{\psi}_1(m_x) = n_x$.
So
$$\varphi \circ \tilde{\psi}_0 = \psi_1 \circ p : F \to N_1.$$
Let
$$\psi_0 := \tilde{\psi}_0 \circ e : P \to N_0.$$

$\searrow$ (cont)

(cont)

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$$\varphi \circ \psi_0 = \varphi \circ \tilde{\varphi}_0 \circ e = \psi_1 \circ p \circ e = \psi_1 \circ \mathbb{1}_P = \psi_1 . \quad \square$$

<u>Corollary</u>. If P is <u>projective</u> then it is <u>flat</u>.

<u>pf</u>. The A-module A is flat since $A \otimes_A M \cong M$ canonically ($1 \otimes m \mapsto m$).

Take F free, with basis $\{m_x\}_{x \in X}$. Then $F \cong \bigoplus_{x \in X} A$. By Cor. p. 61, imp. (i) $\Rightarrow$ (ii), get that F is flat.

Finally choose P' s.t. $F := P \oplus P'$ is free. By the implic. (ii) $\Rightarrow$ (i) of Cor. (p. 61) conclude that P is flat. $\square$

<u>Exercise</u>. Prove that $\mathbb{Q}$ is <u>not</u> a projective $\mathbb{Z}$-module. (We will see soon that $\mathbb{Q}$ is flat over A.)

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